Unit 2: Measurement

Introduction

We will discover that numbers can be written in several different ways. For instance, $\frac{1}{2}$ can be written several different ways and still have the same value.

$\frac{1}{2} = \frac{4}{8}$ $\qquad \frac{1}{2} = 0.5$ $\qquad \frac{1}{2} = \frac{5}{10}$ $\qquad \frac{1}{2} = 50\%$

It is important to know that the value of a number has not changed. Also, choosing which form is appropriate for a particular problem is important. With the presentation of numerous "real-world" situations, you will learn to choose among various methods of approaching and solving problems.

Lesson One Purpose

- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, and exponents. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Decimals and Fractions

Before considering the relationship between **decimals** and **fractions**, let's briefly review **place value**.

Place Value

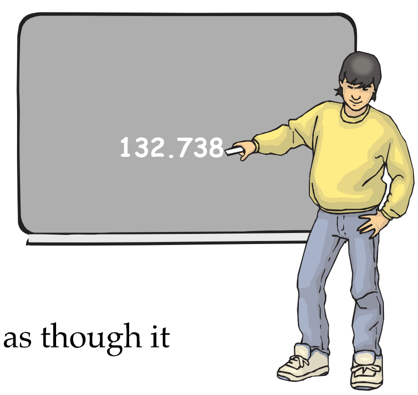
Place value is the position of a single **digit** in a whole number or decimal number. For example, the *decimal* 132.738 has three parts.

132.738	132	.	738
whole number part | decimal part; decimal point	whole number part	decimal point	decimal part

Use the chart below to see the place value of all the *digits* in the decimal 132.738.

1	3	2	.	7	3	8
hundreds	tens	ones	decimal point	tenths	hundredths	thousandths
100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$

Writing or Reading Decimals



To write or read a decimal in words:

- Write or read the **whole number** part in words.
- Write or read *and* for the **decimal point**.
- Write or read the decimal part in words as though it were a *whole number*.
- Write or read the place value of the last digit.

In words, 132.738 is written or said as *one hundred thirty-two* and *seven hundred thirty-eight* thousandths.

As a *fraction*, 132.738 is written as $132\,\frac{738}{1,000}$.

Other examples:

Decimal	Words	Fraction
0.3	three tenths	$\frac{3}{10}$
0.51	fifty-one hundredths	$\frac{51}{100}$
0.07	seven hundredths	$\frac{7}{100}$
0.007	seven thousandths	$\frac{7}{1,000}$

Simplifying Fractions: Writing in Lowest Terms or Simplest Form

Frequently, we write decimals as fractions or as **mixed numbers** consisting of both a whole number and a fraction. Most times we are also asked to **simplify fractions** and show these answers in **simplest form**.

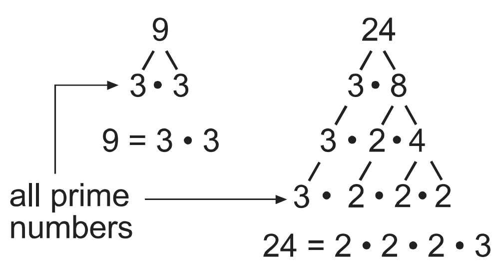
When is a fraction in *simplest form*?

Consider the fraction $\frac{9}{24}$.

Factor the **numerator**, 9, and the **denominator**, 24, of the fraction $\frac{9}{24}$.

Remember: A *factor* is a number that divides evenly into another number.

First find the **prime factorizations** of 9 and 24 to write each number as the product of **prime numbers**. A *prime number* is a whole number that has only two factors, 1 and itself.



numerator → 9, denominator → 24: $\frac{9}{24} = \frac{3 \times \cancel{3}}{2 \times 2 \times 2 \times \cancel{3}} = \frac{3}{8}$ We cancel one 3 in the numerator and one 3 in the denominator and get $\frac{3}{8}$.

Another way to simplify $\frac{9}{24}$ and write in simplest form is to divide the numerator and denominator by their **greatest common factor (GCF)**.

Remember: The *greatest common factor* or *GCF* is the largest of the **common factors** of two or more numbers. *Common factors* are factors of two or more numbers.

$$\frac{9 \div 3}{24 \div 3} = \frac{3}{8}$$

> **To summarize:** A fraction is *reduced* or in *simplest form* when the greatest number that divides both the numerator and denominator is 1. In other words, the fraction's numerator and denominator have no common factor greater than 1.

Another example:

Reduce $\frac{28}{40}$.

- Factor method: $\frac{28}{40} = \frac{\cancel{2} \times \cancel{2} \times 7}{\cancel{2} \times \cancel{2} \times 2 \times 5} = \frac{7}{10}$
- Greatest common factor (GCF) method: $\frac{28 \div 4}{40 \div 4} = \frac{7}{10}$

Converting Decimals to Fractions

Now that we have practiced writing fractions in simplest form, let's start with decimals and convert the decimals to fractions written in simplest form. Either the *factor method* or the *greatest common factor method* can be used to write the fraction in simplest form. Consider the following examples:

$$0.125 = \frac{125}{1{,}000} = \frac{1}{8}$$

$$22.5 = 22\frac{5}{10} = 22\frac{1}{2}$$

$$0.25 = \frac{25}{100} = \frac{1}{4}$$

Converting Fractions to Decimals

To change fractions to decimals we use long division. See below. In the fraction $\frac{3}{8}$, the bar separating the 3 and the 8 is called the *fraction bar*, so $\frac{3}{8}$ is read as 3 *divided by* 8. In $\frac{3}{2}$, it is read as 3 *divided by* 2.

Examples:

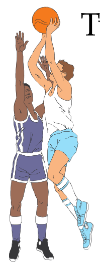
$$\frac{3}{8} = ? \qquad \begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ \underline{24} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array} \qquad\qquad \frac{3}{2} = ? \qquad \begin{array}{r} 1.5 \\ 2\overline{)3.0} \\ \underline{2} \\ 10 \\ \underline{10} \\ 0 \end{array}$$

See the example below. The division does not come out evenly—the remainder is not 0. The **quotient** is a **repeating decimal**. A *repeating decimal* is a decimal in which one digit or a series of digits repeat endlessly. In this example, the answer was **rounded** to the nearest hundredth.

In the example below, $1\frac{1}{4}$ is a *mixed number*. The mixed number was first changed to its **equivalent improper fraction** before converting the fraction into a decimal using long division.

$$\frac{1}{6} = ? \qquad \begin{array}{r} 0.166 = 0.17 \\ 6\overline{)1.000} \\ \underline{6} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array} \qquad\qquad 1\frac{1}{4} = \frac{5}{4} = ? \qquad \begin{array}{r} 1.25 \\ 4\overline{)5.00} \\ \underline{4} \\ 10 \\ \underline{8} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

Comparing Fractions and Decimals



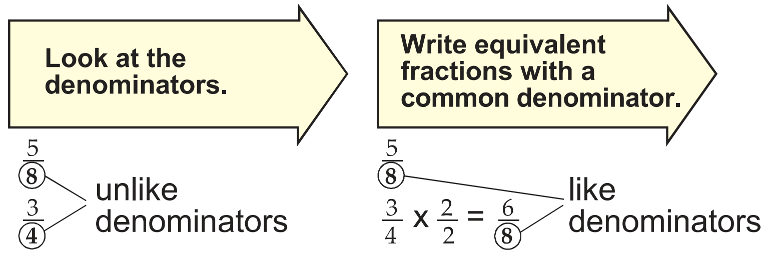
Two boys played basketball over the weekend. George bragged that he sank 5 baskets out of 8. Jamaar boasted that he sank 3 baskets out of 4. Which boy really has "bragging rights"?

This problem can be solved by finding which fraction is larger. Is it $\frac{5}{8}$ or $\frac{3}{4}$? There are a few ways to tell, so let's examine three of them.

Method 1: The Common Denominator Method

The fractions $\frac{5}{8}$ and $\frac{3}{4}$ obviously have different denominators. They need *like* denominators to be able to be easily compared. In this case, 4 is a factor of 8, therefore 8 is a **common denominator** of both fractions. We can change the *appearance* of $\frac{3}{4}$ and keep its value.

Comparing Fractions



$\frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$

Think. What number times 4 equals 8? Answer: 2. To keep the *equivalent* value of the fraction $\frac{3}{4}$ and to change its denominator to 8, multiply it by $\frac{2}{2}$. The fraction $\frac{2}{2}$ is another name for the number 1. If you multiply a number by 1, nothing happens to it—the result is the original number. The number 1 is the *identity element* for multiplication, also called the *property of multiplying by 1*. See properties in Unit 1.

We say that $\frac{6}{8}$ and $\frac{3}{4}$ are equivalent fractions. The fractions have the same value.

Now we have $\frac{5}{8}$ and $\frac{6}{8}$ which can be easily compared.

Compare the numerators.

The fractions compare the same way the numerators compare.

$\frac{⑤}{8}$

$\frac{⑥}{8}$

5 < 6

Since 5 < 6,

then $\frac{5}{8} < \frac{6}{8}$,

therefore $\frac{5}{8} < \frac{3}{4}$.

For fractions with *like* denominators, the *larger fraction* is the one with the *larger numerator*.

Therefore $\frac{5}{8} < \frac{6}{8}$ or $\frac{5}{8} < \frac{3}{4}$ or $\frac{3}{4} > \frac{5}{8}$.

Here is another example using the *common denominator* method:

Which is larger? $\frac{8}{9}$ or $\frac{7}{8}$? The denominators, 9 and 8, are both factors of 72. Let's rewrite both fractions so that they both have a *like* denominator of 72.

$\frac{8}{9} \bullet \frac{8}{8} = \frac{64}{72}$ *and* $\frac{7}{8} \bullet \frac{9}{9} = \frac{63}{72}$

$\frac{8}{8}$ is another name for 1.

$\frac{9}{9}$ is another name for 1.

$\frac{64}{72} > \frac{63}{72}$

therefore

$\frac{8}{9} > \frac{7}{8}$

Tip: You may wish to draw a "big one" around fractions equivalent to one as a reminder of their value.

$$\frac{8}{9} \bullet \frac{8}{8} = \frac{64}{72} = \frac{8}{9}$$

or

$$\frac{7}{8} \bullet \frac{9}{9} = \frac{63}{72} = \frac{7}{8}$$

Method 2: The Change Fractions into Decimals Method

Let's go back to the original problem about George and Jamaar. Another way to compare fractions is to express them as decimals, and then compare the decimals.

Remember: To change a fraction to a decimal, we divide the denominator into the numerator. For example, the fraction $\frac{3}{4}$ is read as *3 divided by 4*, and the fraction $\frac{5}{8}$ is read as *5 divided by 8*.

$$\begin{array}{r} 0.75 \\ 4\overline{)3.0} \\ \underline{2\,8} \\ 20 \end{array} \qquad\qquad \begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{4\,8} \\ 20 \\ \underline{16} \\ 40 \end{array}$$

$$\frac{3}{4} = 0.75 \textit{ and } \frac{5}{8} = 0.625$$

To compare decimals, use place value to compare the digits in the same place. The decimal 0.75 is the same as the decimal 0.750. Just because a decimal is "longer" than another decimal, do *not* assume that it is larger!

$$0.75 = 0.750 \textit{ and}$$
$$0.750 > 0.625 \textit{ because } 7 > 6 \textit{ so } \frac{3}{4} > \frac{5}{8}$$

Comparing Decimal Numbers

Line up the numbers using their decimal points. Start at the left and compare digits in the same place value.

compare place value

0 . 7 5 0
0 . 6 2 5

Find the digits in the first place where they are different, and then compare.

different digit in same place value

0 . ⑦ 5 0
0 . ⑥ 2 5 7 > 6

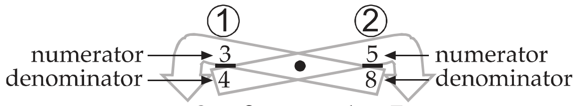
The decimal numbers compare the same way the digits compare.

Since 7 > 6,
then 0.750 > 0.625.

0.750 > 0.625
or
0.625 < 0.750

Method 3: The Cross Products Method

We can compare fractions by using **cross products**. *Cross products* are the **product** of one numerator and the *opposite* denominator in a pair of fractions. If the cross products are equal, the fractions are equivalent.



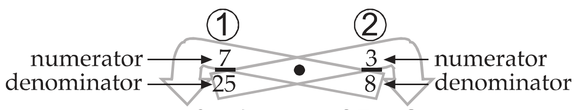
8 x 3   4 x 5

24   20

Since $24 > 20$,

then $\frac{3}{4} > \frac{5}{8}$.

Here is another example using cross products.



8 x 7   25 x 3

56   75

Since $56 < 75$,

then $\frac{7}{25} < \frac{3}{8}$.

Renaming Fractions and Mixed Numbers

Simplifying a Fraction

Previously, we reviewed the process of *reducing* or simplifying a fraction and writing it in *simplest form*. We reduced a fraction to its *lowest terms* so that its numerator and denominator have no common factor greater than 1 and then wrote it in *simplest form*. This extends to simplifying fractions in algebraic *expressions* as well. Reducing fractions is really just looking for ones!

Study these examples:

	Factor method		GCF method
Example:	$\frac{15}{25} = \frac{\cancel{5} \bullet 3}{\cancel{5} \bullet 5} = \frac{3}{5}$	or	$\frac{15}{25} = \frac{15 \div 5}{25 \div 5} = \frac{3}{5}$

Example: $\frac{12x^3y^2}{16x^4y} = \frac{\cancel{4} \bullet 3 \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet y \bullet \cancel{y}}{\cancel{4} \bullet 4 \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet x \bullet \cancel{y}} = \frac{3y}{4x}$

Example: $\frac{-34mn}{68m^2} = \frac{-\cancel{34} \bullet \cancel{m} \bullet n}{\cancel{34} \bullet 2 \bullet \cancel{m} \bullet m} = \frac{-1n}{2m} = \frac{-n}{2m}$

Remember: $-1n = -n$

We *always* reduce fractions and write them in simplest form if we can!

Renaming a Fraction

In the last section we found that we could **change the appearance of fractions by multiplying by one**. The number 1 is the identity element for multiplication. Multiplying a number by 1 does not change a number's value.

Study the following examples:

$$\frac{5}{7} \bullet \frac{2}{2} = \frac{10}{14} \qquad \frac{5}{7} \bullet \frac{3}{3} = \frac{15}{21} \qquad \frac{5}{7} \bullet \frac{4}{4} = \frac{20}{28} \qquad \frac{5}{7} \bullet \frac{x}{x} = \frac{5x}{7x}$$

We know that $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, and $\frac{x}{x}$ are all equal to 1. Multiplying $\frac{5}{7}$ times any of those fractions equal to 1 will not change the value of $\frac{5}{7}$. Therefore, we know that $\frac{5}{7}$, $\frac{10}{14}$, $\frac{15}{21}$, $\frac{20}{28}$, and $\frac{5x}{7x}$ are all equivalent fractions.

Here is another example:

Change $\frac{2x}{9y}$ to an equivalent fraction that has $18y$ in the denominator.

9 x 2 = 18, so the "1" we should use should be $\frac{2}{2}$

$$\frac{2x}{9y} \bullet \frac{2}{2} = \frac{4x}{18y}$$

$\frac{2x}{9y}$ is equivalent to $\frac{4x}{18y}$

Changing Improper Fractions to Mixed Numbers

We say that these fractions are *improper fractions*: $\frac{7}{5}$, $\frac{11}{6}$, $\frac{15}{8}$, $\frac{50}{7}$

Improper fractions are fractions where the numerator is larger than the denominator.

There is nothing wrong with this, but sometimes rewriting these numbers can help us get a *feel* for their size.

Study the following examples:

Example: $\frac{18}{5}$

$$\begin{array}{r} 3 \\ 5\,\overline{)18} \\ \underline{15} \\ 3 \end{array}$$

therefore, $\frac{18}{5} = 3\frac{3}{5}$ — write **remainder** of 3 over **divisor** of 5

3 ← number of fifths left over

Example: $\frac{27}{10}$

$$\begin{array}{r} 2 \\ 10\,\overline{)27} \\ \underline{20} \\ 7 \end{array}$$

therefore, $\frac{27}{10} = 2\,\frac{7}{10}$ — write *remainder* of 7 over *divisor* of 10

7 ← number of tenths left over

Example: $\frac{18}{3}$

$$\begin{array}{r} 6 \\ 3\,\overline{)18} \\ \underline{18} \\ 0 \end{array}$$

therefore, $\frac{18}{3} = 6$

0 ← no number is left over, so there is no remainder

Example: $\frac{38}{8}$

$$\begin{array}{r} 4 \\ 8\,\overline{)38} \\ \underline{32} \\ 6 \end{array}$$

therefore, $\frac{38}{8} = 4\,\frac{6}{8}$

but $\frac{6}{8} = \frac{\cancel{2} \bullet 3}{\cancel{2} \bullet 4} = \frac{3}{4}$

$\frac{38}{8} = 4\frac{3}{4}$

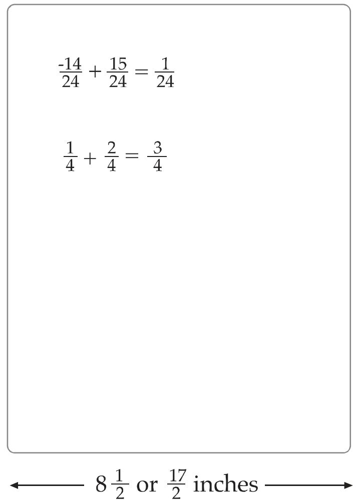
write remainder of 6 over divisor of 8 in simplest form $\frac{6}{8} = \frac{3}{4}$

We must reduce!

6 ← number of eighths left over

Reversing the Process: Writing Mixed Numbers and Whole Numbers as Fractions

Mixed numbers like $8\frac{1}{2}$ are useful. They help us understand the size of a number. Most of us can understand that most letter-sized printer paper is $8\frac{1}{2}$ inches wide. However, we would have trouble visualizing how wide $\frac{17}{2}$ inches would be. Yet both numbers represent the same amount.



In the next few sections we will discover that while mixed numbers have their place in our world, they are difficult to deal with when we have to add, subtract, multiply, or divide them. Here the improper fraction is useful.

Study the example below to review how to change *mixed numbers* to *improper fractions*:

$2\frac{3}{4}$

$2 \leftarrow \times \; \frac{3}{4}$ ($4 \times 2 = 8$) Multiply 2 and 4. (This gives the number of fourths in 2.)

$2 \; + \rightarrow \frac{3}{4}$ ($8 + 3 = 11$) Now add the answer 8 and 3. (This gives the number of fourths in $2\frac{3}{4}$.)

Therefore,

$$2\frac{3}{4} = \frac{11}{4}.$$

Or visualize it like this:

$8 + 3 = 11$

$2\frac{+3}{\times 4} = \frac{11}{4}$ Start with 4 times the 2, plus the 3, equals 11, and put the 11 over the first number.

$4 \times 2 = 8$

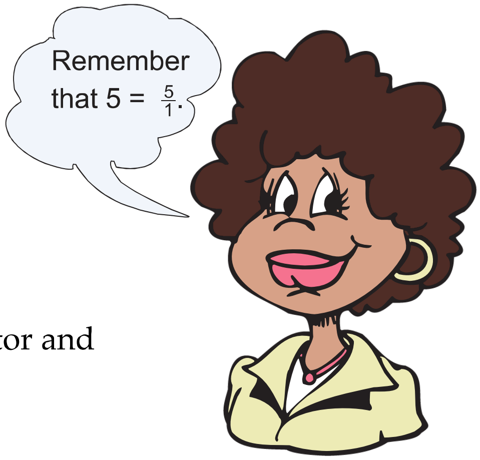
So, $2\frac{3}{4} = \frac{11}{4}$.

Study the example below to review how to change *whole numbers* to *fractions*.

Suppose that we want to change the whole number 5 to fourths.

Remember that $5 = \frac{5}{1}$

Since we do *not* change the value of the fraction, if we multiply the fraction by 1, we can multiply $\frac{5}{1}$ times $\frac{4}{4}$.



$$\frac{5}{1} \bullet \frac{4}{4} = \frac{20}{4}$$

Notice we multiply both numerator and denominator by 4.

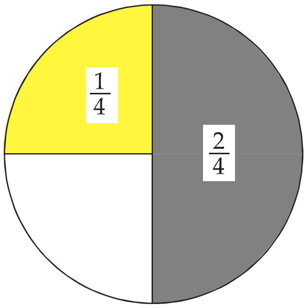
Therefore,

$$5 = \frac{20}{4}.$$

Adding and Subtracting Fractions

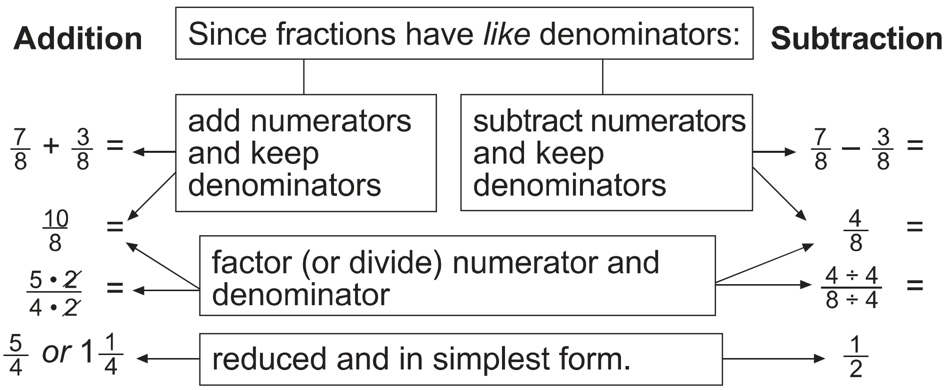
Adding and Subtracting Fractions with Like Denominators

We know that one quarter (one fourth) plus two quarters (two fourths) adds to three quarters (three fourths). Mathematically, this sentence translates to $\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$.



To add (or subtract) fractions with *like* denominators:

- add (or subtract) the numerators
- write the **sum** (or **difference**) over the denominator
- write final answer in simplest form.



Remember: $\frac{5}{4}$ is an *improper fraction* which has been simplified and reduced and is in simplest form. However, it may also be renamed by writing it as $1\frac{1}{4}$, which is called a *mixed number* in simplest form.

The rule for adding (or subtracting) fractions with *like* denominators can be thought of in this way:

> Add (or subtract) the top.
> Keep the bottom.
> *Always* reduce to simplest form.

Study the next two examples and see if you can understand how the rule is used.

How to Add Fractions with Like Denominators

Example:

$\frac{5}{8} + \frac{1}{8} =$	←	denominators are the same, add numerators	
$\frac{6}{8} =$	←		
$\frac{3 \bullet \cancel{2}}{4 \bullet \cancel{2}} =$	←	factor (or divide) numerator and denominator →	$\frac{6 \div 2}{8 \div 2} =$
$\frac{3}{4}$	←	reduced and in simplest form →	$\frac{3}{4}$

Example:

$\frac{7}{14} + \frac{11}{14} =$	←	denominators are the same, add numerators	
$\frac{18}{14} =$	←		
$\frac{9 \bullet \cancel{2}}{7 \bullet \cancel{2}} =$	←	factor (or divide) numerator and denominator →	$\frac{18 \div 2}{14 \div 2} =$
$\frac{9}{7}$ *or* $1\frac{2}{7}$	←	reduced and in simplest form →	$\frac{9}{7}$ *or* $1\frac{2}{7}$

Here are several more examples that use the rule.

How to Subtract Fractions with Like Denominators

Example:

$\frac{10}{7} - \frac{3}{7} =$ ← subtract numerators

$\frac{7}{7}$ *or* 1 ← reduced and in simplest form

How to Add a Negative and a Positive Fraction

Example:

$\frac{-14}{24} + \frac{15}{24} =$ ← add numerators

$\frac{1}{24}$ ← reduced and in simplest form

How to Add Mixed Numbers with Like Denominators

Example:

$1\frac{5}{7} + 3\frac{3}{7} =$

$\frac{12}{7} + \frac{24}{7} =$

$\frac{36}{7}$ *or* $5\frac{1}{7}$

rewrite as an improper fraction

add numerators

reduced and in simplest form

How to Subtract Mixed Numbers with Like Denominators

Example:

$6\frac{1}{4} - 2\frac{3}{4} =$

$\frac{25}{4} - \frac{11}{4} =$

$\frac{14}{4} =$

$\frac{\cancel{2} \bullet 7}{\cancel{2} \bullet 2} =$

$\frac{7}{2}$ *or* $3\frac{1}{2}$

rewrite as an improper fraction

subtract numerators

factor (or divide) numerator and denominator → $\frac{14 \div 2}{4 \div 2} =$

reduced and in simplest form → $\frac{7}{2}$ *or* $3\frac{1}{2}$

Fractions and Equations

Sometimes we have to deal with fractions when we **solve equations**. *Equations* are mathematical sentences that equate one expression to another. Study this example and see if you remember all the steps from Unit 1.

How to Handle Fractions in Solving Equations

Example:

$$x + \frac{1}{5} = \frac{7}{5}$$

$$x + \frac{1}{5} - \frac{1}{5} = \frac{7}{5} - \frac{1}{5} \quad \leftarrow \text{subtract } \frac{1}{5} \text{ from both sides}$$

$$x = \frac{6}{5} \text{ or } 1\frac{1}{5} \quad \leftarrow \text{reduced and in simplest form}$$

Does the **solution** of $\frac{6}{5}$ or $1\frac{1}{5}$ make the equation true? To check, **substitute** $\frac{6}{5}$ for the variable x in the original equation.

$$x + \frac{1}{5} = \frac{7}{5}$$

$$\frac{6}{5} + \frac{1}{5} = \frac{7}{5}$$

$$\frac{7}{5} = \frac{7}{5}$$

or

$$1\frac{1}{5} = 1\frac{1}{5}$$

It checks!

Writing Negative Fractions

Many times in algebra we have to deal with **negative numbers** and *negative* fractions. You need to be aware of the following:

Negative fractions can be written in three ways.

$\frac{5}{-7}$, $\frac{-5}{7}$, and $-\frac{5}{7}$

All of these fractions are equivalent.

Here is an example which contains negative fractions:

How to Add Negative Fractions with Like Denominators

Example:

$\frac{5}{-7} + \frac{-3}{7} =$ ← since denominators are *not* the same, rewrite the fractions so the signs are in the numerators

$\frac{-5}{7} + \frac{-3}{7} =$ ← add numerators

$\frac{-8}{7}$ *or* $-1\frac{1}{7}$ ← reduced and in simplest form

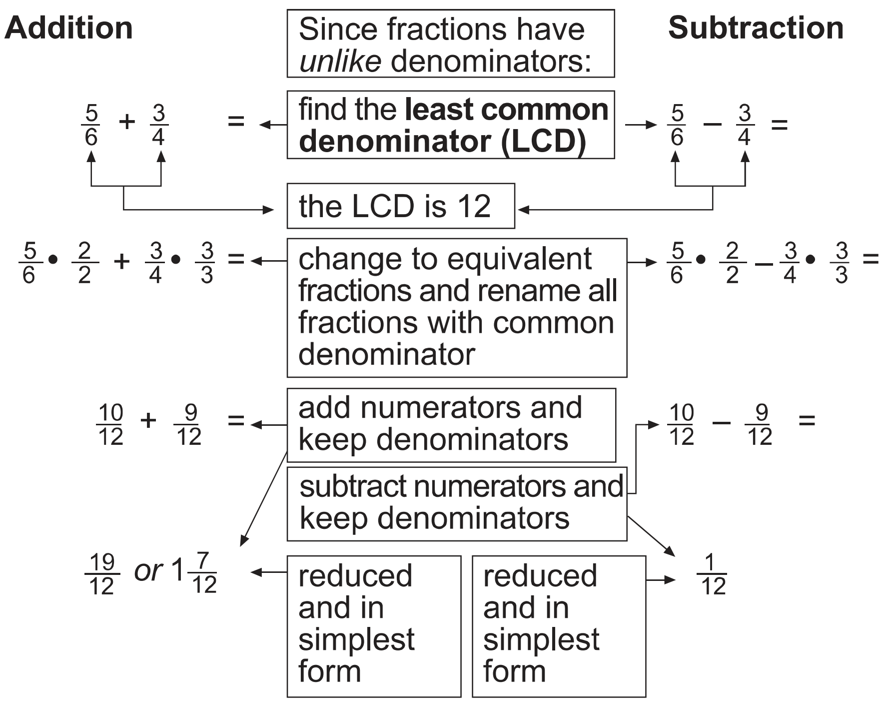
Note: When a negative improper fraction is written as a mixed number, the entire mixed number takes the negative sign.

$$\frac{-8}{7} = -1\frac{1}{7}$$

Adding and Subtracting Fractions with Unlike Denominators

To add (or subtract) fractions with *unlike* denominators:

- rename all fractions so that there is a common denominator
- add (or subtract) the numerators
- write the *sum* (or *difference*) over the denominator
- always reduce to simplest form.



Study the following examples.

How to Add Fractions with Unlike Denominators

Example:

$$\frac{3}{4} + \frac{2}{5} =$$

find LCD

$$\frac{3}{4} \bullet \frac{5}{5} + \frac{2}{5} \bullet \frac{4}{4} =$$

rewrite with common denominators

$$\frac{15}{20} + \frac{8}{20} =$$

add numerators

$$\frac{23}{20} \textit{ or } 1\frac{3}{20}$$

reduced and in simplest form

How to Subtract Mixed Numbers with Unlike Denominators

Example:

$$8\frac{1}{2} - 3\frac{3}{4} =$$

rewrite as an improper fraction

$$\frac{17}{2} - \frac{15}{4} =$$

find LCD

$$\frac{17}{2} \bullet \frac{2}{2} - \frac{15}{4} =$$

rewrite with common denominators

$$\frac{34}{4} - \frac{15}{4} =$$

subtract numerators

or

rewrite to add using **additive inverse** of second fraction

$$\frac{34}{4} + \frac{\mathbf{-15}}{\mathbf{4}} =$$

$$\frac{19}{4} \textit{ or } 4\frac{3}{4}$$

reduced and in simplest form

How to Combine Negative Mixed Numbers

Example:

$-3\frac{3}{7} - 1\frac{1}{3} =$ ← rewrite as improper fractions

$\frac{-24}{7} - \frac{4}{3} =$ ← find LCD

$\frac{-24}{7} \bullet \frac{3}{3} - \frac{4}{3} \bullet \frac{7}{7} =$ ← rewrite with common denominators

$\frac{-72}{21} - \frac{28}{21} =$ ← subtract numerators

or

rewrite to add using *additive inverse* of second fraction → $\frac{-72}{21} + \mathbf{\frac{-28}{21}} =$

$\frac{-100}{21}$ *or* $-4\frac{16}{21}$ ← reduced and in simplest form → $\frac{-100}{21}$ *or* $-4\frac{16}{21}$

How to Subtract Whole Numbers and Fractions

Example:

$4 - \frac{5}{6} =$ ← rewrite the whole number as improper fraction

$\frac{4}{1} - \frac{5}{6} =$ ← find LCD

$\frac{4}{1} \bullet \frac{6}{6} - \frac{5}{6} =$ ← rewrite with common denominators

$\frac{24}{6} - \frac{5}{6} =$ ← subtract numerators

or

rewrite to add using additive inverse of second fraction → $\frac{24}{6} + \mathbf{\frac{-5}{6}} =$

$\frac{19}{6}$ *or* $3\frac{1}{6}$ ← reduced and in simplest form → $\frac{19}{6}$ *or* $3\frac{1}{6}$

On the following page, the last example requires us to use many skills. See if you can follow all the steps in solving the equation. You will have to remember skills from Unit 1.

How to Solve Equations with Mixed Numbers Using Additive Inverse

Remember: To solve an equation, you must get the *variable*—or any symbol that can represent a number—alone on one side of the equals sign. To do this you need to *undo* any operations on the variable.

Example:

$$x + 5\tfrac{1}{6} = 3\tfrac{2}{3}$$ ← undo the addition of $5\tfrac{1}{6}$

$$x + 5\tfrac{1}{6} - 5\tfrac{1}{6} = 3\tfrac{2}{3} - 5\tfrac{1}{6}$$ ← subtract $5\tfrac{1}{6}$ from both sides

$$x = \frac{11}{3} - \frac{31}{6}$$ ← rewrite mixed numbers as improper fractions and find LCD

$$x = \frac{11}{3} \bullet \frac{2}{2} - \frac{31}{6}$$ ← rewrite with common denominators

$$x = \frac{22}{6} - \frac{31}{6}$$ ← subtract numerators

or

rewrite to add using additive inverse of second fraction → $x = \frac{22}{6} + \frac{-31}{6}$ → $x = \frac{-9}{6}$

$$x = \frac{-9}{6}$$

$$x = \frac{\cancel{3} \bullet -3}{\cancel{3} \bullet 2}$$ ← factor (or divide) numerator and denominator to write in lowest terms → $x = \frac{-9 \div 3}{6 \div 3}$ → $x = \frac{-3}{2}$

$$x = \frac{-3}{2}$$

$$x = -1\tfrac{1}{2}$$ ← reduced and in simplest form → $x = -1\tfrac{1}{2}$

To check solution:

$$x + 5\tfrac{1}{6} = 3\tfrac{2}{3}$$

$$-1\tfrac{1}{2} + 5\tfrac{1}{6} = 3\tfrac{2}{3}$$ ← substitute the solution for the variable

$$-\frac{3}{2} + \frac{31}{6} = 3\tfrac{2}{3}$$ ← rewrite mixed numbers as improper fractions and find LCD

$$-\frac{3}{2} \bullet \frac{3}{3} + \frac{31}{6} = 3\tfrac{2}{3}$$ ← rewrite with common denominators

$$-\frac{9}{6} + \frac{31}{6} = 3\tfrac{2}{3}$$ ← add numerators

$$\frac{22}{6} = 3\tfrac{2}{3}$$ ← reduced and in simplest form

$$3\tfrac{4}{6} = 3\tfrac{2}{3}$$

$$3\tfrac{2}{3} = 3\tfrac{2}{3}$$ ← *It checks!*

Multiplying and Dividing Fractions

Multiplying Fractions

To multiply two fractions—(if need be, rewrite mixed numbers as improper fractions)—multiply the numerators and multiply the denominators. Always reduce!

Study the following three examples to review multiplying fractions:

How to Multiply Fractions

Example:

$\frac{4}{9} \bullet \frac{2}{3} =$ ← multiply numerators, multiply denominators

$\frac{8}{27}$ ← reduced and in simplest form

Example:

$\frac{5}{6} \bullet \frac{2}{5} =$ ← multiply numerators, multiply denominators

$\frac{10}{30} =$ ← factor (or divide) numerator and denominator to write in lowest term → $\frac{\overset{1}{\cancel{10 \div 10}}}{\underset{3}{\cancel{30 \div 10}}} =$

$\frac{\cancel{10} \times 1}{\cancel{10} \times 3} =$

$\frac{1}{3}$ ← reduced and in simplest form → $\frac{1}{3}$

How to Multiply Mixed Numbers

Example:

$(2\frac{1}{4}) \bullet (1\frac{1}{3}) =$ ← rewrite as improper fractions

$\frac{9}{4} \bullet \frac{4}{3} =$ ← multiply numerators, multiply denominators

$\frac{36}{12}$ *or* 3 ← reduced and in simplest form

There is a shortcut called **canceling** that can be used when multiplying fractions. *Canceling* is dividing a numerator and a denominator by a *common factor* to write a fraction in lowest terms before multiplying fractions. You may use canceling to divide the numerator and the denominator of the same or different fractions.

Study the following two examples of canceling fractions.

How to Multiply Fractions

Example:

$\frac{3}{4} \bullet \frac{8}{9} =$ ← cancel by dividing the opposite numerator and denominator (by 3) to write in lowest terms

$\frac{\overset{1}{\cancel{3}}}{4} \bullet \frac{8}{\underset{3}{\cancel{9}}} =$ ← cancel by dividing the opposite numerator and denominator (by 4) to write in lowest terms

$\frac{1}{\underset{1}{\cancel{4}}} \bullet \frac{\overset{2}{\cancel{8}}}{3} =$ ← multiply numerators, multiply denominators

$\frac{2}{3}$ ← reduced and in simplest form

$$\frac{\overset{1}{\cancel{3 \div 3}}}{4 \div 4} \bullet \frac{8 \div 4}{\underset{3}{\cancel{9 \div 3}}} = \frac{1}{\underset{1}{\cancel{4 \div 4}}} \bullet \frac{\overset{2}{\cancel{8 \div 4}}}{3} = \frac{2}{3}$$

How to Multiply Negative Mixed Numbers

Example:

$(-3\frac{1}{2})\,(4\frac{2}{3}) =$ ← rewrite mixed numbers as improper fractions

$\frac{-7}{2} \bullet \frac{14}{3} =$ ← cancel by dividing numerator and denominator (by 2) to write in lowest terms

$\frac{-7}{\underset{1}{\cancel{2}}} \bullet \frac{\overset{7}{\cancel{14}}}{3} =$ ← multiply numerators, multiply denominators (negative • **positive** = negative)

$\frac{-49}{3}$ *or* $-16\frac{1}{3}$ ← reduced and in simplest form

$$\frac{-7}{\underset{1}{\cancel{2 \div 2}}} \bullet \frac{\overset{7}{\cancel{14 \div 2}}}{3} = \frac{-49}{3} \text{ or } -16\frac{1}{3}$$

Dividing Fractions

To divide fractions, multiply by the **reciprocal** of the *divisor*. Two numbers are *reciprocals* if their product is one.

Since

$\frac{{}^{1}\cancel{3}}{{}_{1}\cancel{5}} \bullet \frac{\cancel{5}^{1}}{\cancel{3}_{1}} = 1,$

$\frac{3}{5}$ is the reciprocal of $\frac{5}{3}$ and

$\frac{5}{3}$ is the reciprocal of $\frac{3}{5}$.

For a fraction not equal to 0, find its reciprocal by *inverting* or turning the fraction upside down, like $\frac{3}{5}$ and $\frac{5}{3}$. In everyday language you could say the following: *Flip* the *second* fraction and multiply.

Study the following three examples and see if you can see how the rule works.

How to Divide Fractions and Whole Number

Example:

$\frac{9}{10} \div 2 =$	remember, $2 = \frac{2}{1}$
$\frac{9}{10} \div \frac{2}{1} =$	invert the second fraction (reciprocal) and multiply
$\frac{9}{10} \bullet \frac{1}{2} =$	multiply numerator multiply denominator
$\frac{9}{20}$	reduced and in simplest form

How to Divide Fractions

Example:

$\frac{3}{4} \div \frac{2}{3} =$	invert the second fraction (reciprocal) and multiply
$\frac{3}{4} \bullet \frac{3}{2} =$	multiply numerator multiply denominator
$\frac{9}{8}$ *or* $1\frac{1}{8}$	reduced and in simplest form

How to Divide Mixed Numbers and Fractions

Example:

$2\frac{2}{5} \div \frac{8}{15} =$ ← change mixed numbers to improper fraction

$\frac{12}{5} \div \frac{8}{15} =$ ← invert second fraction and multiply

$\frac{12}{5} \bullet \frac{15}{8} =$ ← cancel by dividing the opposite numerator and denominator (by 4) to write in lowest terms → $\frac{{}^{3}\cancel{12 \div 4}}{5 \div 5} \bullet \frac{15 \div 5}{\cancel{8 \div 4}_{2}} =$

$\frac{{}^{3}\cancel{12}}{5} \bullet \frac{15}{\cancel{8}_{2}} =$ ← cancel by dividing the opposite numerator and denominator (by 5) to write in lowest terms → $\frac{3}{{}_{1}\cancel{5 \div 5}} \bullet \frac{\cancel{15 \div 5}^{3}}{2} =$

$\frac{3}{{}_{1}\cancel{5}} \bullet \frac{\cancel{15}^{3}}{2} =$

$\frac{9}{2}$ *or* $4\frac{1}{2}$ ← multiply numerator, multiply denominator; reduced and in simplest form → $\frac{9}{2}$ *or* $4\frac{1}{2}$

How to Solve Equations by Mulitplying Using Reciprocals

Example:

$\frac{3}{4}x = \frac{7}{8}$ ← multiply both sides by the reciprocal of $\frac{3}{4}$

${}^{1}\frac{\cancel{4}}{3} \bullet \frac{3}{\cancel{4}_{1}} x = {}^{1}\frac{\cancel{4}}{3} \bullet \frac{7}{\cancel{8}_{2}}$ ← cancel by dividing the opposite numerator and denominator (by 4) to write in lowest terms → ${}^{1}\frac{\cancel{4 \div 4}}{3} \bullet \frac{3}{\cancel{4 \div 4}_{1}} x = {}^{1}\frac{\cancel{4 \div 4}}{3} \bullet \frac{7}{\cancel{8 \div 4}_{2}}$

${}_{1}\frac{1}{\cancel{3}} \bullet \frac{\cancel{3}^{1}}{1} x = \frac{1}{3} \bullet \frac{7}{2}$ ← cancel by dividing the opposite numerator and denominator (by 3) to write in lowest terms → ${}_{1}\frac{1}{\cancel{3 \div 3}} \bullet \frac{\cancel{3 \div 3}^{1}}{1} x = \frac{1}{3} \bullet \frac{7}{2}$

$x = \frac{7}{6}$ *or* $1\frac{1}{6}$ ← multiply numerator, multiply denominator; reduced and in simplest form → $x = \frac{7}{6}$ *or* $1\frac{1}{6}$

Adding and Subtracting Decimals

The following is important to remember when we add or subtract decimals:

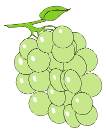
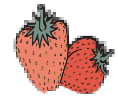
- Write the decimals so that the decimals line up vertically (↕).
- Write whole numbers as decimal numbers by adding a decimal point and zeros.
- Place the decimal point in the sum or difference so that it lines up vertically with the decimal points in the problem.
- Add or subtract as with whole numbers.
- Bring the decimal point straight down into the sum or difference.

Add: 3.29 + 1.34	3.29 + 1.34 4.63	Line up decimals vertically.
Add: 25 + 3.06	25.00 + 3.06 28.06	Write the decimal point and two zeros. Line up decimals vertically.
Subtract: 8.8 – 2.5	8.8 – 2.5 6.3	Line up decimals vertically.
Subtract: 25 – 3.8	25.0 – 3.8 21.2	Write the decimal point and one zero. Line up decimals vertically.

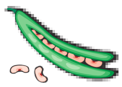
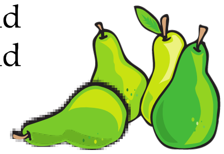
Multiplying and Dividing Decimals

You may not think about it, but decimals are used a lot at a grocery store. For example, at the ABC Produce Market, fruits and vegetables are priced per pound.

Use these typical prices per pound to work through the following examples:

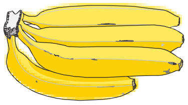
strawberries	$1.25	per pound
bananas	$0.49	per pound
grapes	$0.99	per pound
pears	$1.15	per pound
pole beans	$1.29	per pound

Example:

To find the cost of 4 pounds of bananas, we can *estimate* by rounding $0.49 to $0.50 and think:

4 x $0.50 is $2.00.

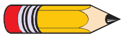


or

We can calculate the *exact cost* by multiplying 4 x $0.49. Then we count the number of digits to the right of the decimal point.

$0.49		2 digits to the right of the decimal point
x 4	+	0 digits to the right of the decimal point
$1.96		2 digits to the right of the decimal point

Then count off the same number of digits in the product.



Remember: Each place to the right of the decimal point is a *decimal place*. The number of decimal places in the product must equal the *sum* of the number of decimal places in the factors.

When multiplying decimals:

1. Multiply as if the factors were whole numbers.

2. Count the number of digits (decimal places) to the right of the decimal point in each factor and *add*.

3. Count off the *sum* of the digits (decimal places) in the factors (from right to left) and insert a decimal point. (Sometimes you will need to add zeros as place holders in the product.)

Example:

Suppose we buy 3.5 pounds of pears. Find the cost by multiplying 3.5 x $1.15.

$1.15	2 digits to the right of the decimal point
x 3.5	+ 1 digit to the right of the decimal point
575	
345	
$4.025	3 digits to the right of the decimal point

$4.025 will have to be *rounded* to the nearest cent. So our cost would be $4.03.

Example:

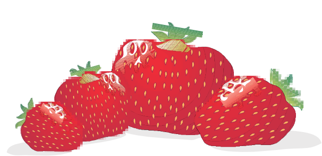
If Katie has $5.00, how many pounds of strawberries can she buy? Since each pound of strawberries is $1.25, we need to determine how many times $1.25 will divide into $5.00.

Remember: When dividing by decimals, you always want to divide by a whole number. So sometimes you must multiply the *dividend* and the *divisor* by a power of 10, moving the decimal point to the right. Then divide as if you were dividing by whole numbers.

$$1.25\overline{)5.00} = 4.$$

500

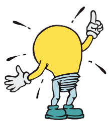
Move both decimal points 2 places to the *right* and then divide.



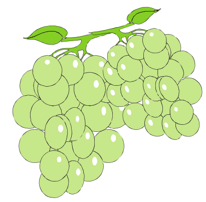
Katie can buy 4 pounds of strawberries.

Notice that some items are priced at $ 0.49, $ 0.99, and $1.29 per pound. We can estimate the number of pounds we can buy with a certain amount of money to the nearest 10 cents.

Think: $0.49 .... $0.50
$0.99 .... $1.00
$1.29 .... $1.30



If we have $3.00, a quick way to determine how many pounds of grapes (at $0.99 a pound) we can buy is to divide $3.00 by $1.00.



```
        3. pounds of grapes
1.00)3.00
      300
```

Lesson Two Purpose

- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, and exponents. (MA.A.1.4.4)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)
- Use concrete and graphic models to derive formulas for finding rate, distance, and time. (MA.B.1.4.2)
- Relate the concepts of measurement to similarity and proportionality in real-world situations. (MA.B.1.4.3)
- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)

Writing Numbers in Scientific Notation

In Unit 1, we learned to use **exponents**. When you multiply 10s together, the product is called a *power of 10*. Powers of 10 are used to represent very large quantities and extremely small quantities. *Exponents* can be used to show a power of 10. The exponent tells the number of times that 10 is a factor.

Look at powers of 10 below and notice that we have extended our study to include negative exponents.

$$10^4 = 10{,}000$$
$$10^3 = 1{,}000$$
$$10^2 = 100$$
$$10^1 = 10$$
$$10^0 = 1$$
$$10^{-1} = \frac{1}{10^1} = \frac{1}{10} = 0.1$$
$$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$$
$$10^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000} = 0.001$$
$$10^{-4} = \frac{1}{10^4} = \frac{1}{10{,}000} = 0.0001$$

Scientific notation is a *shorthand* method of writing very large or very small numbers using exponents. The number in *scientific notation* is expressed as a product of a power of 10 *and* a number that is greater than or equal to 1 and less than 10.

To change a number written in decimal notation to scientific notation, write it as a *product* of two *factors*:

(a decimal greater than or equal to 1 and less than 10) x (a power of 10)

Where a is a decimal ≥ 1 and < 10 — $a \cdot 10^n$ — A power of 10—where n is an integer {…, -2, -1, 0, 1, 2, …}

Example 1: speed of light = 300,000,000 miles per second, written in scientific notation = 3.0×10^8

Example 2: light travels 1 meter in 0.0000000033 seconds, written in scientific notation = 3.3×10^{-9}

Example:

Write 51,000,000 and 0.000865 in scientific notation.

Large Number	**Small Number**
51,000,000	0.000865

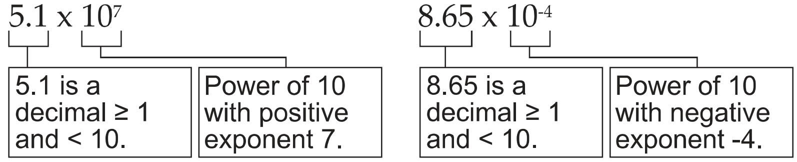
1. Count the number of places we move the decimal point to get a number between 1 and 10.

Remember: Only *one* nonzero digit (a number from 1 to 9) can be to the *left* of the decimal point.

51,000,000.	0.000865
7 places to the *left*	4 places to the *right*

2. Write the number as a *product* of a number between 1 and 10 and a power of 10.

5.1×10^{7} $\qquad$ 8.65×10^{-4}



Consider these large quantities:	In scientific notation:
700,000,000	7×10^{8}
980,000	9.8×10^{5}
40,000,000	4×10^{7}
250,000,000,000	2.5×10^{11}

And these very small quantities:	In scientific notation:
0.0085	8.5×10^{-3}
0.000009	9×10^{-6}
0.000000556	5.56×10^{-7}
0.0000302	3.02×10^{-5}

Ratios and Rates

The first question we ask is "What is the difference between a **ratio** and a **rate**?"

A *ratio* is a comparison of *two like quantities*. A ratio is actually a fraction which can be written in the following ways.

The ratio of 1 to 4 can be written

$\frac{1}{4}$ *or*
1 to 4 *or*
1:4.

Each one of these is read as *one to four*.

When writing ratios, *order* is important. All of these expressions are read *one to four*. If the *colon notation* is used, like in the last example above, the first number is divided by the second.

The ratio of 5 to 11 is

$\frac{5}{11}$, not $\frac{11}{5}$ *or*
5:11, not 11:5.

Also, we usually express ratios in *lowest terms*. A ratio is said to be in lowest terms if the two numbers are relatively *prime*. You do *not* change an improper fraction to a mixed number if the improper fraction is expressed as a ratio. For example:

The ratio of \$25 to \$15 is written

$$\frac{\$25}{\$15} = \frac{25 \div 5}{15 \div 5} = \frac{5}{3}.$$

Note: We leave *ratios* as *improper fractions* in lowest terms and do *not* rewrite as mixed numbers.

Rates are used to compare *different kinds of quantities*.

Note: We write *rates* as *fractions in simplest form*.

Example 1:

Suppose Jake ran 4 miles in 44 minutes. Find Jake's rate (or average speed).

As a rate, we write Jake's distance (4 miles) over his time (11 minutes) as a fraction. Then we reduce and write in simplest form.

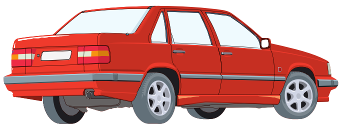
$$\text{Rate} = \frac{\text{distance}}{\text{time}} = \frac{\text{4 miles}}{\text{44 minutes}} = \frac{\text{1 mile}}{\text{11 minutes}} \textit{ or } \text{1 mile per 11 minutes}$$

(Always write the units—or fixed quantity—in your answer.)

So, Jake's average rate of speed was 1 mile in 11 minutes. In reflection, this answer is *reasonable* because if Jake's average speed was 11 minutes per mile and he traveled 4 miles, it would have taken him 44 minutes.

Example 2:

You drive 300 miles in 5 hours. Find your rate (or average speed). Use the same formula of rate equals distance over time. Write in your distance (300 miles) over your time (5 hours) and reduce the fraction to simplest form.

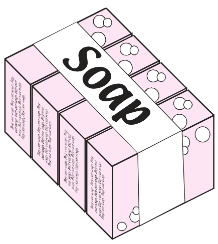


$$\text{Rate} = \frac{\text{distance}}{\text{time}} = \frac{\text{300 miles}}{\text{5 hours}} = \frac{\text{60 miles}}{\text{1 hour}} \textit{ or } \text{60 miles per hour}$$

So your average rate of speed was 60 miles per hour. In reflection, this answer is *reasonable* because if your average rate of speed was 60 miles per hour and you traveled for 5 hours, you would have gone 300 miles.

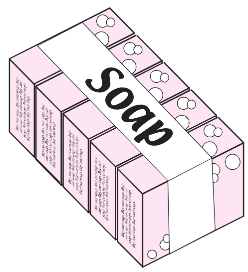
We actually found a **unit rate** since we read 60 mi/hr as 60 miles *per* hour. A *unit rate* is a rate for one **unit** of a given quantity. A unit rate has a denominator of 1.

Unit rates are used a lot in **unit pricing**. A *unit price* is the cost of a particular item, expressed in the unit in which the product is generally measured. A unit price enables us to find the *better buy*. For example, which package of soap below is the better buy?



4 bars of soap

$0.99



5 bars of soap

$1.15

First, determine the price per bar.

4 bars: $\frac{\$0.99 \div 4}{4 \div 4} = \frac{\$0.25}{1 \text{ bar}}$ *or* $0.25 per bar

Answer is reasonable because 4 bars at $0.25 equals $1.00. (The $0.25 was rounded to the nearest cent, that is why 4 bars did not equal $0.99.)

5 bars: $\frac{\$1.15 \div 5}{5 \div 5} = \frac{\$0.23}{1 \text{ bar}}$ *or* $0.23 per bar

Answer is reasonable because 5 bars at $0.23 equals $1.15.

The package containing 5 bars of soap is the better buy. Each bar of soap costs $0.23, which is less than each bar in the 4 bar-package. But be careful—buying a package with more items in the package will *not* always result in the better buy. You must *do the math* to be sure.

Writing and Solving Proportions

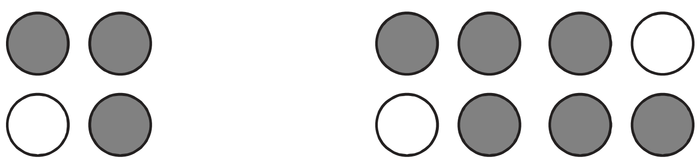
An equation showing that two ratios (or *rates*) are equal is called a **proportion**.

$\frac{3}{4} = \frac{6}{8}$ is a *proportion*.

We read this as *3 is to 4 as 6 is to 8*.

Also, we can see that the proportion $\frac{3}{4} = \frac{6}{8}$ is *true*.

Look below. In the first set of circles, 3 out of 4 circles are shaded. In the second set, 6 out of 8 circles are shaded. And although the second set has more circles, the ratio of shaded circles to total circles is the same. That is, $\frac{3}{4} = \frac{6}{8}$. One ratio is *equal* to the other ratio, and is therefore called a *proportion*.



The ability to compare and produce equal ratios involves *proportional reasoning*. A common use of proportions is to make or use maps and scale models.

There are two ways to find out whether a proportion is true.

- One method is to write each fraction in simplest form and compare them.

  Since $\frac{6}{8}$ simplifies to $\frac{3}{4}$, we note that $\frac{3}{4} = \frac{6}{8}$ is true.

- Another way to determine if the proportion is true is to use *cross products* with **cross multiplication**. *Cross multiplication* is a method for solving and checking proportions.

$\frac{3}{4} = \frac{6}{8}$

$8(3) = 4(6)$ When cross products are *equal*, the proportion *is true*.

$24 = 24$

If cross products are *not equal*, the proportion is *false*.

We ask the following question.

Is $\frac{2}{3} = \frac{4}{9}$ a proportion?

$\frac{2}{3} \times \frac{4}{9}$

Is $9(2) = 3(4)$?

Does $18 = 12$? Of course not!

Therefore, $\frac{2}{3} = \frac{4}{9}$ is *not* a proportion.

Examples:

We use cross products to state whether each proportion is true or false.

a. $\frac{2}{11} = \frac{1}{5}$

Is $5(2) = 11(1)$?

10 is *not* equal to 11
Answer: *false*

b. $\frac{2}{3} = \frac{5}{6}$

Is $6(2) = 3(5)$?

12 is *not* equal to 15
Answer: *false*

c. $\frac{1}{2} = \frac{3}{6}$

Is $6(1) = 2(3)$?

6 is equal to 6
Answer: *true*

As we've already seen, a proportion states that two ratios are equal. Frequently, proportions are presented as follows:

$\frac{a}{b} = \frac{c}{d}$ In words: *a* is to *b* as *c* is to *d*.

Also:

$d(a) = b(c)$

To solve a proportion when three parts are known, we use cross multiplication or the *cross product property*.

Solve:

$\frac{x}{3} = \frac{1}{6}$ ← use cross product property

$6(x) = 3(1)$

$\frac{6x}{6} = \frac{3}{6}$ ← divide each side by 6

$x = \frac{1}{2}$ ← reduced and in simplest form

We can check our solution by substituting the $\frac{1}{2}$ in the original proportion.

Is $\frac{\frac{1}{2}}{3} = \frac{1}{6}$?

Is $\frac{1}{2}(6) = 3(1)$?

Is $(3) = 3$?

Yes, our solution is correct.

or

Since $\frac{1}{2} = 0.5$

Is $\frac{0.5}{3} = \frac{1}{6}$?

Is $0.5(6) = 3(1)$?

Is $3.0 = 3$?

Yes, our solution is correct.

The cross product property *only* works when solving a proportion. It does not apply to doing operations with fractions, such as multiplying or dividing.

Using Proportions

Writing proportions is one of the most important skills we can learn. Courses in science, business, health, engineering and, of course, mathematics, require a solid foundation in handling proportions. Additionally, we use proportions in everyday situations.

When we are given a specified ratio (or rate) of two quantities, a proportion can be used to find an unknown quantity.

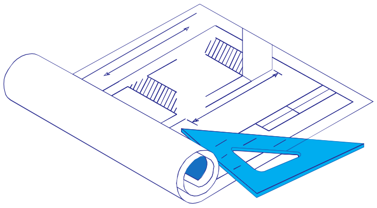
Example 1:

On an architect's blueprint, 1 inch corresponds to 6 feet. How long is a deck represented by a $3\frac{1}{2}$ inch line on the blueprint?

$$\frac{1 \text{ inch}}{6 \text{ feet}} = \frac{3\frac{1}{2} \text{ inches}}{x \text{ feet}}$$

Note: If the *left* side of our proportion compares blueprint *length to actual length*, then the *right* side must also compare blueprint *length to actual deck length*.

Use cross products:



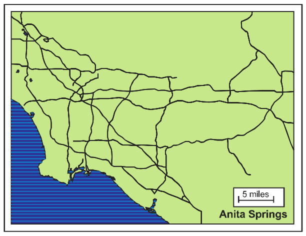
$$\begin{aligned} \frac{1}{6} &= \frac{3\frac{1}{2}}{x} \\ 1x &= 6(3\tfrac{1}{2}) \\ x &= 21 \text{ feet} \end{aligned} \qquad \textit{or} \qquad \begin{aligned} \frac{1}{6} &= \frac{3.5}{x} \\ 1x &= 6(3.5) \\ x &= 21 \text{ feet} \end{aligned}$$

Therefore, 21 feet is represented by $3\,\frac{1}{2}$ *or* 3.5 inches on the blueprint.

Example 2:

On a map of Anita Springs, 5 miles corresponds to 2 inches. How many miles correspond to 7 inches?

Translate: Let n represent our unknown miles. Since 5 miles corresponds to 2 inches as n miles corresponds to 7 inches, we can write the proportion:



$$\begin{aligned} \frac{\text{miles}}{\text{inches}}\ \frac{5}{2} &= \frac{n}{7}\ \frac{\text{miles}}{\text{inches}} \\ 5(7) &= 2n \\ 35 &= 2n \\ 17\tfrac{1}{2} &= n \qquad \textit{or} \qquad 17.5 = n \end{aligned}$$

So, $17\frac{1}{2}$ *or* 17.5 miles corresponds to 7 inches on the map.

Example 3:

The instructions on a bottle of cough syrup state that the patient should take $\frac{1}{2}$ teaspoon for every 40 pounds (lbs) of body weight. At this rate, find the appropriate dose for a 120-pound woman.

$$\begin{aligned} \frac{\text{teaspoons}}{\text{pounds}}\ \frac{\frac{1}{2}}{40} &= \frac{x}{120}\ \frac{\text{teaspoons}}{\text{pounds}} \\ \tfrac{1}{2}(120) &= 40x \\ 60 &= 40x \\ 1\tfrac{1}{2} &= x \end{aligned}$$

or

$$\begin{aligned} \frac{0.5}{40} &= \frac{x}{120} \\ 0.5(120) &= 40x \\ 60 &= 40x \\ 1.5 &= x \end{aligned}$$

Thus, the 120-pound woman needs $1\frac{1}{2}$ *or* 1.5 teaspoons for each dose of cough syrup.

Lesson Three Purpose

- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, and exponents. (MA.A.1.4.4)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Percents

In previous sections, we reviewed fractions and decimals.

- To change a *fraction to a decimal*, we divide the denominator into the numerator.

Note: When setting up the division problem for the fraction $\frac{2}{5}$, it is helpful to remember that the fraction with its fraction bar reads *2 divided by 5*.

$\frac{2}{5}$ fraction	$\begin{array}{r} 0.4 \\ 5\overline{)2.0} \end{array}$	so $\frac{2}{5} = 0.4$ decimal
$\frac{1}{2}$ fraction	$\begin{array}{r} 0.5 \\ 2\overline{)1.0} \end{array}$	so $\frac{1}{2} = 0.5$ decimal

- To change a *decimal to a fraction*, simply count the decimal places and use the same number of zeros in the denominator.

$0.4 = \frac{4}{10} = \frac{2}{5}$

one decimal place (0.4); one zero (10)

$0.002 = \frac{2}{1{,}000} = \frac{1}{500}$

three decimal places (0.002); three zeros (1,000)

We know that we can express fractions as decimals, and decimals as fractions. We can also express fractions and decimals as **percents**. *Percent* literally means *per 100*, *hundredths*, or *out of every hundred*. The symbol *%* means *percent.* Its use dates back to the 1400s.

Look at the percents below. Each was changed from a percent to a fraction and then to a decimal.

$$5\% = \frac{5}{100} = 0.05 \qquad 6.2\% = \frac{6.2}{100} = \frac{62}{1{,}000} = 0.062$$

$$16\% = \frac{16}{100} = 0.16 \qquad 5\frac{1}{4}\% = 5.25\% = \frac{5.25}{100} = \frac{525}{10{,}000} = 0.0525$$

$$200\% = \frac{200}{100} = 2$$

Look for a pattern! Note how the decimal point *shifts* places when we change a percent to a decimal.

Rule: To change a *percent to a decimal*,

- drop the percent sign,
- move the decimal point *two places* to the *left*, and
- insert zeros as placeholders if necessary.

Remember: If you do *not* see a decimal point, it is at the end of the number.

Examples:

$$17.\% = 0.17$$

$$175.\% = 1.75$$

$$10.5\% = 0.105$$

$$30\frac{1}{2}\% = 30.5\% = 0.305$$

$$2\frac{3}{4}\% = 2.75\% = 0.0275$$

Rule: To change a *decimal to a percent*,

- move the decimal point *two places* to the *right*,
- insert zeros as placeholders as necessary, and
- then write the % sign.

Examples:

$0.03 = 3\%$

$0.375 = 37.5\%$

$0.66\frac{2}{3} = 66\frac{2}{3}\%$

$7. = 700\%$

Memory Trick: Here is an easy way to remember the two previous rules:

Decimal starts with a *D*.
Percent starts with a *P*.
D comes before P in the alphabet.

D —right→ *P* (To move from *D* to *P* in the alphabet, move to the right)

To change a *decimal to a percent*, move the decimal point to the *right*

$0.15 = 15\%$

D ←left— *P* (To move from *P* to *D* in the alphabet, move to the left)

To change a *percent to a decimal*, move the decimal point to the *left*.

$85\% = 0.85$

Solving Percent Problems with Equations

We will see that many problems involving percents require that we take some *percent of a quantity*.

To do these problems, use the *fundamental rule for percentage*.

$PB = A$

percent of base= amount
$PB = A$

Note: P is the *percent*,
B is the *base*, and
A is the *amount*.

Example 1: Solve for the *amount* (A).

5 percent of 20 is what number?

Equation:	5% x 20 =	A	←	change 5% to a decimal and multiply
	0.05 (20) =	A	←	
Answer:	1 =	A	←	So: 5% of 20 is 1.

Example 2: Solve for the *base*.

5 percent of what number is 20?

Equation:	5% x B =	20	←	change 5% to a decimal and multiply
	$0.05B$ =	20	←	
	$\frac{0.05B}{0.05}$ =	$\frac{0.20}{0.05}$	←	divide both sides by 0.05
Answer:	B =	400	←	So: 5% of 400 is 20.

Example 3: Solve for the *percent*.

What percent of 20 is 5?

Equation:	$P(20)$	$= 5$	←	use commutative property
	$20P$	$= 5$		
	$\frac{20P}{20}$	$= \frac{5}{20}$	←	divide both sides by 20
	P	$= 0.25$	←	change 0.25 to a percent for final answer
Answer:	P	$= 25\%$	←	So: 25% of 20 is 5.

Notice: Since our answer needs to be a percent, we write 0.25 as 25%, our final answer.

Let's look at three more examples.

Example 4:

What number is 15% of 60?

Equation:	A	$=$	PB	
	A	$=$	$15\%(60)$	
	A	$=$	$0.15(60)$	← change 15% to a decimal and multiply
	A	$=$	9	
Answer:	9 is 15% of 60			

Example 5:

15% of what number is 60?

Equation:	PB	$= A$	
	$15\%B$	$= A$	← change 15% to a decimal
	$0.15B$	$= 60$	
	$\frac{0.15B}{0.15}$	$= \frac{60}{0.15}$	← divide both sides by 0.15
	B	$= 400$	
Answer:	15% of 400 is 60		

Example 6:

15 is what percent of 60?

Equation: $A = PB$

$15 = P(60)$

$\frac{15}{60} = \frac{P(60)}{60}$ ← divide both sides by 60

$0.25 = P$

$25\% = P$ ← change 0.25 to percent for final answer

Answer: 15 is 25% of 60

Example 7:

Be careful with *fractional percents*. Instead of converting these percents to decimals, our work will be easier if we convert fractional percents to fractions.

Equation: $16\frac{2}{3}\%$ of $72 = A$

First, change the fractional percent to an improper fraction.

$16\frac{2}{3}\% = \frac{50}{3}$ hundredths or $\frac{\frac{50}{3}}{100}$ ← means $\frac{50}{3} \div 100$

$= \frac{50}{3} \times \frac{1}{100}$ ← multiply by reciprocal

$= \frac{\overset{1}{\cancel{50}}}{3} \times \frac{1}{\underset{2}{\cancel{100}}}$ ← cancel by dividing the opposite numerator and denominator (by 50) to write in lowest terms

$= \frac{1}{6}$ ← multiply numerator, multiply denominator

Therefore: $16\frac{2}{3}\%$ of $72 = A$ ← rewrite equation with fraction and multiply

$\frac{1}{6}(72) = A$

$\frac{1}{\underset{1}{\cancel{6}}} \times \frac{\overset{12}{\cancel{72}}}{1} = A$ ← cancel by dividing the opposite numerator and denominator (by 6) to write in lowest terms

$\frac{12}{1} = A$ ← multiply numerator, multiply denominator

$12 = A$ ← reduced and in simplest form

In advertising, we may see signs stating $33\frac{1}{3}\%$ off. Again, we need to convert our percent to a fraction rather than a decimal.

Suppose a \$480 stereo system is reduced $33\frac{1}{3}\%$. We want to know how much money we would save by taking advantage of the discount.

$33\frac{1}{3}\% =$ ← change percent to a fraction

$\frac{33\frac{1}{3}}{100} =$ ← change mixed number to an improper fraction and multiply by reciprocal

$33\frac{1}{3} \div 100 =$

$\frac{100}{3} \times \frac{1}{100} =$ ← cancel by dividing the opposite numerator and denominator (by 100)

$\frac{\cancel{100}^{1}}{3} \times \frac{1}{\cancel{100}_{1}} =$ ← multiply numerator, multiply denominator

$\frac{1}{3}$ ← reduced and in simplest form

Therefore: $33\frac{1}{3}\%$ of $\$480 = A$ ← rewrite equation with fraction and multiply

$\frac{1}{3}(480) = A$

$\frac{1}{\cancel{3}_{1}} \bullet \frac{\cancel{480}^{160}}{1} = A$ ← cancel by dividing the opposite numerator and denominator (by 3) to write in lowest terms

$\frac{160}{1} = A$ ← multiply numerator, multiply denominator

$160 = A$ ← reduced and in simplest form

We would save \$160.

Therefore,

\$480	= original cost
– 160	= $33\frac{1}{3}\%$ off original cost
\$320	= sale cost

In the last assignment, we saw that fractions like $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{1}{5}$ convert to **terminating decimals** like 0.5, 0.25, 0.75, and 0.2 respectively. *Terminating decimals* contain a *finite* (limited) number of digits. Therefore, fractional percents like $6\frac{1}{2}\%$, $3\frac{1}{4}\%$, and $12\frac{3}{4}\%$ can easily be converted to decimals in our computations.

Examples:

$$6\tfrac{1}{2}\% = 6.5\% = 0.065$$
$$3\tfrac{1}{4}\% = 3.25\% = 0.0325$$
$$12\tfrac{3}{4}\% = 12.75\% = 0.1275$$
$$18\tfrac{1}{5}\% = 18.2\% = 0.182$$

However: $16\frac{2}{3}\% = \frac{1}{6}$ (as explained on page 218)

$33\frac{1}{3}\% = \frac{1}{3}$

Likewise: $66\frac{2}{3}\% = \frac{2}{3}$ because

$66\frac{2}{3}\% =$ ← change percent to a fraction

$\frac{66\frac{2}{3}}{100} =$ ← change mixed number to an improper fraction and multiply

$66\frac{2}{3} \div 100 =$

$\frac{200}{3} \times \frac{1}{100} =$ ← cancel by dividing the opposite numerator and denominator (by 100) to write in lowest terms

$\frac{{}^{2}\cancel{200}}{3} \times \frac{1}{\cancel{100}_{1}} =$ ← multiply numerator / multiply denominator

$\frac{2}{3}$ ← reduced and in simplest form

You may find it helpful to memorize the following chart of equivalent percents, decimals, and fractions. (See practice on pages 213 and 214 where you computed these equivalents.)

Equivalent Percents, Decimals, and Fractions			
$20\% = 0.2 = \frac{1}{5}$	$25\% = 0.25 = \frac{1}{4}$	$12\frac{1}{2}\% = 0.125 = \frac{1}{8}$	$16\frac{2}{3}\% = 0.1\overline{6} = \frac{1}{6}$
$40\% = 0.4 = \frac{2}{5}$	$50\% = 0.5 = \frac{1}{2}$	$37\frac{1}{2}\% = 0.375 = \frac{3}{8}$	$33\frac{1}{3}\% = 0.\overline{3} = \frac{1}{3}$
$60\% = 0.6 = \frac{3}{5}$	$75\% = 0.75 = \frac{3}{4}$	$62\frac{1}{2}\% = 0.625 = \frac{5}{8}$	$66\frac{2}{3}\% = 0.\overline{6} = \frac{2}{3}$
$80\% = 0.8 = \frac{4}{5}$		$87\frac{1}{2}\% = 0.875 = \frac{7}{8}$	$83\frac{1}{3}\% = 0.8\overline{3} = \frac{5}{6}$
$100\% = 1$			

Most calculators have a percent key—[%]. Depending upon your calculator, you may be able to use one of the following two key sequences to calculate the percent of a number.

Example:

64% of 75
64 [%] [x] 75 [=]
or
75 [x] 64 [%]

The result should be 48.

Using What We Know to Examine Percent Equations

Sometimes it can help to examine mathematical steps using an equation to which we already know the answer. For example, if we know that

100% of 48 is 48,

we also know that half or

50% of 48 is 24.

Let's use the statement 50% of 48 equals 24 and reexamine the *fundamental rule for percentage*:

percent of base = amount
$PB = A$

and look at solving for *amount*, *base*, and then for *percent*.

First, solving for the *amount* (A)—which we already know from above is **24**:

- What number is 50% of 48?

$$A = P \times B$$
$$A = 0.50 \times 48$$
$$A = \mathbf{24}$$

Answer: **24** is 50% of 48

Second, solving for the *base* (B)—which we already know from above is **48**:

- 50% of what number is 24?

$$P \times B = A$$
$$0.50B = 24$$
$$\frac{0.50B}{0.50} = \frac{24}{0.50}$$
$$B = \mathbf{48}$$

Answer: 50% of **48** is 24

Last, solving for the *percent* (P)—which we already know from the previous page is **50%**.

- 24 is what percent of 48?

$$A = PB$$
$$24 = P(48)$$
$$\frac{24}{48} = \frac{(P)(48)}{48}$$
$$0.50 = P$$
$$\mathbf{50\%} = P$$

Answer: 24 is **50%** of 48

If you come across an equation you are unsure of how to solve, go back to an equation you do know the answers to and *plug* those numbers in to help you retrace the steps to the solution.

Discount and Sales Tax

Most of us shop enough to know that stores frequently will have sales where they *discount* or mark down items. The discount is the part that you do *not* pay.

Most percent problems can be worked using the *fundamental rule for percentage problems* (*PBA*).

percent	of	base	=	amount
↓	↓	↓	↓	↓
percent	x	base	=	amount
			or	
		P x *B*	=	*A*

Example:

A local shoe store offers a 15% discount on shoes if you bring a newspaper coupon. How much would the discount be on a $75 pair of running shoes?

In this case,

the *percent* (or rate) is 15% or 0.15
the *base* (total) is $75, the cost of the shoes.
the *amount* you *don't* pay is the discount.

$$P \times B = A$$
$$(0.15) \times (\$75) = A$$

$11.25 is the *discount* or amount you do *not* pay.

If the shoes once cost $75, and you don't pay $11.25, then the *sales price* will be

cost – discount = sales price

$75 – $11.25 = $63.75

Taxes

It has been said that there are only two things certain in life—death and taxes. In some counties in Florida the residents pay a 7% sales tax. (This rate varies around the state.)

Example:

Let's go back to the running shoes in the last example. To figure the tax, we will use the same *fundamental rule for percentage problems* (*PBA*).

percent x base = amount

In this case,

the *percent* is 7 % or 0.07
the *base* is \$63.75, the cost of the shoes
the *amount* is the tax that will be paid to the state of Florida.

P x *B*	=	*A*
(0.07) x (\$63.75)	=	amount (tax)
\$4.4625	= ←	tax (round to the nearest cent)
\$4.46	= ↙	tax (to be paid)

Not only do we have to pay the store for the shoes (\$63.75), but we also have to pay tax (\$4.46). Our *total bill* will be

cost	+	tax	=	total bill
\$63.75	+	\$4.46	=	\$68.21

What if our county's sales tax was 6.5% or 0.065? How much would we have to pay on \$63.75?

P x *B*	=	*A*
(0.065) x (\$63.75)	=	amount (tax)
\$4.14375	= ←	tax (round to the nearest cent)
\$4.14	= ↙	tax (to be paid)

cost	+	tax	=	total bill
\$63.75	+	\$4.14	=	\$67.89

Interest

Borrowing Money

Banks and individuals lend money to people who qualify because they feel the money will most likely be repaid. The person who borrows money must pay a fee. **Interest** is the fee paid for the use of borrowed money. Usually the *interest* is some percent or rate of the borrowed money. The borrowed money is called the **principal**. The longer a person keeps the money, the bigger fee or greater interest he will pay.

To *compute interest*, we use the following formula:

$$\boxed{I = p \times r \times t}$$

p is the *principal*; r is the *rate* (percent); t is the *time* in years

interest = principal • rate • time

Remember: Change the *rate* from a *percent* to a *decimal*.

Example:

Suppose that you need $6,000 for a used car. Your mom agrees to give it to you if you pay her back in 1 year. She also charges you 5% (0.05) annually (per year). How much will the car cost you?

$$I = p \times r \times t$$
$$I = (\$6{,}000) \times (0.05) \times (1)$$
$$I = \$300$$

Remember: You still must repay your mom the original $6,000 plus the fee (interest) of $300:

$6,000 + $300 = $6,300
$6,300 is the total amount repaid to your mom.

Money Saving

Also, when you deposit money into a savings account, you are actually agreeing to let the bank use your money. For this privilege, the bank pays *you* interest on your money. To figure this interest, we use the same formula as before:

interest (I) = principal (p) • annual rate of interest (r) • time in years (t)

$$I = p \times r \times t$$
or
$$I = prt$$

Example:

Suppose your mom decides to put her $6,300 in a savings account for 1 year. The bank that she uses pays a rate of $5\frac{3}{4}\%$ (0.0575) on the principal, the amount in the savings account.

$I = p \times r \times t$
$I = (\$6{,}300) \times (0.0575) \times (1)$
$I = \$362.25$

When you deposit money into a savings account, you are actually agreeing to let the bank use your money.

At the end of the year your mom still has the original $6,300 plus the $362.25 that she earned in interest.

$6,300 + $362.25 = $6,662.25
principal + interest = *new* principal

Example:

Mr. Powell deposited $1,000 in his savings account. His account earns $6\frac{1}{2}\%$ (0.065) annually. Unfortunately, Mr. Powell has to withdraw his money after 6 months. How much interest will he earn?

To work this problem we have to deal with the time very carefully. If we use 6 for our *t* in the formula, we will get a large number which is not correct. This would be the answer for allowing the money to stay in the bank 6 years! To get the correct answer, we have to realize that 6 months is $\frac{1}{2}$ year.

$$\frac{\text{6 months}}{\text{12 months in one year}} = \frac{1}{2}$$

$I = \quad p \quad \text{x} \quad r \quad \text{x} \quad t$

$I = (\$1{,}000) \quad \text{x} \quad (0.065) \quad \text{x} \quad (\frac{1}{2})$

$I = 32.50$

$32.50 is the interest that Mr. Powell will earn.

Percent of Increase or Decrease

In 2001, United States businesses sold 9.4 billion dollars worth of video games. This was up from 6.9 billion dollars worth sold in 1999. The new amount is more than the original amount.

Find the **percent of change**—in this case, the **percent of increase**. This is a *percent of increase* because the new amount is more than the original amount.

To find the *percent of change*—*percent of increase* or *decrease*—follow these steps:

Step 1: Subtract to find the *amount of change* or difference between the two amounts.

Step 2: Write the amount of change over the original amount as a fraction and divide.

$$\frac{\text{amount of change}}{\text{original amount}}$$

Remember: The *denominator* in the formula above is *always* the *original amount*, whether less than or greater than the new amount.

Step 3: Change the answer to a percent.

Example 1:

Find out the *percent of increase* to solve the problem above.

Step 1:	$9.4 - 6.9 = 2.5$	← subtract to find the amount of change
Step 2:	$\frac{2.5}{6.9} \approx$	← write as a fraction: $\frac{\text{amount of change (increase)}}{\text{original amount}}$
	$0.36 =$	← divide and round to the nearest hundred
Step 3:	36%	← rewrite the answer as a percent

Answer: 36% increase

Here is another example: Janet weighed 120 pounds when she entered 9th grade. At the end of the year she weighed 105 pounds. The new amount is less than the original amount.

Find the *percent of change*—in this case, the **percent of decrease**. This is a *percent of decrease* because the new amount is less than the original amount. In business, the percent of decrease is often called the *discount*.

Example 2:

Find the *percent of decrease* to solve the problem above.

Step 1: $120 - 105 = 15$ ← subtract to find the amount of change

Step 2: $\frac{15}{120} =$ ← write as a fraction: $\frac{\text{amount of change (decrease)}}{\text{original amount}}$

$0.125 =$ ← divide and round to the nearest hundredth

Step 3: 12.5% ← rewrite the answer as a percent

Answer: 12.5% decrease