Systems of Equations

When we look at an equation like $x + y = 5$, we see that because there are two **variables**, there are many possible solutions. For instance,

- if $x = 5$, then $y = 0$
- if $x = 2$, then $y = 3$
- if $x = -4$, then $y = 9$
- if $x = 2.5$, then $y = 2.5$, etc.

Another equation such as $x - y = 1$ allows a specific solution to be determined. Taken together, these two equations help to limit the possible solutions.

When taken together, we call this a **system of equations**. A *system of equations* is a group of two or more equations that are related to the same situation and share the same variables. Look at the equations below.

$$x + y = 5$$
$$x - y = 1$$

One possible way to solve the system of equations above is to **graph each equation** on the same set of **axes**. Use a **table** *of values* like those on the following page to help determine two possible **points** for each **line ($\longleftrightarrow$)**.

Table of Values

$x + y = 5$	
x	y
0	5
5	0

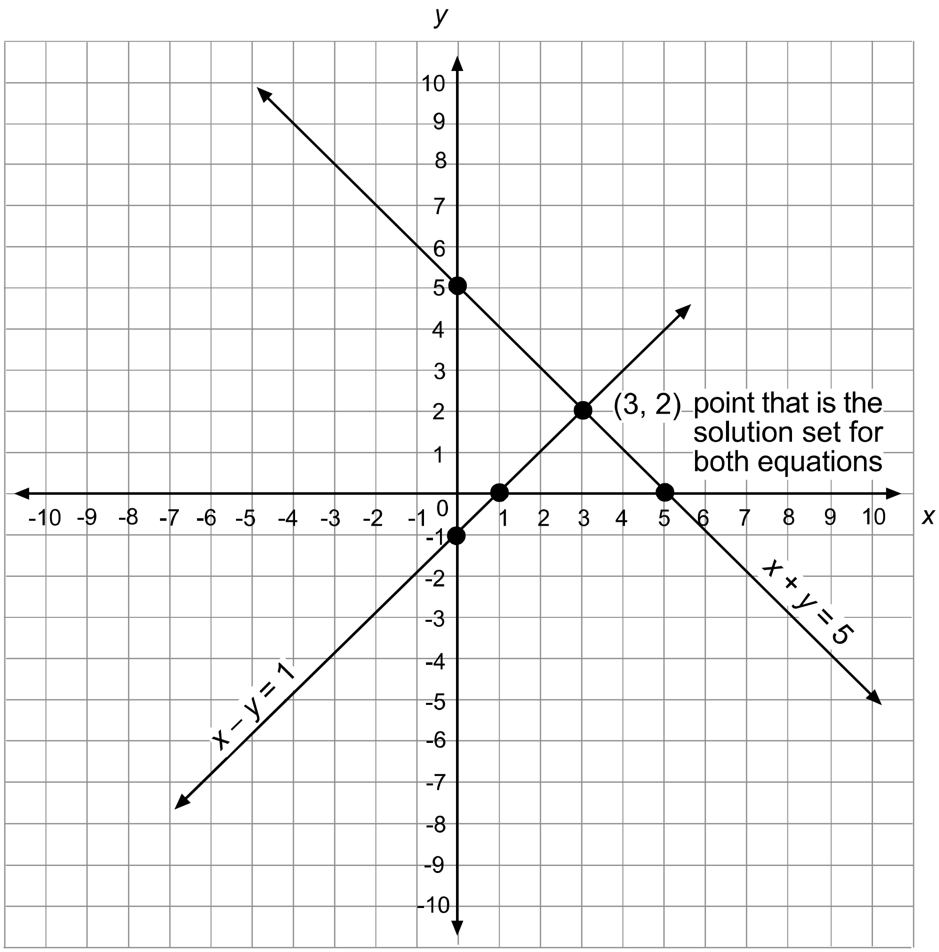
Table of Values

$x - y = 1$	
x	y
0	-1
1	0

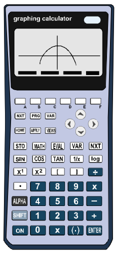
Notice that the values in the table represent the ***x*-intercepts** and ***y*-intercepts**.

Plot the *points* for the first equation on the **coordinate grid** or **plane** below, then draw a *line* connecting them. Do the same for the second set of points.

Graph of $x + y = 5$ and $x - y = 1$



We see from the **graph** above that the two lines **intersect** or *cross at a point*. That point (3, 2) is the solution set for both equations. It is the only point that makes both equations true. You can check your work by replacing x with 3 and y with 2 in both equations to see if they produce true statements.



You can also produce this graph on your *graphing calculator*. To closely estimate the **coordinates** of the points of the graph, move the cursor, the blinking dot, along one line until it gets to the point of **intersection**.

Although graphing is one way to deal with systems of equations; however, it is *not* always the most accurate method. If our graph paper is *not* perfect, our pencil is *not* super-sharp, or the point of *intersection* is *not* at a corner on the grid, we may *not* get the correct answer.

The system can also be solved algebraically with more accuracy. Let's see how that works.

We know from past experience that we can solve problems more easily when there is only one *variable*. So, our job is to eliminate a variable. If we look at the two equations **vertically** (straight up and down), we see that, by adding in columns, the y's will disappear.

$$\begin{array}{rcl} x + y & = & 5 \\ x - y & = & 1 \\ \hline 2x + 0 & = & 6 \end{array}$$

This leaves us with a new equation to solve:

$$\begin{array}{rcl} 2x + 0 & = & 6 \\ 2x & = & 6 \\ \frac{2x}{2} & = & \frac{6}{2} \\ x & = & 3 \end{array}$$

← divide both sides by 2

We've found the value for x; now we must find the value of y. Use either of the original equations and replace the x with 3. The example below uses the first one.

$$\begin{array}{rcl} x + y & = & 5 \\ 3 + y & = & 5 \\ 3 - 3 + y & = & 5 - 3 \\ y & = & 2 \end{array}$$

← subtract 3 from both sides

So, our solution set is {3, 2}.

Let’s try another! We’ll solve and then graph this time.

$$\begin{aligned} 2x + y &= 6 \\ \underline{-2x + 2y} &\underline{= -12} && \leftarrow \text{add to eliminate the } x\text{'s} \\ 0 + 3y &= -6 && \leftarrow \text{solve} \\ \frac{\cancel{3}y}{\cancel{3}} &= \frac{-\cancel{6}}{\cancel{3}} = \frac{-2}{1} \\ y &= -2 \end{aligned}$$

$$\begin{aligned} 2x + -2 &= 6 && \leftarrow \text{replace } y \text{ with -2 in one equation and solve for } x \\ 2x + -2 + 2 &= 6 + 2 \\ 2x &= 8 \\ \frac{\cancel{2}x}{\cancel{2}} &= \frac{\cancel{8}}{\cancel{2}} = \frac{4}{1} \\ x &= 4 \end{aligned}$$

Our solution set is {4, -2}.

Now let’s see how graphing the two equations is done on the following page.

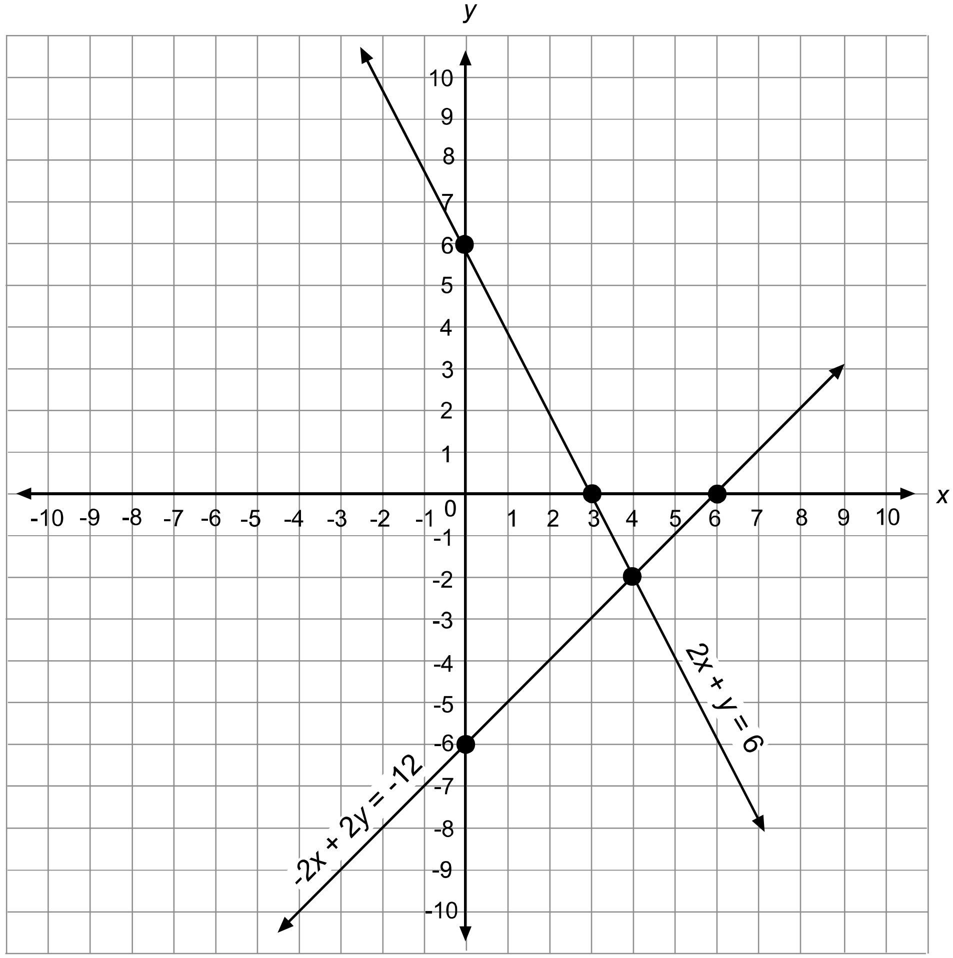
Table of Values

$2x + y = 6$	
x	***y***
0	6
3	0

Table of Values

$-2x + 2y = -12$	
x	***y***
0	-6
6	0

Graph of $2x + y = 6$ and $-2x + 2y = -12$



Note: Watch for these special situations.

- If the graphs of the equations are the same line, then the two equations are *equivalent* and have an **infinite** (that is, limitless) number of possible solutions.

- If the graphs do not *intersect* at all, they are **parallel (||)**, and are an equal distance at every point. They have *no* possible solutions. The solution set would be empty—{ }.

Using Substitution to Solve Equations

There are other processes we can use to solve systems of equations. Let's take a look at some of the options.

Example 1

Suppose our two equations are as follows.

$$\begin{aligned} 2x + 3y &= 14 \\ x &= 4 \end{aligned}$$

To solve this system, we could use a method called **substitution**. We simply put the value of x from the second equation in for the x in the first equation.

$$\begin{aligned} 2x + 3y &= 14 && \text{substitute 4 for } x \\ 2(4) + 3y &= 14 && \text{simplify} \\ 8 + 3y &= 14 \\ 8 - 8 + 3y &= 14 - 8 && \text{subtract} \\ 3y &= 6 && \text{divide} \\ \frac{3y}{3} &= \frac{6}{3} \\ y &= 2 \end{aligned}$$

The solution set is {4, 2}.

Example 2

This one is a little more complex.

Below are our two equations.

$$4x - y = -2$$
$$x = y + 4$$

We can substitute $(y + 4)$ from the second equation in for x in the first equation.

$4x - y = -2$	substitute $(y + 4)$ for
$4(y + 4) - y = -2$	**distribute**
$4y + 16 - y = -2$	simplify
$3y + 16 = -2$	subtract
$3y = -18$	divide
$y = -6$	

Notice that $(y + 4)$ is in parentheses. This helps us remember to *distribute* when the time comes.

Now we must find the value of x. Use an original equation and substitute -6 for y and then solve for x.

$4x - y = -2$	original equation
$4x - (-6) = -2$	substitute (-6) for y
$4x + 6 = -2$	simplify
$4x = -8$	subtract
$x = -2$	divide

Now try the practice on the following page.

Using Magic to Solve Equations

There are times when neither the algebraic or substitution method seems like a good option. If the equations should look similar to these, we have another option.

Example 1

$$5x + 12y = 41$$
$$9x + 4y = 21$$

We have to perform a little "math-magic" to solve this problem. When looking at these equations, you should see that if the $4y$ were $-12y$ instead, we could add vertically and the y's would disappear from the equation.

So, our job is to make that $4y$ into $-12y$. We could do that by multiplying $4y$ by -3. The only catch is that we must multiply the whole equation by -3 to keep everything balanced.

$9x + 4y = 21$	← original equation
$-3(9x + 4y = 21)$	← multiply equation by -3
$-27x + (-12y) = -63$	← new 2nd equation

Now line up the equations, replacing the second one with the new equation.

$5x + 12y = 41$	← original 1st equation
$-27x + (-12y) = -63$	← new 2nd equation
$-22x + 0 = -22$	← subtract vertically
$-22x = -22$	← simplify
$x = 1$	← divide

Now that we know the value of x, we can replace x with 1 in the original equation and solve for y.

$5x + 12y = 41$	← original 1st equation
$5(1) + 12y = 41$	← substitute (1) for x
$5 + 12y = 41$	← simplify
$12y = 36$	← subtract
$y = 3$	← divide

Our solution set is {1, 3}. Be sure to put the answers in the *correct order* because they are an **ordered pair**, where the first and second value represent a position on a *coordinate grid* or *system*.

Sometimes you may have to perform "math-magic" on both equations to get numbers to "disappear."

Example 2

$$3x - 4y = 2$$
$$2x + 3y = 7$$

After close inspection, we see that this will take double magic. If the **coefficients** of the x's could be made into a $6x$ and a $-6x$, this problem might be solvable. Let's try!

Multiply the first equation by 2 and the second equation by -3.

$$2(3x - 4y = 2) \longrightarrow 6x - 8y = 4 \longleftarrow 1^{st} \text{ equation} \bullet 2$$
$$-3(2x + 3y = 7) \longrightarrow -6x - 9y = -21 \longleftarrow 2^{nd} \text{ equation} \bullet -3$$
$$0 - 17y = -17$$
$$-17y = -17$$
$$y = 1$$

Use $y = 1$ to find the value of x using an original equation.

$$3x - 4y = 2 \longleftarrow \text{original } 1^{st} \text{ equation}$$
$$3x - 4(1) = 2 \longleftarrow \text{substitute (1) for } y$$
$$3x - 4 = 2$$
$$3x = 6$$
$$x = 2$$

The solution set is {2, 1}.

Now it's your turn to practice on the next page.

Solving More Word Problems

Let's see how we might use the methods we've learned to solve word problems.

Example 1

Twice the **sum** of two integers is 20. The larger integer is 1 more than twice the smaller. Find the integers.

Let S = the smaller integer
Let L = the larger integer

Now, write equations to fit the wording in the problem.

$2(S + L) = 20$ and $L = 2S + 1$

$2S + 2L = 20$	← simplify
$2S + 2(2S + 1) = 20$	← substitute (2*S* + 1) for *L*
$2S + 4S + 2 = 20$	← distribute
$6S + 2 = 20$	← simplify
$6S = 18$	← subtract
$S = 3$	← divide

Since the smaller integer is 3, the larger one is 2(3) + 1 or 7. The integers are 3 and 7.

Example 2

Three tennis lessons and three golf lessons cost $60. Nine tennis lessons and six golf lessons cost $147. Find the cost of one tennis lesson and one golf lesson.

Let T = the cost of 1 tennis lesson
Let G = the cost of 1 golf lesson

Use the variables to interpret the sentences and make equations.

$$3T + 3G = 60$$
$$9T + 6G = 147$$

Make the *coefficients* of G match by multiplying the first equation by -2.

$-6T + -6G = -120$	← -2(3T + 3G = 60)
$9T + 6G = 147$	← bring in the 2nd equation
$3T + 0 = 27$	← subtract
$3T = 27$	← divide
$T = 9$	

We know that one tennis lesson costs $9, so let's find the cost of one golf lesson.

$3T + 3G = 60$	← original 1st equation
$3(9) + 3G = 60$	← substitute (9) for T
$27 + 3G = 60$	← simplify
$3G = 33$	← divide
$G = 11$	

So, one tennis lesson costs $9 and one golf lesson costs $11.

Now you try a few items on the next page.