Solving Equations

A mathematical sentence that contains an equal sign (=) is called an **equation**. An *equation* is a mathematical sentence stating that the two **expressions** have the same value. An *expression* is a mathematical phrase, or part of a number sentence that contains numbers, operation signs, and sometimes **variables**.

We also learned the rules to add and subtract and to multiply and divide **positive numbers** and **negative numbers**.

<table>
<tr><th colspan="10">Rules for Adding and Subtracting Positive and Negative Integers</th></tr>
<tr><td>(+)</td><td>+</td><td>(+)</td><td>=</td><td>+</td><td>(+)</td><td>–</td><td>(+)</td><td>=</td><td>positive if first number is greater, otherwise it is negative</td></tr>
<tr><td>(–)</td><td>+</td><td>(–)</td><td>=</td><td>–</td><td>(–)</td><td>–</td><td>(–)</td><td>=</td><td>positive if first number is greater, otherwise it is negative</td></tr>
<tr><td>(+)</td><td>+</td><td>(–)</td><td>=</td><td rowspan="2">} use sign of integer with greater absolute value</td><td>(+)</td><td>–</td><td>(–)</td><td>=</td><td>+</td></tr>
<tr><td>(–)</td><td>+</td><td>(+)</td><td>=</td><td>(–)</td><td>–</td><td>(+)</td><td>=</td><td>–</td></tr>
</table>

<table>
<tr><th colspan="10">Rules for Multiplying and Dividing Positive and Negative Integers</th></tr>
<tr><td>(+)</td><td>•</td><td>(+)</td><td>=</td><td>+</td><td>(+)</td><td>÷</td><td>(+)</td><td>=</td><td>+</td></tr>
<tr><td>(–)</td><td>•</td><td>(–)</td><td>=</td><td>+</td><td>(–)</td><td>÷</td><td>(–)</td><td>=</td><td>+</td></tr>
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</table>

To **solve** the equation is to find the number that we can **substitute** for the *variable* to make the equation true.

Study these examples. Each equation has been *solved* and then checked by substituting the answer for the variable in the original equation. If the answer makes the equation a true sentence, it is called the **solution** of the equation.

Solve:

$$n + 14 = -2$$
$$n + 14 - 14 = -2 - 14$$
$$n = -2 + -14$$
$$n = -16$$

Check:

$$n + 14 = -2$$
$$-16 + 14 = -2$$
$$-2 = -2$$ *It checks!*

Solve:

$$y - (-6) = 2$$
$$y + 6 - 6 = 2 - 6$$
$$y = 2 + -6$$
$$y = -4$$

Check:

$$y - (-6) = 2$$
$$-4 - (-6) = 2$$
$$-4 + 6 = 2$$
$$2 = 2$$ *It checks!*

Solve:

$$-6x = -66$$
$$\frac{-6x}{-6} = \frac{-66}{-6}$$
$$x = 11$$

Check:

$$-6x = -66$$
$$-6(11) = -66$$
$$-66 = -66$$ *It checks!*

Solve:

$$\frac{y}{-10} = 5$$
$$(-10)\frac{y}{-10} = 5(-10)$$
$$y = -50$$

Check:

$$\frac{y}{-10} = 5$$
$$\frac{-50}{-10} = 5$$
$$5 = 5$$ *It checks!*

Interpreting Words and Phrases

Words and phrases can suggest relationships between numbers and mathematical operations. In Unit 1 we learned how words and phrases can be translated into mathematical expressions. Appendix B also contains a list of mathematical symbols and their meanings.

Relationships between numbers can be indicated by words such as **consecutive**, *preceding*, *before*, and *next*. Also, the same mathematical expression can be used to translate many different word expressions.

Below are some of the words and phrases we associate with the four mathematical operations and with powers of a number.

Mathematical Symbols and Words

+	−	x	÷	power
add	subtract	multiply	divide	power
sum	difference	product	quotient	square
plus	minus	times		cube
total	remainder	of		
more than	less than	twice		
increased by	decreased by	doubled		

Solving Two-Step Equations

When solving an equation, you want to get the *variable* by itself on one side of the equal sign. You do this by *undoing* all the operations that were done on the variable. In general, undo the addition or subtraction first. Then undo the multiplication or division.

Study the following examples.

A. Solve:

$$2y + 2 = 30$$

$$2y + 2 - 2 = 30 - 2 \quad \longleftarrow \text{subtract 2 from each side}$$

$$\frac{2y}{2} = \frac{28}{2} \quad \longleftarrow \text{divide each side by 2}$$

$$y = 14$$

Check:

$$2y + 2 = 30$$

$$2(14) + 2 = 30 \quad \longleftarrow \text{replace } y \text{ with 14}$$

$$28 + 2 = 30$$

$$30 = 30 \quad \textit{It checks!}$$

B. Solve:

$$2x - 7 = -29$$

$$2x - 7 + 7 = -29 + 7 \quad \longleftarrow \text{add 7 to each side}$$

$$\frac{2x}{2} = \frac{-22}{2} \quad \longleftarrow \text{divide each side by 2}$$

$$x = -11$$

Check:

$$2x - 7 = -29$$

$$2(-11) - 7 = -29 \quad \longleftarrow \text{replace } x \text{ with -11}$$

$$-22 - 7 = -29$$

$$-29 = -29 \quad \textit{It checks!}$$

C. Solve:

$$\frac{n}{7} + 18 = 20$$

$$\frac{n}{7} + 18 - 18 = 20 - 18 \quad \longleftarrow \text{ subtract 18 from each side}$$

$$(\cancel{7})\frac{n}{\cancel{7}} = 2(7) \quad \longleftarrow \text{ multiply each side by 7}$$

$$n = 14 \quad \longleftarrow \text{ simplify both sides}$$

Check:

$$\frac{n}{7} + 18 = 20$$

$$\frac{14}{7} + 18 = 20 \quad \longleftarrow \text{ replace } n \text{ with 14}$$

$$2 + 18 = 20$$

$$20 = 20 \quad \textit{It checks!}$$

D. Solve:

$$\frac{t}{-2} + 4 = -10$$

$$\frac{t}{-2} + 4 - 4 = -10 - 4 \quad \longleftarrow \text{ subtract 4 from each side}$$

$$\frac{\cancel{(-2)}\, t}{\cancel{-2}} = -14(-2) \quad \longleftarrow \text{ multiply each side by -2}$$

$$t = 28 \quad \longleftarrow \text{ simplify both sides}$$

Check:

$$\frac{t}{-2} + 4 = -10$$

$$\frac{28}{-2} + 4 = -10 \quad \longleftarrow \text{ replace } t \text{ with 28}$$

$$-14 + 4 = -10$$

$$-10 = -10 \quad \textit{It checks!}$$

Special Cases

Reciprocals: Two Numbers Whose Product is 1

Note: $5 \bullet \frac{1}{5} = 1$ and $\frac{5}{5} = 1$

When you multiply 5 by $\frac{1}{5}$ and divide 5 by 5, both equations yield 1.

We see that 5 is the **reciprocal** of $\frac{1}{5}$ and $\frac{1}{5}$ is the *reciprocal* of 5. Every number but zero has a reciprocal. (Division by zero is undefined.) Two numbers are reciprocals if their product is 1.

Below are some examples of numbers and their reciprocals.

Number	Reciprocal
$-\frac{1}{4}$	-4
1	1
$\frac{-2}{3}$	$\frac{-3}{2}$
$\frac{7}{8}$	$\frac{8}{7}$
-2	$-\frac{1}{2}$
$\frac{1}{7}$	7
x	$\frac{1}{x}$

Multiplication Property of Reciprocals

any nonzero number times its reciprocal is 1

$$x \bullet \frac{1}{x} = 1$$

If $x \neq 0$

Remember: When two numbers are reciprocals of each other, they are also called **multiplicative inverses** of each other.

Study the following two examples.

Method 1: Division Method

$$\begin{aligned} 5x - 6 &= 9 \\ 5x - 6 + 6 &= 9 + 6 \\ 5x &= 15 \\ \frac{5x}{5} &= \frac{15}{5} \\ x &= 3 \end{aligned}$$

Method 2: Reciprocal Method

$$\begin{aligned} 5x - 6 &= 9 \\ 5x - 6 + 6 &= 9 + 6 \\ 5x &= 15 \\ \frac{1}{5} \bullet 5x & \quad \frac{1}{5} \bullet 15 \\ x &= 3 \end{aligned}$$

Both methods work well. However, the *reciprocal method* is probably easier in the next two examples, which have fractions.

$$\begin{aligned} -\frac{1}{5}x - 1 &= 9 \\ -\frac{1}{5}x - 1 + 1 &= 9 + 1 \\ -\frac{1}{5}x &= 10 \\ \mathbf{-5} \bullet -\frac{1}{5}x &= \mathbf{-5} \bullet 10 \quad \longleftarrow \text{multiply by reciprocal of } -\tfrac{1}{5} \text{ which is } -5 \\ x &= -50 \end{aligned}$$

Here is another equation with fractions.

$$\begin{aligned} -\frac{3}{4}x + 12 &= 36 \\ -\frac{3}{4}x + 12 - 12 &= 36 - 12 \\ -\frac{3}{4}x &= 24 \\ -\frac{\mathbf{4}}{\mathbf{3}} \bullet -\frac{3}{4}x &= -\frac{\mathbf{4}}{\mathbf{3}} \bullet 24 \quad \longleftarrow \text{multiply by reciprocal of } -\tfrac{3}{4} \text{ which is } -\tfrac{4}{3} \\ 1 \bullet x &= -32 \\ x &= -32 \end{aligned}$$

Multiplying by -1

Here is another equation which sometimes gives people trouble.

$5 - x = -10$

Remember: $5 - x$ is not the same thing as $x - 5$. To solve this equation we need to make the following observation.

Property of Multiplying by -1
-1 times a number equals the opposite of that number
$-1 \bullet x = -x$

This property is also called the **multiplicative property of -1**, which says the *product* of any number and -1 is the opposite or **additive inverse** of the number.

See the following examples.

$-1 \bullet 5 = -5$
$-1 \bullet (-6) = 6$

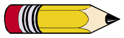
Now let's go back to $5 - x = -10$ using the property of multiplying by -1. We can rewrite the equation as follows.

$$
\begin{aligned}
5 - 1x &= -10 \\
5 - 1x \mathbf{- 5} &= -10 \mathbf{- 5} && \longleftarrow \text{subtract 5 from both sides to isolate the variable} \\
-1x &= -15 \\
\frac{-1x}{-1} &= \frac{-15}{-1} \\
x &= 15
\end{aligned}
$$

This example requires great care with the positive numbers and negative signs.

$$
\begin{aligned}
11 - \tfrac{1}{9}x &= -45 \\
11 - \tfrac{1}{9}x \mathbf{- 11} &= -45 \mathbf{- 11} && \longleftarrow \text{subtract 11 from both sides} \\
-\tfrac{1}{9}x &= -56 && \text{to isolate the variable} \\
\mathbf{-9} \bullet -\tfrac{1}{9}x &= \mathbf{-9} \bullet -56 && \longleftarrow \text{multiply by reciprocal of } -\tfrac{1}{9} \text{ which is -9} \\
x &= 504
\end{aligned}
$$

Consider the following example.



Remember: *Decreased by* means *subtract*, *product* means *multiply*, and *is* translates to the = sign.

Five decreased by the product of 7 and x is -6. Solve for x.

Five decreased by the product of 7 and x is -6.

$$
\begin{aligned}
5 - 7x &= -6 \\
5 - 7x \mathbf{- 5} &= -6 \mathbf{- 5} && \longleftarrow \text{subtract 5 from both sides to isolate the variable} \\
-7x &= -11 \\
\frac{-7x}{-7} &= \frac{-11}{-7} \\
x &= \frac{11}{7} \text{ or } 1\tfrac{4}{7}
\end{aligned}
$$