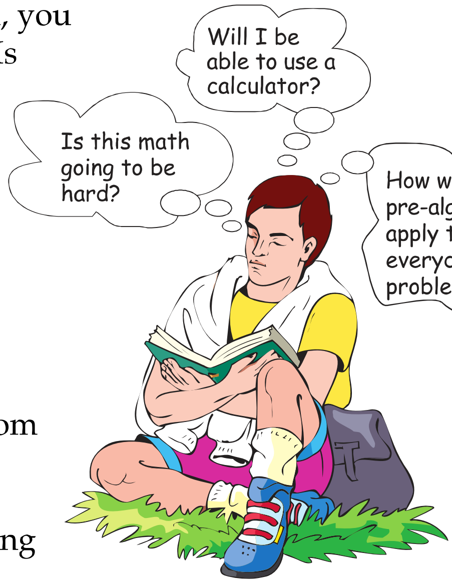
Unit 1: Number Sense, Concepts, and Operations

Introduction

As you get ready to study pre-algebra, you probably are thinking several things. Is this math going to be very different from what I've already had? Is it going to be hard? Will I be able to use a calculator? How will pre-algebra apply to everyday problems?



Pre-algebra is best described as a link between arithmetic and algebra.

- You will make the transition from arithmetic, which is mostly numerical, to problems that require more advanced reasoning skills.
- You will be working with *variables*, or symbols that represent numbers.
- You will be learning how to handle very large and very small numbers in an efficient fashion.
- You will have many opportunities to become proficient with a calculator.

Mastery of these skills is essential in our ever-changing technical world. Proficiency in pre-algebra will help you make the leap from numerical thinking to the more abstract thinking required in algebra and geometry.

Lesson One Purpose

- Associate verbal names, written word names, and standard numerals with integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.1)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)

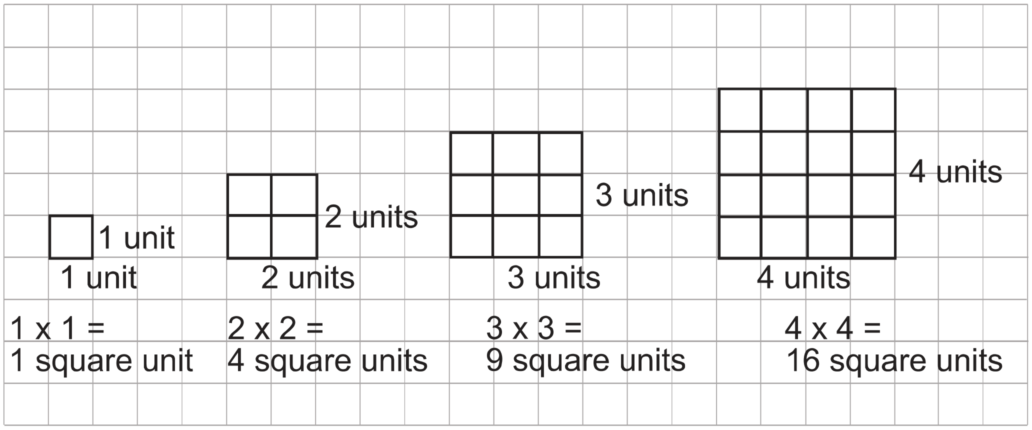
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, exponents, and radicals. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Writing Whole Numbers

There are many ways to write **whole numbers**. A *whole number* is any number in the set {0, 1, 2, 3, 4, …}. For example, to write a whole number, you may sometimes use an **exponent**, or the **power of a number**.

Using Exponents

Study the **squares** below. Each small *square* has four **sides** of the same **unit length (*l*)**.

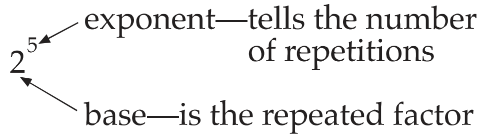


One way to describe the **area (*A*)** of the squares above is by using *exponents*. The *area* is the inside region of a two-dimensional figure. The area is measured in **square units**.

The area of each square above can be described using exponents. An exponent tells how many times the **base** occurs as a **factor**. See below.

1 x 1 is 1 • 1 *or* 1^2
2 x 2 is 2 • 2 *or* 2^2
3 x 3 is 3 • 3 *or* 3^2
4 x 4 is 4 • 4 *or* 4^2

The numbers 1^2, 2^2, 3^2, and 4^2 on the previous page are numerical **expressions** in *exponential form*. Exponential form has two parts—the *base* and the *exponent*. For example, 2^5 is the exponential form of 2 x 2 x 2 x 2 x 2. The numeral two (2) is called the *base*, and the numeral five (5) is called the *exponent*.



The value is 32 because $2^5 = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 = 32$.

Exponential form can also be defined as the *power of a number* written as an exponent. The exponent is the number that tells how many times a base is used as a *factor*. A factor divides evenly into another number.

For example:

32 can be written 2^5, or 2 to the fifth power.

How to Read Powers

These *powers* are read as follows:

6^2 six **squared** *or* six to the second power

10^3 ten **cubed** *or* ten to the third power

7^4 seven to the fourth power

1^{10} one to the tenth power

Remember:
The meaning of 10^3 is $10 \bullet 10 \bullet 10$. Ten is used as a factor 3 times. The value is 1,000.

The meaning of 4^3 is $4 \bullet 4 \bullet 4$. Four is used as a factor 3 times. The value is 64.

Special Numbers—Perfect Squares

Numbers like 1, 4, 9, 16, 25, 36, 49, and 64 are called **perfect squares** because

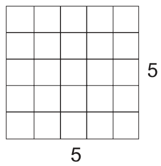
$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
etc.

are the result of squaring a whole number.

The *square of a number* is the result when a number is used as a factor twice.

25 is the square of 5 because

$5^2 = 5 \text{ x } 5 = 25$



Here is another way to think of a *perfect square*. A perfect square is the **product** when a number is multiplied by itself.

25 is the square of 5 and a *perfect square* because it is the *product* of an **integer** {… , -2, -1, 0, 1, 2, …} multiplied by itself: 5 times 5. Here are other examples.

Perfect Squares

Number	1	2	3	4	5	6	7	8	← natural numbers
Square	1	4	9	16	25	36	49	64	← perfect squares

Numbers in the second row are called *perfect squares*. Perfect squares each have a square root that is a whole number.

Powers of 10

If you multiply 10s together, the product is called a *power of 10*. An exponent can be used to show a power of 10. The exponent tells the number of times that 10 is a factor.

10^2 has a value of 100

10^3 has a value of 1,000

10^4 has a value of 10,000

Remember: The **standard form** of a number is a method of writing the common symbol for a numeral. The *standard form* for eight is 8. When the base of a number is 10, the exponent and the number of zeros for the standard form are the same.

This is a nice shortcut. For example:

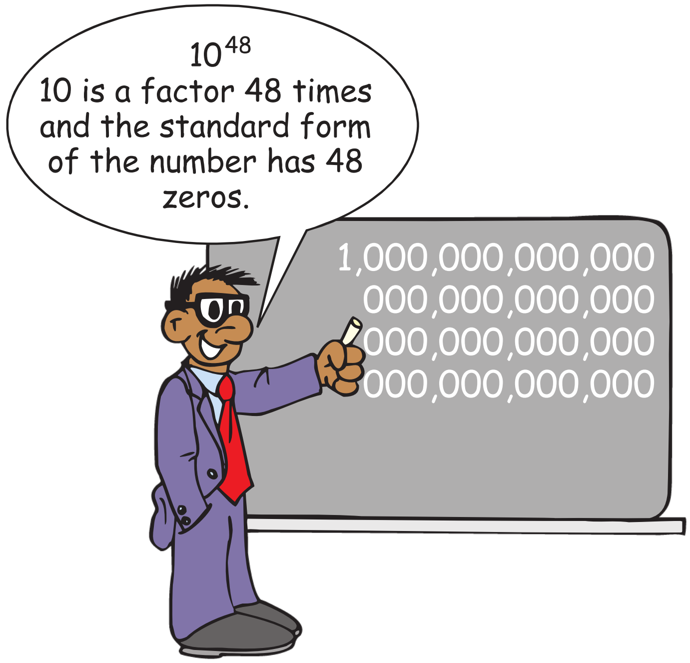
exponent

base → $10^4 = 10 \bullet 10 \bullet 10 \bullet 10 = 10{,}000$

4 factors of 10

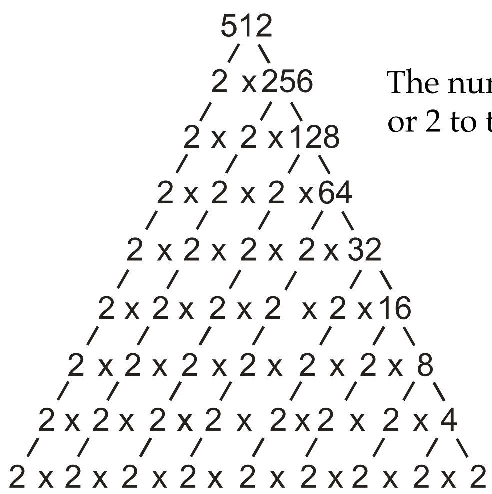
4 zeros to represent the exponent 4

exponential form **factor form** **standard form**



Exponential Form

What if you are asked to write 512 in *exponential form*? You would need to do a *factor tree* to find the answer.



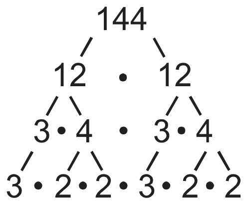
The number 512 can be written as 2^9, or 2 to the ninth power.

This is 2^9.

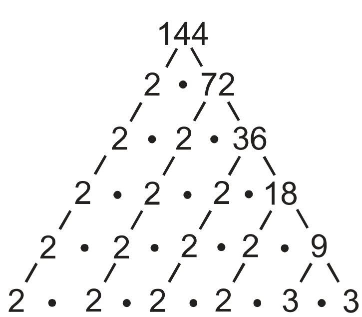
2^9 is the exponential form of 512.

Remember: A **prime number** is any whole number with only two factors, 1 and itself. 2 is a *prime number*. Don't stop until you have *all* prime numbers on the bottom of the factor tree.

Here is a second example done two ways. You can start with any factor of 144.



This is $2^4 \bullet 3^2$.



This is $2^4 \bullet 3^2$.

$2^4 \bullet 3^2$ is the exponential form of 144.

You may also try "upside down" dividing. In this method, begin dividing by the smallest *prime number* that is a factor of the original number. Continue the process until the last **quotient** or the result of the division problem below is 1.

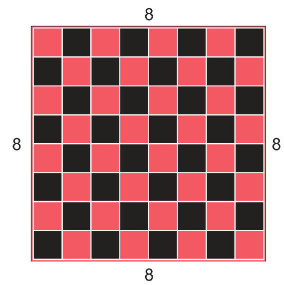
For example:

2)2,000	The smallest prime number factor of 2,000 is 2.
2)1,000	The smallest prime number factor of 1,000 is 2.
2)500	The smallest prime number factor of 500 is 2.
2)250	The smallest prime number factor of 250 is 2.
5)125	The smallest prime number factor of 125 is 5.
5)25	The smallest prime number factor of 25 is 5.
5)5	The smallest prime number factor of 5 is 5.
1	

The **prime factorization** is the writing of a number as the product of prime numbers. The *prime factorization* of 2,000 includes each of these prime divisors: $2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 2^4 \times 5^3$.

$2^4 \bullet 5^3$ is the exponential form of 2,000.

Using Square Roots



A checkerboard is a perfect square containing 64 little squares. Each side of a square has the same length and width. We know that $8^2 = 64$, so each side of the checkerboard is 8 *units*. The opposite of squaring a number is called *finding the* **square root (of a number)**. The *square root* of 64 or $\sqrt{64}$ is 8.

The square root of a number is shown by the symbol $\sqrt{\ }$, which is called a **radical sign** or *square root sign*. The number underneath is called a **radicand**. The **radical** is an expression that has a **root**. A *root* is an equal factor of a number.

radical sign → $\sqrt{100}$ ← radicand

radical

$\sqrt{100} = 10$ because $10^2 = 100$

$\sqrt{9} = 3$ because $3^2 = 9$

$\sqrt{121} = 11$ because $11^2 = 121$

All the numbers used in this lesson so far have been easy. They have all been *perfect squares*.

For example, 1, 4, 9, 16, 36, 49, 64, 81, and 100 are all perfect squares because their square roots are whole numbers. What do we do if the *radicand* is *not* a *perfect square*? We have three options to use to find the square root of a number:

Option 1: We can refer to a *square root chart*. Below is a partial table of squares and square roots. See Appendix A for a more complete table.

$\sqrt{6} \approx 2.449$

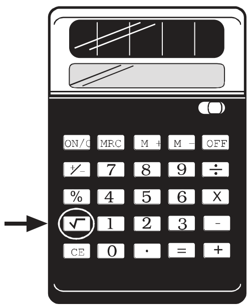
This answer is **rounded** to the nearest thousandth.

Remember: The symbol, ≈, means *is approximately equal to*. The symbol, ≈, is used with a number that describes another number without specifying it exactly.

Table of Squares and Approximate Square Roots

n	n^2	$\sqrt{n}$
1	1	1.000
2	4	1.414
3	9	1.732
4	16	2.000
5	25	2.236
6	36	2.449
7	49	2.646
8	64	2.828
9	81	3.000
10	100	3.162

Option 2: We can use a *calculator*. Look for a key with the $\sqrt{\ }$ symbol. Enter 6, hit this key, and you will get 2.44948974278. This result is a decimal *approximation* of the $\sqrt{6}$. You will have to *round* the number to the nearest thousandth.



$$\sqrt{6} = 2.44\underline{9}48974278$$
$$\sqrt{6} \approx 2.449$$

Option 3: We can **estimate**. We know

$$\sqrt{4} = 2$$
$$\sqrt{6} \approx ?$$
$$\sqrt{9} = 3$$

$\sqrt{6}$ is about half way between $\sqrt{4}$ and $\sqrt{9}$, so a good guess would be 2.5.

Note: Appendix B contains a list of mathematical symbols and their meanings and Appendix C contains formulas and conversions.

Evaluating Algebraic Expressions—Order of Operations

Consider the following.

Evaluate 5 + 4 • 3 =

Is the answer 27 or is the answer 17? You could argue that both answers are valid, although 17 is the universally accepted answer. Mathematicians have agreed on the following **order of operations**.

Rules for Order of Operations

Always start on the *left* and move *to* the *right*.

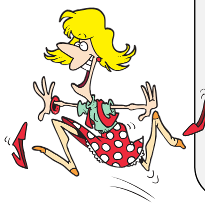
1. Do operations inside *parentheses* first. ( ), [ ], **or** $\frac{x}{y}$
2. Then do all *powers* **or** *roots*. x^2 **or** $\sqrt{x}$
3. Next do *multiplication* **or** *division*, whichever comes first from left to right. x **or** ÷
4. Finally, do *addition* **or** *subtraction*, whichever comes first from left to right. + **or** –

Note: The fraction bar sometimes comes in handy to show grouping.

Example: $\frac{3x^2 + 8}{2} = (3x^2 + 8) \div 2$

The *order of operations* makes sure everyone doing the problem will get the same answer.

Some people remember these rules by using this mnemonic device to help their memory.



Please **P**ardon **M**y **D**ear **A**unt **S**ally*

Please .... **P**arentheses
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My **D**ear .... **M**ultiplication or **D**ivision
Aunt **S**ally .... **A**ddition or **S**ubtraction

* Also known as **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally—**P**arentheses, **E**xponents, **M**ultiplication or **D**ivision, **A**ddition or **S**ubtraction.

Remember: You do multiplication **or** division—*whichever* comes first from *left to right*, and then addition **or** subtraction—*whichever* comes first from *left to right*.

Study the following.

$$25 - 3 \bullet 2 =$$

There are no parentheses. There are no powers or roots. We look for multiplication or division and find multiplication. We multiply. We look for addition or subtraction and find subtraction. We subtract.

$$\begin{aligned} 25 - 3 \bullet 2 &= \\ 25 - 6 &= \\ 19 & \end{aligned}$$

Study the following.

$$12 \div 3 + 6 \div 2 =$$

There are no parentheses. There are no powers or roots. We look for multiplication or division and find division. We divide. We look for addition or subtraction and find addition. We add.

$$\begin{aligned} 12 \div 3 + 6 \div 2 &= \\ 4 + 3 &= \\ 7 & \end{aligned}$$

If the rules were ignored, one might divide 12 by 3 and get 4, then add 4 and 6 to get 10, then divide 10 by 2 to get 5. Agreement is needed—using the agreed-upon *order of operations*.

Study the following.

$$30 - 3^3 =$$

There are no parentheses. We look for powers and roots and find powers, 3^3. We calculate this. We look for multiplication or division and find none. We look for addition or subtraction and find subtraction. We subtract.

$$30 - 3^3 =$$
$$30 - 27 =$$
$$3$$

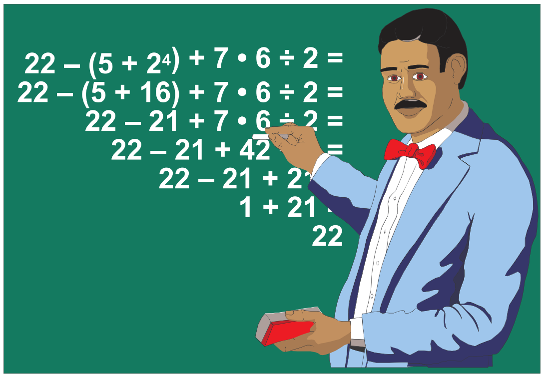
Please	**P**arentheses
Pardon	**P**owers
My	**M**ultiplication *or*
Dear	**D**ivision
Aunt	**A**ddition *or*
Sally	**S**ubtraction

Study the following.

$$22 - (5 + 2^4) + 7 \bullet 6 \div 2 =$$

We look for parentheses and find them. We must do what is inside the parentheses first. We find addition and a power. We do the power first and then the addition. There are no roots. We look for multiplication or division and find both. We do them in the order they occur, left to right, so the multiplication occurs first. We look for addition or subtraction and find both. We do them in the order they occur, left to right, so the subtraction occurs first.

$$22 - (5 + 2^4) + 7 \bullet 6 \div 2 =$$
$$22 - (5 + 16) + 7 \bullet 6 \div 2 =$$
$$22 - 21 + 7 \bullet 6 \div 2 =$$
$$22 - 21 + 42 \div 2 =$$
$$22 - 21 + 21 =$$
$$1 + 21 =$$
$$22$$



Lesson Two Purpose

- Associate verbal names, written word names, and standard numerals with integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.1)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Variables and Expressions

Suppose you are n years old today. In 4 years, your age can be described by the expression $n + 4$. Two years ago, your age would have been $n - 2$.

The letter n is a **variable**. A *variable* is any symbol that could represent a number. In this example, the variable represents your current age. Note that *any* letter of the alphabet or symbol can be used as a variable. A combination of operations, variables, and numbers is called a mathematical expression, algebraic expression, or simply an *expression*.

Here are sample phrases used to write mathematical expressions.

	Word Expression	Mathematical Expression
Addition:	5 **increased** by a number *n*	$5 + n$
	a number *y* plus 2	$y + 2$
	a number *t* increased by 4	$t + 4$
	the **sum** of a number *b* and 5	$b + 5$
	10 more than a number *m*	$m + 10$
Subtraction:	a number *x* minus 2	$x - 2$
	a number *n* less 3	$n - 3$
	5 less than a number *t*	$t - 5$
	a number *t* less than 5	$5 - t$
	a number *c* **decreased** by 2	$c - 2$
	the **difference** of a number *x* and 5	$x - 5$

	Word Expression	Mathematical Expression
Multiplication:	4 times a number y (form used most often is $4y$)	$4 \times y$, $4(y)$, $4 \bullet y$, or $4y$
	the product of 3 and a number n	$3n$
	6 multiplied by a number t	$6t$
	twice a number p	$2p$
	$\frac{1}{2}$ a number y	$\frac{1}{2}y$
Division:	a number y divided by 2	$\frac{y}{2}$
	the *quotient* of t and 4	$\frac{t}{4}$
	a number c divided by 3	$\frac{c}{3}$
	3 divided by a number c	$\frac{3}{c}$
Power:	the square of x	x^2
	the cube of a	a^3
	the fourth power of x	x^4

Remember: $5n = 5 \times n$, $5(n)$, $5 \bullet n$

$\frac{x}{3} = x \div 3$

Evaluating Expressions

Here is how to evaluate mathematical expressions.

Suppose you are 16, and we let your age be represented by the variable *a*. The variable *a* now has a given **value** of 16. Calculate your age as follows:

a. in 4 more years $a + 4 = \mathbf{16} + 4 = 20$

b. divided by 2 $\frac{a}{2} = \frac{\mathbf{16}}{2} = 8$

c. twice your age increased by 2 $2a + 2 = 2(\mathbf{16}) + 2 = 32 + 2 = 34$

d. the product of your age and 3 $3a = 3(\mathbf{16}) = 48$

Suppose you are 16, and we let your age be represented by the variable a.

Lesson Three Purpose

- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, and associative, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)

Solving Equations by Guessing

An **equation** is a mathematical sentence that *equates* one expression to another expression.

For example, you know:

$2 + 2 = 4$
$2 \bullet 3^2 = 18$

Now, consider this *equation*:

$2(x + 3) = 14$

What number could I use in place of the variable *x*, so that the left side is equal to the right side? We can guess. It must be a number that when multiplied by 2 equals 14.

2 x ? = 14

We know that

$2 \bullet 7 = 14$,

and we know that

$(4 + 3) = 7$.

Therefore "4" is a **solution** to this equation.

4 is the *value* of the variable *x*.
$x = 4$

The equation is **solved** by **substituting** or *replacing* x in the original equation with the value of 4.

$$
\begin{aligned}
2(x + 3) &= 14 \\
2(4 + 3) &= 14 \\
2(7) &= 14 \\
14 &= 14
\end{aligned}
$$

The *solution* of 4 makes the equation true.

Finding the value of a variable that makes a mathematical sentence true is called *solving the equation*. The value of the variable is called *the solution of the equation*.

Properties

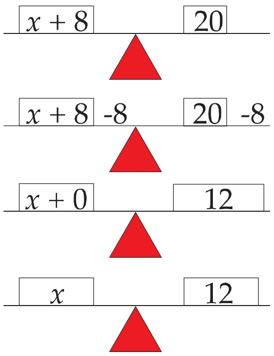
Guessing is an acceptable way to solve simple equations, but we need to develop strategies which will help us solve harder equations. Before we do this, we need to examine some basic *properties* which will help us work with variables. These properties will help us make the leap from simple to more complex equations.

Order (Commutative Property)	
Commutative Property of Addition: Numbers can be added in any order and the sum will be the same. $10 + 2 = 2 + 10$ $x + 2 = 2 + x$	**Commutative Property of Multiplication:** Numbers can be multiplied in any order and the product will be the same. $2 \bullet 10 = 10 \bullet 2$ $2 \bullet x = x \bullet 2$
Grouping (Associative Property)	
Associative Property of Addition: Numbers can be grouped in any order and the sum will be the same. $(5 + 3) + 2 = 5 + (3 + 2)$ $(5 + x) + y = 5 + (x + y)$	**Associative Property of Multiplication:** Numbers can be grouped in any order and the product will be the same. $(5 \bullet 3) \bullet 2 = 5 \bullet (3 \bullet 2)$ $(5 \bullet x) \bullet y = 5 \bullet (x \bullet y)$
Identity Properties	
Additive Identity: The sum of any number and zero is the number. $5 + 0 = 5$ $x + 0 = x$	**Multiplicative Identity:** The product of any number and one is the number. $5 \bullet 1 = 5$ $x \bullet 1 = x$
Inverse Properties	
Additive Inverse: The sum of any number and its additive inverse is 0. $3 + -3 = 0$ 3 and -3 are additive inverses, also called **opposites**.	**Multiplicative Inverse:** The product of any number and its multiplicative inverse (**reciprocal**) is 1. $4 \times \frac{1}{4} = 1$ 4 and $\frac{1}{4}$ are multiplicative inverses, also called *reciprocals*.

Solving One-Step Equations

Solving equations is very similar to keeping a see-saw balanced. For the see-saw to stay balanced, we know that we have to keep each side the same.

Whatever we do to one side of an equation, we have to do to the other.



subtract 8 from both sides

Remember: Our goal is to isolate the variable.

Study the following examples.

Left Right

a.
$$\begin{aligned} x + 8 &= 20 \\ -8 \quad & \quad -8 \\ x + 0 &= 12 \\ x &= 12 \end{aligned}$$

subtract 8 from both sides

Addition and subtraction are **inverse** (or opposite) **operations**. We were able to *undo* the adding by subtracting because 8 and -8 are *additive inverses* or opposites.

b.
$$\begin{aligned} x - 12 &= 13 \\ +12 \quad & \quad +12 \\ x + 0 &= 25 \\ x &= 25 \end{aligned}$$

add 12 to both sides

We were able to undo the subtraction by adding.

c.
$$\begin{aligned} 5x &= 25 \\ \frac{5x}{5} &= \frac{25}{5} \\ \frac{{}^{1}\cancel{5}x}{{}_{1}\cancel{5}} &= \frac{{}^{5}\cancel{25}}{{}_{1}\cancel{5}} \\ 1x &= 5 \\ x &= 5 \end{aligned}$$

divide both sides by 5

Multiplication and division are *inverse operations*. We can undo multiplication by dividing because 5 and $\frac{1}{5}$ are *multiplicative inverses*, or reciprocals.

d.
$$\begin{aligned} \frac{x}{5} &= 10 \\ \frac{5}{1} \bullet \frac{x}{5} &= 5 \bullet 10 \\ \frac{{}^{1}\cancel{5}}{1} \bullet \frac{x}{{}_{1}\cancel{5}} &= 5 \bullet 10 \\ 1 \bullet \frac{x}{1} &= 5(10) \\ 1 \bullet x &= 50 \\ 1x &= 50 \\ x &= 50 \end{aligned}$$

multiply both sides by 5

We can undo division by multiplying.

Remember: Multiplying a number by its *reciprocal* results in a product of 1. Any two numbers with a product of 1 are called *multiplicative inverses*. Here are some examples:

Number	Reciprocal	Product
$\frac{7}{8}$	$\frac{8}{7}$	$\frac{7}{8} \times \frac{8}{7} = 1$
8	$\frac{1}{8}$	$8 \times \frac{1}{8} = 1$
$\frac{1}{3}$	$\frac{3}{1}$	$\frac{1}{3} \times \frac{3}{1} = 1$

Left Right

e. $\boxed{\frac{1}{5}x = 6}$

We can use the reciprocal of a *fraction* to isolate the variable.

$$5 \bullet \frac{1}{5}x = 5 \bullet 6$$

multiply both sides by 5

$$\frac{{}^{1}\cancel{5}}{1} \bullet \frac{1}{{}_{1}\cancel{5}}x = 5 \bullet 6$$

$$\begin{aligned} 1 \bullet 1x &= 30 \\ 1x &= 30 \\ x &= 30 \end{aligned}$$

f. $\boxed{\frac{3}{7}x = 21}$

Use the reciprocal of the fraction to isolate the variable.

$$\frac{7}{3} \bullet \frac{3}{7}x = \frac{7}{3} \bullet 21$$

multiply both sides by $\frac{7}{3}$

$$\frac{{}^{1}\cancel{7}}{{}_{1}\cancel{3}} \bullet \frac{{}^{1}\cancel{3}}{{}_{1}\cancel{7}}x = \frac{7}{{}_{1}\cancel{3}} \bullet \frac{{}^{7}\cancel{21}}{1}$$

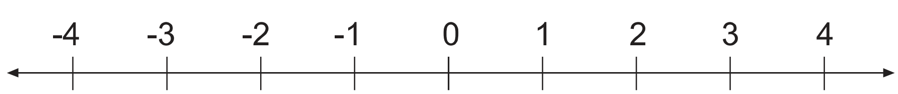
$$\begin{aligned} 1 \bullet 1x &= 49 \\ 1x &= 49 \\ x &= 49 \end{aligned}$$

Lesson Four Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, exponents, and absolute value. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Negative Numbers

Numbers were invented by people. The **positive numbers** 1, 2, 3, 4, 5, … were probably invented first, and were used for counting. Eventually, there was a need for numbers like 4.5 and $\frac{3}{4}$. *Positive numbers* are numbers greater than zero. **Negative numbers** were invented to represent things like temperatures below freezing, overdrawn bank balances, owing money, loss, or going backwards. *Negative numbers* are numbers less than zero. We frequently represent numbers on a **number line**.

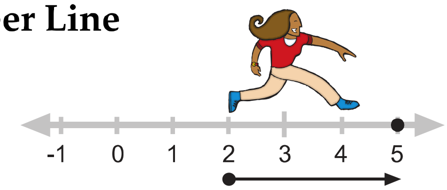


Numbers to the *left* of zero (also called the **origin** or the beginning) are *negative*, numbers to the *right* are *positive*. The set of numbers used on the number line above is called the *set of integers*.

This *set* can be written {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}.

The bigger the number, the farther it is to the right. The smaller the number, the farther it is to the left. A number is considered positive if it does *not* have a sign written in front of it.

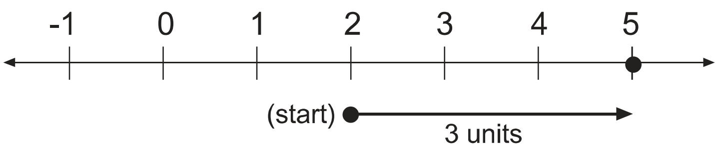
Adding Numbers by Using a Number Line



We are going to get a visual feel for adding integers by using a number line and taking "trips."

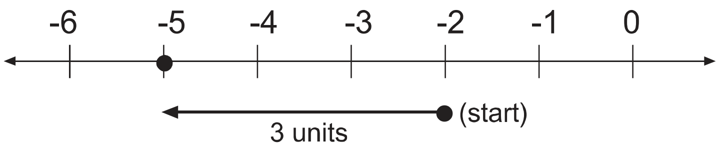
Trip One: Add 2 + 3

1. Start at 2.
2. Move 3 units to the right in the *positive* direction.
3. Finish at 5.
   So, 2 + 3 = 5.



Trip Two: Add -2 + -3

1. Start at -2.
2. Move 3 units to the left in the *negative* direction.
3. Finish at -5.
   So, -2 + -3 = -5.

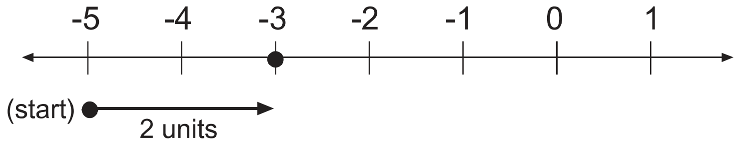


Trip Three: Add -5 + 2

1. Start at -5.

2. Move 2 units to the right in a *positive* direction.

3. Finish at -3.
   So, -5 + 2 = -3.

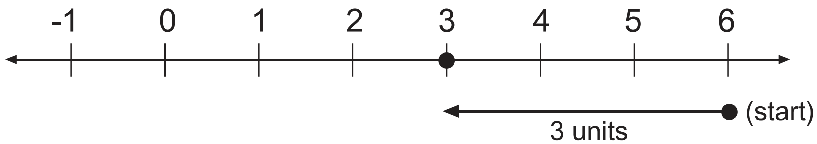


Trip Four: Add 6 + -3

1. Start at 6.

2. Move 3 units to the left in a *negative* direction.

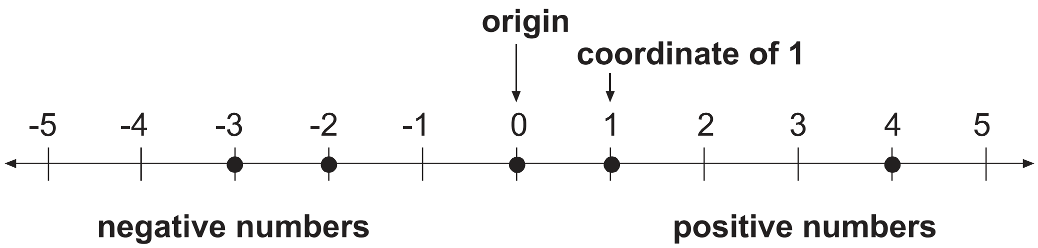
3. Finish at 3.
   So 6 + -3 = 3.



Graph of a Number

To **graph a number** means to draw a dot at the **point** that represents that *integer*. A *point* represents an exact location. The number paired with a point is called the **coordinate** of the point. The graph of zero (0) on a number line is called the *origin*.

Below is a graph of {-3, -2, 0, 1, 4}.



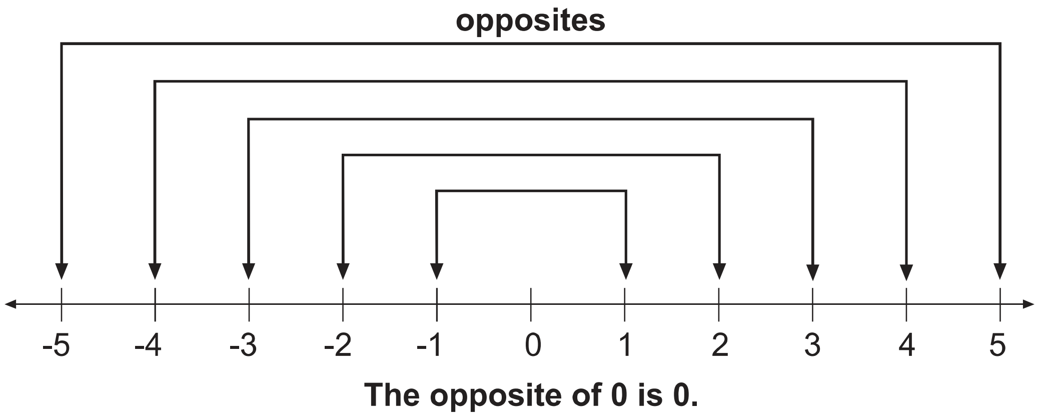
Number line shows coordinates of points -3, -2, 0, 1, and 4.

Opposites and Absolute Value

We can visualize the process of adding by using a number line, but there are faster ways to add. To accomplish this, we must know two things: *opposites* or additive inverses and **absolute value**.

Opposites or Additives Inverses

5 and -5 are called *opposites*. Opposites are two numbers whose points on the number line are the same distance from 0 but in opposite directions.



Every positive integer can be paired with a negative integer. These pairs are called *opposites*. For example, the opposite of 4 is -4 and the opposite of -5 is 5.

The opposite of 4 can be written -(4), so -(4) equals -4.

$$-(4) = -4$$

The opposite of -5 can be written -(-5), so -(-5) equals 5.

$$-(-5) = 5$$

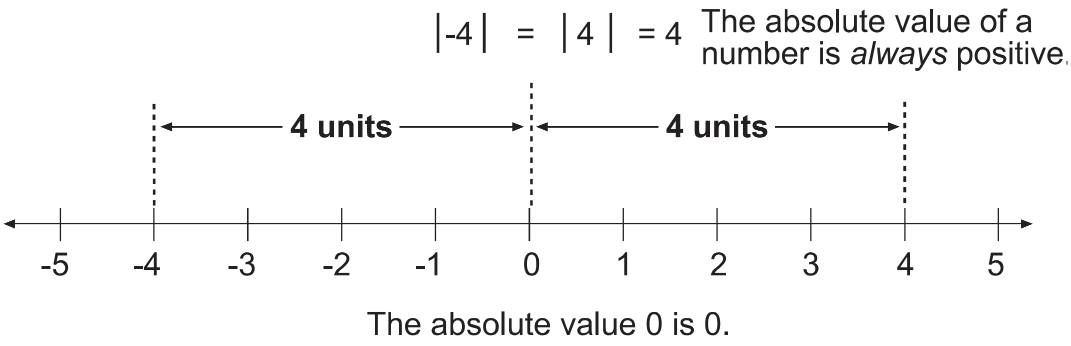
Two numbers are opposites or *additive inverses* of each other if their sum is zero.

For example: $4 + -4 = 0$
$-5 + 5 = 0$

Absolute Value

The *absolute value* of a number is the distance the number is from the *origin* or zero (0) on a number line. The symbol | | placed on either side of a number is used to show absolute value.

Look at the number line below. -4 and 4 are different numbers. However, they are the same distance in number of units from 0. Both have the same *absolute value* of 4. Absolute value is *always* positive because distance is always positive—you cannot go a negative distance.



$|-4|$ denotes the absolute value of -4.

$|-4| = 4$

$|4|$ denotes the absolute value of 4.

$|4| = 4$

The absolute value of 10 is 10. We can use this notation:

$$|10| = 10$$

The absolute value of -10 is also 10. We can use this notation:

$$|-10| = 10$$

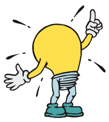
Both 10 and -10 are 10 units away from the origin. Consequently, the absolute value of both numbers is 10.

Now that we have this terminology under our belt, we can introduce two rules for adding numbers which will enable us to add quickly.

Adding Positive and Negative Integers

There are specific rules for adding positive and negative numbers.

1. **If the two integers have the *same sign*, *keep* the *sign* and *add* their absolute values.**



Example: -5 + -7
Think: Both signs are negative.

$|-5| = 5$

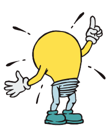
$|-7| = 7$

$5 + 7 = 12$

The sign will be negative because both signs were negative. Therefore, the answer is -12.

-5 + -7 = -12

2. **If the two integers have *opposite signs*, *subtract* the absolute values. The answer has the *sign* of the integer with the *greater* absolute value.**



Example: -8 + 3
Think: Signs are opposite.

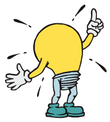
$|-8| = 8$

$|\ 3| = 3$

$8 - 3 = 5$

The sign will be negative because 8 has the greater absolute value. Therefore the answer is -5.

-8 + 3 = -5



Example: -6 + 8
Think: Signs are opposite.

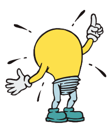
$|-6| = 6$

$|8| = 8$

$8 - 6 = 2$

The sign will be positive because 8 has a greater absolute value. Therefore the answer is 2.

-6 + 8 = 2



Example: 5 + -7
Think: Signs are opposite.

$|5| = 5$

$|-7| = 7$

$7 - 5 = 2$

The sign will be negative because 7 has the greater absolute value. Therefore the answer is -2.

5 + -7 = -2

Rules to Add Integers

- The sum of two positive integers is *positive*.
- The sum of two negative integers is *negative*.
- The sum of a positive integer and a negative integer takes the *sign of the greater absolute value*.
- The sum of a positive integer and a negative integer is zero if numbers have the *same absolute value*.

Subtracting Integers

In the last section, we saw that 8 plus -3 equals 5.

$$8 + (-3) = 5$$

From elementary school we know that 8 minus 3 equals 5.

$$8 - 3 = 5$$

Below are similar examples.

$10 + (-7) = 3$	$12 + (-4) = 8$
$10 - 7 = 3$	$12 - 4 = 8$

These three examples show that there is a connection between adding and subtracting. As a matter of fact we can make any subtraction problem into an addition problem and vice versa.

This idea leads us to the following definition:

Definition of Subtraction
$$a - b = a + (-b)$$

Examples:

$8 - 10 = 8 + (-10) = -2$

$12 - 20 = 12 + (-20) = -8$

$-2 - 3 = -2 + (-3) = -5$

Even if we have $8 - (-8)$, this becomes

8 plus the opposite of -8, which equals 8.

$8 + -(-8) =$

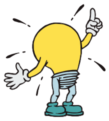
$8 + 8 = 16$

And -9 – (-3), this becomes

-9 plus the opposite of -3, which equals 3.

-9 + -(-3)

-9 + 3 = -6



Shortcut Two negatives become one positive!

10 – (-3) becomes 10 plus 3.

10 + 3 = 13

And -10 – (-3) becomes -10 plus 3.

-10 + 3 = -7

Generalization: Subtracting Integers

Subtracting an integer is the same as adding its opposite.

$a - b = a + (-b)$

Multiplying Integers

What patterns do you notice?

3(4) = 12	3(-4) = -12
2 • 4 = 8	2 • -4 = -8
1(4) = 4	1(-4) = -4
0 • 4 = 0	0 • -4 = 0
-1(4) = -4	-1(-4) = 4
-2 • 4 = -8	-2 • -4 = 8
-3(4) = -12	-3(-4) = 12

Ask yourself:

- What is the sign of the product of two positive integers?
  3(4) = 12   2 • 4 = 8   *positive*
- What is the sign of the product of two negative integers?
  -1(-4) = 4   -2 • -4 = 8   *positive*
- What is the sign of the product of a positive integer and a negative integer or a negative integer and a positive integer?
  3(-4) = -12   -2 • 4 = -8   *negative*
- What is the sign of the product of any integer and 0?
  0 • 4 = 0   0 • -4 = 0   *neither*, zero is neither positive or negative

You can see that the sign of a *product* depends on the signs of the numbers being multiplied. Therefore, you can use the following rules to multiply integers.

Rules to Multiply Integers

- The product of two positive integers is *positive*.
- The product of two negative integers is *positive*.
- The product of two integers with different signs is *negative*.
- The product of any integer and 0 is *0*.

Dividing Integers

Think:

1. What would you multiply $^+6$ by to get $^+42$?

   $^+6 \bullet ? = {}^+42$
   Answer: $^+7$ because $^+6 \bullet {}^+7 = {}^+42$

2. What would you multiply -6 by to get -54?

   $-6 \bullet ? = -54$
   Answer: $^+9$ because $-6 \bullet {}^+9 = -54$

3. What would you multiply -15 by to get 0?

   $-15 \bullet ? = 0$
   Answer: 0 because $-15 \bullet 0 = 0$

Remember: The result of dividing is a *quotient*.

Example:

42 divided by 7 results in a quotient of 6.

$42 \div 7 = 6$ ← quotient

To find the quotient of 12 and 4 we write:

$4\overline{)12}$ *or* $12 \div 4$ *or* $\frac{12}{4}$

Each problem above is read "12 *divided by* 4." In each form, the quotient is 3.

quotient → 3; divisor → 4; dividend → 12: $4\overline{)12}$ *or* $12 \div 4 = 3$ (dividend 12, divisor 4, quotient 3) *or* $\frac{12}{4} = 3$ (dividend 12, divisor 4, quotient 3)

In $\frac{12}{4}$, the bar separating 12 and 4 is called a *fraction bar*. Just as *subtraction* is the *reverse of addition*, *division* is the *reverse of multiplication*. This means that *division* can be *checked by multiplication*.

$$4\overline{)12}^{\,3} \quad \textit{because} \quad 3 \bullet 4 = 12$$

Division of integers is *related to* multiplication of integers. The sign rules for division can be discovered by writing a related multiplication problem.

For example,

same signs → positive quotient (or product)	**different signs → negative quotient (or product)**
$\frac{6}{2}$ = 3 because 3 • 2 = 6	$\frac{-6}{2}$ = -3 because -3 • 2 = -6
$\frac{-6}{-2}$ = 3 because 3 • -2 = -6	$\frac{6}{-2}$ = -3 because -3 • -2 = 6

Below are the rules used to divide integers.

Rules to Divide Integers

- The quotient of two positive integers is *positive*.
- The quotient of two negative integers is *positive*.
- The quotient of two integers with different signs is *negative*.
- The quotient of 0 divided by any nonzero integer is *0*.

Note the special division properties of 0:

$0 \div 9 = 0$	$0 \div -9 = 0$
$\frac{0}{5} = 0$	$\frac{0}{-5} = 0$
$15\overline{)\,0}$ = 0	$-15\overline{)\,0}$ = 0

Remember: Division by 0 is *undefined*. The quotient of any number and 0 is not a number.

We say that $\frac{+9}{0}$, $\frac{+5}{0}$, $\frac{+15}{0}$, $\frac{-9}{0}$, $\frac{-5}{0}$, and $\frac{-15}{0}$ are *undefined*.

Likewise $\frac{0}{0}$ is undefined.

For example, try to divide 134 ÷ 0. To divide, think of the related multiplication problem.

$? \times 0 = 134$

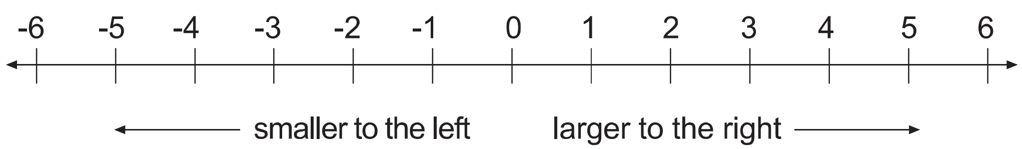
Any number times 0 is 0—so mathematicians say that division by 0 is undefined.

Lesson Five Purpose

- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including appropriate inverse relationships. (MA.A.3.4.1)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

Comparing and Ordering Numbers

Here is how to use the number line to compare two *integers*.



Notice that values *increase* as you move to the *right*.

The number line above shows that

-4 < -1   -1 < 2   0 < 3   (< is less than)

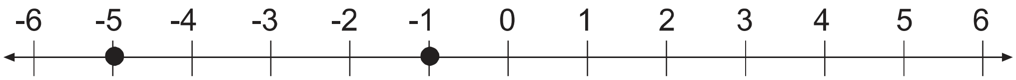
and it shows that

3 > -4   1 > -2   0 > -4   (> is greater than)

Each mathematical sentence above shows the *order* relationship between two quantities. The sentences compare two *unequal* expressions and use < (less than) and > (greater than) symbols.

These sentences state one expression is less than or greater than another expression and are called **inequalities**.

On the number line below, the integers graphed were used to write two *inequalities*.



Since -5 is to the left of -1, we can write -5 < -1.

Since -1 is to the right of -5, we can write -1 > -5.

Remember: The *is greater than* (>) or *is less than* (<) symbols always point to the lesser number.

Solving Inequalities

An inequality is formed when an *inequality symbol* is placed between two expressions. Solving an inequality is similar to solving an equation.

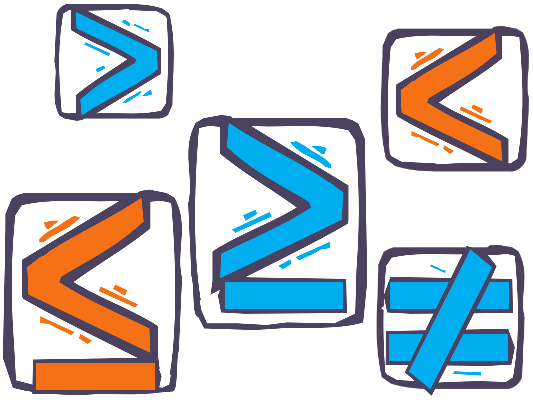
An inequality is any mathematical sentence that compares two unequal expressions using one of the following *inequality symbols*:

Inequality Symbol	Meaning
$>$	is greater than
$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to
$\neq$	is not equal to

In inequalities, numbers or expressions are compared.

These are inequalities.

$2 > 9 - x$	$2 + 2 \geq 4$	$y - 3 < 4$	$8 \times 4 \leq 50$	$6r \neq 8$
↑ is greater than	↑ is greater than or equal to	↑ is less than	↑ is less than or equal to	↑ is not equal to



Examples:

A. $x - 3 < 7$

$x - 3 (+3) < 7 (+3)$ add 3 to both sides

$x < 10$ simplify

B. $y + 4 > 12$

$y + 4 (-4) > 12 (-4)$ subtract 4 from both sides

$y > 8$ simplify

C. $\frac{d}{3} \geq 5$

$(3)\frac{d}{3} \geq 5(3)$ multiply both sides by 3

$(\cancel{3})\frac{d}{\cancel{3}} \geq 5(3)$ 3s cancel on the left side

$d \geq 15$ simplify

D. $2n \leq 14$

$\frac{2n}{2} \leq \frac{14}{2}$ divide both sides by 2

$\frac{\cancel{2}n}{\cancel{2}} \leq \frac{14}{2}$ 2s cancel on left side

$n \leq 7$ simplify

E. $15 > y - 2$ notice that the variable is on the right

$15 (+2) > y - 2 (+2)$ add 2 to both sides

$17 > y$ simplify

This *solution* may also be rewritten with the variable stated first. So instead of $17 > y$, it reads $y < 17$.

Remember: When the variable and the solution switch sides, the inequality symbol also changes to keep the solution the same.

As $y < 17$, the solution states that any number less than 17 is a solution of the problem. In other words, three solutions could be as follows:

-10, or 0, or 12.

However, keep in mind that there are infinitely many more numbers that are solutions.