Unit 4: Making Sense of Rational Expressions

This unit emphasizes performing mathematical operations on rational expressions and using these operations to solve equations and inequalities.

Unit Focus

Reading Process Strand

Standard 6: Vocabulary Development

- LA.910.1.6.1 The student will use new vocabulary that is introduced and taught directly.
- LA.910.1.6.2 The student will listen to, read, and discuss familiar and conceptually challenging text.
- LA.910.1.6.5 The student will relate new vocabulary to familiar words.

Writing Process Strand

Standard 3: Prewriting

- LA.910.3.1.3 The student will prewrite by using organizational strategies and tools (e.g., technology, spreadsheet, outline, chart, table, graph, Venn diagram, web, story map, plot pyramid) to develop a personal organizational style.

Algebra Body of Knowledge

Standard 3: Linear Equations and Inequalities

- MA.912.A.3.2
  Identify and apply the distributive, associative, and commutative properties of real numbers and the properties of equality.

- MA.912.A.3.4
  Solve and graph simple and compound inequalities in one variable and be able to justify each step in a solution.

Standard 4: Polynomials

- MA.912.A.4.1
  Simplify monomials and monomial expressions using the laws of integral exponents.

- MA.912.A.4.2
  Add, subtract, and multiply polynomials.

- MA.912.A.4.3
  Factor polynomial expressions.

- MA.912.A.4.4
  Divide polynomials by monomials and polynomials with various techniques, including synthetic division.

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

canceling .... dividing a numerator and a denominator by a common factor to write a fraction in lowest terms or before multiplying fractions

Example: $\frac{15}{24} = \frac{\overset{1}{\cancel{3}} \cdot 5}{2 \cdot 2 \cdot 2 \cdot \underset{1}{\cancel{3}}} = \frac{5}{8}$

common denominator ...... a common multiple of two or more denominators

Example: A common denominator for $\frac{1}{4}$ and $\frac{5}{6}$ is 12.

common factor .... a number that is a factor of two or more numbers

Example: 2 is a common factor of 6 and 12.

common multiple .... a number that is a multiple of two or more numbers

Example: 18 is a common multiple of 3, 6, and 9.

cross multiplication .... a method for solving and checking proportions; a method for finding a missing numerator or denominator in equivalent fractions or ratios by making the cross products equal

Example: Solve this proportion by doing the following:

$$\frac{n}{9} = \frac{8}{12}$$

$$\frac{n}{9} = \frac{8}{12}$$

$$12 \times n = 9 \times 8$$

$$12n = 72$$

$$n = \frac{72}{12}$$

$$n = 6$$

Solution:

$$\frac{6}{9} = \frac{8}{12}$$

decimal number .... any number written with a decimal point in the number

Example: A decimal number falls between two whole numbers, such as 1.5, which falls between 1 and 2. Decimal numbers smaller than 1 are sometimes called *decimal fractions*, such as five-tenths, or $\frac{5}{10}$, which is written 0.5.

denominator .... the bottom number of a fraction, indicating the number of equal parts a whole was divided into

Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

difference .... a number that is the result of subtraction

Example: In 16 – 9 = 7, the difference is 7.

distributive property ....the product of a number and the sum or difference of two numbers is equal to the sum or difference of the two products *Examples*: $x(a + b) = ax + bx$

$$5(10 + 8) = 5 \cdot 10 + 5 \cdot 8$$

equation ....a mathematical sentence stating that the two expressions have the same value *Example*: $2x = 10$

equivalent (forms of a number) ....the same number expressed in different forms *Example*: $\frac{3}{4}$, 0.75, and 75%

expression..............................a mathematical phrase or part of a number sentence that combines numbers, operation signs, and sometimes variables *Examples*: $4r^2$; $3x + 2y$; $\sqrt{25}$ An expression does *not* contain equal (=) or inequality (<, >, ≤, ≥, or ≠) signs.

factor ....a number or expression that divides evenly into another number; one of the numbers multiplied to get a product *Example*: 1, 2, 4, 5, 10, and 20 are factors of 20 and $(x + 1)$ is one of the factors of $(x_2 - 1)$.

factoring ....expressing a polynomial expression as the product of monomials and polynomials *Example*: $x_2 - 5x + 4 = 0$ $(x - 4)(x - 1) = 0$

fraction ....any part of a whole
Example: One-half written in fractional form is $\frac{1}{2}$.

inequality ....a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression *Examples*: $a \neq 5$ *or* $x < 7$ *or* $2y + 3 \geq 11$

integers ....the numbers in the set
{… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

inverse operation ....an action that undoes a previously applied action
Example: Subtraction is the inverse operation of addition.

irrational number ....a real number that cannot be expressed as a ratio of two integers *Example*: $\sqrt{2}$

least common denominator (LCD) ....the smallest common multiple of the denominators of two or more fractions
Example: For $\frac{3}{4}$ and $\frac{1}{6}$, 12 is the least common denominator.

least common multiple (LCM) ....the smallest of the common multiples of two or more numbers *Example*: For 4 and 6, 12 is the least common multiple.

like terms ....terms that have the same variables and the same corresponding exponents *Example*: In $5x^2 + 3x^2 + 6$, the like terms are $5x^2$ and $3x^2$.

minimum ....the smallest amount or number allowed or possible

multiplicative identity .....the number one (1); the product of a number and the multiplicative identity is the number itself

Example: 5 x 1 = 5

multiplicative property of -1 ....the product of any number and -1 is the opposite or additive inverse of the number *Example*: $-1(a) = -a$ and $a(-1) = -a$

negative numbers ....numbers less than zero

numerator............................the top number of a fraction, indicating the number of equal parts being considered

Example: In the fraction $\frac{2}{3}$, the numerator is 2.

order of operations ....the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and/or division (as read from left to right), then addition and/or subtraction (as read from left to right); also called *algebraic order of operations*

Example: 5 + (12 – 2) ÷ 2 – 3 x 2 = 5 + 10 ÷ 2 – 3 x 2 =

5 + 5 – 6 =

10 – 6 =

4

polynomial ....a monomial or sum of monomials; any rational expression with no variable in the denominator *Examples*: $x^3 + 4x^2 - x + 8$ $5mp^2 - 7x^2y^2 + 2x^2 + 3$

positive numbers ....numbers greater than zero

product ....the result of multiplying numbers together
Example: In 6 x 8 = 48, the product is 48.

quotient ....the result of dividing two numbers
Example: In 42 ÷ 7 = 6, the quotient is 6.

ratio ....the comparison of two quantities
Example The ratio of a and b is $a:b$ or $\frac{a}{b}$, where $b \neq 0$.

rational expression ....a fraction whose numerator and/or denominator are polynomials
Examples: $\frac{x}{8}$ $\frac{5x+2}{x^2+1}$ $\frac{4x^2+1}{}$

rational number ....a number that can be expressed as a ratio $\frac{a}{b}$, where a and b are integers and $b \neq 0$

real numbers ....the set of all rational and irrational numbers

reciprocals ....two numbers whose product is 1; also called *multiplicative inverses*
Examples: 4 and $\frac{1}{4}$ are reciprocals because $\frac{4}{1} \times \frac{1}{4} = 1$; $\frac{3}{4}$ and $\frac{4}{3}$ are reciprocals because $\frac{3}{4} \times \frac{4}{3} = 1$; zero (0) has no multiplicative inverse

simplest form (of a fraction) ....a fraction whose numerator and denominator have no common factor greater than 1

Example: The simplest form of $\frac{3}{6}$ is $\frac{1}{2}$.

simplify an expression .....to perform as many of the indicated operations as possible

solution ....any value for a variable that makes an equation or inequality a true statement *Example*: In $y = 8 + 9$ $y = 17$ 17 is the solution.

substitute ....to replace a variable with a numeral
Example: $8(a) + 3$ $8(5) + 3$

sum ....the result of adding numbers together
Example: In $6 + 8 = 14$, the sum is 14.

term ....a number, variable, product, or quotient in an expression

Example: In the expression $4x_2 + 3x + x$, the terms are $4x_2$, $3x$, and x.

variable ....any symbol, usually a letter, which could represent a number

Unit 4: Making Sense of Rational Numbers

Introduction

Algebra students must be able to add, subtract, multiply, divide, and simplify rational expressions efficiently. These skills become more important as you progress in using mathematics. As an algebra student, you will have the opportunity to work with methods you will need for future mathematical success.

Lesson One Purpose

Reading Process Strand

Standard 6: Vocabulary Development

- LA.910.1.6.1 The student will use new vocabulary that is introduced and taught directly.

- LA.910.1.6.2 The student will listen to, read, and discuss familiar and conceptually challenging text.

- LA.910.1.6.5 The student will relate new vocabulary to familiar words.

Writing Process Strand

Standard 3: Prewriting

- LA.910.3.1.3 The student will prewrite by using organizational strategies and tools (e.g., technology, spreadsheet, outline, chart, table, graph, Venn diagram, web, story map, plot pyramid) to develop a personal organizational style.

Algebra Body of Knowledge

Standard 4: Polynomials

- MA.912.A.4.1 Simplify monomials and monomial expressions using the laws of integral exponents.

- MA.912.A.4.2 Add, subtract, and multiply polynomials.

- MA.912.A.4.3 Factor polynomial expressions.

- MA.912.A.4.4
Divide polynomials by monomials and polynomials with various techniques, including synthetic division.

Simplifying Rational Expressions

An **expression** is a mathematical phrase or part of a number sentence that combines numbers, operation signs, and sometimes **variables**. A **fraction**, or any part of a whole, is an *expression* that represents a **quotient**—the result of dividing two numbers. The same *fraction* may be expressed in many different ways.

1 2 = 2 4 = 3 6 = 10 5

If the **numerator** (top number) and the **denominator** (bottom number) are both **polynomials**, then we call the fraction a **rational expression**. A *rational expression* is a fraction whose *numerator* and/or *denominator* are *polynomials*. The fractions below are all rational expressions.

$$\frac{x}{x+y} \qquad \frac{a^2 - 2a + 1}{a} \qquad \frac{1}{y^2 + 4} \qquad \frac{a}{b - 3}$$

When the *variables* or any symbols which could represent numbers (usually letters) are replaced, the result is a numerator and a denominator that are **real numbers**. In this case, we say the entire *expression* is a *real number*. Real numbers are all **rational numbers** and **irrational numbers**. *Rational numbers* are numbers that can be expressed as a **ratio** *ba* , where a and b are **integers** and $b \neq 0$. *Irrational numbers* are real numbers that *cannot* be expressed as a *ratio* of two *integers*. Of course, there is an exception: when a denominator is equal to 0, we say the fraction is *undefined*.

Note: In this unit, we will agree that *no* denominator equals 0.

Fractions have some interesting properties. Let's examine them.

- If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. $\frac{4}{8} = \frac{6}{12}$ therefore $4 \cdot 12 = 8 \cdot 6$

  $\frac{a}{b} = \frac{c}{d}$

  $a \cdot d = b \cdot c$

  $ad = bc$

  In other words, if two fractions are equal, then the **products** are equal when you **cross multiply**.

  $\frac{4}{8} = \frac{6}{12}$

  $4 \cdot 12 = 8 \cdot 6$

  $48 = 48$

- $\frac{a}{b} = \frac{ac}{bc}$ $\frac{4}{7} = \frac{4 \cdot 3}{7 \cdot 3}$ therefore $\frac{4}{7} = \frac{12}{21}$

  Simply stated, if you *multiply* both the numerator and the denominator by the *same* number, the new fraction will be **equivalent** to the original fraction.

- $\frac{ac}{bc} = \frac{a}{b}$ $\frac{9}{21} = \frac{9 \div 3}{21 \div 3}$ therefore $\frac{9}{21} = \frac{3}{7}$

  In other words, if you *divide* both the numerator and the denominator by the *same* number, the new fraction will be *equivalent* to the original fraction. The same rules are true for **simplifying** rational expressions by performing as many indicated operations as possible. Many times, however, it is necessary to **factor** and find numbers or expressions that divide the numerator or the denominator, or both, so that the **common factors** become easier to see. Look at the following example:

  $$\frac{3x + 3y}{3} = \frac{\cancel{3}^{1}(x + y)}{\cancel{3}_{1}} = x + y$$

  Notice that, by **factoring** a 3 out of the numerator, we can divide (or **cancel**) the 3s, leaving $x + y$ as the final result.

Before we move on, do the practice on the following pages.

Practice

Simplify *each* **expression**. *Refer to* **properties** *and* **examples** *on the previous pages as needed. Show* **essential steps**.

1. $\frac{4x - 4}{x - 1}$
2. $\frac{4m - 2}{2m - 1}$
3. $6x - 3y\ 3$

Example:

$\frac{4x \; 2x6 - = 3 \; 6}{\;} \; \frac{2\cancel{6}(2\;3x\;3 - 3)}{\;} = \; \frac{1(2x - 3)}{\;} = \; \underline{\qquad\quad}$

4. $\frac{5a - 10}{15}$

5. $\frac{2y - 8}{4}$

6. $\frac{3m + 6n}{3}$

7. $\frac{14r^3s^4 + 28rs^2 - 7rs}{7r^2s^2}$

Practice

Simplify *each* **expression**. *Refer to* **properties** *and* **examples** *on the previous pages as needed. Show* **essential steps**.

Example:

$$\frac{2x^2-8}{x+2} = \frac{2(x^2-4)}{x+2} = \frac{2(x-2)\cancel{(x+2)}}{\cancel{x+2}} = 2(x-2)$$

Note: In the above example, notice the following:

- → After we factored 2 from the numerator,
- → we were left with $x^2 - 4$,
- → which can be factored into $(x + 2)(x - 2)$.
- → Then the $(x + 2)$ is cancelled,
- → leaving $2(x - 2)$ as the final answer.

1. $\dfrac{3y^2 - 27}{y - 3}$

2. $\dfrac{a - b}{a^2 - b^2}$

3. $\frac{b - a}{a^2 - b^2}$

4. $\frac{9x + 3}{6x + 2}$

5. $\frac{9x^2 + 3}{6x + 3}$

Additional Factoring

Look carefully at numbers 2-5 in the previous practice. What do you notice about them?

Alert! You cannot cancel individual **terms** (numbers, variables, products, or quotients in an expression)—you can only cancel *factors* (numbers or expressions that exactly divide another number)!

$$\frac{2x + 4}{4} \neq \frac{2x}{4} \qquad \frac{3x + 6}{3} \neq \frac{x+6}{3} \qquad \frac{9x + 3_2\, 6x}{3 \neq} + \frac{9x_2\, x}{6x}$$

Look at how simplifying these expressions was taken a step further. Notice that additional factoring was necessary.

Example

$$\frac{x^{2} + 5x + 6}{x + 3} = \cancel{(x + 3)(x + 2)} = (x + 2) = x + 2\ x + 3$$

Look at the denominator above. It is one of the factors of the numerator. Often, you can use the problem for hints as you begin to factor.

Practice

Factor *each of these and then* **simplify**. *Look for* **hints** *within the problem. Refer to the previous page as necessary. Show* **essential steps**.

1. $\dfrac{a^2 - 3a + 2}{a - 2}$

2. $\dfrac{b^2 - 2b - 3}{b - 3}$

Sometimes, it is necessary to **factor both** *the numerator and denominator. Examine the example below, then* **simplify** *each of the following* **expressions**.

Example:

$$\frac{x^2 - 4}{x^2 + x - 6} = \frac{(x + 2)\cancel{(x - 2)}^{1}}{(x + 3)\cancel{(x - 2)}_{1}} = \frac{x + 2}{x + 3}$$

Note: The x's do **not** cancel.

3. $\dfrac{2r^2 + r - 6}{r^2 + r - 2}$

4. $\dfrac{x^2 + x - 2}{x^2 - 1}$

Practice

Simplify *each* **expression**. *Show* **essential steps**.

1. $\frac{5b - 10}{b - 2}$

2. $\frac{6a - 9}{10a - 15}$

3. $\frac{9x + 3}{9}$

4. $\frac{6b + 9}{12}$

5. $\frac{3a^2b + 6ab - 9b^2}{3b}$

6. $\frac{x^2 - 16}{x + 4}$

7. $\frac{2a - b}{b^2 - 4a^2}$

8. $\frac{6x^2 + 2}{9x^2 + 3}$

Practice

Factor *each of these* **expressions** *and then* **simplify**. *Show* **essential steps**.

1. $$\frac{y^2 + 5y - 14}{y - 2}$$

2. $$\frac{a^2 - 5a + 4}{a - 4}$$

3. $$\frac{6m^2 - m - 1}{2m^2 + 9m - 5}$$

4. $$\frac{4x^2 - 9}{2x^2 + x - 6}$$

Practice

Use the list below to write the correct term for each definition on the line provided.

denominator	**numerator**	**rational**
expression	**polynomial**	**expression real**
fraction	**quotient**	**numbers variable**

_______________ 1. a mathematical phrase or part of a number sentence that combines numbers, operation signs, and sometimes variables

_______________ 2. the top number of a fraction, indicating the number of equal parts being considered

_______________ 3. the bottom number of a fraction, indicating the number of equal parts a whole was divided into

_______________ 4. the set of all rational and irrational numbers

_______________ 5. any part of a whole

_______________ 6. a fraction whose numerator and/or denominator are polynomials

_______________ 7. any symbol, usually a letter, which could represent a number

_______________ 8. a monomial or sum of monomials; any rational expression with no variable in the denominator

_______________ 9. the result of dividing two numbers

Practice

Use the list below to complete the following statements.

canceling	**integers**
cross multiplication	**product**
equivalent	**simplify an expression**
factor	**terms**

1. If you *multiply* both the numerator and the denominator by the *same* number, the new fraction will be __________ because it is the same number expressed in a different form.

2. The numbers in the set {... , -4, -3, -2, -1, 0, 1, 2, 3, 4, ...} are __________ .

3. If you divide a numerator and a denominator by a common factor to write a fraction in lowest terms, or before multiplying fractions, you are __________ .

4. To __________, you need to perform as many of the indicated operations as possible.

5. Numbers, variables, products, or quotients in an expression are called __________ .

6. A ______________________ is a number or expression that divides evenly into another number.

7. When you multiply numbers together, the result is called the ______________________ .

8. To find a missing numerator or denominator in equivalent fractions or ratios, you can use a method called and make the cross products equal. ______________________