Unit 3: Algebraic Thinking

Introduction

Algebraic thinking provides tools for looking at situations. You can state, simplify, and show relationships through algebraic thinking. Using algebraic thinking and algebraic symbols, you can record ideas and gain insights into situations.

In this lesson you will use what you have learned about solving equations to solve equations involving positive and negative numbers.

You have learned to solve equations such as these:

$y + 12 = 36$ $\quad$ $y - 12 = 36$ $\quad$ $12y = 36$ $\quad$ $\frac{y}{12} = 36$

Lesson One Purpose

- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two-dimensional shapes. (MA.B.1.4.1)
- Use systems of equations and inequalities to solve real-world problems algebraically. (MA.D.2.4.2)

Solving Equations

A mathematical sentence that contains an equal sign (=) is called an **equation**. In Unit 1 and Unit 2, we learned that an *equation* equates one **expression** to another expression.

We also learned the rules to add and subtract and to multiply and divide **positive numbers** and **negative numbers**.

Rules to Add and Subtract Positive and Negative Integers	
(+) + (+) = +	(+) – (+) = positive if first number is greater, otherwise it is negative
(–) + (–) = –	(–) – (–) = negative if first number is greater, otherwise it is positive
(+) + (–) = (–) + (+) = } use sign of **integer** with greatest absolute value	(+) – (–) = + (–) – (+) = –

Rules to Multiply and Divide Positive and Negative Integers	
(+) • (+) = +	(+) ÷ (+) = +
(–) • (–) = +	(–) ÷ (–) = +
(+) • (–) = –	(+) ÷ (–) = –
(–) • (+) = –	(–) ÷ (+) = –

To **solve** the equation is to find the number that we can **substitute** for the **variable** to make the equation true.

Study these examples. Each equation has been *solved* and then checked by substituting the answer for the variable in the original equation. If the answer makes the equation a true sentence, it is called the **solution** of the equation.

Solve:

$$\begin{aligned} n + 14 &= -2 \\ n + 14 - 14 &= -2 - 14 \\ n &= -2 + -14 \\ n &= -16 \end{aligned}$$

Check:

$$\begin{aligned} n + 14 &= -2 \\ -16 + 14 &= -2 \\ -2 &= -2 \quad \textit{It checks!} \end{aligned}$$

Solve:

$$\begin{aligned} y - (-6) &= 2 \\ y + 6 - 6 &= 2 - 6 \\ y &= 2 + -6 \\ y &= -4 \end{aligned}$$

Check:

$$\begin{aligned} y - -6 &= 2 \\ -4 - -6 &= 2 \\ -4 + 6 &= 2 \\ 2 &= 2 \quad \textit{It checks!} \end{aligned}$$

Solve:

$$\begin{aligned} -6x &= -66 \\ \frac{-6x}{-6} &= \frac{-66}{-6} \\ x &= 11 \end{aligned}$$

Check:

$$\begin{aligned} -6x &= -66 \\ -6(11) &= -66 \\ -66 &= -66 \quad \textit{It checks!} \end{aligned}$$

Solve:

$$\begin{aligned} \frac{y}{-10} &= 5 \\ (-10)\frac{y}{-10} &= 5(-10) \\ y &= -50 \end{aligned}$$

Check:

$$\begin{aligned} \frac{y}{-10} &= 5 \\ \frac{-50}{-10} &= 5 \\ 5 &= 5 \quad \textit{It checks!} \end{aligned}$$

Interpreting Words and Phrases

Words and phrases can suggest relationships between numbers and mathematical operations. In Unit 1 and Unit 2, we learned how words and phrases can be translated into mathematical expressions. (See pages 38-39 and page 215.) Appendix B also contains a list of mathematical symbols and their meanings.

Relationships between numbers can be indicated by words such as **consecutive**, *preceding*, *before*, and *next*. Also, the same mathematical expression can be used to translate many different word expressions.

Below are some of the words and phrases we associate with the four mathematical operations and with powers of a number.

Mathematical Symbols and Words

+	–	x	÷	power
add	subtract	multiply	divide	power
sum	difference	product	quotient	square
plus	minus	times		cube
total	remainder	of		
more than	less than	twice		
increased by	decreased by	doubled		

Solving Two-Step Equations

When solving an equation, you want to get the *variable* by itself on one side of the equal sign. You do this by *undoing* all the operations that were done on the variable. In general, undo the addition or subtraction first. Then undo the multiplication or division.

Study the following examples.

A. Solve:

$2y + 2 =$	30	
$2y + 2 - 2 =$	$30 - 2$	subtract 2 from each side
$\frac{2y}{2} =$	$\frac{28}{2}$	divide each side by 2
$y =$	14	

Check:

$2y + 2 =$	30	
$2(14) + 2 =$	30	replace y with 14
$28 + 2 =$	30	
$30 =$	30	*It checks!*

B. Solve:

$2x - 7 =$	-29	
$2x - 7 + 7 =$	$-29 + 7$	add 7 to each side
$\frac{2x}{2} =$	$\frac{-22}{2}$	divide each side by 2
$x =$	-11	

Check:

$2x - 7 =$	-29	
$2(-11) - 7 =$	-29	replace x with -11
$-22 - 7 =$	-29	
$-29 =$	-29	*It checks!*

C. Solve:

$$\frac{n}{7} + 18 = 20$$

$$\frac{n}{7} + 18 - 18 = 20 - 18 \quad \text{subtract 18 from each side}$$

$$(\not{7})\frac{n}{\not{7}} = 2(7) \quad \text{multiply each side by 7 and}$$

$$n = 14 \quad \text{cancel the 7s on the left}$$

Check:

$$\frac{n}{7} + 18 = 20$$

$$\frac{14}{7} + 18 = 20 \quad \text{replace } n \text{ with 14}$$

$$2 + 18 = 20$$

$$20 = 20 \quad \textit{It checks!}$$

D. Solve:

$$\frac{t}{-2} + 4 = -10$$

$$\frac{t}{-2} + 4 - 4 = -10 - 4 \quad \text{subtract 4 from each side}$$

$$\frac{(\not{-2})\,t}{\not{-2}} = -14(-2) \quad \text{multiply each side by -2 and}$$

$$t = 28 \quad \text{cancel the -2s on the left}$$

Check:

$$\frac{t}{-2} + 4 = -10$$

$$\frac{28}{-2} + 4 = -10 \quad \text{replace } t \text{ with 28}$$

$$-14 + 4 = -10$$

$$-10 = -10 \quad \textit{It checks!}$$

Special Cases

Reciprocals: Two Numbers Whose Product is 1

Note: $5 \bullet \frac{1}{5} = 1$ and $\frac{5}{5} = 1$

Multiplying 5 by $\frac{1}{5}$ and dividing 5 by 5, both yield 1.

We see that 5 is the **reciprocal** of $\frac{1}{5}$ and $\frac{1}{5}$ is the *reciprocal* of 5. Every number but zero has a reciprocal. (Division by zero is undefined.) Two numbers are reciprocals if their product is 1.

Here are some examples of numbers and their reciprocals:

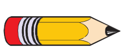
Number	Reciprocal
$-\frac{1}{4}$	-4
1	1
$\frac{-2}{3}$	$\frac{-3}{2}$
$\frac{7}{8}$	$\frac{8}{7}$
-2	$-\frac{1}{2}$
$\frac{1}{7}$	7
x	$\frac{1}{x}$

Multiplication Property of Reciprocals

any nonzero number times its reciprocal is 1

$$x \bullet \frac{1}{x} = 1$$

If $x \neq 0$



Remember: When two numbers are reciprocals of each other, they are also called **multiplicative inverses** of each other.

Study the following two examples:

Method 1: Division Method

$$\begin{aligned} 5x - 6 &= 9 \\ 5x - 6 + 6 &= 9 + 6 \\ 5x &= 15 \\ \frac{5x}{5} &= \frac{15}{5} \\ x &= 3 \end{aligned}$$

Method 2: Reciprocal Method

$$\begin{aligned} 5x - 6 &= 9 \\ 5x - 6 + 6 &= 9 + 6 \\ 5x &= 15 \\ \frac{1}{5} \bullet 5x &= \frac{1}{5} \bullet 15 \\ x &= 3 \end{aligned}$$

Both methods work well. However, the *reciprocal method* is probably easier in the next two examples, which have fractions:

$$\begin{aligned} \text{-}\frac{1}{5}x - 1 &= 9 \\ \text{-}\frac{1}{5}x - 1 + 1 &= 9 + 1 \\ \text{-}\frac{1}{5}x &= 10 \\ \mathbf{-5} \bullet \text{-}\frac{1}{5}x &= \mathbf{-5} \bullet 10 \qquad \text{multiply by reciprocal of } \text{-}\frac{1}{5} \text{ which is -5} \\ x &= \text{-}50 \end{aligned}$$

Here is another equation with fractions:

$$\begin{aligned} \text{-}\frac{3}{4}x + 12 &= 36 \\ \text{-}\frac{3}{4}x + 12 - 12 &= 36 - 12 \\ \text{-}\frac{3}{4}x &= 24 \\ \text{-}\frac{\mathbf{4}}{\mathbf{3}} \bullet \text{-}\frac{3}{4}x &= \text{-}\frac{\mathbf{4}}{\mathbf{3}} \bullet 24 \qquad \text{multiply by reciprocal of -}\frac{3}{4} \text{ which is -}\frac{4}{3} \\ 1 \bullet x &= \text{-}32 \\ x &= \text{-}32 \end{aligned}$$

Multiplying by -1

Here is another equation which sometimes gives people trouble:

$5 - x = -10$

Remember: $5 - x$ is not the same thing as $x - 5$. To work this equation we need to make the following observation:

> **Property of Multiplying by -1**
>
> **-1 times a number equals the opposite of that number**
>
> $\mathbf{-1 \bullet \boldsymbol{x} = -\boldsymbol{x}}$

This property is also called the **multiplicative property of -1** which says the *product* of any number and -1 is the opposite or **additive inverse** of the number.

For example:

$-1 \bullet 5 = -5$
$-1 \bullet (-6) = 6$

Now let's go back to $5 - x = -10$ using the property of multiplying by -1. We can rewrite the equation as

$$\begin{aligned} 5 - 1x &= -10 \\ 5 - 1x \mathbf{- 5} &= -10 \mathbf{- 5} \\ -1x &= -15 \\ \frac{-1x}{-1} &= \frac{-15}{-1} \\ x &= 15 \end{aligned}$$

subtract 5 from both sides to isolate the variable

This example requires great care with the positive numbers and negative signs:

$$\begin{aligned} 11 - \tfrac{1}{9}x &= -45 \\ 11 - \tfrac{1}{9}x \mathbf{- 11} &= -45 \mathbf{- 11} \\ -\tfrac{1}{9}x &= -56 \\ \mathbf{-9} \bullet -\tfrac{1}{9}x &= \mathbf{-9} \bullet -56 \\ x &= 504 \end{aligned}$$

subtract 11 from both sides to isolate the variable

multiply by reciprocal of $-\frac{1}{9}$ which is -9

Consider the following example:

Remember: *Decreased by* means *subtract*, *product* means *multiply*, and *is* translates to the = sign.

Five decreased by the product of 7 and x is -6. Solve for x.

Five decreased by the product of 7 and x is -6.

$$\begin{aligned} 5 - 7x &= -6 \\ 5 - 7x \mathbf{- 5} &= -6 \mathbf{- 5} \\ -7x &= -11 \\ \frac{-7x}{-7} &= \frac{-11}{-7} \\ x &= \frac{11}{7} \text{ or } 1\tfrac{4}{7} \end{aligned}$$

subtract 5 from both sides to isolate the variable

Lesson Two Purpose

- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, distributive, associative, and transitive, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two-dimensional shapes. (MA.B.1.4.1)
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The Distributive Property

Consider 4(2 + 6). The rules for **order of operations** would have us add inside parentheses first.

$$4(2 + 6) =$$
$$4(8) =$$
$$32$$

Remember: Order of operations—

1. parentheses
2. powers (exponents)
3. multiplication *or* division
4. addition *or* subtraction

However, there is a second way to do the problem.

$$\begin{aligned} 4(2+6) &= \\ 4(2)+4(6) &= \\ 8 \;+\; 24 &= \\ 32 & \end{aligned}$$

In this way, the 4 is being *distributed* over the addition. The second way of doing the problem illustrates the **distributive property**.

> **The Distributive Property**
>
> For any numbers a, b, and c,
> $a(b + c) = ab + ac$
>
> Also, it works for subtraction:
> $a(b - c) = ab - ac$

This property is most useful in simplifying expressions that contain variables, such as $2(x + 4)$.

To **simplify an expression** we must perform as many of the indicated operations as possible. However, in the expression $2(x + 4) =$, we can't add first, unless we know what number x represents. The *distributive property* allows us to rewrite the equation:

$$\begin{aligned} 2(x+4) &= \\ 2x + 2(4) &= \\ 2x + 8 & \end{aligned}$$

The distributive property allows you to multiply each term *inside* a set of parentheses by a factor *outside* the parentheses. Multiplication is *distributive over* addition and subtraction.

$$\begin{aligned} 5(3+1) &= (5 \bullet 3) + (5 \bullet 1) \\ 5(4) &= 15 + 5 \\ 20 &= 20 \end{aligned}$$

$$\begin{aligned} 5(3-1) &= (5 \bullet 3) - (5 \bullet 1) \\ 5(2) &= 15 - 5 \\ 10 &= 10 \end{aligned}$$

Not all operations are distributive. You cannot distribute division over addition.

$$\begin{aligned} 14 - (5 + 2) &\neq 14 \div 5 + 14 \div 2 \\ 14 \div 7 &\neq 2.8 + 7 \\ 2 &\neq 9.8 \end{aligned}$$

In Unit 1 we learned about other properties that help us work with variables. See page 51 and study the chart below.

Properties

Addition		**Multiplication**	
Commutative:	$a + b = b + a$	Commutative:	$ab = ba$
Associative:	$(a + b) + c = a + (b + c)$	Associative:	$(ab)c = a(bc)$
Identity:	0 is the identity. $a + 0 = a$ and $0 + a = a$	Identity:	1 is the identity. $a \cdot 1 = a$ and $1 \cdot a = a$
Addition		**Subtraction**	
Distributive:	$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$	Distributive:	$a(b - c) = ab - ac$ and $(b - c)a = ba - ca$

These properties deal with the following:

order—**commutative property** of addition and commutative property of multiplication

grouping—**associative property** of addition and associative property of multiplication

identity—**additive identity** property and **multiplicative identity** property

zero—**multiplicative property of zero**

distributive—**distributive property** of multiplication over addition and over subtraction

Notice in the distributive property that it does *not* matter whether $\boldsymbol{a}$ is placed on the *right* or the *left* of the expression in parentheses.

$$\boldsymbol{a}(b + c) = (b + c)\boldsymbol{a} \text{ or } \boldsymbol{a}(b - c) = (b - c)\boldsymbol{a}$$

The **symmetric property of equality** (if $a = b$, then $b = a$) says that if one quantity equals a second quantity, then the second quantity also equals the first quantity. We use the **substitution property of equality** when replacing a variable with a number or when two quantities are equal and one quantity can be replaced by the other. Study the chart and examples below describing properties of equality.

Properties of Equality

Reflexive:	$a = a$
Symmetric:	If $a = b$, then $b = a$.
Transitive:	If $a = b$ and $b = c$, then $a = c$.
Substitution:	If $a = b$, then a may be replaced by b.

Examples of Properties of Equality	
Reflexive:	$8 – e = 8 – e$
Symmetric:	If $5 + 2 = 7$, then $7 = 5 + 2$.
Transitive:	If $9 – 2 = 4 + 3$ and $4 + 3 = 7$, then $9 – 2 = 7$.
Substitution:	If $x = 8$, then $x \div 4 = 8 \div 4$. x is replaced by 8.
	or
	If $9 + 3 = 12$, then $9 + 3$ may be replaced by 12.

Consider this expression to simplify.

$5(6x + 3) + 8 =$	use the distributive property to
$5(6x) + 5(3) + 8 =$	distribute 5 over $6x$ and 3
$30x + 15 + 8 =$	use the associative property to
$30x + 23$	associate 15 and 8

and

$6 + 2(4x – 3) =$	use order of operations to multiply before adding, then
$6 + 2(4x) + 2(-3) =$	distribute 2 over $4x$ and -3
$6 + 8x + -6 =$	use the associative property to associate 6 and -6
$8x + 0 =$	use the identity property of addition
$8x$	

Simplifying Expressions

Here's how to use the distributive property and the definition of subtraction to simplify the following expressions.

Example 1:

Simplify

$-7a - 3a$

$-7a - 3a$	$=$	$-7a + -3a$	
	$=$	$(-7 + -3)a$	use the distributive property
	$=$	$-10a$	

Example 2:

Simplify

$10c - c$

$10c - c$	$=$	$10c - 1c$	
	$=$	$10c + -1c$	
	$=$	$(10 + -1)c$	use the distributive property
	$=$	$9c$	

The expressions $-7a - 3a$ and $-10a$ are called **equivalent** *expressions*. The expressions $10c - c$ and $9c$ are also called *equivalent* expressions. Equivalent expressions express the same number. An expression is in simplest form when it is replaced by an equivalent expression having no **like terms** and no parentheses.

Study these examples.

$$-5x + 4x = (-5 + 4)x$$
$$= -x$$

$$5y - 5y = 5y + -5y$$
$$= (5 + -5)y$$
$$= 0y \qquad \text{multiplicative property of zero}$$
$$= 0$$

The multiplicative property of 0 says for any number *a*,

$$a \bullet 0 = 0 \bullet a = 0.$$

Frequently, the following shortcut is used to simplify expressions.

First

- change each subtraction to adding the opposite
- then combine *like terms* (terms that have the same variable) by adding.

Simplify

$2a + 3 - 6a$

$2a + 3 - 6a = 2a + 3 + -6a$ (**like terms**: $2a$ and $-6a$) } change $-6a$ to $+ -6a$

$= -4a + 3$ } combine like terms by adding

Simplify

$8b + 7 - b - 6$

$8b + 7 - b - 6 = 8b + 7 + -1b + -6$ (**like terms**: $8b$ and $-1b$; **like terms**: 7 and -6) } change $-b$ to $+ -1b$ and -6 to $+ -6$

$= 7b + 1$ } combine like terms by adding

Simplify

$7x + 5 + 3x = 10x + 5$ (**like terms**: $7x$ and $3x$) combine like terms

Equations with Like Terms

Consider the following equation.

$$2x + 3x + 4 = 19$$

Look at both sides of the equation and see if either side can be simplified.

> Always simplify first by combining like terms.

$2x + 3x + 4 = 19$	
$5x + 4 = 19$	add like terms
$5x + 4 - 4 = 19 - 4$	subtract 4 from each side
$5x = 15$	
$\frac{5x}{5} = \frac{15}{5}$	divide each side by 5
$x = 3$	

Always mentally check your answer by *substituting* the solution for the variable in the original equation.

Substitute 3 for x in the equation.

$2x + 3x + 4 = 19$	
$2(3) + 3(3) + 4 = 19$	
$6 + 9 + 4 = 19$	
$19 = 19$	*It checks!*

Consider this example:

The product of x and 7 plus the product of x and 3 is 45.

Remember: To work a problem like this one, we need to remember two things. The word *product* means *multiply* and the word *is* always translates to =.

The product of x and 7 plus the product of x and 3 is 45.

$$
\begin{aligned}
7x + 3x &= 45 \\
10x &= 45 && \text{add like terms} \\
\frac{10x}{10} &= \frac{45}{10} && \text{divide both sides by 10} \\
x &= 4.5
\end{aligned}
$$

Check by substituting 4.5 for x in the original equation.

$$
\begin{aligned}
7x + 3x &= 45 \\
7(4.5) + 3(4.5) &= 45 \\
31.5 + 13.5 &= 45 \\
45 &= 45 && \textit{It checks!}
\end{aligned}
$$

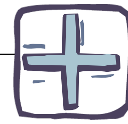
Here is another example which appears to be a little harder:

$$
\begin{aligned}
3x - 2 - x + 10 &= -12 \\
3x - 2 - 1x + 10 &= -12 && \text{remember: } 1 \bullet x = x \\
3x - 1x - 2 + 10 &= -12 && \text{add like terms} \\
2x + 8 &= -12 \\
2x + 8 - 8 &= -12 - 8 && \text{subtract 8 from both sides} \\
2x &= -20 \\
\frac{2x}{2} &= \frac{-20}{2} && \text{divide both sides by 2} \\
x &= -10
\end{aligned}
$$

Check by substituting -10 into the original equation.

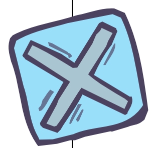
$$
\begin{aligned}
3x - 2 - x + 10 &= -12 \\
3(-10) - 2 - (-10) + 10 &= -12 \\
-30 - 2 + 10 + 10 &= -12 \\
-32 + 20 &= -12 \\
-12 &= -12 && \textit{It checks!}
\end{aligned}
$$

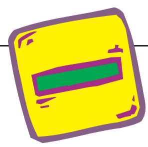
Putting It All Together

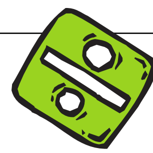


Guidelines for Solving Equations

1. Use the distributive property to clear parentheses.
2. Combine like terms. We want to isolate the variable.
3. Undo addition or subtraction using **inverse operations**.
4. Undo multiplication or division using *inverse operations*.
5. Check by substituting the solution in the original equation.







SAM = **S**implify (steps 1 and 2) then
Add (or subtract)
Multiply (or divide)

Example 1

Solve:

$$\begin{aligned} 6y + 4(y + 2) &= 88 && \\ 6y + 4y + 8 &= 88 && \text{use distributive property} \\ 10y + 8 - 8 &= 88 - 8 && \text{combine like terms and undo addition} \\ & && \text{by subtracting 8 from each side} \\ \frac{10y}{10} &= \frac{80}{10} && \text{undo multiplication by dividing by 10} \\ y &= 8 && \end{aligned}$$

Check solution in the original equation:

$$\begin{aligned} 6y + 4(y + 2) &= 88 \\ 6(8) + 4(8 + 2) &= 88 \\ 48 + 4(10) &= 88 \\ 48 + 40 &= 88 \\ 88 &= 88 && \textit{It checks!} \end{aligned}$$

Example 2

Solve:

$-\frac{1}{2}(x+8)$	=	10	
$-\frac{1}{2}x-4$	=	10	use distributive property
$-\frac{1}{2}x-4+4$	=	$10+4$	undo subtraction by adding 4 to both sides
$-\frac{1}{2}x$	=	14	
$(-2)-\frac{1}{2}x$	=	14(-2)	isolate the variable by multiplying
x	=	-28	each side by the reciprocal of $-\frac{1}{2}$

Check solution in the original equation:

$-\frac{1}{2}(x+8)$	=	10	
$-\frac{1}{2}(-28+8)$	=	10	
$-\frac{1}{2}(-20)$	=	10	
10	=	10	*It checks!*

Example 3

Solve:

26	=	$\frac{2}{3}(9x - 6)$	
26	=	$\frac{2}{3}(9x)-\frac{2}{3}(6)$	use distributive property
26	=	$6x-4$	
$26+4$	=	$6x-4+4$	undo subtraction by adding 4 to each side
$\frac{30}{6}$	=	$\frac{6x}{6}$	undo multiplication by dividing each side by 6
5	=	x	

Check solution in the original equation:

26	=	$\frac{2}{3}(9x - 6)$	
26	=	$\frac{2}{3}(9 \bullet 5-6)$	
26	=	$\frac{2}{3}(39)$	
26	=	26	*It checks!*

Example 4

Solve:

$x - (2x + 3)$	$=$	4	
$x - 1(2x + 3)$	$=$	4	use the multiplication property of -1
$x - 2x - 3$	$=$	4	use the multiplicative identity of 1 and use the distributive property
$-1x - 3$	$=$	4	combine like terms
$-1x - 3 + 3$	$=$	$4 + 3$	undo subtraction
$\frac{-1x}{-1}$	$=$	$\frac{7}{-1}$	undo multiplication
x	$=$	-7	

Examine the solution steps above. See the use of the *multiplicative property of -1* in front of the parentheses on line two.

line 1: $x - (2x + 3) = 4$
line 2: $x \mathbf{- 1}(2x + 3) = 4$

Also notice the use of *multiplicative identity* on line three.

line 3: $\mathbf{1}x - 2x - 3 = 4$

The simple variable x was multiplied by 1 $(1 \bullet x)$ to equal $1x$. The $1x$ helped to clarify the number of variables when combining like terms on line four.

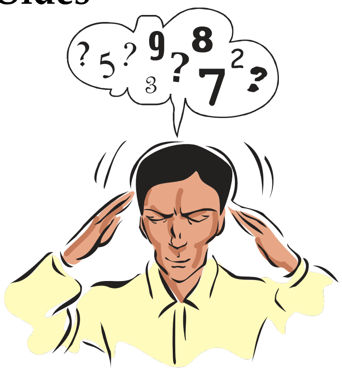
Check solution in the original equation:

$x - (2x + 3)$	$=$	4	
$-7 - (2 \bullet -7 + 3)$	$=$	4	
$-7 - (-11)$	$=$	4	
4	$=$	4	*It checks!*

Lesson Three Purpose

- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, distributive, associative, and transitive, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Use systems of equations and inequalities to solve real-world problems algebraically. (MA.D.2.4.2)
- Interpret data that has been collected, organized, and displayed in tables. (MA.E.1.4.1)

Solving Equations with Variables on Both Sides



I am thinking of a number. If you multiply my number by 3 and then subtract 2, you get the same answer that you do when you add 4 to my number. What is my number?

To solve this riddle, begin by translating these words into an algebraic sentence. Let x represent my number.

If you multiply my number by 3 and then subtract 2
$3x - 2$

you get the same answer
$=$

that you do when you add 4 to my number
$4 + x$

Putting it all together, we get the equation $3x - 2 = 4 + x$. Note that this equation is different from equations in previous units. There is a *variable* on both sides. To solve such an equation, we do what we've done in the past. Make sure both sides are simplified, and that there are no parentheses.

Strategy: Collect the variables on one side. Collect the numbers on the other side.

Now let's go back to the equation which goes with our riddle.

Solve:

$3x - 2 = 4 + x$	both sides are simplified
$3x - 2 = 4 + 1x$	multiplicative identity of 1
$3x - 2 - 1x = 4 + 1x - 1x$	collect variables on the left
$2x - 2 = 4$	combine like terms; simplify
$2x - 2 + 2 = 4 + 2$	collect numbers on the right
$\frac{2x}{2} = \frac{6}{2}$	divide both sides by 2
$x = 3$	

Check solution in the original equation and the original riddle:

$3x - 2 = 4 + x$	
$3 \bullet 3 - 2 = 4 + 3$	
$9 - 2 = 7$	
$7 = 7$	*It checks!*

Study the equation below.

Solve:

$2(3x + 4) = 5(x - 2)$	
$6x + 8 = 5x - 10$	distributive property
$6x + 8 - 5x = 5x - 10 - 5x$	variables on the left
$x + 8 = -10$	simplify
$x + 8 - 8 = -10 - 8$	numbers on the right
$x = -18$	

Check solution in the original equation:

$2(3x + 4) = 5(x - 2)$	
$2(3 \bullet -18 + 4) = 5(-18 - 2)$	
$2(-50) = 5(-20)$	
$-100 = -100$	*It checks!*

Let's work this next example in two different ways.

1. Collect the *variables* on the *left* and the *numbers* on the *right*.

Solve:

$6y = 4(5y - 7)$	
$6y = 20y - 28$	distributive property
$6y - 20y = 20y - 28 - 20y$	variables on the left
$\frac{-14y}{-14} = \frac{-28}{-14}$	divide both sides by -14
$y = 2$	

Check solution in the original equation:

$6y = 4(5y - 7)$	
$6 \bullet 2 = 4(5 \bullet 2 - 7)$	
$12 = 4 \bullet 3$	
$12 = 12$	*It checks!*

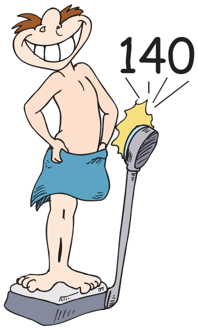
2. Collect the *variables* on the *right* and *numbers* on the *left*.

Solve:

$6y = 4(5y - 7)$	
$6y = 20y - 28$	distributive property
$6y - 6y = 20y - 28 - 6y$	variables on the right
$0 = 14y - 28$	simplify
$0 + 28 = 14y - 28 + 28$	numbers on the left
$\frac{28}{14} = \frac{14y}{14}$	divide both sides by 14
$2 = y$	

We get the *same* answer, so the choice of which side you put the variable on is up to you!

Problems That Lead to Equations



Joshua presently weighs 100 pounds, but is on a diet where he gains 2 pounds per week. When will he weigh 140 pounds?

The answer is 20 weeks. Let's use this simple problem to help us think algebraically.

Step 1: Read the problem and label the variable. Underline all clues.

Joshua presently weighs 100 pounds, but is on a diet where he gains 2 pounds per week. When will he weigh 140 pounds?

Let x represent the number of weeks.

Step 2: Plan.

Let $2x$ represent the weight Joshua will gain.

Step 3: Write the equation.

present weight	+ gain	= desired weight
100	+ $2x$	= 140

Step 4: Solve the equation.

$$100 + 2x = 140$$
$$100 + 2x - 100 = 140 - 100$$ add -100 to both sides
$$2x = 40$$
$$\frac{2x}{2} = \frac{40}{2}$$ divide both sides by 2
$$x = 20$$

Step 5: Check your solution. Does your answer make sense?

100 (now) + 2(20) weight gain = 140

We will use this 5-step approach on the following problems. You will find that many times a picture or chart will also help you arrive at an answer. Remember, we are learning to think algebraically and to do that the procedure is as important as the final answer!

5-Step Plan for Thinking Algebraically

Step 1: **Read** the problem and **label** the variable. **Underline** all clues.
Decide what *x* represents.

Step 2: **Plan.**

Step 3: **Write** an equation.

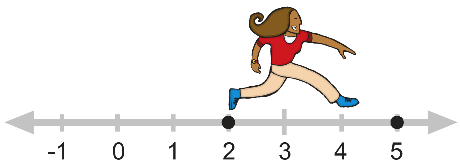
Step 4: **Solve** the equation.

Step 5: **Check** your solution. Does your answer make sense?

Lesson Four Purpose

- Understand the relative size of integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.2)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, and absolute value. (MA.A.1.4.4)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Use systems of equations and inequalities to solve real-world problems algebraically. (MA.D.2.4.2)

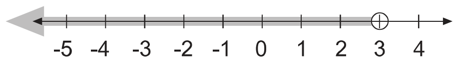
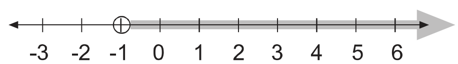
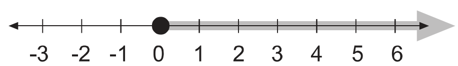
Graphing Inequalities on a Number Line



In Unit 1 we compared numbers using **inequality** symbols. We also solved simple *inequalities* and added numbers using a **number line**. In this unit we will graph inequalities on a number line. A **graph of a number** is the point on a number line paired with the number. Graphing solutions on a number line will help you visualize solutions.

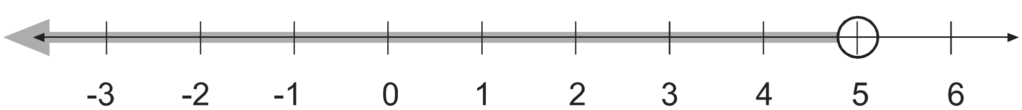
Here are some examples of inequalities, their verbal meanings, and their graphs.

Inequalities

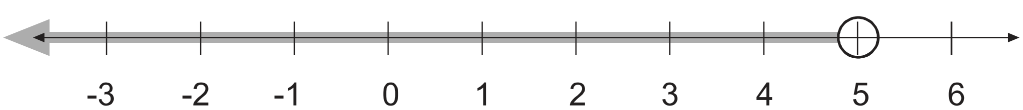
Inequality	Meaning	Graph
a. $x < 3$	All real numbers less than 3.	 Open circle means that 3 is *not* a solution. Shade to left.
b. $x > -1$	All real numbers greater than -1.	 Open circle means that -1 is *not* a solution. Shade to right.
c. $x \leq 2$	All real numbers less than or equal to 2.	 Solid circle means that 2 is a solution. Shade to left.
d. $x \geq 0$	All real numbers greater than or equal to 0.	 Solid circle means that 0 is a solution. Shade to right.

For each example, the inequality is written with the variable on the left. Inequalities can also be written with the variable on the right. However, graphing is easier if the variable is on the left.

Consider $x < 5$, which means the same as $5 > x$. Note that the graph of $x < 5$ is all real numbers less than 5.



The graph of $5 > x$ is all real numbers that 5 is greater than.



To write an inequality that is *equivalent* to (or the same as) $x < 5$, move the number and variable to the opposite side of the inequality, and then reverse the inequality.

$$x < 5$$
$$5 > x$$

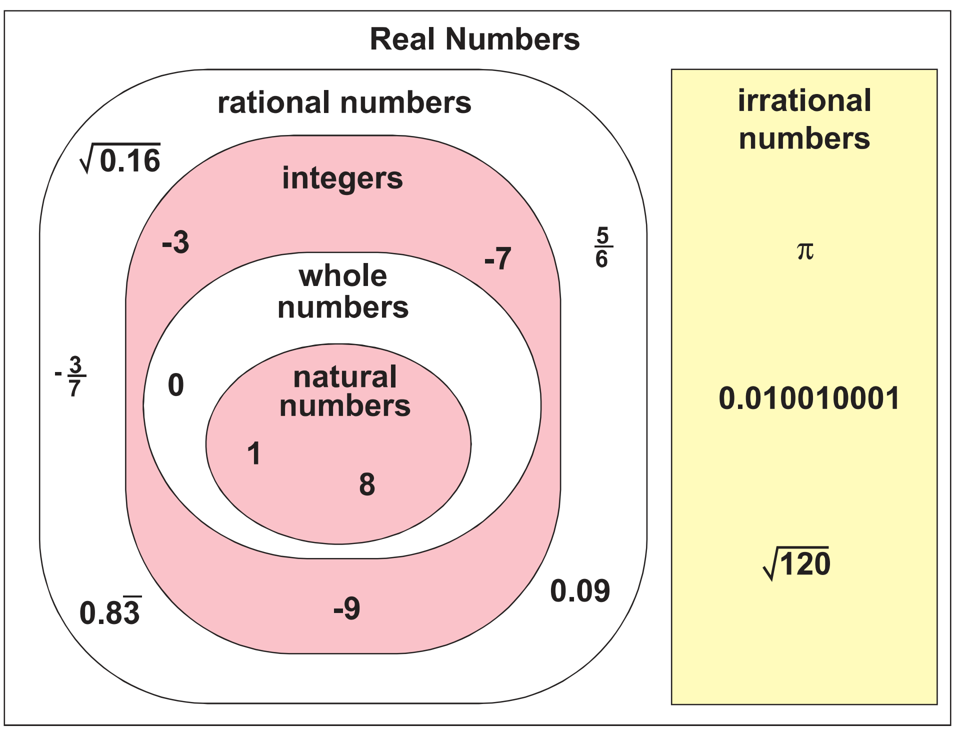
$x < 5$ means the same as

$5 > x$

The inequality $y \geq -2$ is equivalent to $-2 \leq y$. Both inequalities can be written as the *set of all* **real numbers** that are *greater than or equal to* -2.

The inequality $0 \leq x$ is equivalent to $x \geq 0$. Each can be written as the *set of all real numbers* that are *greater than or equal to* zero.

Remember: *Real numbers* are all **rational numbers** and all **irrational numbers**.



Rational numbers can be expressed as a **ratio** of two *integers*.

rational numbers	4	$-3\frac{3}{4}$	0.250	0	$0.33\bar{3}$
expressed as ratio of two integers	$\frac{4}{1}$	$-\frac{15}{4}$	$\frac{1}{4}$	$\frac{0}{1}$	$\frac{1}{3}$

Note: All integers are rational numbers.

A *ratio* is a *quotient* (the result of a division) of two numbers used to compare two quantities. For example, a ratio of 8 to 11 is $\frac{8}{11}$.

Solving Inequalities

We have been solving *equations* since Unit 1. When we solve inequalities, the procedures are the same except for one important difference.

> **When we multiply or divide both sides of an inequality by the same *negative number*, we *reverse* the direction of the inequality symbol.**

Example: Solve by *dividing* by a *negative number* and *reversing* the inequality sign.

$$-3x < 6$$

$$\frac{-3x}{-3} > \frac{6}{-3}$$ divide each side by -3 and reverse the inequality symbol

$$x > -2$$

To check this solution, pick any number *greater than* -2 and substitute your choice into the original inequality. For instance, -1, 0, or 3, or 3,000 could be substituted into the original problem.

Check with different solutions of numbers *greater than* -2:

substitute -1

$-3x < 6$
$-3(-1) < 6$
$3 < 6$ *It checks!*

substitute 3

$-3x < 6$
$-3(3) < 6$
$-9 < 6$ *It checks!*

substitute 0

$-3x < 6$
$-3(0) < 6$
$0 < 6$ *It checks!*

substitute 3,000

$-3x < 6$
$-3(3{,}000) < 6$
$-9{,}000 < 6$ *It checks!*

Notice that -1, 0, 3, and 3,000 are all *greater than* -2 and each one *checks* as a solution.

Study the following examples.

Example: Solve by *multiplying* by a *negative number* and *reversing* the inequality sign.

$$-\frac{1}{3}y \geq 4$$

$$(-3) -\frac{1}{3}y \leq 4(-3)$$ multiply each side by -3 and reverse the inequality symbol

$$y \leq -12$$

Example: Solve by first adding, then *dividing* by a *negative number*, and *reversing* the inequality sign.

$$-3a - 4 > 2$$

$$-3a - 4 + 4 > 2 + 4$$ add 4 to each side

$$-3a > 6$$

$$\frac{-3a}{-3} < \frac{6}{-3}$$ divide each side by -3 and reverse the inequality symbol

$$a < -2$$

Example: Solve by first subtracting, then *multiplying* by a *negative number*, and *reversing* the inequality sign.

$$\frac{y}{-2} + 5 \leq 0$$

$$\frac{y}{-2} + 5 - 5 \leq 0 - 5$$ subtract 5 from each side

$$\frac{y}{-2} \leq -5$$

$$\frac{(-2)y}{-2} \geq (-5)(-2)$$ multiply each side by -2 and reverse the inequality symbol

$$y \geq 10$$

Example: Solve by first subtracting, then *multiplying* by a *positive number* and **not** *reversing* the inequality sign.

$\frac{n}{2} + 5 \leq 2$	
$\frac{n}{2} + 5 - 5 \leq 2 - 5$	subtract 5 from each side
$\frac{n}{2} \leq -3$	
$\frac{(2)n}{2} \leq -3(2)$	multiply each side by 2, *but*
$n \leq -6$	do *not* reverse the inequality symbol because we multiplied by a positive number

When multiplying or dividing both sides of an inequality by the same *positive number*, do *not* reverse the inequality symbol—leave it alone.

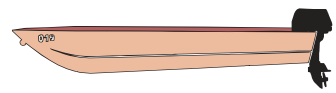
Example: Solve by first adding, then *dividing* by a positive number, and **not** *reversing* the inequality sign.

$7x - 3 > -24$	
$7x - 3 + 3 > -24 + 3$	add 3 to each side
$7x > -21$	
$\frac{7x}{7} > \frac{-21}{7}$	divide each side by 7
$x > -3$	do *not* reverse the inequality symbol because we divided by a positive number

Study the following.

Many problems in everyday life involve inequalities.

Example: A summer camp needs a boat with a motor. A local civic club will donate the money on the condition that the camp will spend *less than* $1,500 for both. The camp decides to buy a boat for $1,050. How much can be spent on the motor?



Choose a variable. *Let*	x	= cost of the motor,
then let	$x + 1{,}050$	= cost of motor and boat,
and cost of motor +	cost of boat	< total money.
Write as an inequality:	$x + 1{,}050$	$< 1{,}500$
solve	$x + 1{,}050 - 1{,}050$	$< 1{,}500 - 1{,}050$
	x	$<$ \$450

Interpretation of solution: The camp can spend *any* amount *less than* $450 for the motor. (**Note:** The motor *cannot* cost $450.)

For the following:

- **choose a variable**
- **set up an inequality**
- **solve**
- **interpret your solution**.