enVision® Florida

B.E.S.T. ALGEBRA 1

Student Edition

enVision® Florida

B.E.S.T. ALGEBRA 1

SAVVAS
LEARNING COMPANY

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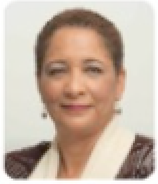

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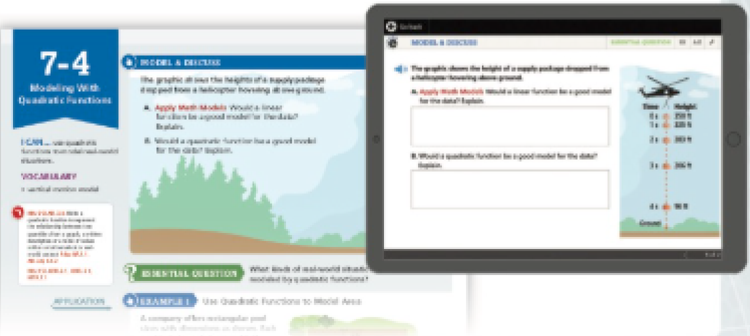
- Consultant to the writers of the PARCC Content Frameworks for high school mathematics

enVision® Florida B.E.S.T. Algebra 1 offers a carefully constructed lesson design to help you succeed in math.

Step 1 At the start of each lesson, you and your classmates will work together to come up with a solution strategy for the problem or task posed. After a class discussion, you'll be asked to reflect back on the processes and strategies you used in solving the problem.

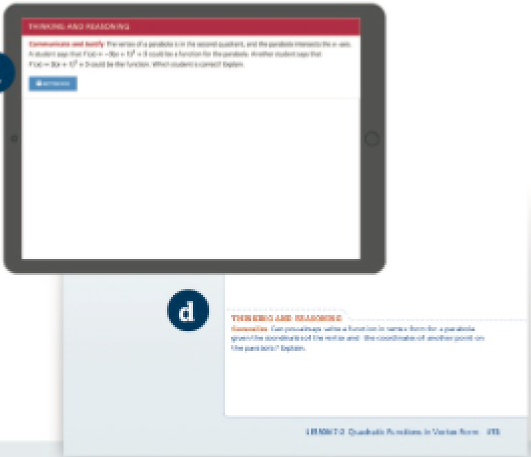


Step 2 Next, your teacher will guide you through new concepts and skills for the lesson.

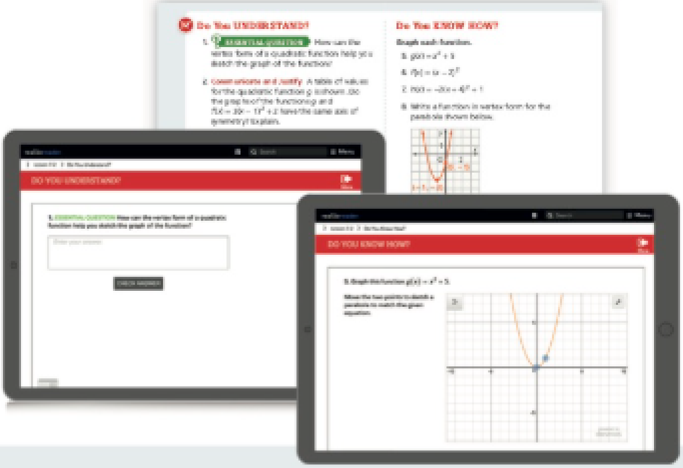
After each example ⓐ, you work out a problem called the **Try It!** ⓑ to solidify your understanding of these concepts.

Side notes ⓒ help you with study tips, suggestions for avoiding common errors, and questions that support learning together and having a growth mindset.

In addition, you will periodically answer **Thinking and Reasoning** ⓓ questions to refine your thinking and problem-solving skills.



Step 2 cont. This part of the lesson concludes with a Lesson Check that helps you to know how well you are understanding the new content presented in the lesson. With the exercises in the **Do You Understand?** and **Do You Know How?**, you can gauge your understanding of the lesson concepts.


Do You UNDERSTAND?

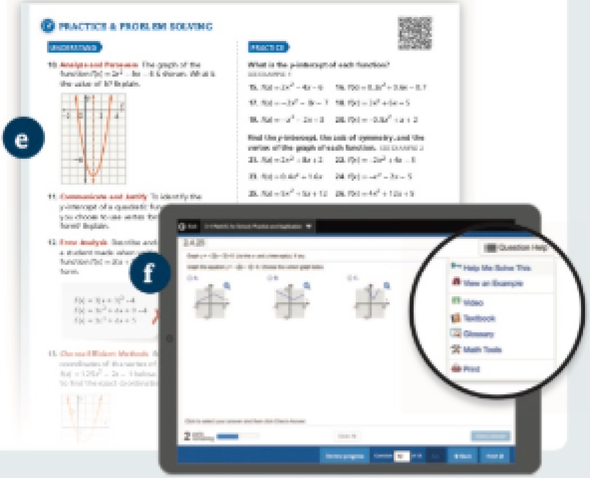
Do You KNOW HOW?

DO YOU UNDERSTAND?

DO YOU KNOW HOW?

Step 3 In Step 3, you will find a balanced exercise set with **Understand** exercises that focus on conceptual understanding, **Practice** exercises that target procedural fluency, and **Apply** exercises for which you apply concept and skills to real-world situations ⓔ.

The **Assessment and Practice** ⓕ exercises offer practice for high stakes assessments. Your teacher may have you complete the assignment in your Student Edition, Student Companion, or online at SavvasRealize.com.

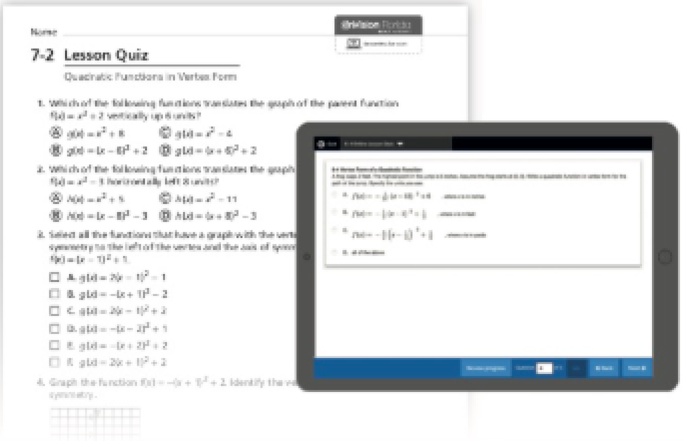


PRACTICE & PROBLEM SOLVING

UNDERSTAND

PRACTICE

Step 4 Your teacher may have you take the Lesson Quiz after each lesson. You can take the quiz online or in print. To do your best on the quiz, review the lesson problems in that lesson.


Name

7-2 Lesson Quiz

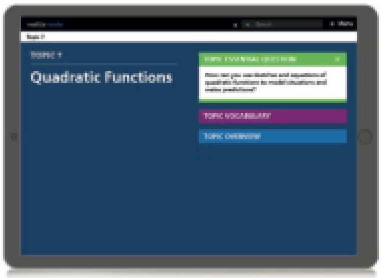
Quadratic Functions in Vertex Form

Digital Resources

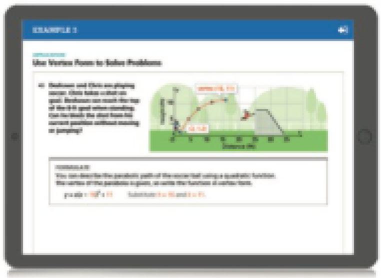
Everything you need for math, anytime, anywhere.

SavvasRealize.com is your gateway to all of the digital resources for **enVision® Florida B.E.S.T. Algebra 1.**

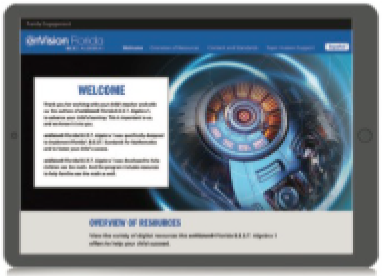
INTERACTIVE STUDENT EDITION Log in to access your interactive student edition, called Realize Reader.



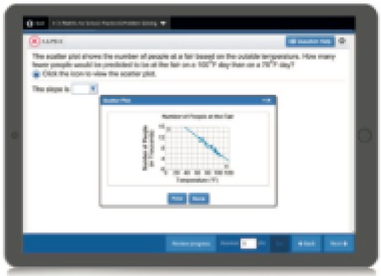
ANIMATION View and interact with real-world applications.



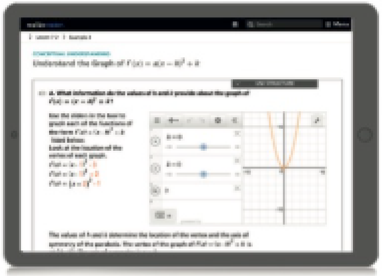
FAMILY ENGAGEMENT Involve family in your learning.



PRACTICE Practice what you've learned.

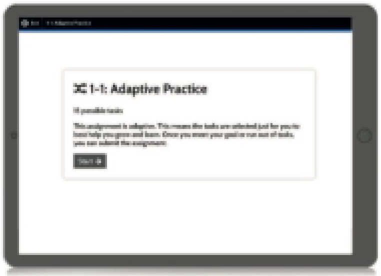


ACTIVITIES Complete *Explore & Reason*, *Model & Discuss*, *Critique & Explain* activities. Interact with *Examples* and *Try Its*.

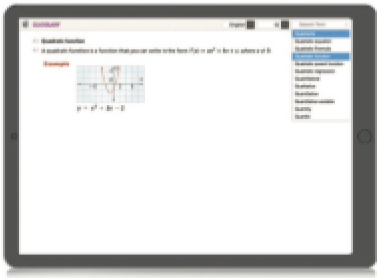


VIDEOS Watch clips to support Mathematical Modeling in 3 Acts Lessons and **enVision**® STEM Projects.

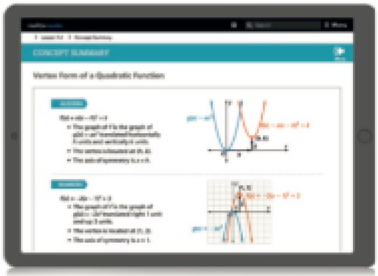
ADAPTIVE PRACTICE
Practice that is *just right* and *just for you.*


1-1: Adaptive Practice

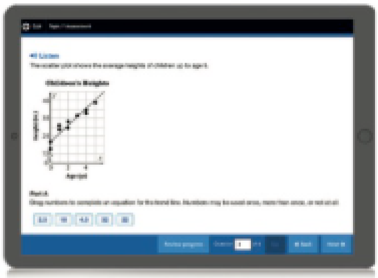
GLOSSARY Read and listen to English and Spanish definitions.



CONCEPT SUMMARY Review key lesson content through multiple representations.



ASSESSMENT
Show what you've learned.



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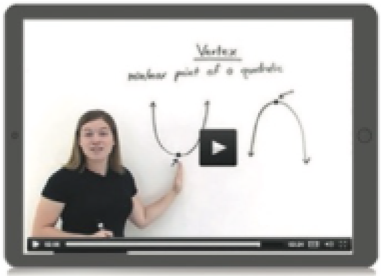

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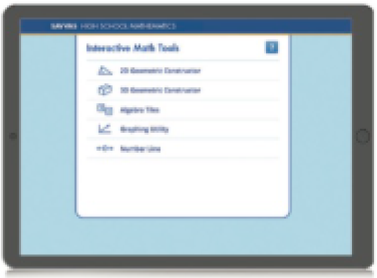
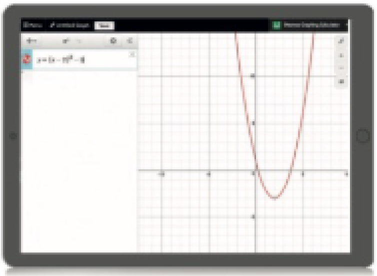

Interactive Math Tools

DESMOS Use Anytime and as embedded Interactives in Lesson content.



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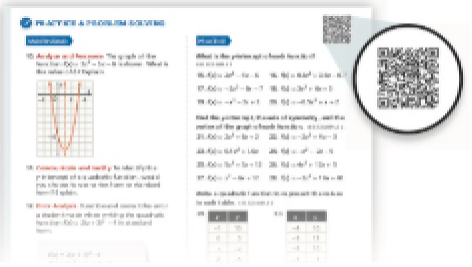

PRACTICE & PROBLEM SOLVING

Florida's B.E.S.T. Standards and Benchmarks

Number Sense and Operations

MA.912.NSO.1 Generate equivalent expressions and perform operations with expressions involving exponents, radicals, or logarithms.

MA.912.NSO.1.1 Extend previous understanding of the Laws of Exponents to include rational exponents. Apply the Laws of Exponents to evaluate numerical expressions and generate equivalent numerical expressions involving rational exponents.

Clarification 1: Instruction includes the use of technology when appropriate.

Clarification 2: Refer to the K-12 Formulas (Appendix E) for the Laws of Exponents.

Clarification 3: Instruction includes converting between expressions involving rational exponents and expressions involving radicals.

Clarification 4: Within the Mathematics for Data and Financial Literacy course, it is not the expectation to generate equivalent numerical expressions.

MA.912.NSO.1.2 Generate equivalent algebraic expressions using the properties of exponents.

Example: The expression 1.5^{3t+2} is equivalent to the expression $2.25(1.5)^{3t}$ which is equivalent to $2.25(3.375)^{t}$.

MA.912.NSO.1.4 Apply previous understanding of operations with rational numbers to add, subtract, multiply and divide numerical radicals.

Algebra 1 Example: The expression $\frac{\sqrt{136}}{\sqrt{2}}$ is equivalent to $\sqrt{\frac{136}{2}}$ which is equivalent to $\sqrt{68}$ which is equivalent to $2\sqrt{17}$.

Clarification 1: Within the Algebra 1 course, expressions are limited to a single arithmetic operation involving two square roots or two cube roots.

Algebraic Reasoning

MA.912.AR.1 Interpret and rewrite algebraic expressions and equations in equivalent forms.

MA.912.AR.1.1 Identify and interpret parts of an equation or expression that represent a quantity in terms of a mathematical or real-world context, including viewing one or more of its parts as a single entity.

Algebra 1 Example: Derrick is using the formula $P = 1000(1 + .1)^{t}$ to make a prediction about the camel population in Australia. He identifies the growth factor as $(1 + .1)$, or 1.1, and states that the camel population will grow at an annual rate of 10% per year.

Example: The expression 1.15^{t} can be rewritten as $\left(1.15\tfrac{1}{12}\right)^{12t}$ which is approximately equivalent to 1.012^{12t}. This latter expression reveals the approximate equivalent monthly interest rate of 1.2% if the annual rate is 15%.

Clarification 1: Parts of an expression include factors, terms, constants, coefficients and variables.

Clarification 2: Within the Mathematics for Data and Financial Literacy course, problem types focus on money and business.

MA.912.AR.1.2 Rearrange equations or formulas to isolate a quantity of interest.

Algebra 1 Example: The Ideal Gas Law $PV = nRT$ can be rearranged as $T = \frac{PV}{nR}$ to isolate temperature as the quantity of interest.

Example: Given the Compound Interest formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, solve for P.

Mathematics for Data and Financial Literacy Honors Example: Given the Compound Interest formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, solve for t.

Clarification 1: Instruction includes using formulas for temperature, perimeter, area and volume; using equations for linear (standard, slope-intercept and point-slope forms) and quadratic (standard, factored and vertex forms) functions.

Clarification 2: Within the Mathematics for Data and Financial Literacy course, problem types focus on money and business.

MA.912.AR.1.3 Add, subtract and multiply polynomial expressions with rational number coefficients.

Clarification 1: Instruction includes an understanding that when any of these operations are performed with polynomials the result is also a polynomial.

Clarification 2: Within the Algebra 1 course, polynomial expressions are limited to 3 or fewer terms.

MA.912.AR.1.4 Divide a polynomial expression by a monomial expression with rational number coefficients.

Clarification 1: Within the Algebra 1 course, polynomial expressions are limited to 3 or fewer terms.

MA.912.AR.1.7 Rewrite a polynomial expression as a product of polynomials over the real number system.

Example: The expression $4x^3y - 3x^2y^4$ is equivalent to the factored form $x^2y\,(4x - 3y^3)$.

Example: The expression $16x^2 - 9y^2$ is equivalent to the factored form $(4x - 3y)\,(4x + 3y)$.

Clarification 1: Within the Algebra 1 course, polynomial expressions are limited to 4 or fewer terms with integer coefficients.

MA.912.AR.2 Write, solve and graph linear equations, functions and inequalities in one and two variables.

MA.912.AR.2.1 Given a real-world context, write and solve one-variable multi-step linear equations.

MA.912.AR.2.2 Write a linear two-variable equation to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.

Clarification 1: Instruction includes the use of standard form, slope-intercept form and point-slope form, and the conversion between these forms.

MA.912.AR.2.3 Write a linear two-variable equation for a line that is parallel or perpendicular to a given line and goes through a given point.

Clarification 1: Instruction focuses on recognizing that perpendicular lines have slopes that when multiplied result in –1 and that parallel lines have slopes that are the same.

Clarification 2: Instruction includes representing a line with a pair of points on the coordinate plane or with an equation.

Clarification 3: Problems include cases where one variable has a coefficient of zero.

MA.912.AR.2.4 Given a table, equation or written description of a linear function, graph that function, and determine and interpret its key features.

Clarification 1: Key features are limited to domain, range, intercepts and rate of change.

Clarification 2: Instruction includes the use of standard form, slope-intercept form and point-slope form.

Clarification 3: Instruction includes cases where one variable has a coefficient of zero.

Clarification 4: Instruction includes representing the domain and range with inequality notation, interval notation or set-builder notation.

Clarification 5: Within the Algebra 1 course, notations for domain and range are limited to inequality and set-builder notations.

MA.912.AR.2.5 Solve and graph mathematical and real-world problems that are modeled with linear functions. Interpret key features and determine constraints in terms of the context.

Algebra 1 Example: Lizzy's mother uses the function $C(p) = 450 + 7.75p$, where $C(p)$ represents the total cost of a rental space and p is the number of people attending, to help budget Lizzy's 16th birthday party. Lizzy's mom wants to spend no more than \$850 for the party. Graph the function in terms of the context.

Clarification 1: Key features are limited to domain, range, intercepts and rate of change.

Clarification 2: Instruction includes the use of standard form, slope-intercept form and point-slope form.

Clarification 3: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

Clarification 4: Within the Algebra 1 course, notations for domain, range and constraints are limited to inequality and set-builder.

Clarification 5: Within the Mathematics for Data and Financial Literacy course, problem types focus on money and business.

MA.912.AR.2.6 Given a mathematical or real-world context, write and solve one-variable linear inequalities, including compound inequalities. Represent solutions algebraically or graphically.

Algebra 1 Example: The compound inequality $2x \leq 5x + 1 < 4$ is equivalent to $-1 \leq 3x$ and $5x < 3$, which is equivalent to $\frac{-1}{3} \leq x \leq \frac{3}{5}$.

MA.912.AR.2.7 Write two-variable linear inequalities to represent relationships between quantities from a graph or a written description within a mathematical or real-world context.

Clarification 1: Instruction includes the use of standard form, slope-intercept form and point-slope form and any inequality symbol can be represented.

Clarification 2: Instruction includes cases where one variable has a coefficient of zero.

MA.912.AR.2.8 Given a mathematical or real-world context, graph the solution set to a two-variable linear inequality.

Clarification 1: Instruction includes the use of standard form, slope-intercept form and point-slope form and any inequality symbol can be represented.

Clarification 2: Instruction includes cases where one variable has a coefficient of zero.

MA.912.AR.3 Write, solve and graph quadratic equations, functions and inequalities in one and two variables.

MA.912.AR.3.1 Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real number system.

Clarification 1: Within the Algebra 1 course, instruction includes the concept of non-real answers, without determining non-real solutions.

Clarification 2: Within this benchmark, the expectation is to solve by factoring techniques, taking square roots, the quadratic formula and completing the square.

MA.912.AR.3.4 Write a quadratic function to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.

Algebra I Example: Given the table of values below from a quadratic function, write an equation of that function.

x	-2	-1	0	1	2
$f(x)$	2	-1	-2	-1	2

Clarification 1: Within the Algebra 1 course, a graph, written description or table of values must include the vertex and two points that are equidistant from the vertex.

Clarification 2: Instruction includes the use of standard form, factored form and vertex form.

Clarification 3: Within the Algebra 2 course, one of the given points must be the vertex or an x-intercept.

MA.912.AR.3.5 Given the x-intercepts and another point on the graph of a quadratic function, write the equation for the function.

MA.912.AR.3.6 Given an expression or equation representing a quadratic function, determine the vertex and zeros and interpret them in terms of a real-world context.

MA.912.AR.3.7 Given a table, equation or written description of a quadratic function, graph that function, and determine and interpret its key features.

Clarification 1: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; vertex; and symmetry.

Clarification 2: Instruction includes the use of standard form, factored form and vertex form, and sketching a graph using the zeros and vertex.

Clarification 3: Instruction includes representing the domain and range with inequality notation, interval notation or set-builder notation.

Clarification 4: Within the Algebra 1 course, notations for domain and range are limited to inequality and set-builder.

MA.912.AR.3.8 Solve and graph mathematical and real-world problems that are modeled with quadratic functions. Interpret key features and determine constraints in terms of the context.

Algebra 1 Example: The value of a classic car produced in 1972 can be modeled by the function $V(t) = 19.25t^2 - 440t + 3500$, where t is the number of years since 1972. In what year does the car's value start to increase?

Clarification 1: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; vertex; and symmetry.

Clarification 2: Instruction includes the use of standard form, factored form and vertex form.

Clarification 3: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

Clarification 4: Within the Algebra 1 course, notations for domain, range and constraints are limited to inequality and set-builder.

MA.912.AR.4 **Write, solve and graph absolute value equations, functions and inequalities in one and two variables.**

MA.912.AR.4.1 Given a mathematical or real-world context, write and solve one-variable absolute value equations.

MA.912.AR.4.2 Given a mathematical or real-world context, write and solve one-variable absolute value inequalities. Represent solutions algebraically or graphically.

MA.912.AR.4.3 Given a table, equation or written description of an absolute value function, graph that function and determine its key features.

Clarification 1: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; vertex; end behavior and symmetry.

Clarification 2: Instruction includes representing the domain and range with inequality notation, interval notation or set-builder notation.

Clarification 3: Within the Algebra 1 course, notations for domain and range are limited to inequality and set-builder.

MA.912.AR.5 **Write, solve and graph exponential and logarithmic equations and functions in one and two variables.**

MA.912.AR.5.3 Given a mathematical or real-world context, classify an exponential function as representing growth or decay.

Clarification 1: Within the Algebra 1 course, exponential functions are limited to the forms $f(x) = ab^x$, where b is a whole number greater than 1 or a unit fraction, or $f(x) = a(1 \pm r)^x$, where $0 < r < 1$.

MA.912.AR.5.4 Write an exponential function to represent a relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.

Clarification 1: Within the Algebra 1 course, exponential functions are limited to the forms $f(x) = ab^x$, where b is a whole number greater than 1 or a unit fraction, or $f(x) = a(1 \pm r)^x$, where $0 < r < 1$.

Clarification 2: Within the Algebra 1 course, tables are limited to having successive nonnegative integer inputs so that the function may be determined by finding ratios between successive outputs.

MA.912.AR.5.6 Given a table, equation or written description of an exponential function, graph that function and determine its key features.

Clarification 1: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; constant percent rate of change; end behavior and asymptotes.

Clarification 2: Instruction includes representing the domain and range with inequality notation, interval notation or set-builder notation.

Clarification 3: Within the Algebra 1 course, notations for domain and range are limited to inequality and set-builder.

Clarification 4: Within the Algebra 1 course, exponential functions are limited to the forms $f(x) = ab^x$, where b is a whole number greater than 1 or a unit fraction, or $f(x) = a(1 \pm r)^x$, where $0 < r < 1$.

MA.912.AR.9 **Write and solve a system of two- and three-variable equations and inequalities that describe quantities or relationships.**

MA.912.AR.9.1 Given a mathematical or real-world context, write and solve a system of two-variable linear equations algebraically or graphically.

Clarification 1: Within this benchmark, the expectation is to solve systems using elimination, substitution and graphing.

Clarification 2: Within the Algebra 1 course, the system is limited to two equations.

MA.912.AR.9.4 Graph the solution set of a system of two-variable linear inequalities.

Clarification 1: Instruction includes cases where one variable has a coefficient of zero.

Clarification 2: Within the Algebra 1 course, the system is limited to two inequalities.

MA.912.AR.9.6 Given a real-world context, represent constraints as systems of linear equations or inequalities. Interpret solutions to problems as viable or non-viable options.

Clarification 1: Instruction focuses on analyzing a given function that models a real-world situation and writing constraints that are represented as linear equations or linear inequalities.

Functions

MA.912.F.1 Understand, compare and analyze properties of functions.

MA.912.F.1.1 Given an equation or graph that defines a function, determine the function type. Given an input-output table, determine a function type that could represent it.

Clarification 1: Within the Algebra 1 course, functions represented as tables are limited to linear, quadratic and exponential.

Clarification 2: Within the Algebra 1 course, functions represented as equations or graphs are limited to vertical or horizontal translations or reflections over the x-axis of the following parent functions: $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) = |x|$, $f(x) = 2^x$, and $f(x) = \left(\frac{1}{2}\right)^x$.

MA.912.F.1.2 Given a function represented in function notation, evaluate the function for an input in its domain. For a real-world context, interpret the output.

Algebra 1 Example: The function $f(x) = \frac{x}{2} - 8$ models Alicia's position in miles relative to a water stand x minutes into a marathon. Evaluate and interpret for a quarter of an hour into the race.

Clarification 1: Problems include simple functions in two-variables, such as $f(x, y) = 3x - 2y$.

Clarification 2: Within the Algebra 1 course, functions are limited to one-variable such as $f(x) = 3x$.

MA.912.F.1.3 Calculate and interpret the average rate of change of a real-world situation represented graphically, algebraically or in a table over a specified interval.

Clarification 1: Instruction includes making the connection to determining the slope of a particular line segment.

MA.912.F.1.5 Compare key features of linear functions each represented algebraically, graphically, in tables or written descriptions.

Clarification 1: Key features are limited to domain; range; intercepts; slope and end behavior.

MA.912.F.1.6 Compare key features of linear and nonlinear functions each represented algebraically, graphically, in tables or written descriptions.

Clarification 1: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior and asymptotes.

Clarification 2: Within the Algebra 1 course, functions other than linear, quadratic or exponential must be represented graphically.

Clarification 3: Within the Algebra 1 course, instruction includes verifying that a quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically.

MA.912.F.1.8 Determine whether a linear, quadratic or exponential function best models a given real-world situation.

Clarification 1: Instruction includes recognizing that linear functions model situations in which a quantity changes by a constant amount per unit interval; that quadratic functions model situations in which a quantity increases to a maximum, then begins to decrease or a quantity decreases to a minimum, then begins to increase; and that exponential functions model situations in which a quantity grows or decays by a constant percent per unit interval.

Clarification 2: Within this benchmark, the expectation is to identify the type of function from a written description or table.

MA.912.F.2 Identify and describe the effects of transformations on functions. Create new functions given transformations.

MA.912.F.2.1 Identify the effect on the graph or table of a given function after replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$ and $f(x + k)$ for specific values of k.

Clarification 1: Within the Algebra 1 course, functions are limited to linear, quadratic and absolute value.

Clarification 2: Instruction focuses on including positive and negative values for k.

MA.912.F.2.3 Given the graph or table of $f(x)$ and the graph or table of $f(x) + k$, $kf(x)$, $f(kx)$ and $f(x + k)$, state the type of transformation and find the value of the real number k.

Clarification 1: Within the Algebra 1 course, functions are limited to linear, quadratic and absolute value.

MA.912.F.3 Create new functions from existing functions.

MA.912.F.3.1 Given a mathematical or real-world context, combine two functions, limited to linear and quadratic, using arithmetic operations. When appropriate, include domain restrictions for the new function.

Example: The quotient of the functions $f(x) = 3x^2 - 7x + 3$ and $g(x) = 6x - 1$ can be expressed as $h(x) = \frac{(3x^2 - 7x + 3)}{(6x - 1)}$, where the domain of $h(x)$ is $-\infty \leq x < \frac{1}{6}$ and $\frac{1}{6} < x \leq \infty$.

Clarification 1: Instruction includes representing domain restrictions with inequality notation, interval notation or set-builder notation.

Clarification 2: Within the Algebra 1 Honors course, notations for domain and range are limited to inequality and set-builder.

Financial Literacy

MA.912.FL.3 Describe the advantages and disadvantages of short-term and long-term purchases.

MA.912.FL.3.2 Solve real-world problems involving simple, compound and continuously compounded interest.

Example: Find the amount of money on deposit at the end of 5 years if you started with $500 and it was compounded quarterly at 6% interest per year.

Example: Joe won $25,000 on a lottery scratch-off ticket. How many years will it take at 6% interest compounded yearly for his money to double?

Clarification 1: Within the Algebra 1 course, interest is limited to simple and compound.

MA.912.FL.3.4 Explain the relationship between simple interest and linear growth. Explain the relationship between compound interest and exponential growth and the relationship between continuously compounded interest and exponential growth.

Clarification 1: Within the Algebra 1 course, exponential growth is limited to compound interest.

Data Analysis and Probability

MA.912.DP.1 Summarize, represent and interpret categorical and numerical data with one and two variables.

MA.912.DP.1.1 Given a set of data, select an appropriate method to represent the data, depending on whether it is numerical or categorical data and on whether it is univariate or bivariate.

Clarification 1: Instruction includes discussions regarding the strengths and weaknesses of each data display.

Clarification 2: Numerical univariate includes histograms, stem-and-leaf plots, box plots and line plots; numerical bivariate includes scatter plots and line graphs; categorical univariate includes bar charts, circle graphs, line plots, frequency tables and relative frequency tables; and categorical bivariate includes segmented bar charts, joint frequency tables and joint relative frequency tables.

Clarification 3: Instruction includes the use of appropriate units and labels and, where appropriate, using technology to create data displays.

MA.912.DP.1.2 Interpret data distributions represented in various ways. State whether the data is numerical or categorical, whether it is univariate or bivariate and interpret the different components and quantities in the display.

Clarification 1: Within the Probability and Statistics course, instruction includes the use of spreadsheets and technology.

MA.912.DP.1.3 Explain the difference between correlation and causation in the contexts of both numerical and categorical data.

Algebra 1 Example: There is a strong positive correlation between the number of Nobel prizes won by country and the per capita chocolate consumption by country. Does this mean that increased chocolate consumption in America will increase the United States of America's chances of a Nobel prize winner?

MA.912.DP.1.4 Estimate a population total, mean or percentage using data from a sample survey; develop a margin of error through the use of simulation.

Algebra 1 Example: Based on a survey of 100 households in Twin Lakes, the newspaper reports that the average number of televisions per household is 3.5 with a margin of error of ±0.6. The actual population mean can be estimated to be between 2.9 and 4.1 television per household. Since there are 5,500 households in Twin Lakes the estimated number of televisions is between 15,950 and 22,550.

Clarification 1: Within the Algebra 1 course, the margin of error will be given.

MA.912.DP.2 Solve problems involving univariate and bivariate numerical data.

MA.912.DP.2.4 Fit a linear function to bivariate numerical data that suggests a linear association and interpret the slope and *y*-intercept of the model. Use the model to solve real-world problems in terms of the context of the data.

Clarification 1: Instruction includes fitting a linear function both informally and formally with the use of technology.

Clarification 2: Problems include making a prediction or extrapolation, inside and outside the range of the data, based on the equation of the line of fit.

MA.912.DP.2.5 Given a scatter plot that represents bivariate numerical data, assess the fit of a given linear function by plotting and analyzing residuals.

Clarification 1: Within the Algebra 1 course, instruction includes determining the number of positive and negative residuals; the largest and smallest residuals; and the connection between outliers in the data set and the corresponding residuals.

MA.912.DP.2.6 Given a scatter plot with a line of fit and residuals, determine the strength and direction of the correlation. Interpret strength and direction within a real-world context.

Clarification 1: Instruction focuses on determining the direction by analyzing the slope and informally determining the strength by analyzing the residuals.

MA.912.DP.3 **Solve problems involving categorical data.**

MA.912.DP.3.1 Construct a two-way frequency table summarizing bivariate categorical data. Interpret joint and marginal frequencies and determine possible associations in terms of a real-world context.

Algebra 1 Example: Complete the frequency table below.

	Has an A in math	Doesn't have an A in math	Total
Plays an instrument	20		90
Doesn't play an instrument	20		
Total			350

Using the information in the table, it is possible to determine that the second column contains the numbers 70 and 240. This means that there are 70 students who play an instrument but do not have an A in math and the total number of students who play an instrument is 90. The ratio of the joint frequencies in the first column is 1 to 1 and the ratio in the second column is 7 to 24, indicating a strong positive association between playing an instrument and getting an A in math.

MA.912.DP.3.2 Given marginal and conditional relative frequencies, construct a two-way relative frequency table summarizing categorical bivariate data.

Algebra 1 Example: A study shows that 9% of the population have diabetes and 91% do not. The study also shows that 95% of the people who do not have diabetes, test negative on a diabetes test while 80% who do have diabetes, test positive. Based on the given information, the following relative frequency table can be constructed.

	Positive	Negative	Total
Has diabetes	7.2%	1.8%	9%
Doesn't have diabetes	4.55%	86.45%	91%

Clarification 1: Construction includes cases where not all frequencies are given but enough are provided to be able to construct a two-way relative frequency table.

Clarification 2: Instruction includes the use of a tree diagram when calculating relative frequencies to construct tables.

MA.912.DP.3.3 Given a two-way relative frequency table or segmented bar graph summarizing categorical bivariate data, interpret joint, marginal and conditional relative frequencies in terms of a real-world context.

Algebra 1 Example: Given the relative frequency table below, the ratio of true positives to false positives can be determined as 7.2 to 4.55, which is about 3 to 2, meaning that a randomly selected person who tests positive for diabetes is about 50% more likely to have diabetes than not have it.

	Positive	Negative	Total
Has diabetes	7.2%	1.8%	9%
Doesn't have diabetes	4.55%	86.45%	91%

Clarification 1: Instruction includes problems involving false positive and false negatives.

Mathematical Thinking and Reasoning Standards

MA.K12.MTR.1.1 Actively participate in effortful learning both individually and collectively.

Mathematicians who participate in effortful learning both individually and with others:

- Analyze the problem in a way that makes sense given the task.
- Ask questions that will help with solving the task.
- Build perseverance by modifying methods as needed while solving a challenging task.
- Stay engaged and maintain a positive mindset when working to solve tasks.
- Help and support each other when attempting a new method or approach.

MA.K12.MTR.2.1 Demonstrate understanding by representing problems in multiple ways.

Mathematicians who demonstrate understanding by representing problems in multiple ways:

- Build understanding through modeling and using manipulatives.
- Represent solutions to problems in multiple ways using objects, drawings, tables, graphs and equations.
- Progress from modeling problems with objects and drawings to using algorithms and equations.
- Express connections between concepts and representations.
- Choose a representation based on the given context or purpose.

MA.K12.MTR.3.1 Complete tasks with mathematical fluency.

Mathematicians who complete tasks with mathematical fluency:

- Select efficient and appropriate methods for solving problems within the given context.
- Maintain flexibility and accuracy while performing procedures and mental calculations.
- Complete tasks accurately and with confidence.
- Adapt procedures to apply them to a new context.
- Use feedback to improve efficiency when performing calculations.

MA.K12.MTR.4.1 Engage in discussions that reflect on the mathematical thinking of self and others.

Mathematicians who engage in discussions that reflect on the mathematical thinking of self and others:

- Communicate mathematical ideas, vocabulary and methods effectively.
- Analyze the mathematical thinking of others.
- Compare the efficiency of a method to those expressed by others.
- Recognize errors and suggest how to correctly solve the task.
- Justify results by explaining methods and processes.
- Construct possible arguments based on evidence.

MA.K12.MTR.5.1 Use patterns and structure to help understand and connect mathematical concepts.

Mathematicians who use patterns and structure to help understand and connect mathematical concepts:

- Focus on relevant details within a problem.
- Create plans and procedures to logically order events, steps or ideas to solve problems.
- Decompose a complex problem into manageable parts.
- Relate previously learned concepts to new concepts.
- Look for similarities among problems.
- Connect solutions of problems to more complicated large-scale situations.

MA.K12.MTR.6.1 Assess the reasonableness of solutions.

Mathematicians who assess the reasonableness of solutions:

- Estimate to discover possible solutions.
- Use benchmark quantities to determine if a solution makes sense.
- Check calculations when solving problems.
- Verify possible solutions by explaining the methods used.
- Evaluate results based on the given context.

MA.K12.MTR.7.1 Apply mathematics to real-world contexts.

Mathematicians who apply mathematics to real-world contexts:

- Connect mathematical concepts to everyday experiences.
- Use models and methods to understand, represent and solve problems.
- Perform investigations to gather data or determine if a method is appropriate.
- Redesign models and methods to improve accuracy or efficiency.

ELA Expectations

ELA.K12.EE.1.1 Cite evidence to explain and justify reasoning.

Clarifications:
K-1 Students include textual evidence in their oral communication with guidance and support from adults. The evidence can consist of details from the text without naming the text. During 1st grade, students learn how to incorporate the evidence in their writing.

2-3 Students include relevant textual evidence in their written and oral communication. Students should name the text when they refer to it. In 3rd grade, students should use a combination of direct and indirect citations.

4-5 Students continue with previous skills and reference comments made by speakers and peers. Students cite texts that they've directly quoted, paraphrased, or used for information. When writing, students will use the form of citation dictated by the instructor or the style guide referenced by the instructor.

6-8 Students continue with previous skills and use a style guide to create a proper citation.

9-12 Students continue with previous skills and should be aware of existing style guides and the ways in which they differ.

ELA.K12.EE.2.1 Read and comprehend grade-level complex texts proficiently.

Clarifications:
See Text Complexity for grade-level complexity bands and a text complexity rubric.

ELA.K12.EE.3.1 Make inferences to support comprehension.

Clarifications:
Students will make inferences before the words infer or inference are introduced. Kindergarten students will answer questions like "Why is the girl smiling?" or make predictions about what will happen based on the title page. Students will use the terms and apply them in 2nd grade and beyond.

ELA.K12.EE.4.1 Use appropriate collaborative techniques and active listening skills when engaging in discussions in a variety of situations.

Clarifications:
In kindergarten, students learn to listen to one another respectfully.

In grades 1-2, students build upon these skills by justifying what they are thinking. For example: "I think ________ because _______." The collaborative conversations are becoming academic conversations.

In grades 3-12, students engage in academic conversations discussing claims and justifying their reasoning, refining and applying skills. Students build on ideas, propel the conversation, and support claims and counterclaims with evidence.

ELA.K12.EE.5.1 Use the accepted rules governing a specific format to create quality work.

Clarifications:
Students will incorporate skills learned into work products to produce quality work. For students to incorporate these skills appropriately, they must receive instruction. A 3rd grade student creating a poster board display must have instruction in how to effectively present information to do quality work.

ELA.K12.EE.6.1 Use appropriate voice and tone when speaking or writing.

Clarifications:
In kindergarten and 1st grade, students learn the difference between formal and informal language. For example, the way we talk to our friends differs from the way we speak to adults. In 2nd grade and beyond, students practice appropriate social and academic language to discuss texts.

English Language Development for English Language Learners

ELD.K12.ELL.MA.1 English language learners communicate information, ideas and concepts necessary for academic success in the content area of Mathematics.

TOPIC

1

Solving Equations and Inequalities

TOPIC

2

Linear Equations

TOPIC
3
Linear and Absolute Value Functions

TOPIC

4

Systems of Linear Equations and Inequalities

TOPIC

5

Exponents and Exponential Functions

TOPIC

6

Polynomials and Factoring

TOPIC

7

Quadratic Functions

TOPIC

8

Solving Quadratic Equations

TOPIC

9

Working With Functions

TOPIC

10

Analyzing Data

TOPIC

1

Solving Equations and Inequalities

TOPIC ESSENTIAL QUESTION

What general strategies can you use to solve simple equations and inequalities?

Topic Overview

enVision® STEM Project

Design a Smartphone

1-1 Solving Linear Equations
AR.2.1, MTR.1.1, MTR.4.1, MTR.6.1

1-2 Solving Equations with a Variable on Both Sides
AR.2.1, MTR.3.1, MTR.4.1, MTR.5.1

1-3 Literal Equations and Formulas
AR.1.2, MTR.1.1, MTR.2.1, MTR.5.1

1-4 Solving Inequalities in One Variable
AR.2.6, MTR.2.1, MTR.4.1, MTR.7.1

Mathematical Modeling in 3 Acts:

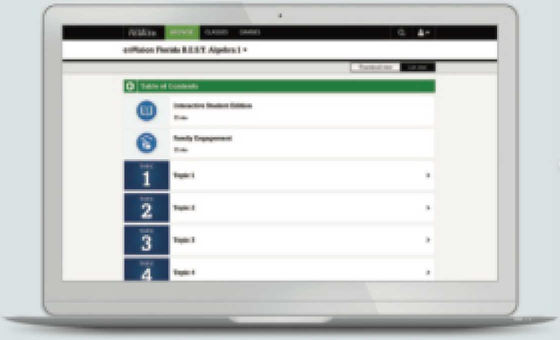
Collecting Cans
AR.2.1, AR.2.6, MTR.7.1

1-5 Compound Inequalities
AR.2.6, MTR.5.1, MTR.6.1, MTR.7.1

1-6 Absolute Value Equations and Inequalities
AR.4.1, AR.4.2, MTR.1.1, MTR.3.1, MTR.5.1

Topic Vocabulary

- compound inequality
- element of a set
- formula
- identity
- literal equation
- set
- subset



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Digital Experience

INTERACTIVE STUDENT EDITION
Access online or offline.

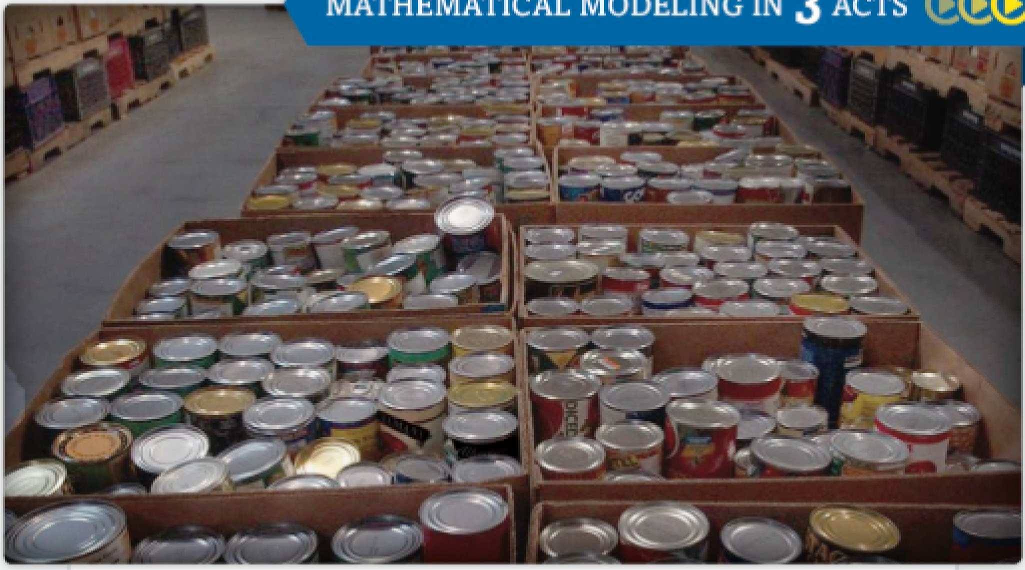
FAMILY ENGAGEMENT
Involve family in your learning.

ACTIVITIES Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its***.

ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Collecting Cans

Many schools and community centers organize canned food drives and donate the food collected to area food pantries or homeless shelters.

A teacher may hold a contest to see which student collects the most cans. The teacher will track the number of cans each student brings in. Sometimes students have their own ways of keeping track. You'll see how some students kept track in the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision® *STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from *Virtual Nerd*, right when you need it.

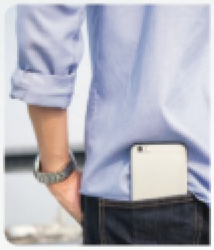
MATH TOOLS Explore math with digital tools and manipulatives.

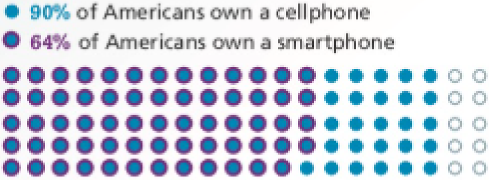
DESMOS Use Anytime and as embedded Interactives in Lesson content.

QR CODES Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

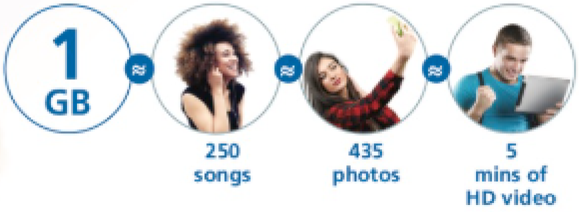
Did You Know?

The average American teenager spends about **9 hours each day** on a digital device.

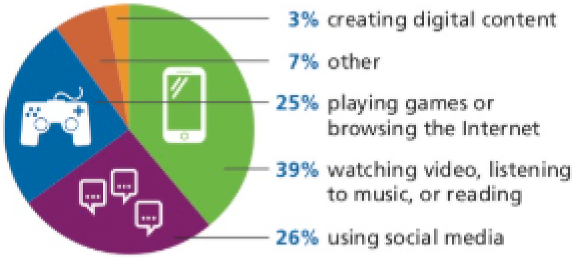




In general, people keep their cellphones on and with them at all times.



How Teens Spend Their Screen Time



Your Task: Design a Smartphone

Smartphones are many things to many people. You and your classmates will decide what a new smartphone will be able to do, how you want it to look and feel, and how much storage it should have.

1-1

Solving Linear Equations

I CAN… create and solve linear equations with one variable.

MA.912.AR.2.1–Given a real-world context, write and solve one-variable multi-step linear equations.

MA.K12.MTR.1.1, MTR.4.1, MTR.6.1

MODEL & DISCUSS

Joshua is going kayaking with a group during one of his vacation days. In his vacation planning, he budgeted \$50 for a kayak rental.

Rental Rates

	Per hour
single kayak	\$15
single sea kayak	\$18
double kayak	\$25

A. How can Joshua determine the number of hours he can rent a kayak for himself? Describe two different options.

B. Joshua found out that there is a \$25 nonrefundable equipment fee in addition to the hourly rates. How does this requirement change the mathematics of the situation?

C. **Choose Efficient Methods** How do the processes you used for parts A and B differ? How are they the same?

ESSENTIAL QUESTION

How do you create equations and use them to solve problems?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Solve Linear Equations

What is the value of x in the equation $\frac{2(x+4)}{3} - 8 = 32$?

VOCABULARY
Remember, a *variable* is an unknown quantity, or a quantity that can vary. An *equation* is a mathematical statement with two expressions set equal to each other. A *solution of an equation* is a value for the variable that makes the equation a true statement.

Method 1

$$\frac{2(x+4)}{3} - 8 = 32$$
$$2(x+4) - 24 = 96$$
$$2x + 8 - 24 = 96$$
$$2x - 16 = 96$$
$$2x = 112$$
$$x = 56$$

Multiply each side by 3 first.

OR

Method 2

$$\frac{2(x+4)}{3} - 8 = 32$$
$$\frac{2(x+4)}{3} = 40$$
$$2(x+4) = 120$$
$$x + 4 = 60$$
$$x = 56$$

Add 8 to each side first.

Each solving method yields the same solution. Is one method better than the other?

Look at how the expression on the left side of the original equation is built up from x.

$$x \rightarrow x+4 \rightarrow 2(x+4) \rightarrow \frac{2(x+4)}{3} \rightarrow \frac{2(x+4)}{3} - 8$$

Notice how Method 2 applies these steps in reverse to isolate x. This is often a good strategy and can lead to simpler solution methods.

Try It! 1. Solve the equation $4 + \frac{3x-1}{2} = 9$. Explain the reasons why you chose your solution method.

EXAMPLE 2 Solve Consecutive Integer Problems

The sum of three consecutive integers is 132. What are the three integers?

Write an equation to model the problem. Then solve.

$$x + (x + 1) + (x + 2) = 132$$

The three integers are consecutive, so each is 1 greater than the previous.

Combine like terms.

$$3x + 3 = 132$$

$$3x + 3 - 3 = 132 - 3$$

$$\frac{3x}{3} = \frac{129}{3}$$

$$x = 43$$

The first of the three consecutive numbers is 43.
The three consecutive numbers whose sum is 132 are 43, 44, 45.

REPRESENT AND CONNECT
How would the equation and solution be different if you let x be the middle number?

Try It! 2. The sum of three consecutive odd integers is 57. What are the three integers?

APPLICATION

EXAMPLE 3 Use Linear Equations to Solve Mixture Problems

A lab technician needs 25 liters of a solution that is 15% acid for a certain experiment, but she has only a solution that is 10% acid and a solution that is 30% acid. How many liters of the 10% and the 30% solutions should she mix to get what she needs?

Formulate Write an equation relating the number of liters of acid in each solution. Represent the total number of liters of one solution with a variable, like x. Then the total number of liters of the other solution must be $25 - x$.

$$\text{25 L of 15\% solution} = x \text{ L of 10\% solution} + (25 - x) \text{ L of 30\% solution}$$

$$(0.15)(25) = 0.10x + 0.30(25 - x)$$

Compute

$$3.75 = 0.1x + 7.5 - 0.3x$$

$$3.75 - 7.5 = 0.1x - 0.3x + 7.5 - 7.5$$

Subtract 7.5 from each side.

$$-3.75 = -0.2x$$

$$(-1\ )(-\ 3.75) = (-1)(-0.2x)$$

$$3.75 = 0.2x$$

$$\frac{3.75}{0.2} = \frac{0.2x}{0.2}$$

Multiply each side by −1, then divide each side by 0.2.

$$18.75 = x$$

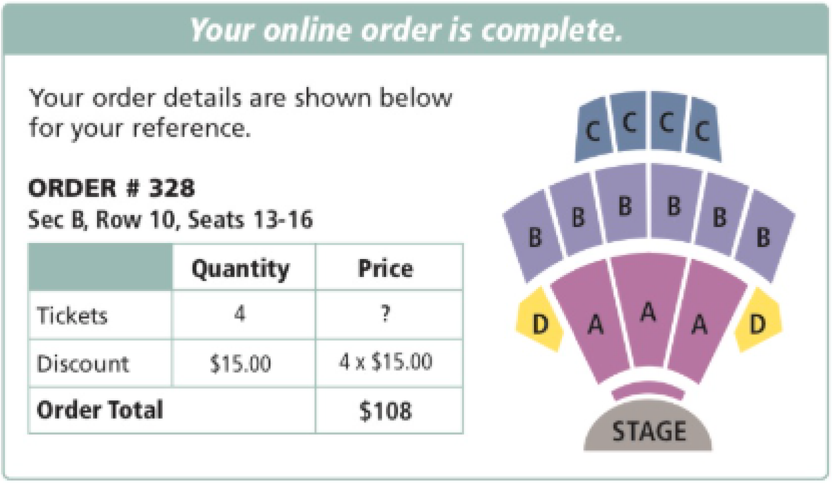
Interpret Since x represents the number of liters of the 10% acid solution, the lab technician should use 18.75 liters of the 10% solution. Since $25 - x$ represents the number of liters of the 30% acid solution, she should use 25 – 18.75, or 6.25 liters of the 30% solution.

Try It! 3. If the lab technician needs 30 liters of a 25% acid solution, how many liters of the 10% and the 30% acid solutions should she mix to get what she needs?

APPLICATION

EXAMPLE 4 Use Linear Equations to Solve Problems

Four friends use an online coupon to get discounts on concert tickets. They spent $108 for the four tickets. What was the price of one ticket without the discount?



Your online order is complete.

Your order details are shown below for your reference.

ORDER # 328
Sec B, Row 10, Seats 13-16

	Quantity	**Price**
Tickets	4	?
Discount	$15.00	4 x $15.00
Order Total		**$108**

Step 1 Write an equation to represent the problem situation.

Let p represent the original ticket price.

4 • original ticket price minus $15 = $108

$$4(p - 15) = 108$$

COMMON ERROR
Subtract 15 from the price of each ticket, not from the total cost of four undiscounted tickets.

Step 2 Solve the equation.

$$4(p - 15) = 108$$

$$\frac{4(p - 15)}{4} = \frac{108}{4}$$

$$p - 15 = 27$$

$$p - 15 + 15 = 27 + 15$$

$$p = 42$$

The ticket price without the discount was $42.

Try It! **4.** The same four friends buy tickets for two shows on consecutive nights. They use a coupon for $5 off each ticket. They pay a total of $416 for 8 tickets. Write and solve an equation to find the original price of the tickets.

APPLICATION

EXAMPLE 5 Solve Work and Time Problems

LaTanya will walk her bike from her house to the bike shop, which is 1.5 mi from her house, to get the bike fixed. After getting her bike repaired, she rides home. If the whole errand took one hour, how much time did LaTanya spend at the bike shop?

ANALYZE AND PERSEVERE
Look for relationships between the distance traveled and the rate when you write the equation.

Step 1 Write an equation to represent the situation.

Time walking + Time at the shop + Time biking = Total time

$$\frac{1.5 \text{ miles}}{3 \text{ miles per hour}} + t + \frac{1.5 \text{ miles}}{10 \text{ miles per hour}} = 1 \text{ hour}$$

The equation $\frac{1.5}{3} + t + \frac{1.5}{10} = 1$ represents the situation.

Step 2 Solve for t.

$$\frac{1.5}{3} + t + \frac{1.5}{10} = 1$$

$$(30)\frac{1.5}{3} + 30t + (30)\frac{1.5}{10} = 30$$

Multiply each side the least common denominator.

$$15 + 30t + 4.5 = 30$$

$$30t + 19.5 = 30$$

$$30t + 19.5 - 19.5 = 30 - 19.5$$

$$30t = 10.5$$

$$\frac{30t}{30} = \frac{10.5}{30}$$

$$t = 0.35$$

LaTanya spent 0.35 h, or 21 min at the bike shop.

Try It! **5.** LaTanya leaves her house at 12:30 P.M. and bikes at 12 mi/h to Marta's house. She stays at Marta's house for 90 min. Both girls walk back to LaTanya's house along the same route at 2.5 mi/h. They arrive at LaTanya's house at 3:30 P.M. How far is Marta's house from LaTanya's house?

CONCEPT SUMMARY Create and Solve Linear Equations

Use the following information about Kelsey's visit to the flower shop.

- Kelsey bought some roses and tulips.
- She bought twice as many tulips as roses.
- Roses cost \$5 each.
- Tulips cost \$2 each.
- Kelsey spent \$36 total.

How many of each kind of flower did Kelsey buy?

WORDS Write an equation to represent the situation.

Cost of Roses + Cost of Tulips = Total Cost

(Cost of One Rose)(Number of Roses) + (Cost of One Tulip)(Number of Tulips) = Total Cost

ALGEBRA $\$5 \cdot x + \$2 \cdot 2x = \$36$

$$5x + 4x = 36$$

$$9x = 36$$

$$x = 4$$

Kelsey bought 4 roses and 8 tulips.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do you create equations and use them to solve problems?

2. **Communicate and Justify** What is a first step to solving for x in the equation $9x - 7 = 10$? How would you check your solution?

3. **Use Patterns and Structure** For an equation with fractions, why is it helpful to multiply both sides of the equation by the LCD?

4. **Error Analysis** Venetta knows that 1 mi ≈ 1.6 km. To convert 5 mi/h to km/h, she multiplies 5 mi/h by $\frac{1 \text{ mi}}{1.6 \text{ km}}$. What error does Venetta make?

Do You KNOW HOW?

Solve each equation.

5. $4b + 14 = 22$

6. $-6k - 3 = 39$

7. $15 - 2(3 - 2x) = 46$

8. $\frac{2}{3}y - \frac{2}{5} = 5$

9. Terrence walks at a pace of 2 mi/h to the theater and watches a movie for 2 h and 15 min. He rides back home, taking the same route, on the bus that travels at a rate of 40 mi/h. The entire trip takes 3.5 h. How far along this route is Terrence's house from the theater? Explain.

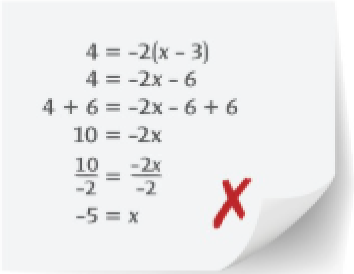
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Communicate and Justify** What could be a first step to solving the equation $3x + -0.5(x + 3) + 4 = 14$? Explain.

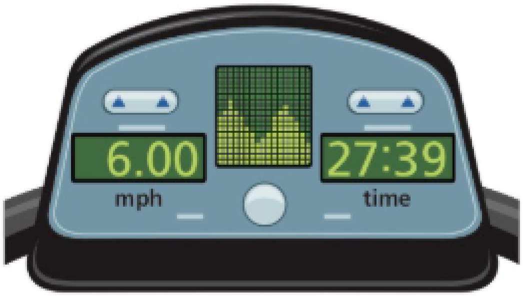
11. **Analyze and Persevere** The sum of four consecutive integers is –18. What is the greatest of these integers?

12. **Error Analysis** Describe and correct the error a student made when solving the equation $4 = -2(x - 3)$. What is the correct solution?



$$4 = -2(x - 3)$$
$$4 = -2x - 6$$
$$4 + 6 = -2x - 6 + 6$$
$$10 = -2x$$
$$\frac{10}{-2} = \frac{-2x}{-2}$$
$$-5 = x$$

13. **Communicate and Justify** Parker ran on a treadmill at a constant speed for the length of time shown. How many miles did Parker run? Explain.



14. **Use Patterns and Structure** The Division Property of Equality says that for every real number a, b, and c, if $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$. Why does the property require that $c \neq 0$?

15. **Higher Order Thinking** Tonya's first step in solving the equation $\frac{1}{2}(2y + 4) = -6$ is to use the Distributive Property on the left side of the equation. Deon's first step is to multiply each side by 2. Which of these methods will result in an equivalent equation? Explain.

PRACTICE

Solve each equation. SEE EXAMPLES 1 AND 2

16. $-4x + 3x = 2$

17. $7 = 5y - 13 - y$

18. $7m - 4 - 9m - 36 = 0$

19. $-2 = -5t + 10 + 2t$

Solve each equation. SEE EXAMPLES 3 AND 4

20. $2(2x + 1) = 26$

21. $-2(2z + 1) = 26$

22. $92 = -4(2r - 5)$

23. $10(5 - n) - 1 = 29$

24. $-(7 - 2x) + 7 = -7$

25. $200 = 16(6t - 3)$

Solve each equation. SEE EXAMPLE 5

26. $\frac{1}{2}x + 2 = 1$

27. $\frac{3}{2}x - \frac{2}{3}x = 2$

28. $\frac{1}{5}(k - 3) = \frac{3}{4}$

29. $\frac{7}{60} = \frac{5}{24}w + \frac{11}{12}$

30. $\frac{3m}{4} - \frac{m}{12} = \frac{7}{8}$

31. $1{,}290 = \frac{h}{10} + \frac{h}{5}$

Solve each equation.

32. $0.1r - 1 = 0.65$

33. $1.2n + 0.68 = 5$

34. $0.025(q + 2) = 2.81$

35. $-0.07p - 0.6 = 5$

36. $1.037x + 0.02x + 25 = 30.285$

37. $-0.85t - 0.85t - 3.9 = -8.15$

38. A bee flies at 20 feet per second directly to an orange grove from its hive. The bee stays at the orange grove for 15 minutes, then flies directly back to the hive at 12 feet per second. It is away from the hive for a total of 20 minutes. SEE EXAMPLE 5

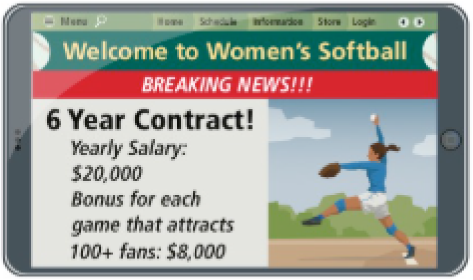
    a. What equation can you use to find the distance of the orange grove from the hive?

    b. How far is the orange grove from the hive?

PRACTICE & PROBLEM SOLVING

APPLY

39. **Apply Math Models** A fastpitch softball player signs a six-year contract. Her agent expects that she will earn $1,000,000 over the next six years. If the agent is right, how many bonus payments, on average, should the pitcher expect each year? Explain.



Welcome to Women's Softball

BREAKING NEWS!!!

6 Year Contract!

Yearly Salary: $20,000

Bonus for each game that attracts 100+ fans: $8,000

40. **Apply Math Models** There are nine water bottles in Devin's refrigerator. He adds three full boxes of water bottles to the refrigerator. Then he adds two more boxes that each have 1 fewer bottle than a full box. When he is done, there are 67 bottles in the refrigerator. Write and solve an equation to find the number of bottles in a full box.

41. **Check for Reasonableness** Yuson used her calculator to solve the equation $\frac{4}{5}x - 8 = 3$. She entered the following on her screen and got an incorrect answer. How could she use parentheses to find the correct answer? Explain. What is the correct answer?

```
3+8*5/4
                    13
```

42. **Apply Math Models** A scientist makes an acid solution by adding drops of acid to 1.2 L of water. The final volume of the acid solution is 1.202 L. Assuming the volume of each drop is 0.05 mL, how many drops were added to the water? About what percent of the solution is acid? Round to the nearest hundredth of a percent.

ASSESSMENT PRACTICE

43. Anna bought 8 tetras and 2 rainbow fish for her aquarium. The rainbow fish cost $6 more than the tetras. She paid a total of $37. Which of the following are true? Select all that apply.

AR.2.1

☐ **A.** The cost of 4 tetras is the same as the cost of a rainbow fish.

☐ **B.** One rainbow fish plus 5 tetras cost $21.

☐ **C.** An equation to find the cost r, in dollars, of a rainbow fish is $8r + 2(r + 6) = 37$

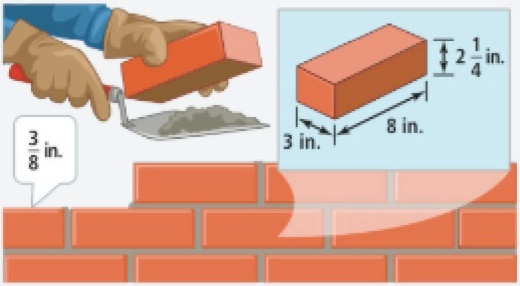
☐ **D.** Reducing the number of rainbow fish by 1 would result in a total cost of $28.50.

☐ **E.** An equation to find the cost t, in dollars, of a tetra t is $8t + 2t + 6 = 37$.

44. **SAT/ACT** What is the solution of $1{,}200 - 5(3x + 30) = 600$?

Ⓐ 30 Ⓑ 50 Ⓒ 150 Ⓓ 200 Ⓔ 250

45. **Performance Task** A mason will lay rows of bricks to build a wall. The mason will spread $\frac{3}{8}$ inch of mortar on top of all but the last row of bricks. The finished wall will be $1\frac{1}{8}$ inch less than 4 feet high.



Part A The mason wants to lay the bricks so that the shortest edges of each brick are vertical. How many rows of bricks are needed? Show your work.

Part B Suppose the mason decides to lay bricks so that the 3-inch edge is vertical. If the mason lays the same number of rows of bricks that were used for the wall described in Part A, how high will this wall be?

1-2

Solving Equations With a Variable on Both Sides

I CAN… write and solve equations with a variable on both sides to solve problems.

VOCABULARY

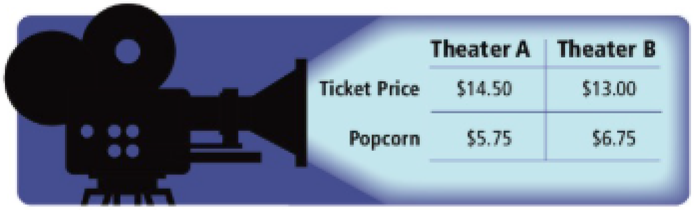
- identity

MA.912.AR.2.1–Given a real-world context, write and solve one-variable multi-step linear equations.

MA.K12.MTR.3.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

Some friends want to see a movie that is showing at two different theaters in town. They plan to share three tubs of popcorn during the movie.



	Theater A	Theater B
Ticket Price	\$14.50	\$13.00
Popcorn	\$5.75	\$6.75

A. **Communicate and Justify** Which movie theater should the friends choose? Explain.

B. For what situation would the total cost at each theater be exactly the same? Explain.

C. There are different methods to solving this problem. Which do you think is the best? Why?

ESSENTIAL QUESTION How do you create equations with a variable on both sides and use them to solve problems?

EXAMPLE 1 Solving Equations With a Variable on Both Sides

STUDY TIP
It does not matter if you add 10 to each side first or add $2x$ to each side first. Either order will result in the same equation, $9x = 27$.

A. What is the value of x in the equation shown?

$$\underbrace{3x - 10 + 4x} = \underbrace{-2(x - 4)} + 9$$

Distribute the –2.

Combine like terms.

$$7x - 10 = -2x + 8 + 9$$

$$7x + 2x = 8 + 9 + 10$$

Collect like terms on the same side of the equation.

$$9x = 27$$

$$\frac{9x}{9} = \frac{27}{9}$$

$$x = 3$$

B. What is the value of n in the equation shown?

$$\frac{1}{2}(n - 4) - 7 = -2n + 6$$

$$\frac{1}{2}(n - 4) = -2n + 13$$

$$2\left(\frac{1}{2}(n - 4)\right) = 2(-2n + 13)$$

Multiply each side by 2 to eliminate the fraction.

$$n - 4 = -4n + 26$$

$$n + 4n = 26 + 4$$

Collect like terms on the same side of the equation.

$$5n = 30$$

$$n = 6$$

Try It! 1. Solve each equation.

a. $100(z - 0.2) = -10(5z + 0.8)$

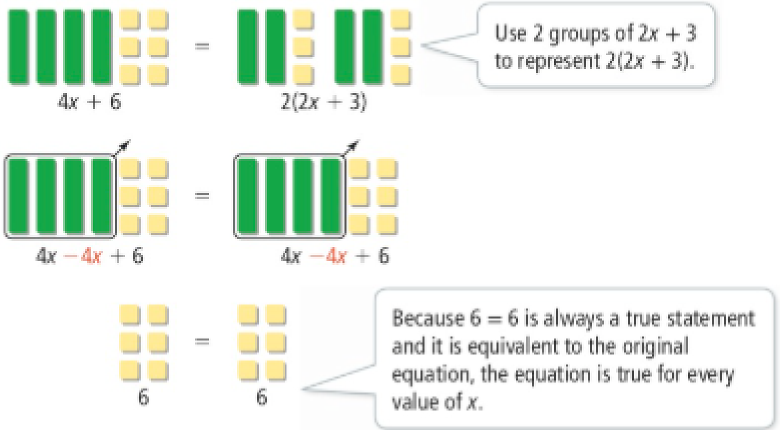
b. $\frac{5}{8}(16d + 24) = 6(d - 1) + 1$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Understand Equations With Infinitely Many or No Solutions

A. What is the value of x in $4x + 6 = 2(2x + 3)$?

Use algebra tiles to represent and solve $4x + 6 = 2(2x + 3)$.



Use 2 groups of $2x + 3$ to represent $2(2x + 3)$.

Because $6 = 6$ is always a true statement and it is equivalent to the original equation, the equation is true for every value of x.

The equation $4x + 6 = 2(2x + 3)$ is an *identity*. You can establish that equation is an **identity** by applying the rules of arithmetic and showing that two sides are identical.

An identity is true all possible values of the variable.

VOCABULARY
Since this equation is true for all values of the variable, it is sometimes referred to as having *infinitely many solutions.*

B. What is the value of x in $6x – 5 = 2(3x + 4)$?

Solve for x.

$$6x - 5 = 2(3x + 4)$$
$$6x - 5 = 6x + 8$$
$$6x - 6x - 5 = 6x - 6x + 8$$
$$-5 = 8$$

Maintain the equality by subtracting $6x$ from each side.

There is no value of x that makes the equation true. Therefore, the equation has no solution.

STUDY TIP
Recall that you can assign a value to the variable in an equation to check whether the equation is true.

Try It! **2.** Solve each equation. Is the equation an identity? Explain.

a. $t - 27 = -(27 - t)$

b. $16(4 - 3m) = 96\left(-\frac{m}{2} + 1\right)$

APPLICATION

EXAMPLE 3 Solve Mixture Problems

Arabica coffee costs \$28 per pound and Robusta coffee costs \$8.75 per pound. How many pounds of Arabica coffee must you mix with 3 pounds of Robusta coffee to make a blend that costs \$15.50 per pound?

Organize the information in a table.

REPRESENT AND CONNECT
How are tables helpful in organizing quantities to represent situations algebraically?

	Price (\$/lb)	·	Amount (lb)	=	Total cost (\$)
Arabica coffee	28.00		a		$28a$
Robusta coffee	8.75		3		26.25
Coffee blend	15.50		$a + 3$		$15.5(a + 3)$

Write an equation to represent the situation.

$$28a + 26.25 = 15.5(a + 3)$$
$$28a + 26.25 = 15.5a + 46.5$$
$$28a - 15.5a = 46.5 - 26.25$$
$$12.5a = 20.25$$
$$a = 1.62$$

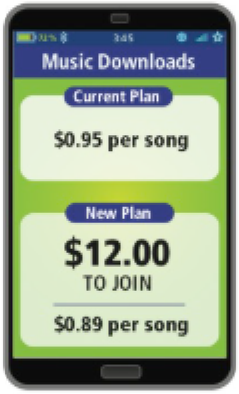
You must mix 1.62 pounds of Arabica coffee with 3 pounds of Robusta coffee to make a blend that costs \$15.50 per pound.

Try It! 3. How many pounds of Arabica coffee should you mix with 5 pounds of Robusta coffee to make a coffee blend that costs \$12.00 per pound?

APPLICATION

EXAMPLE 4 Use Equations to Solve Problems

Cameron pays \$0.95 per song with his current music service. A new music download service charges \$0.89 per song with a \$12 joining fee. Should Cameron switch to the new service?



Formulate Write an equation to represent when the cost for any number of songs, s, is the same for both services.

New music service = Cameron's current music service

$$0.89s + 12 = 0.95s$$

Compute Solve the equation to find the number of songs at which the cost for each option will be the same.

$$0.89s - 0.89s + 12 = 0.95s - 0.89s$$
$$12 = 0.06s$$
$$\frac{12}{0.06} = \frac{0.06s}{0.06}$$
$$200 = s$$

Interpret The cost of the two options will be the same for 200 songs.

If Cameron plans to purchase more than 200 songs, he should switch to the new service because it will cost less than his current service.

CONTINUED ON THE NEXT PAGE

Try It! **4.** Cameron's friend tells him of another service that has a \$15 joining fee but charges \$0.80 per song. At what number of songs does this new service become a less expensive option than Cameron's current service?

CONCEPT SUMMARY Create and Solve Equations with a Variable on Both Sides

Use the following information about Aki's and Hakeem's walkathon.

- Aki walks at a rate of 3 mi/h.
- Hakeem walks at a rate of 4 mi/h.
- Hakeem begins 0.5 hour after Aki.

When will Hakeem catch up to Aki?

WORDS Write an equation to represent the situation.

Hakeem's Distance = Aki's Distance

(Hakeem's Rate)(Hakeem's Time) = (Aki's Rate)(Aki's Time)

ALGEBRA $4(h - 0.5) = 3h$

$4h - 2 = 3h$

$h = 2$

Hakeem will catch up with Aki after 2 hours.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do you create equations with a variable on both sides and use them to solve problems?

2. **Vocabulary** How does the word *identity* relate to the two sides of an equation such as $3x = 2x + x$?

3. **Error Analysis** Isabel says that the equation $x - 2 = -(x - 2)$ has no solution because a number can never be equal to its opposite. Explain the error Isabel made.

4. **Choose Efficient Methods** You are solving an equation with a variable on each side. Does the side on which you choose to isolate the variable affect the solution? Why might you choose one side over the other?

Do You KNOW HOW?

Solve each equation.

5. $10x = 8x + 18$

6. $-28 - 9h = 5h$

7. $2(y - 6) = 3(y - 4) - y$

8. $8x - 4 = 2(4x - 4)$

9. For how many games is the total cost of bowling for one person equal for the two bowling establishments?

Family Bowling

Cost (dollars)	Game	4.00
	Shoes	1.00

Knight Owl Bowling

Cost (dollars)	Game	3.75
	Shoes	2.00

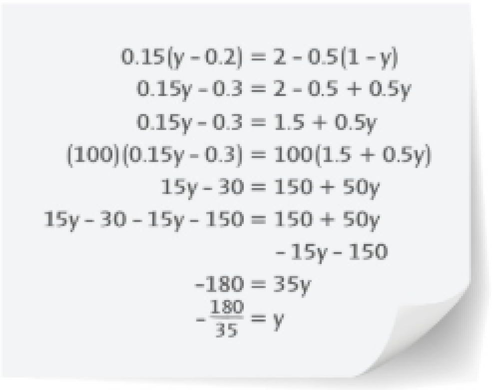
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Use Patterns and Structure** Do only equations with variables on both sides ever have no solution? Or can an equation with the variable on one side have no solution? Justify your answer.

11. **Generalize** How do you know whether an equation is an identity?

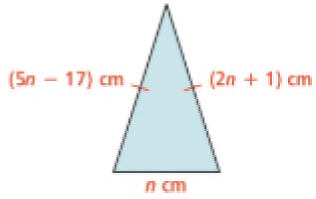
12. **Error Analysis** Describe and correct any error a student may have made when solving the equation $0.15(y - 0.2) = 2 - 0.5(1 - y)$.



$$0.15(y - 0.2) = 2 - 0.5(1 - y)$$
$$0.15y - 0.3 = 2 - 0.5 + 0.5y$$
$$0.15y - 0.3 = 1.5 + 0.5y$$
$$(100)(0.15y - 0.3) = 100(1.5 + 0.5y)$$
$$15y - 30 = 150 + 50y$$
$$15y - 30 - 15y - 150 = 150 + 50y - 15y - 150$$
$$-180 = 35y$$
$$-\frac{180}{35} = y$$

13. **Use Patterns and Structure** When Nicky tried to solve an equation using properties of equality, she ended up with the equation $-3 = -3$. What equation might she have tried to solve? What is the solution of the equation?

14. **Mathematical Connections** The triangle shown is isosceles. Find the length of each side and the perimeter.



15. **Higher Order Thinking** The equation shown has a missing value.

$$-2(2x - \blacksquare) + 1 = 17 - 4x$$

a. For what missing value is the equation an identity?

b. For what missing value(s), if any, does the equation have exactly one solution?

c. For what missing value(s), if any, does the equation have no solution?

PRACTICE

Solve each equation. SEE EXAMPLES 1–3

16. $5x - 4 = 4x$
17. $7x = 8x + 12$
18. $27 - 3x = 3x + 27$
19. $34 - 2x = 7x$
20. $5r - 7 = 2r + 14$
21. $-x = 7x - 56$
22. $5(n - 7) = 2(n + 14)$
23. $6w - 33 = 3(4w - 5)$
24. $3(x - 2) = 9x$
25. $6(x + 5) = 3x$
26. $\frac{4x + 6}{2} = \frac{3x - 15}{3}$
27. $\frac{q + 1}{2} = \frac{q - 1}{3}$
28. $2c + 3 = 2c + 3$
29. $12b + 9 = 12b + 11$
30. $x - 27 = -(27 - x)$
31. $4(x + 9) = x + 9$
32. $16(4 - 3m) = 96\left(-\frac{m}{2} + 1\right)$
33. $6y - 8 = 2(3y - 4)$
34. $5(5t + 1) = 25t - 7$
35. $-3k + 4 = -2 - 6k$
36. $\frac{1}{4}(2(x - 1) + 10) = x$
37. $\frac{6x + 8}{2} - 4 = 3x$
38. $3y = \frac{8 - 12y}{4} + 2$
39. $0.25t = 0.25 - t$
40. $0.625(x + 10) - 10 = 0$

Solve each problem. SEE EXAMPLE 4

41. Tavon has a \$50 gift card that loses \$2 for each 30-day period it is not used. He has a \$40 card that loses \$1.50 for each 30-day period it is not used.

a. Assuming both cards are unused, write and solve an equation for the number of 30-day periods until the value of the gift cards will be equal.

b. What will the value of each card be when they have equal value?

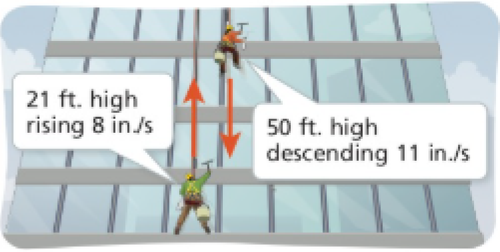
42. A cereal box manufacturer changes the size of the box to increase the amount of cereal it contains. The equations $12 + 7.6n$ and $6 + 8n$, where n is the number of smaller boxes, are both representative of the amount of cereal that the new larger box contains. How many smaller boxes equal the same amount of cereal in the larger box?

PRACTICE & PROBLEM SOLVING

APPLY

43. **Apply Math Models** Arthur wants to buy an item that costs p dollars before tax. Using the Florida sales tax rate of 6%, write two different expressions that represent the price of the item after tax. Show that the two expressions are equal.

44. **Apply Math Models** Two window washers start at the heights shown. One is rising, the other is descending. How long does it take for the two window washers to reach the same height? Explain.



45. **Communicate and Justify** Jamie will choose between two catering companies for an upcoming party. Company A charges a set-up fee of $500 plus $25 for each guest. Company B charges a set-up fee of $200 plus $30 per guest.

a. Write expressions that you can use to determine the amount each company charges for g guests.

b. Jamie learns that the $500 set-up fee for Company A includes payment for 20 guests. The $25 per guest charge is for every guest over the first 20. If there will be 50 guests, which company will cost the least? Explain.

46. **Communicate and Justify** A two-year prepaid membership at Gym A costs $250 for the first year plus $19 per month for the second year. A two-year prepaid membership at Gym B costs $195 for the first year plus $24 per month for the second year. Leah says the cost for both gym memberships will be the same after the 11th month of the second year. Do you agree? Explain.

47. **Apply Math Models** A red balloon is 40 feet above the ground and rising at 2 ft/s. At the same time, a blue balloon is at 60 feet above the ground and descending at 3 ft/s. What will the height of the balloons be when they are the same height above the ground?

ASSESSMENT PRACTICE

48. Kiara has $520 in her bank account. Each week she deposits $80 and withdraws $35. Gael has $310 in his account. Each week he deposits $95 and withdraws $20. Write and solve an equation to determine the week w when Gael and Kiara have the same amounts of money in their accounts. AR.2.1

49. **SAT/ACT** Which equation is an identity?

Ⓐ $\frac{9x}{15} + 27 = \frac{9x}{15} + \frac{27}{15}$

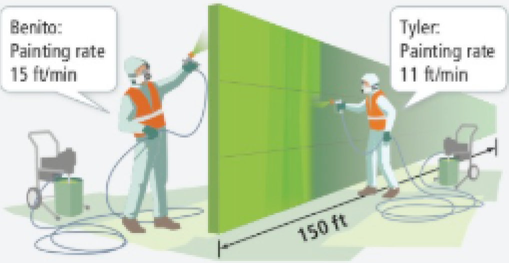
Ⓑ $3\left(\frac{x}{2} + 16\right) - 16 = \frac{3}{2}x$

Ⓒ $-4(3 - 2x) = -12 - 8x$

Ⓓ $-5\left(\frac{x}{15} - 16\right) - 30 = 50 - \frac{1}{3}x$

Ⓔ $36\left(\frac{3}{4}x - 2\right) + 72 = -72 + 27x$

50. **Performance Task** Benito and Tyler are painting opposite sides of the same fence. Tyler has already painted $19\frac{1}{2}$ feet of his side of the fence when Benito starts painting.



Part A How long will it take for the two sides of the fence to have an equal number of feet painted? How many feet will be painted on Benito's side of the fence when the two sides have an equal number of feet painted?

Part B Tyler claims that because he started painting first, he will finish painting his side of the fence before Benito finishes painting his side. Is this true? Explain.

Part C The painter who finishes first gets to rest while the other painter finishes. How long will the painter who finishes first get to rest? Explain.

1-3

Literal Equations and Formulas

I CAN… rewrite and use literal equations to solve problems.

VOCABULARY

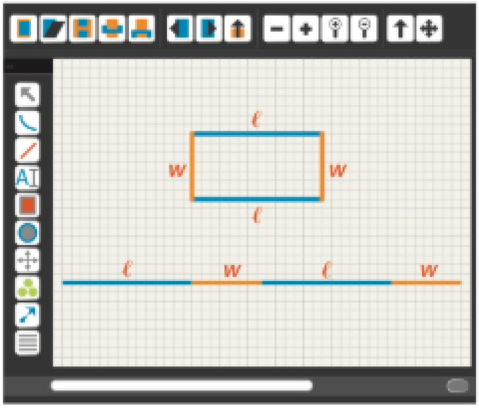
- formula
- literal equation

MA.912.AR.1.2–Rearrange equations or formulas to isolate a quantity of interest.

MA.K12.MTR.1.1, MTR.2.1, MTR.5.1

CRITIQUE & EXPLAIN

Nora drew a rectangle. Then she drew the length of each side from end to end to make a line segment to represent the perimeter P. Nora relates the perimeter and sides with the equation $P = \ell + w + \ell + w$. Her friend Darren uses the equation $\ell = \frac{1}{2}(P - 2w)$.



A. Do the two equations represent the same relationship? Construct a mathematical argument to support your answer.

B. **Choose Efficient Methods** Which equation is more useful? Explain.

ESSENTIAL QUESTION

How is rewriting literal equations useful when solving problems?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Rewrite Literal Equations

Julieta wants to calculate the time it takes to earn a certain amount of interest on a principal amount in an investment with simple interest. What equation can she use?

A **formula** is an equation that states a relationship between one quantity and one or more other quantities. Use the simple interest formula, $I = prt$, and solve for t. The formula $I = prt$ is a **literal equation** because letters represent both variables and known constants.

I = interest
p = principal
r = interest rate
t = time

$$I = prt$$

$$\frac{I}{pr} = \frac{prt}{pr}$$

$$\frac{I}{pr} = t$$

You use properties of equality to solve literal equations for a variable just as you do linear equations.

When she writes the equation this way, she can use what she knows (I, p, and r) to calculate what she needs (t).

VOCABULARY
One definition of *literal* is *of, relating to, or expressed in letters*. A literal equation is an equation expressed in letters, or variables.

Try It! **1.** What equation can Julieta use to calculate the principal amount?

EXAMPLE 2 Use Literal Equations to Solve Problems

In a half hour, Jaylen is meeting his friends at the lake, 6 mi from his house. At what average speed must he ride his bike to get there on time?

Step 1 Solve the formula for r.

$$d = rt$$

Remember, distance = rate • time.

$$\frac{d}{t} = r$$

Step 2 Find the average speed, or rate, at which Jaylen must ride his bike to be on time.

$$\frac{d}{t} = r$$

$$\frac{6}{0.5} = r$$

Substitute 6 for d and 0.5 for t.

$$12 = r$$

Jaylen needs to ride his bike at an average speed of 12 mi/h to get to the lake on time.

USE PATTERNS AND STRUCTURE
How is the structure of the literal equation related to units for rate?

Try It! 2. Jaylen is going to the store 2.5 mi away. He has only 15 min to get there before they close. At what average speed must he ride to get to the store before they close?

EXAMPLE 3 Rewrite a Formula

A worker at a framing store is making a rectangular frame. He knows that the perimeter of the frame is 144 in. and the length is 40 in. How can he determine the width of the frame?

Step 1 Rewrite the perimeter formula $P = 2\ell + 2w$ to solve for w.

$$P - 2\ell = 2\ell + 2w - 2\ell$$

$$\frac{P - 2\ell}{2} = \frac{2w}{2}$$

$$\frac{P - 2\ell}{2} = w$$

The perimeter formula solved for w is $w = \frac{P - 2\ell}{2}$.

Step 2 Use the literal equation to solve for w when P is 144 and ℓ is 40.

$$w = \frac{P - 2\ell}{2}$$

$$w = \frac{144 - 2(40)}{2}$$

$$w = \frac{144 - 80}{2} = 32$$

The width of the frame is 32 in.

COMMON ERROR
Do not divide just 2ℓ by 2 on the left side. Divide the entire expression $P - 2\ell$ by 2. The entire left side of the equation is one expression and must be divided by 2.

Try It! 3. Write the formula for the area of a triangle, $A = \frac{1}{2}bh$ solving for h. Find the height of a triangle when $A = 18$ in.^{2} and $b = 9$ in.

APPLICATION

EXAMPLE 4 Apply Formulas

According to Teo's bread recipe, he should bake the bread at 190°C for 30 minutes. His oven measures temperature in °F. To what temperature in °F should he set his oven?

Formulate ◀ Rewrite the formula to find the Fahrenheit temperature that is equal to 190°C.

Compute ◀ **Step 1** Solve for F.

$$C = \frac{5}{9}(F - 32)$$

$$\frac{9}{5} \cdot C = \frac{9}{5} \cdot \frac{5}{9}(F - 32)$$

Dividing by a fraction is the same as multiplying by its reciprocal.

$$\frac{9}{5}C = F - 32$$

$$\frac{9}{5}C + 32 = F - 32 + 32$$

$$\frac{9}{5}C + 32 = F$$

Step 2 Use the formula for F to find the Fahrenheit temperature equivalent to 190°C.

$$\frac{9}{5}C + 32 = F$$

$$\frac{9}{5}(190) + 32 = 374$$

Interpret ◀ Teo should set the oven to 374°F.

Try It! **4.** The Kelvin scale is used to measure very cold temperatures. The formula $K = C + 273.15$ relates the Celsius and Kelvin scales. Rewrite the formula to solve for C. If the temperature one morning was 12°C what was the temperature in Kelvin (K)?

CONCEPT SUMMARY Literal Equations and Formulas

WORDS Literal equations can use letters for both constants and variables. A formula is a kind of literal equation where one quantity is related to one or more other quantities.

To solve for a particular variable in a literal equation, you rewrite the equation, isolating the variable.

ALGEBRA The volume of a rectangular prism is given by the following formula.

$$V = \ell wh$$

To find a formula for h, the height of the prism, solve for h.

$$V = \ell hw$$

$$\frac{V}{\ell w} = \frac{\ell hw}{\ell w}$$

Divide each side by ℓw.

$$\frac{V}{\ell w} = h$$

$$h = \frac{V}{\ell w}$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is rewriting literal equations useful when solving problems?

2. **Choose Efficient Methods** How is solving $2x + c = d$ similar to solving $2x + 1 = 9$ for x? How are they different? How can you use $2x + c = d$ to solve $2x + 1 = 9$?

3. **Vocabulary** Explain how literal equations and formulas are related.

4. **Error Analysis** Dyani began solving the equation $g = \frac{x-1}{k}$ for x by adding 1 to each side. Explain Dyani's error. Then describe how to solve for x.

Do You KNOW HOW?

Solve each literal equation for the given variable.

5. $y = x + 12$; x

6. $n = \frac{4}{5}(m + 7)$; m

7. Use your equation from Exercise 6 to find m when $n = 40$.

8. William got scores of q_1, q_2, and q_3 on three quizzes.

   a. Write a formula for the average x of all three quizzes.

   b. William got an 85 and an 88 on the first two quizzes. What formula can William use to determine the score he needs on the third quiz to get an average of 90? What score does he need?

PRACTICE & PROBLEM SOLVING

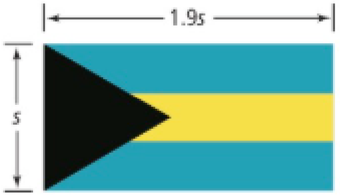
UNDERSTAND

9. **Mathematical Connections** Some two-step equations can be written in the form $ax + b = c$, where a, b, and c are constants and x is the variable.

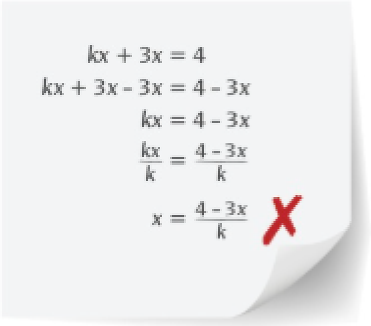
   a. Solve the equation $ax + b = c$ for x.

   b. Use the formula to solve $3x + 7 = 19$ and $\frac{1}{2}x - 1 = 5$.

10. **Analyze and Persevere** The flag of the Bahamas includes an equilateral triangle. The perimeter of the triangle is $P = 3s$, where s is the side length. Solve for s. Use your formula to find the dimensions of the flag in feet and the area in square feet when the perimeter of the triangle is 126 inches.



11. **Error Analysis** Describe and correct the error a student made when solving $kx + 3x = 4$ for x.



$$
\begin{aligned}
kx + 3x &= 4\\
kx + 3x - 3x &= 4 - 3x\\
kx &= 4 - 3x\\
\frac{kx}{k} &= \frac{4 - 3x}{k}\\
x &= \frac{4 - 3x}{k}
\end{aligned}
$$

12. **Higher Order Thinking** Given the equation $ax + b = c$, solve for x. Describe each statement as *always*, *sometimes*, or *never* true. Explain your answer.

   a. If a, b, c, are whole numbers, x is a whole number.

   b. If a, b, c, are integers, x is an integer.

   c. If a, b, c, are rational numbers, x is a rational number.

PRACTICE

Solve each equation for the indicated variable.
SEE EXAMPLES 1 AND 2

13. $A = bh$; h

14. $k = a - y$; y

15. $F = \frac{9}{5}C + 32$; C

16. $w = \frac{x}{a - b}$; x

17. $A = \frac{(b_1 + b_2)h}{2}$; h

18. $y = mx + b$; m

19. $PV = nRT$; R

20. $Ax + By = C$; y

21. $y - y_1 = m(x - x_1)$; m

22. $12(m + 3x) = 18(x - 3m)$; m

23. $V = \frac{1}{3}\pi r^2 h$; h

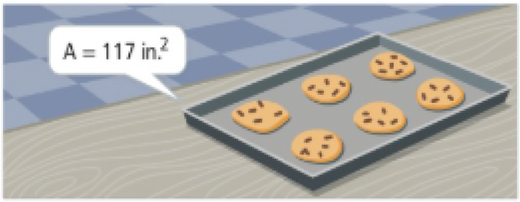
24. $V = \frac{1}{3}\pi r^2(h - 1)$; h

25. $y(a - b) = c(y + a)$; y

26. $x = \frac{3(y - b)}{m}$; y

27. $F = -\frac{Gm}{r^2}$; G

28. Use the area formula $A = \ell w$ to write a formula for the length ℓ of the baking sheet shown. SEE EXAMPLE 3



29. You can determine the approximate temperature in degrees Fahrenheit by counting the number of times a cricket chirps in one minute. Then multiply that by 7, divide by 30, and add 40. SEE EXAMPLE 4

   a. Write a formula for estimating the temperature based on the number of cricket chirps.

   b. Write a new formula for the number of chirps you would expect in one minute at a given Fahrenheit temperature.

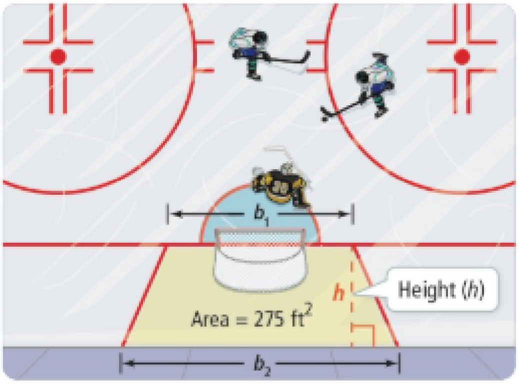
   c. Use the formula to find the number of chirps in one minute when the temperature is 89°F.

PRACTICE & PROBLEM SOLVING

APPLY

30. **Apply Math Models** Water boils at different temperatures at different elevations. The boiling temperature of water is 212°F at sea level (0 ft) but drops about 1.72°F for every 1,000 feet of elevation. Write a formula for the boiling point at a given elevation. Then solve the formula for the elevation when the boiling point for water is 190°F.

31. **Represent and Connect** In the National Hockey League, the goalie may not play the puck outside the isosceles trapezoid behind the net. The formula for the area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$.

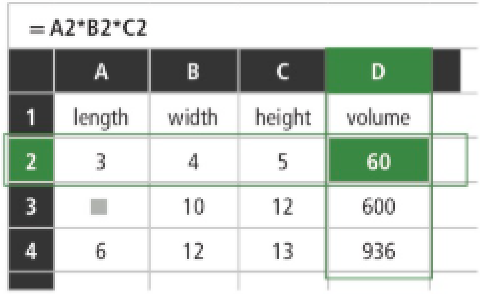


a. Solve the formula for either base, b_1 or b_2.

b. Use the formula to find the length of the base next to the goal given that the height of the trapezoid is 11 ft and the base farthest from the goal is 28 ft.

c. How can you find the distance d of each side of the base that extends from the goal given that the goal is 6 ft long? What is the distance?

32. **Represent and Connect** The formula for cell D2 is shown in the spreadsheet. Use the data shown in row 3 to write a formula for cell A3.



= A2*B2*C2

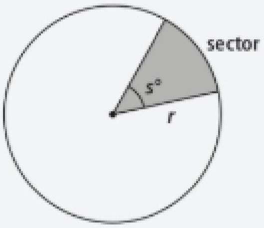
	A	B	C	D
1	length	width	height	volume
2	3	4	5	60
3	■	10	12	600
4	6	12	13	936

ASSESSMENT PRACTICE

33. Brandon wants to find the width of his rectangular neighborhood park. How can he rearrange the formula for perimeter of a rectangle, $p = 2(\ell + w)$, to solve for w? AR.1.2

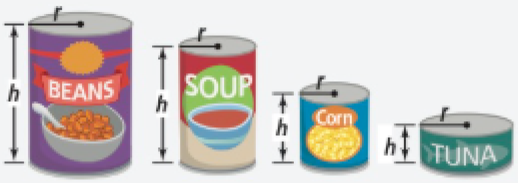
Ⓐ $\ell = p - 2w$
Ⓑ $w = \frac{p - 2\ell}{2}$
Ⓒ $w = \frac{2\ell + p}{2}$
Ⓓ $w = p - 2\ell$

34. **SAT/ACT** The formula for the area of a sector of a circle is $A = \frac{\pi r^2 s}{360}$. Which formula shows s expressed in terms of the other variables?



Ⓐ $s = \frac{\pi r^2 A}{360}$
Ⓑ $s = \frac{360}{\pi r^2 A}$
Ⓒ $s = 360\pi r^2 A$
Ⓓ $s = \frac{360A}{\pi r^2}$
Ⓔ $s = \frac{A}{360\pi r^2}$

35. **Performance Task** A manufacturer can save money by making a can that minimizes the amount of metal used to contain a given volume. For a can with radius r and height h, this goal is reached when $2\pi r^3 = \pi r^2 h$.



Part A Solve the equation for h. How is the height related to the radius for a can that meets the manufacturer's goal?

Part B The area of a label for a can is $A = 2\pi rh$. Use your result from Part A to write a formula giving the area of a label for a can that meets the manufacturer's goals.

1-4
Solving Inequalities in One Variable

I CAN… solve and graph inequalities.

VOCABULARY

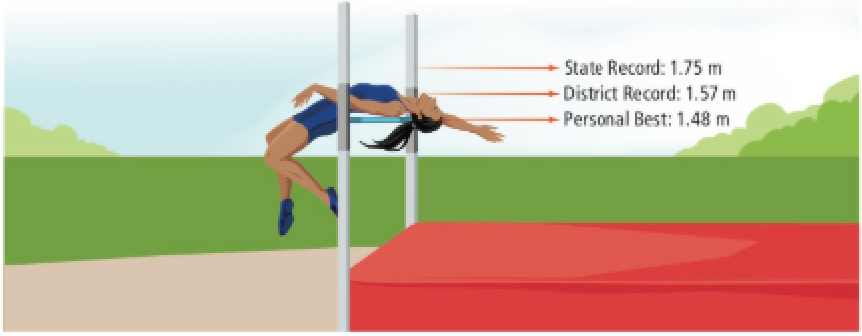
- set

MA.912.AR.2.6–Given a mathematical or real-world context, write and solve one-variable linear inequalities, including compound inequalities. Represent solutions algebraically or graphically.

MA.K12.MTR.2.1, MTR.4.1, MTR.7.1

MODEL & DISCUSS

Nina competes in the high jump event at her school. She hopes to tie or break some records at the next meet.



A. Write and solve an equation to find x, the number of meters Nina must add to her personal best to tie the district record.

B. **Represent and Connect** Rewrite your equation as an inequality to represent the situation where Nina *breaks* the district record. How is the value of x in the inequality related to the value of x in the equation?

C. How many meters does Nina need to add to her personal best to break the state record?

ESSENTIAL QUESTION

How are the solutions of an inequality different from the solutions of an equation?

EXAMPLE 1 Solve Inequalities

Solve $-4(3x - 1) + 6x \geq 16$ and graph the solution.

$$-4(3x - 1) + 6x \geq 16$$

$$-12x + 4 + 6x \geq 16$$

$$-12x + 4 - 4 + 6x \geq 16 - 4$$

Apply the properties of inequalities to solve for x.

$$-12x + 6x \geq 12$$

$$-6x \geq 12$$

$$\frac{-6x}{-6} \leq \frac{12}{-6}$$

$$x \leq -2$$

Remember that the direction of the inequality symbol is reversed when both sides of the inequality are multiplied or divided by negative values.

STUDY TIP
Recall that when you graph the solution of an inequality on a number line, you use an open circle if the inequality symbol is $<$ or $>$, and a closed circle if the inequality symbol is $\leq$ or $\geq$.

Graph the solution.

Recall that a **set** is a collection of objects such as numbers. The solution of the inequality is the set of all real numbers that are less than or equal to −2.

Try It! **1.** Solve each inequality and graph the solution.

a. $-3(2x + 2) < 10$

b. $2(4 - 2x) > 1$

EXAMPLE 2 Solve an Inequality With Variables on Both Sides

Solve $3.5x + 19 \geq 1.5x - 7$. Then graph the solution.

LEARN TOGETHER
How do you listen actively as others share ideas?

Solve the inequality.

$$3.5x + 19 \geq 1.5x - 7$$

$$3.5x - 1.5x + 19 - 19 \geq 1.5x - 1.5x - 7 - 19$$

Collect like terms on the same side of the inequality.

$$2x \geq -26$$

$$x \geq -13$$

Graph the solution.

Try It! **2.** Solve $2x - 5 < 5x - 22$. Then graph the solution.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Understand Inequalities With Infinitely Many or No Solutions

A. Solve $-3(2x - 5) > -6x + 9$.

$$-3(2x - 5) > -6x + 9$$

$$-6x + 15 > -6x + 9$$

$$-6x + 6x + 15 > -6x + 6x + 9$$

$$15 > 9$$

The original inequality is equivalent to $15 > 9$, a true statement. What does this mean?

Using the same steps above, you can show that the inequality is true for any value of x. So all real numbers are solutions of the inequality.

B. Solve $4x - 5 < 2(2x - 3)$.

$$4x - 5 < 2(2x - 3)$$

$$4x - 5 < 4x - 6$$

$$4x - 4x - 5 < 4x - 4x - 6$$

$$-5 < -6$$

USE PATTERNS AND STRUCTURE
Consider the definition of the solution to an inequality. What would the graph of an inequality with no solution look like?

Since the inequality results in a false statement ($-5 < -6$), any value of x you substitute in the original inequality will also result in a false statement.

This inequality has no solution.

Try It! **3.** Solve each inequality.

a. $-2(4x - 2) < -8x + 4$ **b.** $-6x - 5 < -3(2x + 1)$

APPLICATION

EXAMPLE 4 Use Inequalities to Solve Problems

Derek wants to order some roses online. For what number of roses is it less expensive to order from Florist A? From Florist B?

Florist A:
$4.75 per blue rose
plus $40
delivery charge.

Florist B:
$5.15 per red rose
plus $25
delivery charge.

Formulate Write an inequality to compare the total cost of x roses from each florist.

The cost of x roses at Florist A is less than the cost of x roses at Florist B.

$$4.75x + 40 \quad < \quad 5.15x + 25$$

Set up the inequality to find the number of roses it would take for Florist A to be less expensive.

Compute Solve for x.

$$4.75x + 40 < 5.15x + 25$$

$$4.75x - 4.75x + 40 < 5.15x - 4.75x + 25$$

$$40 - 25 < 0.4x + 25 - 25$$

$$\frac{15}{0.4} < \frac{0.4x}{0.4}$$

$$37.5 < x$$

Interpret The solution is all real numbers greater than 37.5. However, the number of roses must be a whole number.

If Derek plans to buy 38 or more roses, then Florist A is less expensive.
If Derek plans to buy 37 or fewer roses, then Florist B is less expensive.

Try It! **4.** If Florist B increases the cost per rose to $5.20, for what number of roses is it less expensive to order from Florist A? From Florist B?

CONCEPT SUMMARY Solving Inequalities in One Variable

WORDS To solve inequalities, use the Properties of Inequalities to isolate the variable.

The solution of an inequality is the set of all real numbers that makes the inequality true. Some inequalities are true for all real numbers (such as $x + 3 < x + 7$), but others have no solutions (such as $x + 7 < x + 3$).

ALGEBRA

$$-5(2x - 3) \leq 34$$

$$-10x + 15 \leq 34$$

$$-10x + 15 - 15 \leq 34 - 15$$

$$-10x \leq 19$$

$$\frac{-10x}{-10} \geq \frac{19}{-10}$$

Reverse the inequality when multiplying or dividing by a negative number.

$$x \geq -1.9$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the solutions of an inequality different from the solutions of an equation?

2. **Communicate and Justify** How is dividing each side of $x > 0$ by a negative value different from dividing each side by a positive value?

3. **Vocabulary** Give an example of two inequalities that are *equivalent inequalities.* Explain your reasoning.

4. **Error Analysis** Rachel multiplied each side of $x \geq 2$ by 3. She wrote the result as $3x \leq 6$. Explain the error Rachel made.

Do You KNOW HOW?

Solve each inequality and graph the solution.

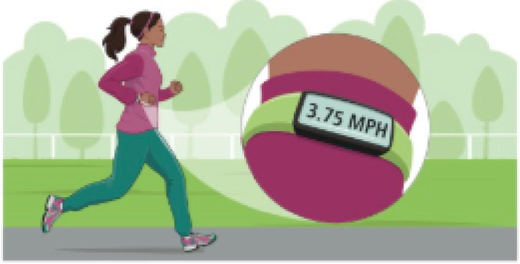
5. $\frac{1}{2}x < 6$

6. $-4x \geq 20$

7. $8 \leq -4(x - 1)$

8. $3x - 2 > 4 - 3x$

9. Lourdes plans to jog at least 1.5 miles. Write and solve an inequality to find x, the number of hours Lourdes will have to jog.



PRACTICE & PROBLEM SOLVING

UNDERSTAND

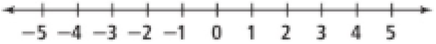
10. **Communicate and Justify** Let a, b, and c be real numbers. Show that each of the following statements is true.

a. If $a > b$ and $c < 0$, then $ca < cb$.

b. If $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$.

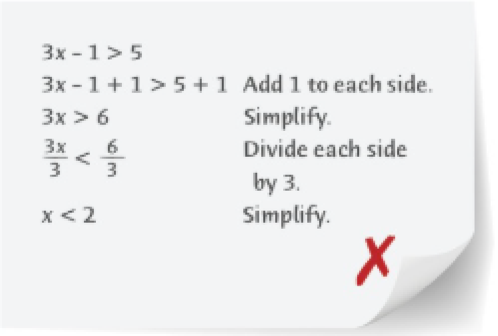
11. **Represent and Connect** For each of the graphs below, write an inequality that the graph represents. Explain your reasoning.

a.



b.

12. **Error Analysis** Describe and correct the error a student made when solving the inequality shown.



$3x - 1 > 5$

$3x - 1 + 1 > 5 + 1$ Add 1 to each side.

$3x > 6$ Simplify.

$\frac{3x}{3} < \frac{6}{3}$ Divide each side by 3.

$x < 2$ Simplify.

13. **Mathematical Connections** Jake's solution to the equation $-4(2x - 3) = 36$ is shown.

$$\begin{aligned} -4(2x - 3) &= 36 \\ -8x + 12 &= 36 \\ -8x + 12 - 12 &= 36 - 12 \\ -8x &= 24 \\ \frac{-8x}{-8} &= \frac{24}{-8} \\ x &= -3 \end{aligned}$$

How is the solution to $-4(2x - 3) > 36$ similar to and different from the solution shown?

14. **Higher Order Thinking** Suppose each side of the inequality $a - b < 0$ is multiplied by c.

a. If $c < 0$ and $c(a - b) > 0$, write an inequality to represent the relationship between a and b.

b. If $c < 0$, is $c(a - b)$ always greater than 0? Explain your reasoning.

PRACTICE

Solve each inequality and graph the solution. SEE EXAMPLES 1 AND 4

15. $x + 9 > 15$

16. $-\frac{1}{5}x > -10$

17. $5x + 15 \leq -10$

18. $-0.3x < 6$

19. $6x \geq -0.3$

20. $-3x > 15$

21. $\frac{1}{4}x > \frac{1}{2}$

22. $x - 8.4 \leq 2.3$

23. $2.1x \geq 6.3$

24. $-2.1x + 2.1 < 6.3$

25. $-\frac{3}{8}x - 20 + 2x > 6$

26. $\frac{2}{3}x + 14 - 3x > -7$

27. $0.5x - 4 - 2x \leq 2$

28. $4x + 1 + 2x \geq 5$

Match each inequality to the graph that represents its solution. Explain your reasoning. SEE EXAMPLE 1

29. $-2(3x - 1) > 20$

30. $2(1 - 3x) < 20$

31. $-2(1 - 3x) > 16$

32. $2(3x - 1) < 16$

A.

B.

C.

D.

Solve each inequality. SEE EXAMPLE 2

33. $2x + 5 < 3x + 4$

34. $2(7x - 2) > 9x + 6$

Solve each inequality and tell whether it has infinitely many or no solutions. SEE EXAMPLE 3

35. $\frac{3}{4}x + \frac{3}{4} - \frac{1}{2}x \geq -1$

36. $\frac{1}{4}x + 3 - \frac{7}{8}x < -2$

37. $-5(2x + 1) < 24$

38. $4(3 - 2x) \geq -4$

39. $7.2x + 6 \leq 2.4x$

40. $-2x - 5 \geq 3x - 25$

41. $2x + 12 > 2(x + 6)$

42. $0.5x + 8 < 2x - 4$

A solution is graphed for each inequality below. Describe the changes that need to be made, if any, to each graph. SEE EXAMPLE 3

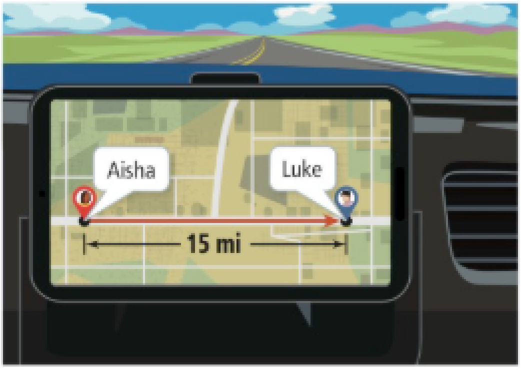
43. $3x - 24 \leq -2(2x - 30)$

44. $-2(x - 5) \geq -2x + 10$

PRACTICE & PROBLEM SOLVING

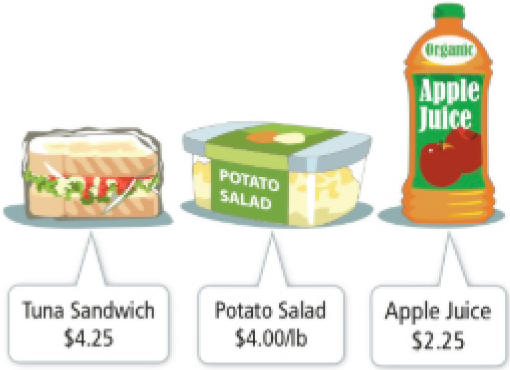
APPLY

45. **Apply Math Models** Luke and Aisha are traveling on the same road from Lakeland to Winter Haven, in the same direction. Luke is driving at a rate of 50 mi/h, and Aisha is driving at a rate of 55 mi/h. Write and solve an inequality to find when Aisha will be ahead of Luke on the highway. Let x represent time in hours.



46. **Analyze and Persevere** An office manager is selecting a water delivery service. Acme H_2O charges a \$15 fee and \$7.50 per 5-gallon jug. Best Water charges a \$24 fee and \$6.00 per 5-gallon jug. How many 5-gallon jugs will the office have to buy each month for the cost of Best Water to be less than that of Acme H_2O?

47. **Apply Math Models** Charlie can spend up to \$8 on lunch. He wants to buy a tuna sandwich, a bottle of apple juice, and x pounds of potato salad. Write and solve an inequality to find the possible numbers of pounds of potato salad he can buy.



ASSESSMENT PRACTICE

48. Select all inequalities that have the same solution, $x < 3$. AR.2.6

☐ **A.** $-\frac{1}{2}x > -\frac{3}{2}$

☐ **B.** $2x > 6$

☐ **C.** $5x + x < 3x + 9$

☐ **D.** $9x < 3$

☐ **E.** $4x - 7 < 5$

49. **SAT/ACT** Which of the following is the solution of $0.125x + 1 - 0.25x < -3$?

Ⓐ $x < -0.5$

Ⓑ $x < 0.5$

Ⓒ $x > 0.5$

Ⓓ $x < 32$

Ⓔ $x > 32$

50. **Performance Task** Students have organized a three-day walkathon to raise money for charity. The average walking speeds of four participants are given in the table below.

Name	Walking Speed (mi/h)
Elijah	3.2
Aubrey	3
Mercedes	2.4
Steve	3.5

Part A Write and solve an inequality to determine how many hours it would take Steve to walk at least 21 mi on Day 1.

Part B At the beginning of Day 2, Mercedes is 2 mi ahead of Elijah. Write and solve an inequality to determine the hours x when Elijah will be behind Mercedes.

Part C At the beginning of Day 3, Elijah starts walking at the marker for Mile 42, and Aubrey starts walking at the marker for Mile 42.5. Write and solve an inequality to determine the hours when Elijah is ahead of Aubrey.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.2.1–Given a real-world context, write and solve one-variable multi-step linear equations.

Also AR.2.6

MA.K12.MTR.7.1

Collecting Cans

Many schools and community centers organize canned food drives and donate the food collected to area food pantries or homeless shelters.

A teacher may hold a contest to see which student collects the most cans. The teacher will track the number of cans each student brings in. Sometimes students have their own ways of keeping track. You'll see how some students kept track in the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. Write a number that you know is too small.
6. Write a number that you know is too large.

ACT 2 Develop a Model

7. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

8. Is your refined conjecture between the highs and lows you set up earlier?

1-5

Compound Inequalities

I CAN… write and solve compound inequalities.

VOCABULARY

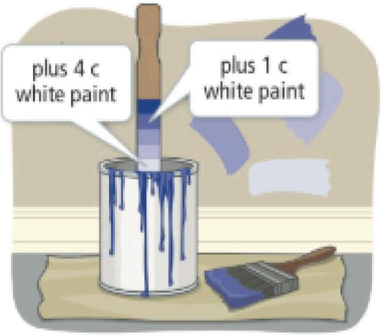
- compound inequality
- element
- subset

MA.912.AR.2.6–Given a mathematical or real-world context, write and solve one-variable linear inequalities, including compound inequalities. Represent solutions algebraically or graphically.

MA.K12.MTR.5.1, MTR.6.1, MTR.7.1

EXPLORE & REASON

Hana has some blue paint. She wants to lighten the shade, so she mixes in 1 cup of white paint. The color is still too dark, so Hana keeps mixing in 1 cup of white paint at a time. After adding 4 cups, she decides the color is too light.



A. Explain in words how much paint Hana should have added initially to get the shade she wants.

B. **Apply Math Models** Represent your answer to part A with one or more inequalities.

C. Hana decides that she likes the shades of blue that appear in between adding 1 cup and 4 cups of white paint. How can you represent the number of cups of white paint that yield the shades Hana prefers?

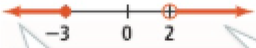
ESSENTIAL QUESTION What are compound inequalities and how are their solutions represented?

EXAMPLE 1 Understand Compound Inequalities

CONCEPTUAL UNDERSTANDING

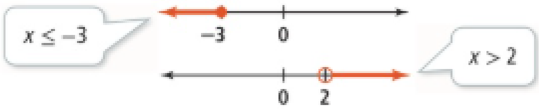
How can you use inequalities to describe the sets of numbers graphed below?

A.



The graph shows the solutions of two inequalities. The two inequalities form a *compound inequality*. A **compound inequality** is made up of two or more inequalities.

Write an inequality to represent the solutions shown in each part of the graph.

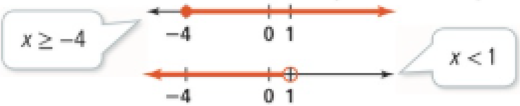


The compound inequality that describes the graph is $x \leq -3$ or $x > 2$.

CHECK FOR REASONABLENESS
There is no number that can be less than −3 AND greater than 2. So it makes sense to use OR to write the compound inequality.

B.

The solutions shown in the graph are greater than or equal to –4. They are also less than 1. Write two inequalities to represent this.



The compound inequality that describes the graph is $-4 \leq x$ and $x < 1$. You can also write this as $-4 \leq x < 1$.

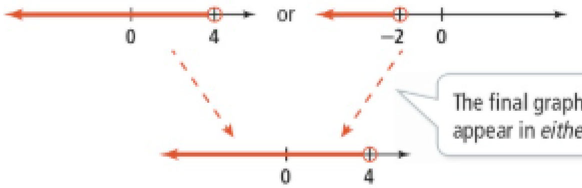
Try It! 1. Write a compound inequality for the graph.

EXAMPLE 2 Solve a Compound Inequality Involving *Or*

Solve the compound inequality $5x - 7 < 13$ or $-4x + 3 > 11$. Graph the solution.

Solve each inequality.

$$\begin{array}{ccc}
5x - 7 < 13 & \text{or} & -4x + 3 > 11 \\
5x - 7 + 7 < 13 + 7 & & -4x + 3 - 3 > 11 - 3 \\
5x < 20 & & -4x > 8 \\
\frac{5x}{5} < \frac{20}{5} & & \frac{-4x}{-4} < \frac{8}{-4} \\
x < 4 & & x < -2
\end{array}$$



The final graph is all points that appear in *either* solution above.

REPRESENT AND CONNECT
Are there any values of x which are elements of the solution set $x < 4$ but not elements of the solution set $x < -2$?

An **element** of a set is an object that is in the set. A **subset** of a set consists of elements from the given set. The solution $x < -2$ is a subset of the solution $x < 4$, so $x < 4$, is the complete solution. The solution is the set of all real numbers less than 4.

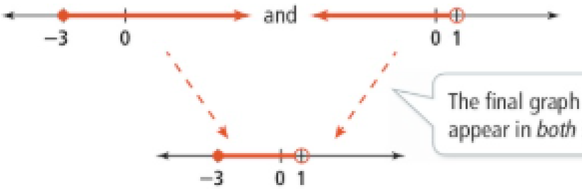
Try It! 2. Solve the compound inequality $-3x + 2 > -7$ or $2(x - 2) \geq 6$. Graph the solution.

EXAMPLE 3 Solve a Compound Inequality Involving *And*

What is the solution of $-12 \leq 7x + 9 < 16$?

Solve each inequality.

$$\begin{array}{ccc}
-12 \leq 7x + 9 & \text{and} & 7x + 9 < 16 \\
-12 - 9 \leq 7x + 9 - 9 & & 7x + 9 - 9 < 16 - 9 \\
-21 \leq 7x & & 7x < 7 \\
\frac{-21}{7} \leq \frac{7x}{7} & & \frac{7x}{7} < \frac{7}{7} \\
-3 \leq x & & x < 1
\end{array}$$



The final graph is all points that appear in *both* solutions above.

The solution is $x \geq -3$ and $x < 1$, or $-3 \leq x < 1$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

Try It! **3.** Solve the compound inequality $2x < 4x - 5 < 15$. Graph the solution.

APPLICATION

EXAMPLE 4 Solve Problems Involving Compound Inequalities

Enrique plans a diet for his dog, River. River consumes between 510 and 540 Calories per day.

If River eats $1\frac{1}{2}$ servings of dog food each day, how many treats can she have?

Formulate Model the situation with a compound inequality.

Let x represent the number of treats River can have each day.

Write an expression to represent River's total daily Calories.

$1\frac{1}{2}$ servings at 320 Cal. per serving plus x treats at 15 Cal. per treat

$$480 \quad + \quad 15x$$

Write a compound inequality for the number of dog treats each day.

at least 510 Calories at most 540 Calories

$$510 \leq 480 + 15x \leq 540$$

Compute Solve the compound inequality.

$$\begin{aligned} 510 &\leq 480 + 15x \leq 540 \\ 510 - 480 &\leq 480 + 15x - 480 \leq 540 - 480 \\ 30 &\leq 15x \leq 60 \\ \frac{30}{15} &\leq \frac{15x}{15} \leq \frac{60}{15} \\ 2 &\leq x \leq 4 \end{aligned}$$

The solution is $2 \leq x \leq 4$.

Interpret River can have at least 2 and at most 4 treats each day.

Try It! **4.** Suppose River has new treats that are 10 Calories each. How many of the new treats can she have and remain in her Calorie range?

CONCEPT SUMMARY Compound Inequalities

WORDS	The solution of a compound inequality involving or includes the solutions of one inequality as well as the solutions of the other inequality.	The solution of a compound inequality involving and includes only solutions that are common to both inequalities.
ALGEBRA	$x < a$ or $x > b$	$x > a$ and $x < b$ $a < x < b$
GRAPHS		

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are compound inequalities and how are their solutions represented?

2. **Use Patterns and Structure** When $a < b$, how is the graph of $x > a$ and $x < b$ similar to the graph of $x > a$? How is it different?

3. **Vocabulary** A *compound* is defined as a *mixture*. Make a conjecture as to why the term *compound inequality* includes the word *compound*.

4. **Error Analysis** Kona graphed the compound inequality $x > 2$ or $x > 3$ by graphing $x > 3$. Explain Kona's error.

Do You KNOW HOW?

Write a compound inequality for each graph.

5.

6.

Solve each compound inequality and graph the solution.

7. $4x - 1 > 3$ and $-2(3x - 4) \geq -16$

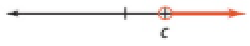
8. $2(4x + 3) \geq -10$ or $-5x - 15 > 5$

9. Nadeem plans to ride her bike between 12 mi and 15 mi. Write and solve an inequality to find how many hours Nadeem will be riding.

PRACTICE & PROBLEM SOLVING

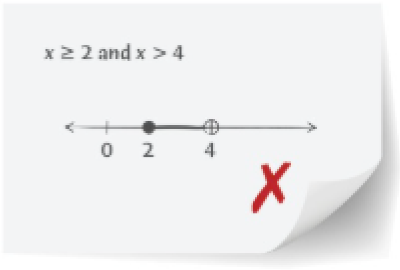
UNDERSTAND

10. **Represent and Connect** The compound inequality $x > a$ and $x > b$ is graphed below. How is the point labeled c related to a and b?



$x > a$ and $x > b$

11. **Error Analysis** Describe and correct the error a student made graphing the compound inequality $x \geq 2$ and $x > 4$.



12. **Generalize** Suppose that $a < b$. Select from the symbols $>$, $<$, $\geq$, and $\leq$, as well as the words *and* and *or*, to complete the compound inequality below so that its solution is all real numbers.

x ▨ a ▨ x ▨ b

13. **Higher Order Thinking** Let a and b be real numbers.

a. If $a > b$, how is the graph of $x > a$ and $x > b$ different from the graph of $x > a$ or $x > b$?

b. If $a < b$, how is the graph of $x > a$ and $x > b$ different from the graph of $x > a$ or $x > b$?

c. If $a = b$, how is the graph of $x > a$ and $x > b$ different from the graph of $x > a$ or $x > b$?

14. **Mathematical Connections** Consider the solutions of the compound inequalities.

$4 < x < 8 \qquad 2 < x < 11$

Describe each solution as a set. Is one set a subset of the other? Explain your answer.

PRACTICE

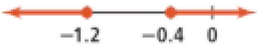
Write a compound inequality for each graph. SEE EXAMPLE 1

15.

16.

17.

18.



Solve each compound inequality and graph the solution. SEE EXAMPLES 2 AND 3

19. $2x + 5 > -3$ and $4x + 7 < 15$

20. $2x - 5 > 3$ or $-4x + 7 < -25$

21. $2x - 5 > 3$ and $-4x + 7 < -25$

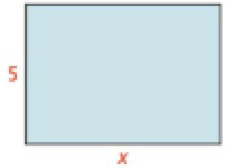
22. $-x + 1 > -2$ or $6(2x - 3) \geq -6$

23. $3 \leq 7x - 4 < 5$

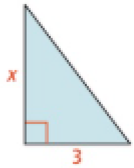
24. $-\frac{5}{8}x + 2 + \frac{3}{4}x > -1$ or $-3(x + 25) > 15$

The value for the area A of each figure is given. Write and solve a compound inequality for the value of x in each figure. SEE EXAMPLE 4

25. $35 \geq A \geq 25$



26. $9 \leq A \leq 12$



Write a compound inequality to represent each sentence below. SEE EXAMPLE 4

27. A quantity x is at least 10 and at most 20.

28. A quantity x is either less than 10 or greater than 20.

29. A quantity x is greater than 10 and less than 20.

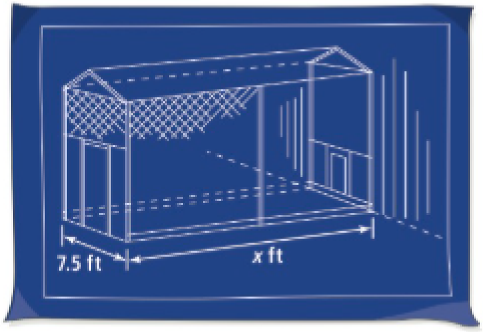
PRACTICE & PROBLEM SOLVING

APPLY

30. **Apply Math Models** Fatima plans to spend at least \$15 and at most \$20 on sketch pads and pencils. If she buys 2 sketch pads, how many pencils can she buy while staying in her price range?

31. **Analyze and Persevere** A peanut company ships its product in a carton that weighs 20 oz when empty. Twenty bags of peanuts are shipped in each carton. The acceptable weight for one bag of peanuts is between 30.5 oz and 33.5 oz, inclusive. If a carton weighs too much or too little, it is opened for inspection. Write and solve a compound inequality to determine *x*, the weights of cartons that are opened for inspection.

32. **Apply Math Models** Volunteers at an animal shelter are building a rectangular dog run so that one shorter side of the rectangle is formed by the shelter building as shown. They plan to spend between \$100 and \$200 on fencing for the sides at a cost of \$2.50 per ft. Write and solve a compound inequality to model the possible length of the dog run.



ASSESSMENT PRACTICE

33. Which of the following compound inequalities have the solution $x < 3$? Select all that apply.
AR.2.6

☐ A. $3x + 5 < 6$ or $-2x + 9 > 3$

☐ B. $3x + 5 < 6$ and $-2x + 9 > 3$

☐ C. $3x - 5 < 10$ and $-2x + 9 > 3$

☐ D. $3x + 5 < 6$ or $-2x + 9 < 3$

☐ E. $6x < 5x + 3 < 4x + 5$

34. **SAT/ACT** What is the solution of $0.2x - 4 - 2x < -0.4$ and $3x + 2.7 < 3$?

Ⓐ $x < -2$

Ⓑ $x < 0.1$

Ⓒ $x < 1$

Ⓓ $x > -2$ and $x < 0.1$

Ⓔ $x > -2$ and $x < 1$

35. **Performance Task** An animal shelter categorizes donors based on their total yearly donation, as shown in the table.

Donor Category	Total Yearly Donation
Bronze	< \$100
Silver	≥ \$100 and < \$500
Gold	≥ \$500 and < \$1,000
Platinum	≥ \$1,000

Part A Keenan donates the same amount each month. Write and solve a compound inequality for the monthly donation that will put him in the Gold category.

Part B Libby donated \$50 during the first month of the year. If she makes three additional donations of equal amounts during the year, how much will she need to donate each time to be in the Silver category?

Part C Paula originally planned to donate \$50 each month. After reviewing her budget, she decides that she must reduce her planned donation. By what amount can she reduce her original planned monthly donation of \$50 so that she will be in the Silver category?

1-6

Absolute Value Equations and Inequalities

I CAN… write and solve absolute value equations and inequalities.

MA.912.AR.4.1–Given a mathematical or real-world context, write and solve one-variable absolute value equations.
Also AR.4.2
MA.K12.MTR.1.1, MTR.3.1, MTR.5.1

MODEL & DISCUSS

Amelia is participating in a 60-mile spin-a-thon. Her spin bike keeps track of the simulated number of miles she travels. She plans to take a 15-minute break within 5 miles of riding 30 miles.


Amelia spins at a constant 22 mph.

Spin-a-thon Schedule	
Event	Time
Start spinning	10:00 A.M.
Stop for break	■
Resume spinning	■

A. Write a compound inequality that models the number of miles Amelia spins before taking a break.

B. How is the number of miles Amelia spins before she takes a break related to the amount of time before she takes a break?

C. **Choose Efficient Methods** About how many hours will Amelia spin before she takes a break? Discuss how you could use your mathematical model to complete the spin-a-thon schedule.

ESSENTIAL QUESTION

Why does the solution for an absolute value equation or inequality typically result in a pair of equations or inequalities?

EXAMPLE 1 Understand Absolute Value Equations

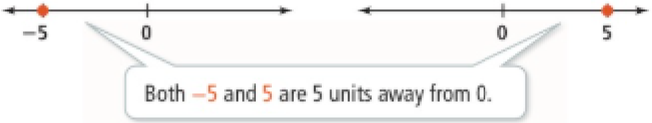
A. What is the value of x in $7 = |x| + 2$?

Solve for x by isolating the absolute value expression on one side of the equation.

$$7 = |x| + 2$$
$$7 - 2 = |x| + 2 - 2$$
$$5 = |x|$$

Use the Subtraction Property of Equality.

STUDY TIP
The absolute value of a number is its distance from 0 on a number line.



Both –5 and 5 are 5 units away from 0.

The solutions are $x = -5$ and $x = 5$.

Check the solutions.

$$7 \overset{?}{=} |-5| + 2 \qquad 7 \overset{?}{=} |5| + 2$$
$$\overset{?}{=} 5 + 2 \qquad \overset{?}{=} 5 + 2$$
$$= 7✓ \qquad = 7✓$$

CONTINUED ON THE NEXT PAGE

USE PATTERNS AND STRUCTURE
When solving an absolute value equation in the form $|ax + b| = c$, use two different equations to find the solutions, $ax + b = c$ and $ax + b = -c$.

EXAMPLE 1 CONTINUED

B. What is the value of x in $|2x - 3| = 1$?

Write and solve equations for the two possibilities:

$2x - 3$ is positive.

$$\begin{aligned} 2x - 3 &= 1 \\ 2x - 3 + 3 &= 1 + 3 \\ 2x &= 4 \\ \frac{2x}{2} &= \frac{4}{2} \\ x &= 2 \end{aligned}$$

$2x - 3$ is negative.

$$\begin{aligned} 2x - 3 &= -1 \\ 2x - 3 + 3 &= -1 + 3 \\ 2x &= 2 \\ \frac{2x}{2} &= \frac{2}{2} \\ x &= 1 \end{aligned}$$

The expression inside the absolute value symbol can be positive or negative. So the expression $2x - 3$ can be equal to 1 or −1.

The solutions are $x = 2$ and $x = 1$.

C. What is the value of x in $3|x + 6| + 8 = 5$?

Step 1 Isolate the absolute value expression.

$$\begin{aligned} 3|x + 6| + 8 - 8 &= 5 - 8 \\ 3|x + 6| &= -3 \\ \frac{3|x + 6|}{3} &= \frac{-3}{3} \end{aligned}$$

Step 2 Solve for x.

$$|x + 6| = -1$$

This equation has no solution.

The absolute value of a number is a distance and cannot be negative.

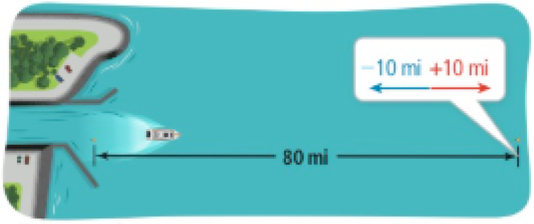
Try It! **1.** Solve each equation.

a. $6 = |x| - 2$ **b.** $2|x + 5| = 4$ **c.** $|3x - 6| = 12$

EXAMPLE 2 Apply an Absolute Value Equation

STUDY TIP
You can use an absolute value equation to model a quantity "plus or minus" another quantity.

The cruising speed of Kennedy's boat is 25 mi/h. She plans to cruise at this speed for a distance either 10 miles before or after the 80-mile point, as shown in the diagram.



Not to scale

A. What equation models the number of hours x that Kennedy will travel?

The distance Kennedy actually travels

$$\underbrace{|25x - 80|} = 10$$

10 miles from the 80-mile point

Final distance from the 80-mile point

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

B. What are the minimum number and maximum number of hours Kennedy will travel?

Write and solve equations for the two possibilities.

HAVE A GROWTH MINDSET
How can you use mistakes as opportunities to learn and grow?

If Kennedy travels 10 miles beyond the 80-mile point, the absolute value expression is positive.

$$25x - 80 = 10$$
$$25x - 80 + 80 = 10 + 80$$
$$25x = 90$$
$$\frac{25x}{25} = \frac{90}{25}$$
$$x = 3.6$$

If Kennedy travels 10 miles before the 80-mile point, the absolute value expression is negative.

$$25x - 80 = -10$$
$$25x - 80 + 80 = -10 + 80$$
$$25x = 70$$
$$\frac{25x}{25} = \frac{70}{25}$$
$$x = 2.8$$

The solutions are $x = 3.6$ and $x = 2.8$.

Kennedy will travel at least 2.8 hours and at most 3.6 hours.

Try It! **2.** What will be the minimum and maximum time that Kennedy will travel if she resets her cruising speed to 20 mi/h?

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Understand Absolute Value Inequalities

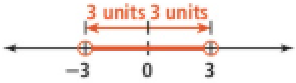
What are the solutions of an absolute value inequality?

COMMON ERROR
Remember to look at the > and < symbols when solving absolute value inequalities. Inequalities with absolute value have to be set up differently to solve if it is an "and" situation vs. an "or" situation.

Solve and graph two absolute value inequalities.

A. $|x| < 3$

The distance between x and 0 must be less than 3, so all values within 3 units to the right and 3 units to the left of 0 are solutions.



$|x| < 3$ is equivalent to the compound inequality $x < 3$ *and* $x > -3$, which can also be written as $-3 < x < 3$.

B. $|x| > 3$

The distance between x and 0 must be greater than 3. Positive values of x must be more than 3 units to the right of 0, and negative values of x must be more than 3 units to the left of 0.

$|x| > 3$ is equivalent to the compound inequality $x < -3$ *or* $x > 3$.

Try It! **3.** Solve and graph the solutions of each inequality.

a. $|x| > 15$

b. $|x| \leq 7$

APPLICATION

EXAMPLE 4 Write an Absolute Value Inequality

Members of the debate team are traveling to a tournament, where they will stay in a hotel for 4 nights. The total cost for each member must be within $20 of $175. Which of the hotels shown can they consider?

Hotel Room Costs*

*Per person per night

Formulate ◀ Write an absolute value inequality to represent the situation.

Let x be the cost per night of a hotel room.

The difference between total cost and $175 is less than or equal to $20.

$$|4x - 175| \leq 20$$

Compute ◀ Solve the inequality to find the maximum and minimum hotel cost for each team member.

Maximum Cost	**Minimum Cost**
$4x - 175 \leq 20$	$4x - 175 \geq -20$
$4x - 175 + 175 \leq 20 + 175$	$4x - 175 + 175 \geq -20 + 175$
$4x \leq 195$	$4x \geq 155$
$\frac{4x}{4} \leq \frac{195}{4}$	$\frac{4x}{4} \geq \frac{155}{4}$
$x \leq 48.75$	$x \geq 38.75$

Interpret ◀ The cost of the hotel room can be between $38.75 and $48.75, inclusive.

The debate team can consider Hotel A or Hotel B.

Try It! **4.** If the debate team increased their limit to $200 plus or minus $20, would they be able to afford Hotel D at $55 per night? Explain.

CONCEPT SUMMARY Absolute Value Equations and Inequalities

WORDS

Absolute Value Equations

To solve an absolute value equation, isolate the absolute value expression. Then write two equations and solve.

Absolute Value Inequalities

If the distance between x and 0 is less than or less than or equal to a value, the solution is two inequalities joined by "and".
If the distance between x and 0 is greater than or greater than or equal to a value, the solution is two inequalities joined by "or".

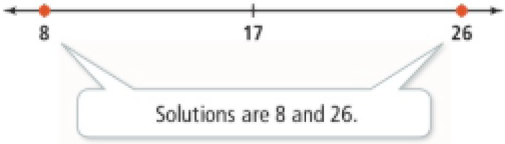
ALGEBRA

$$2|x - 17| = 18$$
$$|x - 17| = 9$$
$$x - 17 = -9 \qquad x - 17 = 9$$
$$x = 8 \qquad x = 26$$

$$|x| \le 1$$
$$x \ge -1 \text{ and } x \le 1$$

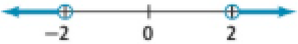
$$|x| > 2$$
$$x < -2 \text{ or } x > 2$$

DIAGRAMS



$|x| \le 1$

$|x| > 2$



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** Why does the solution for an absolute value equation or inequality typically result in a pair of equations or inequalities?

2. **Generalize** How is solving an absolute value equation similar to solving an equation that does not involve absolute value? How is it different?

3. **Vocabulary** Describe how you would explain to another student why the *absolute value* of a number cannot be negative.

4. **Error Analysis** Yumiko solved $|x| > 5$ by solving $x > -5$ and $x < 5$. Explain the error Yumiko made.

Do You KNOW HOW?

Solve each absolute value equation.

5. $5 = |x| + 3$

6. $|2x - 8| = 16$

7. On a road trip, Andrew plans to use his cruise control for 125 mi, plus or minus 20 mi. Write and solve an equation to find the minimum and maximum number of hours for Andrew's road trip.

Solve each absolute value inequality. Graph the solution.

8. $|3x - 6| \ge 9$

9. $|4x - 12| \le 20$

PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. Analyze and Persevere Sasha is solving the absolute value equation $|2x| + 4 = 8$. What is the first step she should take?

11. Represent and Connect The absolute value inequality $5 \leq |x| - n$ is graphed below. What is the value of n?

12. Error Analysis Describe and correct the error a student made when solving $2|x| < 16$.

Solve $2|x| < 16$.

$2|x| < 16$

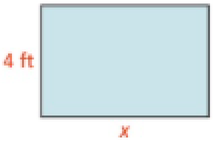
$\frac{2|x|}{2} < \frac{16}{2}$ Divide both sides by 2.

$|x| < 8$ Simplify.

$x < 8$ or $x > -8$ Rewrite using "or."

✗

13. Mathematical Connections Jack wants to model a situation where the perimeter of the rectangle below is 6 ft plus or minus 1.5 ft.



Because he is modeling a length "plus or minus" another length, he decides to use an absolute value equation for his model. Do you agree with his decision? Explain your reasoning.

14. Higher Order Thinking Let a, b, c, and x be real numbers.

a. How is solving $|ax| + b = c$ different from solving $|ax + b| = c$?

b. How is solving $|ax| + b \leq c$ different from solving $|ax + b| \geq c$?

PRACTICE

Solve each absolute value equation. SEE EXAMPLE 1

15. $2 = |x| - 1$

16. $|x| - 4 = 9$

17. $14 = |x| + 2$

18. $|x| + 4 = -9$

19. $|-2x + 8| = 20$

20. $|x - 4| = 9$

21. $2|x + 8| = 20$

22. $2|x - 8| = 20$

23. $5|x + 3| + 8 = 6$

24. $3|x - 2| - 8 = 7$

Write and solve an absolute value equation for the minimum and maximum times for an object moving at the given speed to travel the given distance. (Figures are not to scale.) SEE EXAMPLE 2

25. 5 mi/h

26. 10 ft/s

Solve each absolute value inequality. Graph the solution. SEE EXAMPLES 3 AND 4

27. $2 \leq |x| - 8$

28. $-2 > |x| - 8$

29. $|x| + 5 \geq 10$

30. $|x| + 2.4 < 3.6$

31. $|2x + 5| \geq 9$

32. $|2x - 5| < 9$

33. $-2|x + 4| \leq -6$

34. $-2|2x + 4| + 10 > -6$

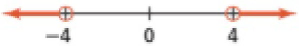
Match each absolute value inequality to the graph that represents its solution. Explain your reasoning. SEE EXAMPLES 3 AND 4

35. $3|x| - 2 \leq 10$

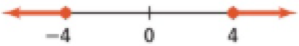
36. $2|x| - 1 < 7$

37. $3|2x| + 1 > 25$

38. $2|4x| - 7 \geq 25$

A.


B.

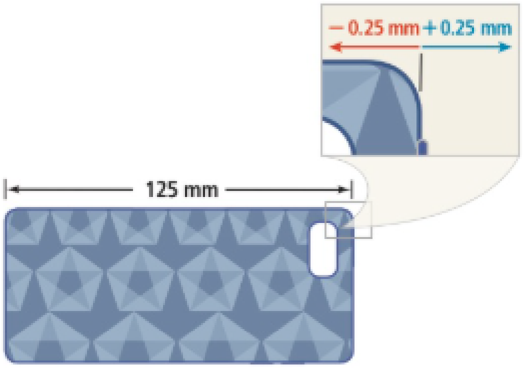

C.


D.


PRACTICE & PROBLEM SOLVING

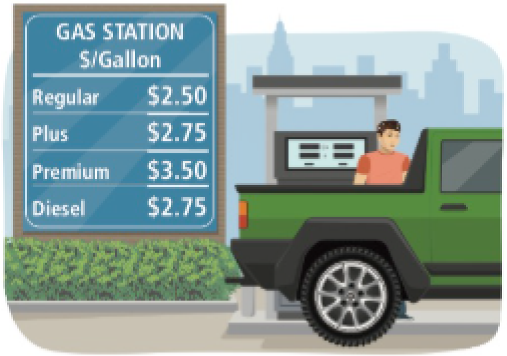
APPLY

39. **Analyze and Persevere** A company manufactures cell phone cases. The length of a certain case must be within 0.25 mm of 125 mm, as shown (figure is not to scale). All cases with lengths outside of this range are removed from the inventory. How could you use an absolute value inequality to represent the lengths of all the cases that should be removed? Explain.



40. **Communicate and Justify** Ashton is hosting a banquet. He plans to spend \$400, plus or minus \$50, at a cost of \$25 per guest. Solve $|25x - 400| = 50$ to find the maximum and minimum number of guests. If there can be up to 7 guests at each table, what is the minimum number of tables Ashton should reserve so that every guest has a seat?

41. **Apply Math Models** Hugo is pumping regular gas into his truck. Write and solve an absolute value equation to find how many gallons of gas will be pumped when the total is \$25 plus or minus \$0.50.



ASSESSMENT PRACTICE

42. Kenzie runs at an average speed of 6.25 mi/h. The length of her typical run is within one mile of 6 miles long.

Write an absolute value equation to represent the situation. Find the shortest or longest amount of time for Kenzie's run. Round to the nearest minute. AR.4.1

43. **SAT/ACT** What is the solution of $|4x - 6| = 2$?

Ⓐ $x = 1, x = 2$

Ⓑ $x = -1, x = 2$

Ⓒ $x = 1, x = -2$

Ⓓ $x = -1, x = -2$

Ⓔ $x = -2, x = 2$

44. **Performance Task** A road sign shows a vehicle's speed as the vehicle passes.

Part A The sign blinks for vehicles traveling within 5 mi/h of the speed limit. Write and solve an absolute value inequality to find the minimum and maximum speeds of an oncoming vehicle that will cause the sign to blink.

Part B Another sign blinks when it detects a vehicle traveling within 2 mi/h of a 35 mi/h speed limit. Write and solve an absolute value inequality to represent the speeds of the vehicles that cause the sign to blink.

Part C The sign is programmed to blink using absolute value inequalities of the form $|x - a| \le b$ and $|x - a| \ge b$. Which of these formulas is used to program the sign for cars traveling either 5 mi/h above or below the 20 mi/h speed limit? What are the values of a and b? Explain.

TOPIC

1

Topic Review

TOPIC ESSENTIAL QUESTION

1. What general strategies can you use to solve simple equations and inequalities?

Vocabulary Review

Choose the correct term to complete each sentence.

2. An equation rule for a relationship between two or more quantities is a(n) ______.
3. A combination of two or more inequalities using the word *and* or the word *or* is a(n) ______.
4. Any of the distinct objects of a set is called a(n) ______.
5. If each element of *B* is also an element of *A*, *B* is a(n) ______ of *A*.
6. A well-defined collection of elements is a(n) ______.
7. An equation in which letters are used for constants and variables is a(n) ______.
8. An equation that is true for all possible values of the variable is a(n) ______.

- compound inequality
- element
- formula
- identity
- literal equation
- set
- subset

Concepts & Skills Review

LESSON 1-1 Solving Linear Equations

Quick Review

You can use properties of equality to solve linear equations. Use the Distributive Property and combine like terms, when needed.

Example

Solve $\frac{2}{3}(6x - 15) + 5x = 26$.

$\frac{2}{3}(6x - 15) + 5x = 26$	
$4x - 10 + 5x = 26$	Distributive Property
$9x - 10 = 26$	Combine like terms.
$9x - 10 + 10 = 26 + 10$	Add 10 to each side.
$9x = 36$	Simplify.
$\frac{9x}{9} = \frac{36}{9}$	Divide each side by 9.
$x = 4$	Simplify.

Practice & Problem Solving

9. **Choose Efficient Methods** What property would you use first to solve $\frac{1}{2}x - 6 = 10$? Explain.

Solve each equation.

10. $3(2x - 1) = 21$
11. $100 = 8(4t - 5)$
12. $\frac{5}{8} = \frac{3}{4}b - \frac{7}{12}$
13. $1.045s + 0.068 = 15.743$
14. **Apply Math Models** The price for an adult movie ticket is $1\frac{1}{3}$ more than a movie ticket for a child. Ines takes her daughter to the movie, buys a box of popcorn for \$5.50, and spends \$26.50. Write and solve an equation to find the prices for each of their movie tickets.

LESSON 1-2 Solving Equations with a Variable on Both Sides

Quick Review

To solve equations with a variable on both sides, rewrite the equation so that all the variable terms are on one side of the equation and the constants are on the other. Then solve for the value of the variable.

Example

Solve $5x - 48 = -3x + 8$.

$5x - 48 = -3x + 8$

$5x - 48 + 3x = -3x + 8 + 3x$ ⋯ Add $3x$ to each side.

$8x - 48 = 8$ ⋯ Simplify.

$8x - 48 + 48 = 8 + 48$ ⋯ Add 48 to each side.

$8x = 56$ ⋯ Simplify.

$\frac{8x}{8} = \frac{56}{8}$ ⋯ Divide each side by 8.

$x = 7$ ⋯ Simplify.

Practice & Problem Solving

15. **Error Analysis** Describe and correct any errors a student may have made when solving the equation $0.6(y - 0.2) = 3 - 0.2(y - 1)$.

$$0.6(y - 0.2) = 3 - 0.2(y - 1)$$
$$0.6y - 0.12 = 3.2 - 0.2y$$
$$100(0.6y - 0.12) = 10(3.2 - 0.2y)$$
$$60y - 12 = 32 - 2y$$
$$60y - 12 + 12 + 2y = 32 + 12 - 2y + 2y$$
$$62y = 42$$
$$y = \frac{21}{31}$$

Solve each equation.

16. $21 - 4x = 4x + 21$

17. $6b - 27 = 3(5b - 2)$

18. $0.45(t + 8) = 0.6(t - 3)$

19. **Communicate and Justify** Aaron can join a gym that charges \$19.99 per month, plus an annual \$12.80 fee, or he can pay \$21.59 per month. He thinks the second option is better because he plans to use the gym for 10 months. Is Aaron correct? Explain.

LESSON 1-3

Literal Equations and Formulas

Quick Review

You can use properties of equality to solve literal equations for a specific variable. You can then use the rewritten equations to solve problems.

Example

Find the height of a cylinder with a volume of 1,650 cm^3 and a radius of 6 cm.

Rewrite the formula for the volume of a cylinder in terms of h.

$$A = \pi r^2 h$$

$$\frac{A}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2}$$

$$\frac{A}{\pi r^2} = h$$

Find the height of the cylinder. Use 3.14 for pi.

$$h = \frac{A}{\pi r^2}$$

$$h = \frac{1{,}650}{(3.14)(6)^2} = \frac{1{,}650}{(3.14)(36)} = \frac{1{,}650}{113.04} \approx 14.60$$

The height of the cylinder is about 14.60 cm.

Practice & Problem Solving

20. **Error Analysis** Describe and correct the error a student made when solving $a = \frac{3}{4}(b + 5)$ for b.

$$a = \frac{3}{4}(b + 5)$$

$$\frac{4}{3}a = \frac{3}{4}(b + 5)\frac{4}{3}$$

$$\frac{4}{3}a = b + 5$$

$$b = \frac{4}{3}a + 5$$

Solve each equation for the given variable.

21. $xy = k$; y

22. $a = \frac{2}{b} + 3c$; c

23. $6(2c + 3d) = 5(4c - 3d)$; d

24. **Apply Math Models** The formula for average acceleration is $a = \frac{V_f - V_i}{t}$, where V_f is the final velocity, V_i is the initial velocity, and t is the time in seconds. Rewrite the equation as a formula for the final velocity, V_f. What is the final velocity when a person accelerates at 2 ft/s^2 for 5 seconds after an initial velocity of 4 ft/s?

LESSON 1-4

Solving Inequalities in One Variable

Quick Review

The same strategies used for solving multistep equations can be used to solve multistep inequalities. When multiplying or dividing by a negative value on both sides, reverse the inequality symbol.

Example

Solve $-2(6x + 5) \leq 74$. Graph the solution.

$-2(6x + 5) \leq 74$

$-12x - 10 \leq 74$ .... Distributive Property

$-12x - 10 + 10 \leq 74 + 10$ .... Add 10 to each side.

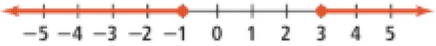
$\frac{-12x}{-12} \geq \frac{84}{-12}$ .... Divide each side by -12.

$x \geq -7$ .... Simplify.

The solution is $x \geq -7$.

Practice & Problem Solving

25. **Represent and Connect** Write an inequality that represents the graph.



Solve each inequality and graph the solution.

26. $x + 8 > 11$

27. $4x + 3 \leq -6$

28. $2.4x - 9 < 1.8x + 6$

29. $3x - 8 \geq 4(x - 1.5)$

30. **Analyze and Persevere** Neil and Yuki run a data entry service. Neil starts at 9:00 A.M. and can type 45 words per minute. Yuki arrives at 10:30 A.M. and can type 60 words per minute. Write and solve an inequality to find at what time Yuki will have typed more words than Neil. Let x represent the time in minutes.

LESSON 1-5 Compound Inequalities

Quick Review

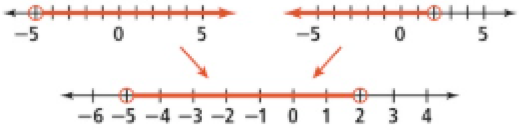
When a compound inequality uses the word *and*, the solution must make both inequalities true. If a compound inequality uses the word *or*, the solution must make at least one of the inequalities true.

Example

Solve $-24 < 4x - 4 < 4$. Graph the solution.

Separate the inequality and solve each separately.

$-24 < 4x - 4$	$4x - 4 < 4$
$-24 + 4 < 4x - 4 + 4$	$4x - 4 + 4 < 4 + 4$
$-20 < 4x$	$4x < 8$
$-5 < x$	$x < 2$



The solution is $x > -5$ and $x < 2$, or $-5 < x < 2$.

Practice & Problem Solving

31. **Communicate and Justify** Describe and correct the error a student made graphing the compound inequality $x > 3$ or $x < -1$.

Solve each compound inequality and graph the solution.

32. $2x - 3 > 5$ or $3x - 1 < 8$

33. $x - 6 \leq 18$ and $3 - 2x \geq 11$

34. $\frac{1}{2}x - 5 > -3$ or $\frac{2}{3}x + 4 < 3$

35. $3(2x - 5) > 15$ and $4(2x - 1) > 10$

36. **Apply Math Models** Lucy plans to spend between \$50 and \$65, inclusive, on packages of beads and packages of charms. If she buys 5 packages of beads at \$4.95 each, how many packages of charms at \$6.55 can Lucy buy while staying within her budget?

LESSON 1-6 Absolute Value Equations and Inequalities

Quick Review

When solving an absolute value equation or inequality you must consider both the positive and negative values of the expression inside the absolute value symbol.

Example

What is the value of x in $|4x + 7| = 43$?

Write and solve inequalities for the two cases.

$4x + 7$ is positive.	$4x + 7$ is negative.
$4x + 7 = 43$	$4x + 7 = -43$
$4x + 7 - 7 = 43 - 7$	$4x + 7 - 7 = -43 - 7$
$4x = 36$	$4x = -50$
$x = 9$	$x = -12.5$

The solutions are $x = 9$ and $x = -12.5$.

Practice & Problem Solving

37. **Choose Efficient Methods** Thato is solving the absolute value equation $|3x| - 5 = 13$. What is the first step he should take?

Solve each absolute value equation or inequality.

38. $3 = |x| + 1$

39. $4|x - 5| = 24$

40. $3 > |x| - 6$

41. $|2x - 3| \leq 12$

42. **Analyze and Persevere** A person's normal body temperature is 98.6°F. According to physicians, a person's body temperature should not be more than 0.5°F from the normal temperature. How could you use an absolute value inequality to represent the temperatures that fall outside of normal range? Explain.

TOPIC

2

Linear Equations

TOPIC ESSENTIAL QUESTION

Why is it useful to have different forms of linear equations?

Topic Overview

enVision® STEM Project:
Design a Pitched Roof

2-1 Slope-Intercept Form
AR.2.2, AR.2.4, MTR.1.1, MTR.6.1, MTR.7.1

2-2 Point-Slope Form
AR.1.2, AR.2.2, AR.2.4, AR.2.5, MTR.2.1, MTR.3.1, MTR.5.1

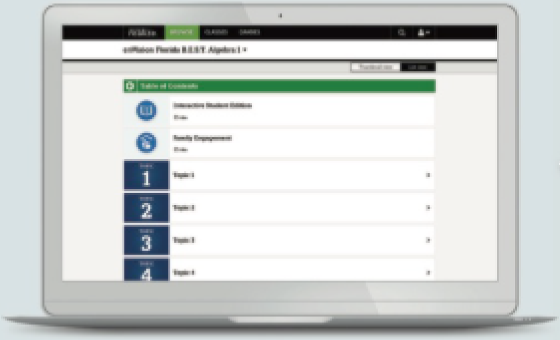
2-3 Standard Form
AR.1.1, AR.1.2, AR.2.2, AR.2.4, AR.2.5, MTR.3.1, MTR.4.1, MTR.7.1

Mathematical Modeling in 3 Acts:
How Tall is Tall?
AR.1.2, AR.2.2, MTR.7.1

2-4 Parallel and Perpendicular Lines
AR.2.3, MTR.1.1, MTR.2.1, MTR.5.1

Topic Vocabulary

- parallel lines
- perpendicular lines
- point-slope form
- reciprocal
- slope-intercept form
- standard form of a linear equation
- *y*-intercept



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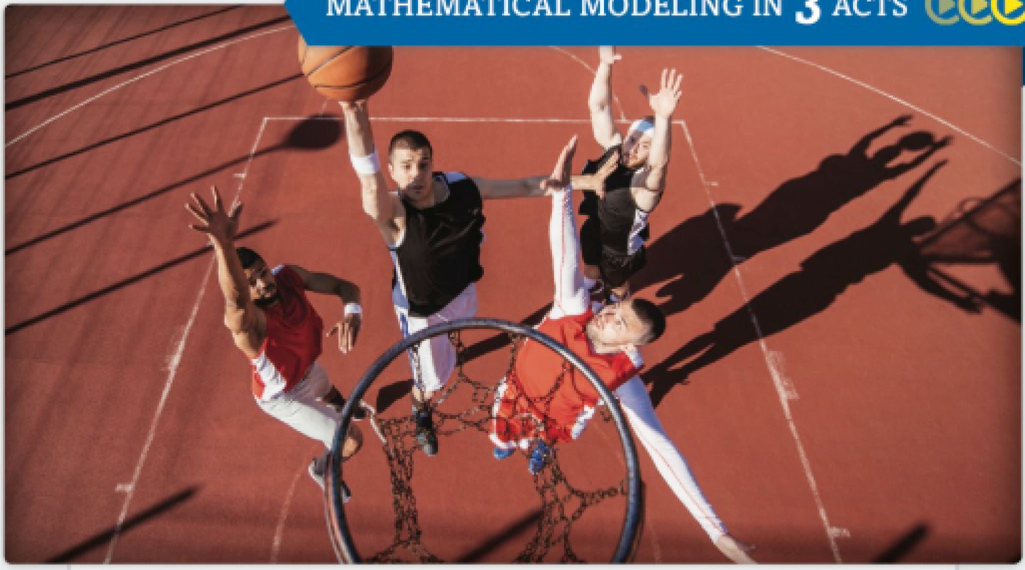
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



How Tall is Tall?

The world's tallest person in recorded history was Robert Wadlow. He was 8 feet 11.1 inches tall! Only 5% of the world population is 6 feet 3 inches or taller. What percent of the population would you guess is 7 feet or taller?

We usually use standard units, such as feet and inches or centimeters, to measure length or height. Did you ever wonder why? In the Mathematical Modeling in 3 Acts lesson you'll consider some interesting alternatives.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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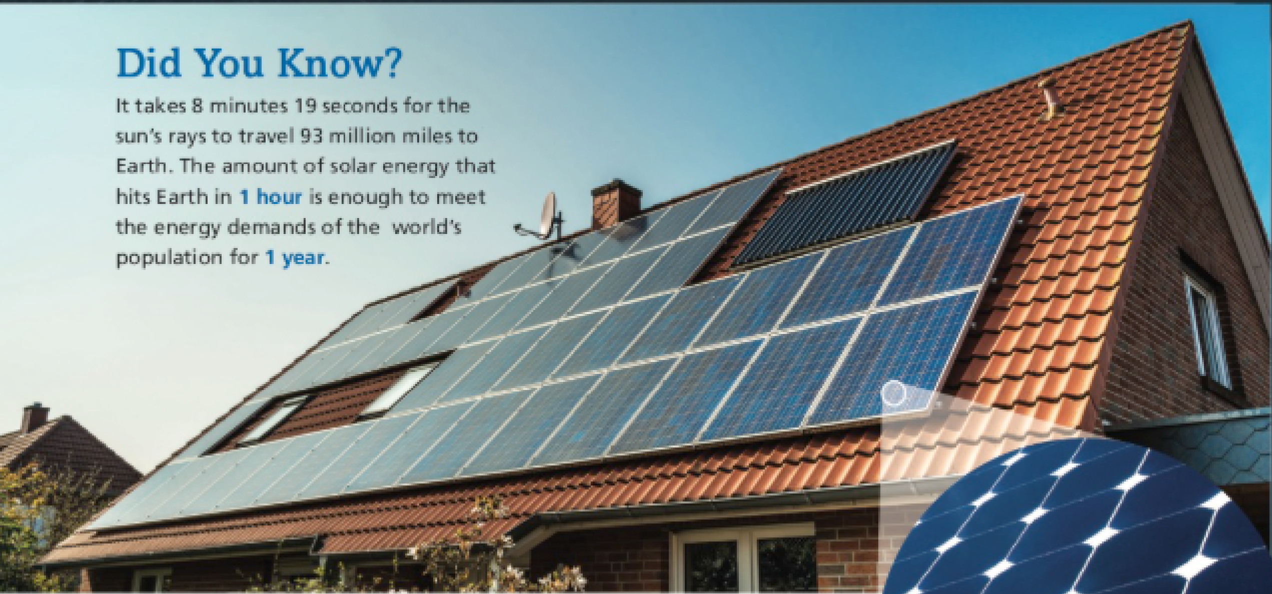
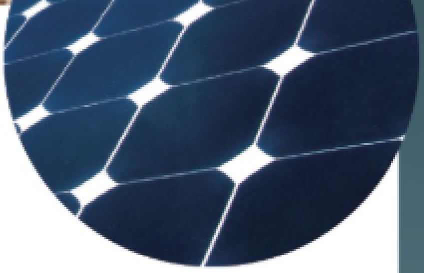
TOPIC 2 **PROJECT** enVision STEM

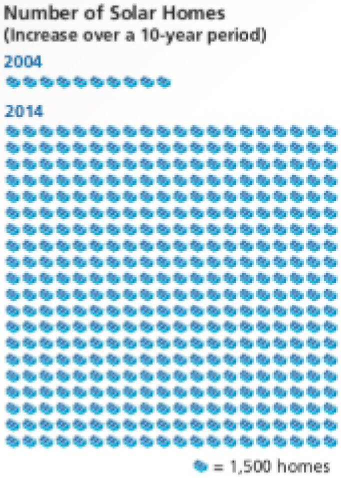


Did You Know?

It takes 8 minutes 19 seconds for the sun's rays to travel 93 million miles to Earth. The amount of solar energy that hits Earth in **1 hour** is enough to meet the energy demands of the world's population for **1 year**.

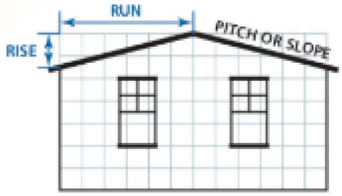


In 2004, about **15,000 homes** in the United States had solar panels. By the end of 2014, about **600,000 homes** had solar panels.



Number of Solar Homes
(Increase over a 10-year period)

What Is **Roof Pitch**?



Roof pitch is closely related to **slope**. This roof's pitch is equivalent to 3 : 12, which means the roof rises (or falls) 3 inches for every horizontal foot.

The steepness, or pitch, of a roof affects many things, including the installation of solar panels and how much snow the roof can handle.

Solar panels are a collection of solar cells. Solar cells **convert sunlight to electricity**. The solar cells in a solar panel are arranged so that solar energy travels along a specific path.

Your Task: Design a Pitched Roof

You and your classmates will analyze roofs to determine their pitch. Then you will design a roof with a pitch that is appropriate for installing solar panels.

2-1

Slope-Intercept Form

I CAN… write and graph linear equations using slope-intercept form.

VOCABULARY

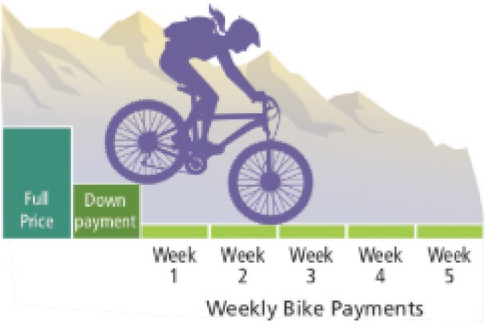
- slope-intercept form
- *y*-intercept

MA.912.AR.2.2–Write a linear two-variable equation to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. Also AR.2.4

MA.K12.MTR.1.1, MTR.6.1, MTR.7.1

MODEL & DISCUSS

Alani wants to buy a $360 bicycle. She is considering two payment options. The image shows Option A, which consists of making an initial down payment then smaller, equal-sized weekly payments. Option B consists of making 6 equal payments over 6 weeks.



A. What factors should Alani take into consideration before deciding between Option A and Option B?

B. **Analyze and Persevere** Suppose Alani could modify Option A and still pay off the bike in 5 weeks. Describe the relationship between the down payment and the weekly payments.

ESSENTIAL QUESTION

What information does the slope-intercept form of a linear equation reveal about a line?

EXAMPLE 1 Graph a Linear Equation

What is the graph of $y = \frac{4}{5}x + 2$?

The equation is in slope-intercept form. You can use the slope and *y*-intercept to graph the line.

$$y = \frac{4}{5}x + 2$$

Step 1 Identify the *y*-intercept in the equation.

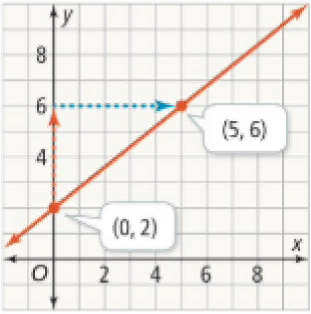
The *y*-intercept is 2, so plot the point (0, 2).

Step 2 Use the slope to plot a second point.

$$m = \frac{4}{5} = \frac{\text{vertical change}}{\text{horizontal change}}$$

Start at (0, 2), move 4 units up and 5 units to the right to locate a second point. Plot the point (5, 6).

Step 3 Draw a line through the points.



USE PATTERNS AND STRUCTURE
Think about the relationship between the value of the coefficient of *x* and the slope of the line.

Try It! **1.** Sketch the graph of $y = -\frac{3}{4}x - 5$.

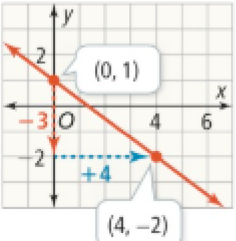
EXAMPLE 2 Write an Equation from a Graph

What is the equation of the line in slope-intercept form?

Step 1 Find the slope between two points on the line.

The line passes through (0, 1) and (4, −2).

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3}{4}$$



Step 2 Find the y-intercept.

The line intersects the y-axis at (0, 1), so the y-intercept is 1.

Step 3 Write the equation in the form $y = mx + b$.

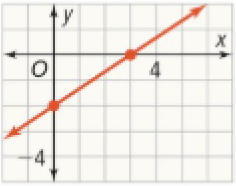
Substitute $-\frac{3}{4}$ for m and 1 for b.

The equation of the line in slope-intercept form is $y = -\frac{3}{4}x + 1$.

STUDY TIP

If you can approximate the y-intercept by looking at the graph, you can use it as one of the two points for finding the slope.

Try It! 2. Write the equation of the line in slope-intercept form.



CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Understand Slope-Intercept Form

How can you find an equation of a line that passes through two points if neither of them is the y-intercept?

Consider the line that passes through the points (−1, −2) and (3, 4).

Step 1 Find the slope of the line.

$$m = \frac{4-(-2)}{3-(-1)} = \frac{3}{2}$$

Step 2 Use the slope and one point to find the y-intercept.

$4 = \frac{3}{2}(3) + b$ ········ Substitute $\frac{3}{2}$ for m and (3, 4) for (x, y) in $y = mx + b$.

$4 = \frac{9}{2} + b$ ········ Simplify.

$-\frac{1}{2} = b$ ········ Solve for b.

Step 3 Use the slope and the y-intercept to write the equation

$y = \frac{3}{2}x + \left(-\frac{1}{2}\right)$ ········ Substitute $\frac{3}{2}$ for m and $-\frac{1}{2}$ for b.

The equation in slope-intercept form of the line that passes through (−1, −2) and (3, 4) is $y = \frac{3}{2}x - \frac{1}{2}$.

COMMON ERROR

You may think that a point with two negative coordinates means that the slope will be negative. Keep in mind that the slope depends on both points, so there is no way to determine the sign of the slope from one point.

Try It! 3. Write the equation in slope-intercept form of the line that passes through the points (5, 4) and (−1, 6).

APPLICATION

EXAMPLE 4 Interpret Slope and *y*-Intercept

Allie received a gift card for her local coffee shop. Every time she goes to the shop, she gets a medium coffee. The graph shows the gift card balance at two points. How can Allie determine the number of medium coffees she can buy with the gift card if she does not know the original value of the card?

HAVE A GROWTH MINDSET
In what ways do you give your best effort and persist?

Step 1 Interpret the meaning of the two points.

(2, 19.7): After buying 2 coffees, Allie had $19.70 left on the gift card.

(4, 14.4): After buying 4 coffees, Allie had $14.40 left on the gift card.

Step 2 Find the slope. Then interpret the meaning of the slope.

Use the points (2, 19.7) and (4, 14.4).

$$m = \frac{19.7 - 14.4}{2 - 4}$$

$$= -2.65$$

The slope is –2.65, which means the rate of change is –$2.65 per coffee. The balance on the gift card decreases by $2.65 each time Allie buys a medium coffee.

Step 3 Use one point and the slope to find the *y*-intercept. Then interpret its meaning.

CHECK FOR REASONABLENESS
Does the line with slope –2.65 and *y*-intercept 25 model the situation for every value of *x*?

$$y = mx + b$$

$$19.7 = -2.65(2) + b$$

$$25 = b$$

The *y*-intercept is 25. It represents the original value of the gift card.

To determine the number of medium coffees she can buy with the gift card, Allie can divide $25 by $2.65. She can purchase 9 medium coffees with the gift card.

Try It! 4. Use information from Example 4 to write the equation in slope-intercept form. Find the *x*-intercept of the graph of the equation. What does the *x*-intercept mean in terms of the situation?

CONCEPT SUMMARY Slope-Intercept Form of a Linear Equation

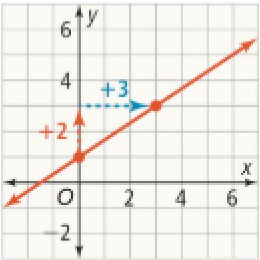
WORDS The slope-intercept form of a linear equation is used when the slope and the *y*-intercept of a line are known.

ALGEBRA The slope-intercept form of a line is $y = \underset{\text{slope}}{m}x + \underset{y\text{-intercept}}{b}$.

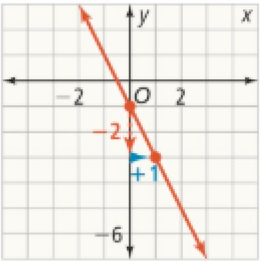
NUMBERS $y = \frac{2}{3}x + 1$

$y = -2x - 1$

GRAPH



The line has a slope of $\frac{2}{3}$.
The *y*-intercept is 1.



The line has a slope of −2.
The *y*-intercept is −1.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What information does the slope-intercept form of a linear equation reveal about a line?

2. **Represent and Connect** How are the graphs of $y = 2x + 1$ and $y = -2x + 1$ similar? How are they different?

3. **Error Analysis** To graph $y = \frac{2}{3}x + 4$, Emaan plots one point at (0, 4) and a second point 2 units right and 3 units up at (2, 7). He then draws a line through (0, 4) and (2, 7). What error did Emaan make?

4. **Analyze and Persevere** When writing the equation of a line in slope-intercept form, how can you determine the value of m in $y = mx + b$ if you know the coordinates of two points on the line?

Do You KNOW HOW?

Sketch the graph of each equation.

5. $y = 2x - 5$

6. $y = -\frac{3}{4}x + 2$

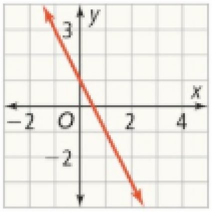
Identify the slope and *y*-intercept of the line for each equation.

7. $y = -5x - \frac{3}{4}$

8. $y = \frac{1}{4}x + 5$

Write the equation of each line in slope-intercept form.

9.



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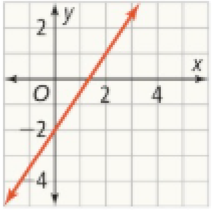


11. A line that passes through (3, 1) and (0, −3)

12. A line that passes through (−1, −5) and (4, −2)

PRACTICE & PROBLEM SOLVING

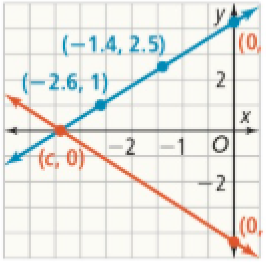
UNDERSTAND

13. Use Patterns and Structure Aisha and Carolina each sketch a graph of the linear equation $y = -\frac{3}{4}x + 2$. Aisha uses the equation $y = \frac{-3}{4}x + 2$ to sketch the graph, and Carolina uses the equation $y = \frac{3}{-4}x + 2$.

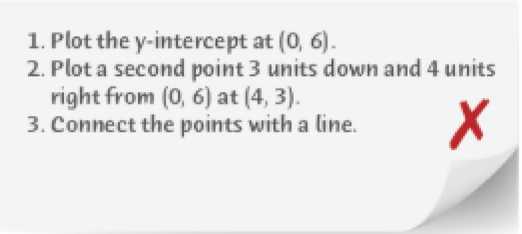
a. Explain how this leads them to use different steps to construct their graphs.

b. Will the two graphs look the same? Explain.

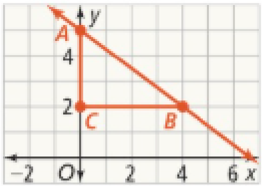
14. Analyze and Persevere Line g passes through the points (–2.6, 1) and (–1.4, 2.5), as shown. Find the equation of the line that passes through $(0, -b)$ and $(c, 0)$.



15. Error Analysis Describe and correct the error a student made when graphing the linear equation $y = -\frac{3}{4}x - 6$.


1. Plot the y-intercept at (0, 6).
2. Plot a second point 3 units down and 4 units right from (0, 6) at (4, 3).
3. Connect the points with a line.

16. Mathematical Connections The points $A(0, 5)$, $B(4, 2)$ and $C(0, 2)$ form the vertices of a right triangle. What is the equation of the line that contains the hypotenuse?



17. Analyze and Persevere Rewrite $y = \frac{1}{4}x + 3$, solving for x. What feature of the line can easily be found using the rearranged equation?

PRACTICE

Sketch the graph of each equation. SEE EXAMPLE 1

18. $y = \frac{3}{8}x + 5$

19. $y = -\frac{1}{2}x + 3$

20. $y = -2x + 3$

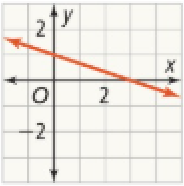
21. $y = 3x - 6$

22. $y = -\frac{3}{5}x + 4$

23. $y = \frac{5}{2}x - \frac{1}{2}$

Write the equation of each line in slope-intercept form. SEE EXAMPLE 2

24.



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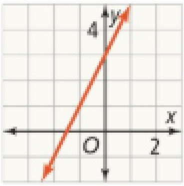

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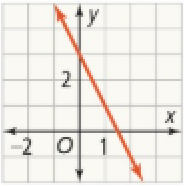

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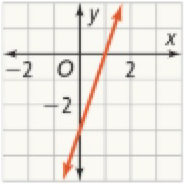


Write the equation of the line that passes through the given points. SEE EXAMPLE 3

28. (0, 1) and (2, 2)

29. (–2, –1) and (0, –5)

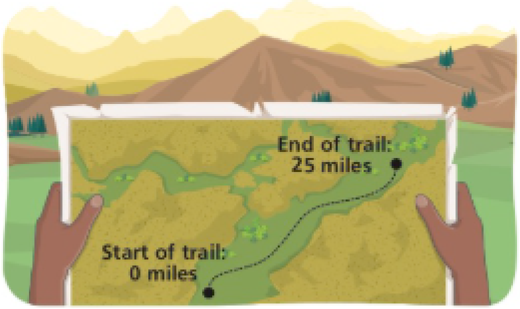
30. (4, 0) and (0, 2)

31. (–2, –6) and (1, 2)

32. $\left(\frac{3}{8}, 0\right)$ and $\left(\frac{5}{8}, \frac{1}{2}\right)$

33. (2, 1.5) and (0, 4.5)

34. Jordan will hike the trail shown at a rate of 4 mi/h. Write a linear equation to represent the distance Jordan still has to walk after x hours. What does the y-intercept of the equation represent? SEE EXAMPLE 4



PRACTICE & PROBLEM SOLVING

APPLY

35. Analyze and Persevere Naomi wants to buy a new computer for $840. She is considering two payment plans that require weekly payments. Which plan will pay for the computer faster? Explain.

36. Apply Math Models Becky is competing in an 8-mi road race. She runs at a constant speed of 6 mi/h. Write an equation in slope-intercept form to represent the distance Becky has left to run.

37. Apply Math Models Luis and Raul are riding their bicycles to the beach from their respective homes. Luis proposes that they leave their respective homes at the same time and plan to arrive at the beach at the same time. The diagram shows Luis's position at two points during his ride to the beach.

Write an equation in slope-intercept form to represent Luis's ride from his house to the beach. If Raul lives 5 miles closer to the beach than Luis, at what speed must Raul ride for the plan to work?

ASSESSMENT PRACTICE

38. The table shows the hourly pay rate for a restaurant employee for a given number of years of experience.

Years	Pay Rate
0	$9.70
1	$10.25
2	$10.80
3	$11.35

Which statement describes a model for the data? Select all that apply.
AR.2.2

☐ **A.** The slope is $9.70 and is the number of years of experience.

☐ **B.** The y-intercept is $0.55 and is the hourly rate of pay.

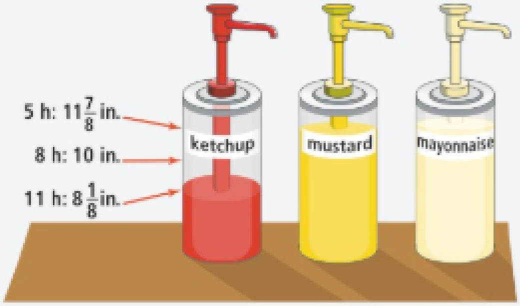
☐ **C.** The slope is $0.55 and is the amount of the pay raise each year.

☐ **D.** The y-intercept = $9.70 and is the starting hourly rate of pay.

39. SAT/ACT What is the equation of the line that has a slope of −3 and a y-intercept of 2?

Ⓐ $y = 2x - 3$
Ⓑ $y = 2x + 3$
Ⓒ $y = -3x + 2$
Ⓓ $y = -3x - 2$

40. Performance Task After filling the ketchup dispenser at the snack bar where she works, Kelley measures the level of ketchup during the day at different hourly intervals.



Part A Assuming the ketchup is used at a constant rate, write a linear equation that can be used to determine the level of ketchup in the dispenser after x hours.

Part B How can you use the equation from Part A to find the level of ketchup when the dispenser is full?

Part C If Kelley fills the ketchup dispenser just before the restaurant opens, and the restaurant is open for 18 hours, will the dispenser need to be refilled before closing time? Explain.

2-2

Point-Slope Form

I CAN… write and graph linear equations in point-slope form.

VOCABULARY

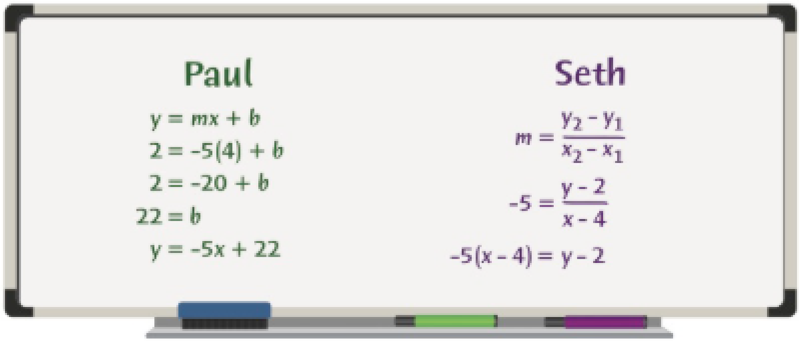
- point-slope form

MA.912.AR.2.2–Write a linear two-variable equation to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. **Also AR.1.2, AR.2.4, AR.2.5**

MA.K12.MTR.2.1, MTR.3.1, MTR.5.1

CRITIQUE & EXPLAIN

Paul and Seth know that one point on a line is (4, 2) and the slope of the line is −5. Each student derived an equation relating *x* and *y*.



Paul

$y = mx + b$

$2 = -5(4) + b$

$2 = -20 + b$

$22 = b$

$y = -5x + 22$

Seth

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$-5 = \frac{y - 2}{x - 4}$

$-5(x - 4) = y - 2$

A. Do the two equations represent the same line? Construct a mathematical argument to support your answer.

B. **Choose Efficient Methods** Generate a table of values for each equation. How can you reconcile the tables with the equations?

ESSENTIAL QUESTION **What information does the point-slope form of a linear equation reveal about a line?**

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Understand Point-Slope Form of a Linear Equation

A. How can you write the equation of a line using any points on a line?

Use the slope formula to find the slope using a specific point (x_1, y_1) and any point (x, y).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{y - y_1}{x - x_1}$$

Substitute x for x_2 and y for y_2.

$$m(x - x_1) = \frac{y - y_1}{x - x_1}(x - x_1)$$

$$m(x - x_1) = y - y_1$$

$$y - y_1 = m(x - x_1)$$

Multiply both sides of the equation by $(x - x_1)$.

You can write the equation of a line using any point, (x_1, y_1), and the slope, m, in **point-slope form**, $y - y_1 = m(x - x_1)$.

REPRESENT AND CONNECT
What mathematical notation is important in this example?

B. Why is it helpful to have point-slope form in addition to slope-intercept form?

Using point-slope form allows you to write the equation of a line without knowing the *y*-intercept. You can use any two points on the line to write the equation.

Try It! **1.** Describe the steps needed to find the *y*-intercept of the graph using point-slope form.

EXAMPLE 2 Write an Equation in Point-Slope Form

A. A line with a slope of $\frac{1}{2}$ passes through the point (3, –2). What form can you use to write the equation of the line? What is the equation in that form?

The slope and a point on the line are known, so use point-slope form.

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = \frac{1}{2}(x - 3)$$

Substitute 3 for x_1, −2 for y_1, and $\frac{1}{2}$ for m.

$$y + 2 = \frac{1}{2}(x - 3)$$

The equation in point-slope form is $y + 2 = \frac{1}{2}(x - 3)$.

CHOOSE EFFICIENT METHODS
Explain why it might not be helpful to apply the Distributive Property to right side of the equation.

B. What is the equation of the line that passes through (–4, 1) and (2, 3).

Find the slope of the line using the two given points.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{3 - 1}{2 - (-4)}$$

Substitute (2, 3) for (x_2, y_2) and (−4, 1) for (x_1, y_1).

$$= \frac{1}{3}$$

Use the slope and one point to write the equation.

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{1}{3}(x - 2)$$

Substitute 2 for x_1, 3 for y_1, and $\frac{1}{3}$ for m.

The equation in point-slope form is $y - 3 = \frac{1}{3}(x - 2)$.

STUDY TIP
You can use either point as (x_1, y_1). You just need to be careful to substitute the *x*- and *y*-coordinates from the same point.

Try It! 2. Write an equation of the line that passes through (2, −1) and (−3, 3).

EXAMPLE 3 Sketch the Graph of a Linear Equation in Point-Slope Form

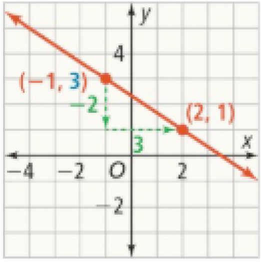
What is the graph of $y - 3 = -\frac{2}{3}(x + 1)$?

Step 1 Identify a point on the line from the equation and plot it.

$$y - 3 = -\frac{2}{3}(x + 1)$$

$$y - 3 = -\frac{2}{3}(x - (-1))$$

The point is (−1, 3).



COMMON ERROR
You may think that the *x*-coordinate of the point is 1. Remember that point-slope form uses $x - x_1$ which in this case is $x - (-1)$.

Step 2 Use the slope to plot a second point.

$$m = \frac{-2}{3} = \frac{\text{vertical change}}{\text{horizontal change}}$$

Move 2 units down and 3 units right from the first point. Plot the point (2, 1).

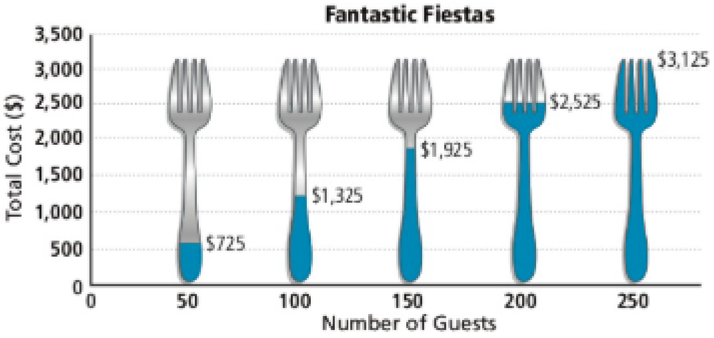
Step 3 Sketch a line through the points.

Try It! 3. Sketch the graph of $y + 2 = \frac{1}{2}(x - 3)$.

APPLICATION

EXAMPLE 4 Apply Linear Equations

An event facility has a banquet hall that can hold up to 250 people. The price for a party includes the cost of the room rental plus the cost of a meal for each guest. Marissa is planning an event for 75 people. She has budgeted $1,200 for the party. Will it be enough?



Formulate Determine which form of a linear equation is more useful.

The number of guests and the total costs represent different data points on a line. The point-slope form is more useful.

Compute The slope represents the cost of each meal. Use the two points (50, 725) and (100, 1,325) to find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1{,}325 - 725}{100 - 50}$$ Substitute (50, 725) for (x_1, y_1) and (100, 1,325) for (x_2, y_2).

$$= 12$$

The slope is 12, meaning that the rate of change is $12 per meal, so each meal costs $12.

Use point-slope form to find the cost of the event for 75 guests.

$$y - y_1 = m(x - x_1)$$

$$y - 725 = 12(x - 50)$$ Substitute 50 for x_1, 725 for y_1, and 12 for m.

$$y - 725 = 12(75 - 50)$$ Substitute 75 for x.

$$y = 300 + 725$$ Simplify and solve for y.

$$y = 1{,}025$$

When $x = 75$, $y = 1{,}025$. The cost of the event for 75 guests is $1,025.

Interpret Since Marissa budgeted $1,200 for her event she will have enough money.

Try It! 4. Rewrite the point-slope form equation from Example 4 in slope-intercept form. What does the y-intercept represent in terms of the situation?

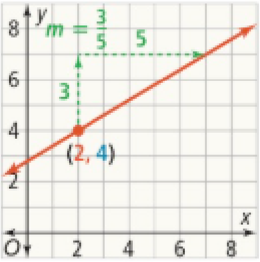
CONCEPT SUMMARY Point-Slope Form of a Linear Equation

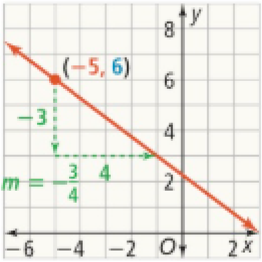
WORDS The point-slope form of a linear equation is useful when you know the slope and at least one point on the line.

ALGEBRA $y - y_1 = m(x - x_1)$.

NUMBERS $y - 4 = \frac{3}{5}(x - 2)$ $y - 6 = -\frac{3}{4}(x + 5)$

GRAPH





Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What information does the point-slope form of a linear equation reveal about a line?

2. **Use Patterns and Structure** If you know a point on a line and the slope of the line, how can you find another point on the line?

3. **Error Analysis** Denzel identified (3, 2) as a point on the line $y - 2 = \frac{2}{3}(x + 3)$. What is the error that Denzel made?

4. **Generalize** You know the slope and one point on a line that is not the *y*-intercept. Why might you write the equation in point-slope form instead of slope-intercept form?

Do You KNOW HOW?

Write the equation of the line in point-slope form that passes through the given point with the given slope.

5. (1, 5); $m = -3$ **6.** (−4, 3); $m = 2$

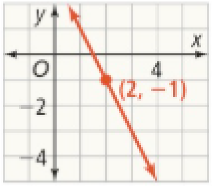
Write an equation of the line in point-slope form that passes through the given points.

7. (4, 2) and (1, 6) **8.** (−2, 8) and (7, −4)

9. Write the equation $y - 6 = -5(x + 1)$ in slope-intercept form.

10. Write the equation of the line in point-slope form.

a.



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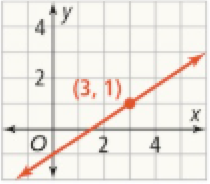
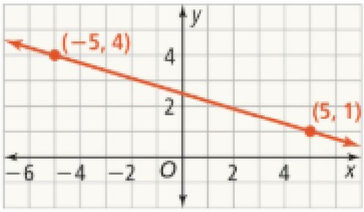


PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. **Use the graph of the line shown.**



a. Write a point-slope form of the equation for the line shown.

b. Estimate the value of the y-intercept of the line.

c. **Communicate and Justify** Use proportional reasoning to support your conjecture about the value of the y-intercept.

d. Rewrite the point-slope form of the equation in slope-intercept form to check your conjecture.

12. **Error Analysis** Describe and correct the error a student made when graphing $y + 5 = -\frac{3}{4}(x - 8)$.

1. Plot a point at (−5, 8).
2. Plot a point 3 units down and 4 units right from (−5, 8) at (−1, 5).
3. Connect the points with a line.

✗

13. **Higher Order Thinking** In slope-intercept form $y = mx + b$, the y-intercept is located at $(0, b)$.

a. What equation do you get when you substitute $(0, b)$ for (x_1, y_1) in point-slope form $y - y_1 = m(x - x_1)$?

b. How are the slope-intercept and the point-slope forms related?

14 **Represent and Connect** Rewrite point-slope form $y - y_1 = m(x - x_1)$ into slope-intercept form. What expression corresponds to the y-intercept b?

PRACTICE

Write the equation in point-slope form of the line that passes through the given point with the given slope. SEE EXAMPLES 1, 2, AND 3

15. $(3, 1)$; $m = 2$

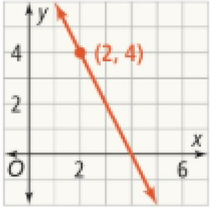
16. $(2, -2)$; $m = -4$

17. $(2, -8)$; $m = -\frac{3}{4}$

18. $(-1, 4)$; $m = \frac{2}{3}$

Write the equation of the line in point-slope form. SEE EXAMPLES 2 AND 3

19.



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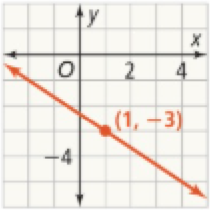


Write an equation of the line in point-slope form that passes through the given points. SEE EXAMPLE 2

21. (2, 4) and (3, 6)

22. (−1, −7) and (2, −4)

23. (3, −5) and (1, −8)

24. (−4, 12) and (−7, −3)

Sketch the graph of each equation. SEE EXAMPLE 3

25. $y + 2 = -3(x + 2)$

26. $y - 2 = 4(x - 1)$

27. $y + 1 = \frac{3}{2}(x - 1)$

28. $y - 3 = \frac{2}{5}(x + 1)$

Rewrite the point-slope form equation into slope-intercept form. SEE EXAMPLE 4

29. $y - 3 = \frac{1}{2}(x - 8)$

30. $y + 4 = -3(x - 5)$

Write an equation of the line in point-slope form that passes through the given points in each table. Then write each equation in slope-intercept form. SEE EXAMPLE 4

31.

x	y
15	100
20	115
25	130
30	145
35	160

32.

x	y
−4	−21
−2	−18
0	−15
2	−12
4	−9

Rewrite the slope-intercept equation into point-slope form.

33. $y = 2x + 7$

34. $y = -\frac{1}{3}x + 9$

PRACTICE & PROBLEM SOLVING

APPLY

35. **Apply Math Models** Liam rented a pedal board for 5.5 hours and paid a total of $93.75. What is an equation in point-slope form that models the cost of renting a pedal board for x hours? How can Liam use the equation to find the one-time service charge?

36. **Choose Efficient Methods** Emery borrowed money from her brother to buy a new phone, and is paying off a fixed amount each week. After 2 weeks, she will owe $456, and after 5 weeks, she will owe $228.

a. What was the original amount Emery borrowed?

b. How much does she pay each week?

c. How useful are equations in point-slope and slope-intercept forms for answering each question?

37. **Choose Efficient Methods** The total price of a printing job at Incredible Invites includes the cost per invitation plus a one-time set-up fee.

Write equations in point-slope and slope-intercept forms to model the situation. What part of the equations represents the cost per invitation? Which form is easier to use to find information about the set-up fee? Explain.

ASSESSMENT PRACTICE

38. Choose the real-world context that matches the data in the table. AR.2.2

x	y
0	75.75
1	91.00
2	106.25
3	121.50

Ⓐ Admission to a theme park is $75.75. The cost of each major attraction within the park is $15.25.

Ⓑ The cost of 3 meals is $121.50. The cost of each additional meal is $75.75.

Ⓒ The weight of a 1 year-old manatee is 91.00 kg. The manatee gains 106.25 in its second year.

Ⓓ The length of 3 sailboats is 121.50 ft. The length of the first sailboat is 75.75 ft.

39. **SAT/ACT** A line with a slope of –2 passes through the point (3, –2). Which of the following is the equation of the line?

Ⓐ $y + 2 = -2(x - 3)$

Ⓑ $y - 2 = -2(x - 3)$

Ⓒ $y - 2 = -2(x + 3)$

Ⓓ $y + 2 = 2(x - 3)$

Ⓔ $y - 2 = 2(x + 3)$

40. **Performance Task** A railway system on a hillside moves passengers at a constant rate to an elevation of 50 m. The elevations of a train are given for 2 different locations.

Part A Write an equation in point-slope form to represent the elevation of the train in terms of time. How can you use the equation to find the rate of increase in elevation of the train in meters per second?

Part B At what elevation does the train start initially? Write a linear equation in a form that gives the information as part of the equation. Explain your reasoning.

2-3

Standard Form

I CAN… write and graph linear equations in standard form.

VOCABULARY

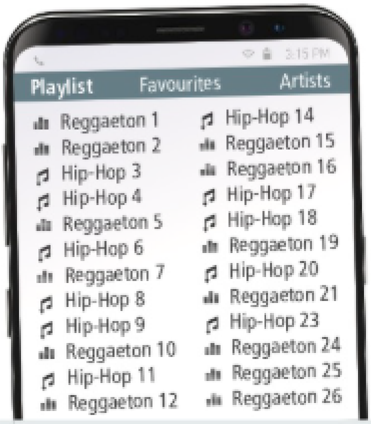
- standard form of a linear equation

MA.912.AR.2.2–Write a linear two-variable equation to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.
Also AR.1.1, AR.1.2, AR.2.4, AR.2.5
MA.K12.MTR.3.1, MTR.4.1, MTR.7.1

CONCEPTUAL UNDERSTANDING

EXPLORE & REASON

Jae makes a playlist of 24 songs for a party. Since he prefers reggaeton and hip-hop music, he builds the playlist from those two types of songs.



Reggaeton 1
Reggaeton 2
Hip-Hop 3
Hip-Hop 4
Reggaeton 5
Hip-Hop 6
Reggaeton 7
Hip-Hop 8
Hip-Hop 9
Reggaeton 10
Hip-Hop 11
Reggaeton 12
Hip-Hop 14
Reggaeton 15
Reggaeton 16
Hip-Hop 17
Hip-Hop 18
Reggaeton 19
Hip-Hop 20
Reggaeton 21
Hip-Hop 23
Reggaeton 24
Reggaeton 25
Reggaeton 26

A. Determine two different combinations of reggaeton and hip-hop songs that Jae could use for his playlist.

B. Plot those combinations on graph paper. Extend a line through the points.

C. **Represent and Connect** Can you use the line to find other meaningful points? Explain.

ESSENTIAL QUESTION

What information does the standard form of a linear equation reveal about a line?

EXAMPLE 1 Understand Standard Form of a Linear Equation

A. Hanna will spend \$150 on music festival tickets. Reserved seat tickets cost \$25 and general admission tickets cost \$10. How can you represent the situation with a linear equation?

Let $x =$ cost of reserved seat tickets Let $y =$ general admission tickets

money spent on reserved seat tickets

money spent on general admission tickets

total budget

$$25 \bullet x \quad + \quad 10 \bullet y \quad = 150$$

The equation, $25x + 10y = 150$ is in standard form. The **standard form of a linear equation** is $Ax + By = C$, where A, B, and C are integers, and A and B are not both equal to 0.

VOCABULARY
Remember, *integers* are rational numbers with no fractional or decimal part.

B. What information does the standard form give you that the slope-intercept form does not?

$$25x + 10y = 150$$
$$10y = -25x + 150$$
$$y = -2.5x + 15$$

Convert the standard form to slope-intercept form and compare the equations.

Slope-Intercept Form

$y = -2.5x + 15$

Hanna can buy 15 general admission tickets if she buys no reserved seat tickets.

When the equation is in slope-intercept form, you can determine the y-intercept by inspection.

AND

Standard Form

$25x + 10y = 150$

Hanna can spend \$150. This is the constraint.

When the equation is in standard form, you can determine the constraint by inspection.

USE PATTERNS AND STRUCTURE
What is the relationship between the sign of the slope and the quantities in the problems?

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Try It! **1.** Is it easier to find the x-intercept of the graph of the equations in Part B using slope-intercept or standard form? Explain.

EXAMPLE 2 Sketch the Graph of a Linear Equation in Standard Form

What is the graph of $3x - 2y = 9$?

To sketch a graph of a linear equation in standard form, find the x- and y-intercepts.

LEARN TOGETHER
How can you share your ideas and communicate your thinking with others?

Step 1 Find the intercepts.

To find the x-intercept, substitute 0 for y and solve for x.

$$3x - 2y = 9$$
$$3x - 2(0) = 9$$
$$3x = 9$$
$$x = 3$$

The x-intercept is 3.

To find the y-intercept, substitute 0 for x and solve for y.

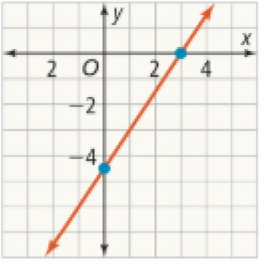
$$3x - 2y = 9$$
$$3(0) - 2y = 9$$
$$-2y = 9$$
$$y = -4.5$$

The y-intercept is –4.5.

Step 2 Sketch a graph of the line.

Plot the x-intercept at (3, 0).
Plot the y-intercept at (0, –4.5).

Sketch the line that passes through the intercepts.



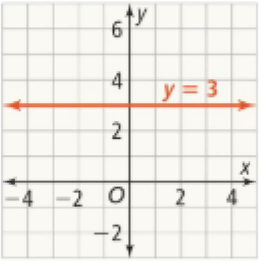
Try It! **2.** Sketch the graph of $4x + 5y = 10$.

EXAMPLE 3 Relate Standard Form to Horizontal and Vertical Lines

A. What does the graph of $Ax + By = C$ look like when $A = 0$?

Graph the line represented by $0x + 2y = 6$.

$$2y = 6$$
$$y = 3$$



The equation $y = 3$ does not include x, so x has no effect on the y-values. The value of y is 3 for every x-value, so the graph of $y = 3$ is a horizontal line.

GENERALIZE
For any real number a, the line represented by $y = a$ has a slope of 0. What is the y-intercept of the line?

In the coordinate plane, an equation in one variable means that the other variable has no effect on the equation or the graph.

When $A = 0$, the graph of $Ax + By = C$ is a horizontal line.

CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

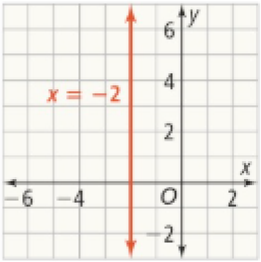
B. What does the graph of $Ax + By = C$ look like when $B = 0$?

Graph the linear equation $3x + 0y = -6$.

$$3x = -6$$

$$x = -2$$

The value of x is -2, regardless of the value of y.



When $B = 0$, the graph of the $Ax + By = C$ is a vertical line.

CHOOSE EFFICIENT METHODS
Can you use slope-intercept or point-slope forms to generate equations for vertical and horizontal lines?

Try It! **3.** Sketch the graph of each equation.

a. $3y = -18$

b. $4x = 12$

APPLICATION

EXAMPLE 4 Use the Standard Form of a Linear Equation

Tamira is making trail mix. She has \$40 to spend on a mixture of almonds and cashews and wants about the same amount of almonds as cashews. How can she determine how many pounds of each kind of nut to buy?

Formulate Write and graph an equation to represent the situation.

price of almonds • x pounds + price of cashews • y pounds = \$40

$$8 \cdot x \quad + \quad 10 \cdot y \quad = 40$$

Compute Find the x- and y-intercepts of $8x + 10y = 40$.

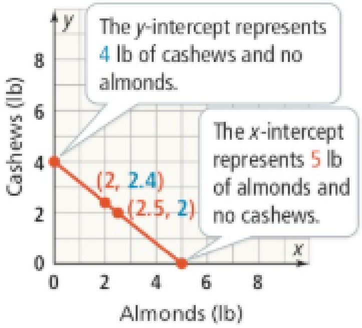
$$8x + 10(0) = 40 \qquad 8(0) + 10y = 40$$

$$x = 5 \qquad y = 4$$

Graph the segment between the intercepts.

Interpret Tamira can use the graph of the equation to help her determine the amount of almonds and cashews to buy. Each point on the line represents a combination of almonds and cashews that costs a total of \$40.

Tamira can buy 2 lb of almonds and 2.4 lb of cashews or 2.5 lb of almonds and 2 lb of cashews for \$40.



The y-intercept represents 4 lb of cashews and no almonds.

The x-intercept represents 5 lb of almonds and no cashews.

Try It! **4.** How does the equation change if Tamira has \$60 to spend on a mixture of almonds and cashews? How many pounds of nuts can she buy if she buys only cashews? Only almonds? A mixture of both?

CONCEPT SUMMARY Standard Form of a Linear Equation

WORDS The standard form of a linear equation is useful

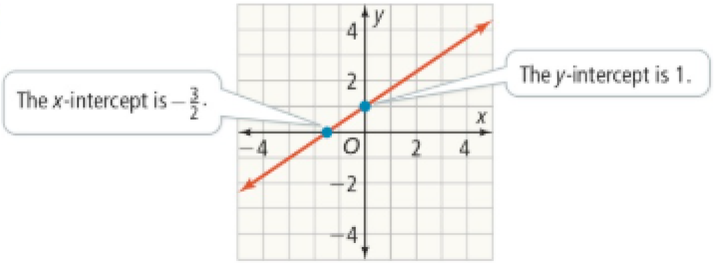
- to find the x- and y-intercepts easily.
- to write the equation of a vertical or horizontal line.

The x-intercept is the value of x when $y = 0$, and the y-intercept is the value of y when $x = 0$.

ALGEBRA $Ax + By = C$, where A, B, and C are integers, and A and B are not both equal to 0.

NUMBERS $2x - 3y = -3$

GRAPH



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What information does the standard form of a linear equation reveal about a line?

2. **Communicate and Justify** How is the standard form of a linear equation similar to and different from the slope-intercept form?

3. **Error Analysis** Malcolm says that $y = -1.5x + 4$ in standard form is $1.5x + y = 4$. What is the error that Malcolm made?

4. **Choose Efficient Methods** Describe a situation in which the standard form of a linear equation is more useful than the slope-intercept form.

Do You KNOW HOW?

Use the x- and y-intercepts to sketch a graph of each equation.

5. $x + 4y = 8$

6. $3x - 4y = 24$

7. $5x = 20$

8. $-3y = 9$

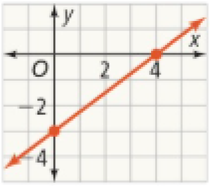
9. Deondra has \$12 to spend on a mixture of green and red grapes. What equation can she use to graph a line showing the different amounts of green and red grapes she can buy for \$12?

PRACTICE & PROBLEM SOLVING

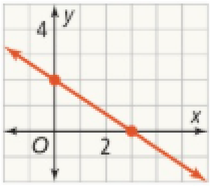
UNDERSTAND

10. **Use Patterns and Structure** If $C = 24$, what values of A and B complete $Ax + By = C$ for each graph? Write the standard form for each equation.

a.



b.



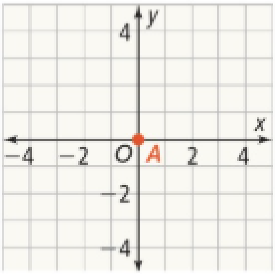
11. **Communicate and Justify** Darren graphs the linear equations $y = -\frac{2}{5}x + 3$ and $2x + 5y = 15$. The graphs look identical so he believes that the equations represent the same line. What mathematical argument can he construct to show that the two forms are equivalent?

12. **Error Analysis** Describe and correct the error a student made finding the intercepts of the graph of the line $4x - 6y = 12$.

1. $4(0) - 6y = 12$
2. $6y = 12$, so $y = 2$; the y-intercept is 2.
3. $4x - 6(0) = 12$
4. $4x = 12$, so $x = 3$; the x-intercept is 3.

✗

13. **Mathematical Connections** Point A is one vertex of triangle ABC. Point B is the x-intercept of $6x - 4y = -12$ and point C is the y-intercept. What are points B and C? Sketch the triangle in the coordinate plane.



14. **Higher Order Thinking** Rewrite the standard form equation $Ax + By = C$ in slope-intercept form. What expression corresponds to the slope of the line represented by $Ax + By = C$?

PRACTICE

Identify the x- and y-intercepts of the graph of each equation. SEE EXAMPLES 1 AND 2

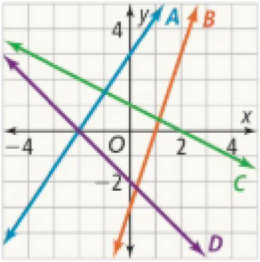
15. $2x + 5y = 10$
16. $3x - 4y = -24$
17. $10x + 5y = 120$
18. $2x - y = 8$

Sketch the graph of each equation. SEE EXAMPLE 2

19. $2x - 4y = 8$
20. $3x + 5y = 15$
21. $3x - 6y = -12$
22. $8x + 12y = -24$

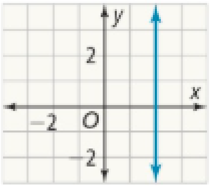
Which line matches each equation? SEE EXAMPLE 2

23. $4x + 4y = -8$
24. $3x - 2y = -6$
25. $x + 2y = 2$
26. $3x - y = 3$

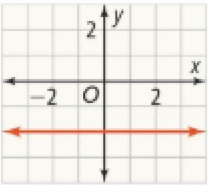


How is the graph of each equation related to standard form $Ax + By = C$? SEE EXAMPLE 3

27.



28.



Sketch the graph of each equation. SEE EXAMPLE 3

29. $4x = 10$
30. $-6y = 3$
31. $3y = -15$
32. $-9x = -27$

Convert each slope-intercept or point-slope equation into standard form.

33. $y = 4x - 18$
34. $y - 3 = \frac{1}{5}(x + 6)$
35. $y = -\frac{1}{2}x - 10$
36. $y - 1 = \frac{2}{3}(x + 6)$

Write an equation in standard form of the line that passes through the given points.

37. (0, 2) and (8, 0)
38. (6, 0) and (0, 4)
39. (3, 0) and (0, −7)
40. (2, −3) and (2, 9)

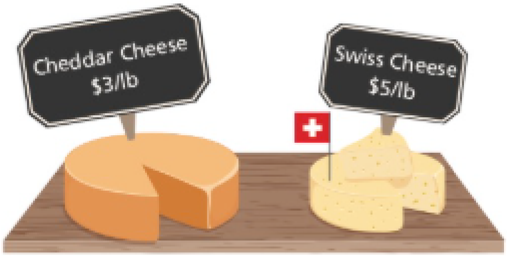
Convert each standard form equation into slope-intercept form.

41. $2x + 3y = 12$
42. $6x - 2y = -20$

PRACTICE & PROBLEM SOLVING

APPLY

43. **Apply Math Models** Keisha is catering a luncheon. She has \$30 to spend on a mixture of Cheddar cheese and Swiss cheese. How many pounds of cheese can Keisha get if she buys only Cheddar cheese? Only Swiss cheese? A mixture of both cheeses? What linear equation in standard form can she use to model the situation?



44. **Apply Math Models** Gregory can buy 4 pounds of wheat flour for \$8 and 5 pounds of rye flour for \$20. He has \$12 to spend on a flour mixture. What linear equation in standard form can Gregory use to model the situation?

45. **Analyze and Persevere** Paxton, a summer camp counselor, has a budget of \$300 to spend on caps and T-shirts for a summer camp.

What equation can Paxton use to determine the number of caps and T-shirts he can order for \$300? If Paxton sketched a graph of the linear equation, would every point on the graph represent a possible solution? Explain.

ASSESSMENT PRACTICE

46. Darren has \$50 to spend on a fruit salad for a party. Oranges are \$2.50/lb and pineapple is \$4.00/lb. Select the equation in standard form that represents how many pounds of each kind of fruit he can buy. AR.2.2

Ⓐ $y - 5 = 100(x - 8)$

Ⓑ $100y = 8x + 5$

Ⓒ $5x + 8y = 100$

Ⓓ $100x + 5y = 8$

47. **SAT/ACT** What is $\frac{3}{8}x + \frac{2}{3}y = 5$ written in standard form?

Ⓐ $y = -\frac{9}{16}x + \frac{15}{2}$

Ⓑ $y + \frac{3}{2} = -\frac{9}{16}(x - 16)$

Ⓒ $\frac{3}{8}x + \frac{2}{3}y = 5$

Ⓓ $3x + \frac{16}{3}y = 40$

Ⓔ $9x + 16y = 120$

48. **Performance Task** Fatima has a total of \$8 to spend to make fruit smoothies. She will use two types of fruit. The table shows the cost of each type of fruit per cup.

Fruit	Cost per cup (\$)
Mango	0.50
Pineapple	0.75
Strawberry	1.00

Part A What are the possible combinations of ingredients that Fatima can buy? Write a linear equation in standard form to model how many cups of fruit she can buy for each possible mixture.

Part B What are the possible amounts of fruit, in cups, that she can buy for each mixture in Part A?

Part C Fatima wants to use at least 12 cups of fruit to make the batch of smoothies. Which mixtures will allow her to make enough and still stay within her budget? Explain your reasoning.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.2.2–Write a linear two-variable equation to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. Also AR.1.2

MA.K12.MTR.7.1

How Tall Is Tall?

The world's tallest person in recorded history was Robert Wadlow. He was 8 feet 11.1 inches tall! Only 5% of the world population is 6 feet 3 inches or taller. What percent of the population would you guess is 7 feet or taller?

We usually use standard units, such as feet and inches or centimeters, to measure length or height. Did you ever wonder why? In the Mathematical Modeling in 3 Acts lesson you'll consider some interesting alternatives.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. Write a number that you know is too small.
6. Write a number that you know is too large.
7. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

8. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

9. Is your refined conjecture between the highs and lows you set up earlier?
10. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

2-4

Parallel and Perpendicular Lines

I CAN… write equations of parallel lines and perpendicular lines.

VOCABULARY

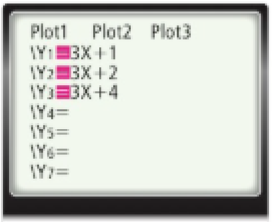
- parallel lines
- perpendicular lines
- reciprocal

MA.912.AR.2.3–Write a linear two-variable equation for a line that is parallel or perpendicular to a given line and goes through a given point.

MA.K12.MTR.1.1, MTR.2.1, MTR.5.1

EXPLORE & REASON

Graph these three equations using a graphing calculator.



```
Plot1  Plot2  Plot3
\Y1=3X+1
\Y2=3X+2
\Y3=3X+4
\Y4=
\Y5=
\Y6=
\Y7=
```

A. **Use Patterns and Structure** Choose any two of the lines you graphed. How are they related to each other?

B. Does your answer to Part A hold for any two lines? Explain.

C. Write another set of three or more equations that have the same relationships as the first three equations.

ESSENTIAL QUESTION

How can the equations of lines help you identify whether the lines are parallel, perpendicular, or neither?

EXAMPLE 1 Write an Equation of a Line Parallel to a Given Line

What is the equation of the line in slope-intercept form that passes through the point (8, 9) and is parallel to the graph of $y = \frac{3}{4}x - 2$?

COMMUNICATE AND JUSTIFY
The horizontal lines given by $y = 3$ and $y = 5$ are parallel and have the same slope, 0. The vertical lines $x = 4$ and $x = 6$ are also parallel. Can you state that their slopes are the same? Explain.

Parallel lines are lines in the same plane that never intersect. Nonvertical lines that are parallel have the same slope but different *y*-intercepts.

Step 1 Identify the slope of the given line.

$$y = \frac{3}{4}x - 2$$

The slope is $\frac{3}{4}$. The slope of a parallel line will be the same.

Step 2 Start with point-slope form. Use the given point and the slope of the parallel line.

$$y - y_1 = m(x - x_1)$$

$$y - 9 = \frac{3}{4}(x - 8)$$

$$y - 9 = \frac{3}{4}x - 6$$

Change point-slope form to slope-intercept form.

$$y = \frac{3}{4}x + 3$$

The equation of the line is $y = \frac{3}{4}x + 3$.

Try It! 1. Write the equation of the line that passes through the given point and is parallel to the given line.

a. $(-3, 5)$; $y = -\frac{2}{3}x$

b. $(1, 4)$; $y = 0x - 3$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Understand the Slopes of Perpendicular Lines

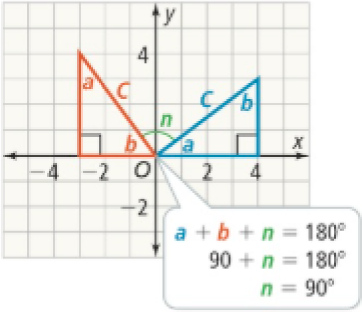
A. How can you create two perpendicular lines?

Perpendicular lines are lines that intersect to form right angles.

Draw two identical right triangles as shown.

Because the angle sum for each triangle is 180° and the right angle is 90°, the sum of angles a and b in each triangle must be 90°. Angles b, n, and a form a straight angle of 180° at the origin, so n must equal 90°.

The hypotenuses (C) of the right triangles intersect at a right angle, so the lines that include them are perpendicular to each other.



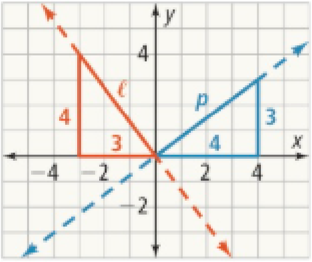
$$a + b + n = 180^\circ$$
$$90 + n = 180^\circ$$
$$n = 90^\circ$$

B. How do the slopes of perpendicular lines compare?

Compute the slopes of lines ℓ and p.

Line ℓ: $m = \frac{4 - 0}{-3 - 0} = -\frac{4}{3}$

Line p: $m = \frac{3 - 0}{4 - 0} = \frac{3}{4}$



The numbers $\frac{4}{3}$ and $\frac{3}{4}$ are reciprocals. The **reciprocal** of a number is 1 divided by that number. The reciprocal of x is $\frac{1}{x}$ when $x \neq 0$.

$$\text{Reciprocal of } \frac{3}{4} \text{ is } 1 \div \frac{3}{4} = 1 \cdot \frac{4}{3}$$
$$= \frac{4}{3}.$$

VOCABULARY

Another way to state the definition is as follows: the product of reciprocals is 1.

$$\frac{3}{4} \cdot \frac{4}{3} = 1$$

So, the slopes of perpendicular lines are *negative reciprocals.* Negative reciprocals have a product of −1.

For example, $-\frac{4}{3} \cdot \frac{3}{4} = -\frac{12}{12} = -1$.

Try It! **2.** Why does it make sense that the slopes of perpendicular lines have opposite signs?

EXAMPLE 3 Write an Equation of a Line Perpendicular to a Given Line

What is the equation of the line that passes through the point (1, 7) and is perpendicular to the graph of $y = -\frac{1}{4}x + 11$?

Step 1 Use the slope of the given line to determine the slope of the line that is perpendicular.

$$y = -\frac{1}{4}x + 11$$

$m = -\frac{1}{4}$

The slope of a line perpendicular to the given line is the negative reciprocal of $-\frac{1}{4}$. Use $\frac{4}{1}$, or 4, as the slope of the new line.

CHOOSE EFFICIENT METHODS
Think about the usefulness of the different forms of a linear equation to decide which form to use.

Step 2 Start with point-slope form. Use the given point and the slope of the perpendicular line.

$$y - y_1 = m(x - x_1)$$
$$y - 7 = 4(x - 1)$$

Substitute 1 for x_1, 7 for y_1 and 4 for m.

The graph of $y - 7 = 4(x - 1)$ passes through the point (1, 7) and is perpendicular to the graph of $y = -\frac{1}{4}x + 11$.

Try It! **3.** Write the equation of the line that passes through the point (4, 5) and is perpendicular to the graph of $y = 2x - 3$.

EXAMPLE 4 Classify Lines

Are the graphs of the equations $3y = -4x + 6$ and $y = -\frac{3}{4}x - 5$ *parallel, perpendicular,* or *neither*?

Step 1 Identify the slope of each line.

Rewrite the equation of the line in slope-intercept form.

$$\begin{array}{ll} 3y = -4x + 6 & y = -\frac{3}{4}x - 5 \\ \frac{3y}{3} = \frac{-4x + 6}{3} & \\ y = -\frac{4}{3}x + 2 & y = -\frac{3}{4}x - 5 \end{array}$$

The slopes of the lines are $-\frac{4}{3}$ and $-\frac{3}{4}$.

COMMON ERROR
You may confuse the slopes of perpendicular lines. The slopes of perpendicular lines are negative reciprocals, not reciprocals.

Step 2 Compare the slopes of the lines.

The slopes of the lines, $-\frac{4}{3}$ and $-\frac{3}{4}$, are neither the same nor negative reciprocals.

The graphs of the equations $3y = -4x + 6$ and $y = -\frac{3}{4}x - 5$ are neither parallel nor perpendicular.

Try It! **4.** Are the graphs of the equations *parallel, perpendicular,* or *neither*?

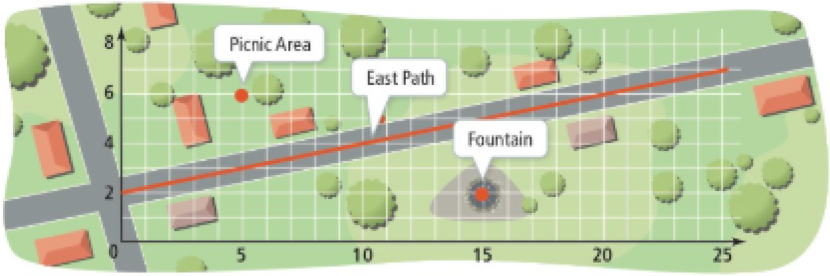
a. $y = 2x + 6$ and $y = \frac{1}{2}x + 3$

b. $y = -5x$ and $25x + 5y = 1$

APPLICATION

EXAMPLE 5 Solve a Real-World Problem

A landscaper plans to install two new paths in a park. The new Fountain Path will be perpendicular to the East Path and lead to the fountain. The new Picnic Path will be parallel to the Fountain Path and pass through the picnic area. What are the equations in point-slope form that represent the new paths?



Formulate Find the slope of the line that represents the East Path. Then determine equations for the two new pathways.

The East Path passes through (0, 2) and (5, 3).

$$m = \frac{3-2}{5-0} = \frac{1}{5}$$

The slope of the line representing the East Path is $\frac{1}{5}$.

Compute Find an equation for the Fountain Path.

The slope is the negative reciprocal of the slope of the East Path.

$$y - 2 = -5(x - 15)$$

The fountain is located at the point (15, 2).

Find the equation of the Picnic Path.

The slope is the same as the slope of the Fountain Path.

$$y - 6 = -5(x - 5)$$

The picnic area is located at the point (5, 6).

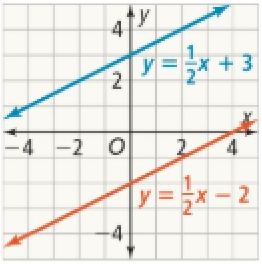
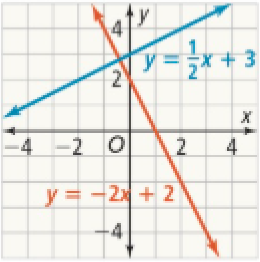
Interpret Equation of the line of the Fountain Path: $y - 2 = -5(x - 15)$

Equation of the line of the Picnic Path: $y - 6 = -5(x - 5)$

Try It! 5. The equation $y = 2x + 7$ represents the North Path on a map.

- **a.** Find the equation for a path that passes through the point (6, 3) and is parallel to the North Path.
- **b.** Find the equation for a path that passes through the same point but is perpendicular to North Path.

CONCEPT SUMMARY Parallel Lines and Perpendicular Lines

	Parallel Lines	Perpendicular Lines
WORDS	The graphs of two equations are parallel if the slopes are the same.	The graphs of two equations are perpendicular if the slopes are negative reciprocals.
NUMBERS	$\frac{1}{2} = \frac{1}{2}$ $y = \frac{1}{2}x + 3$ $y = \frac{1}{2}x - 2$	$\frac{1}{2} \cdot -2 = -1$ $y = \frac{1}{2}x + 3$ $y = -2x + 2$
GRAPHS		

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can the equations of lines help you identify whether the lines are parallel, perpendicular, or neither?

2. **Error Analysis** Dwayne stated that the slope of the line perpendicular to $y = -2x$ is 2. Describe Dwayne's error.

3. **Vocabulary** Describe the difference between the slopes of two parallel lines and the slopes of two perpendicular lines.

4. **Use Patterns and Structure** Is there one line that passes through the point (3, 5) that is parallel to the lines represented by $y = 2x - 4$ and $y = x - 4$? Explain.

Do You KNOW HOW?

The equation $y = -\frac{3}{4}x + 1$ represents a given a line.

5. Write the equation for the line that passes through (–4, 9) and is parallel to the given line.

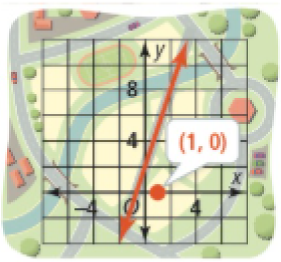
6. Write the equation for the line that passes through (6, 6) and is perpendicular to the given line.

Are the graphs of the equations parallel, perpendicular, or neither?

7. $x - 3y = 6$ and $x - 3y = 9$

8. $y = 4x + 1$ and $y = -4x - 2$

9. What equation represents the road that passes through the point shown and is perpendicular to the road represented by the red line?

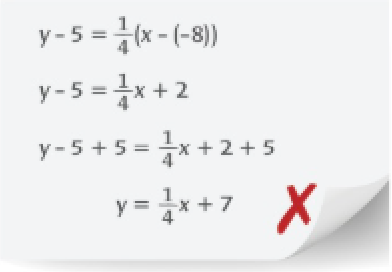


PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Use Patterns and Structure** A line passes through points $A(n, 4)$ and $B(6, 8)$ and is parallel to $y = 2x - 5$. What is the value of n?

11. **Error Analysis** Describe and correct the error the student made when writing the equation of the line that passes through $(-8, 5)$ and is perpendicular to $y = 4x + 2$.



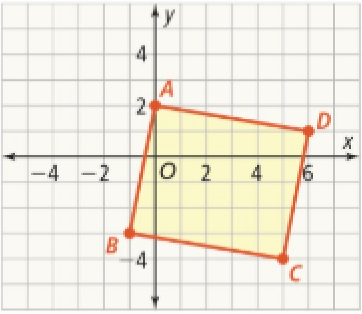
$$y - 5 = \frac{1}{4}(x - (-8))$$
$$y - 5 = \frac{1}{4}x + 2$$
$$y - 5 + 5 = \frac{1}{4}x + 2 + 5$$
$$y = \frac{1}{4}x + 7$$

12. **Analyze and Persevere** The graphs of $4x + 12y = 8$ and $y = mx + 5$ are perpendicular. What is the value of m?

13. **Mathematical Connections** Rectangles have four right angles and opposite sides that are parallel.

    a. Is the figure shown a rectangle? Explain.

    b. If not, how could the points change so it would be a rectangle?



14. **Higher Order Thinking** Explain how you can determine whether the graphs of $5x - 3y = 2$ and $5x - 3y = 8$ are parallel without doing any calculations.

PRACTICE

Write the equation of the line that passes through the given point and is parallel to the given line.
SEE EXAMPLE 1

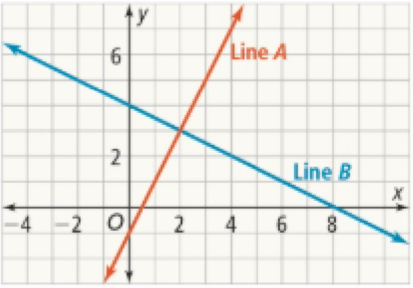
15. $(5, -4)$; $y = \frac{1}{5}x - 4$

16. $(2, 7)$; $3x - y = 5$

17. $(-3, 2)$; $y = -4$

18. $(6, 4)$; $2x + 3y = 18$

19. Use the slopes of lines A and B to show that they are perpendicular to each other.
SEE EXAMPLE 2



Write the equation of the line that passes through the given point and is perpendicular to the given line. SEE EXAMPLES 3 AND 5

20. $(-6, -3)$; $y = -\frac{2}{5}x$

21. $(0, 3)$; $3x - 4y = -8$

22. $(-2, 5)$; $x = 3$

23. $(4, 3)$; $4x - 5y = 30$

Are the graphs of each pair of equations parallel, perpendicular, or neither? SEE EXAMPLE 4

24. $y = 2x + 1$
$2x - y = 3$

25. $y = \frac{1}{2}$
$y = -3$

26. $x = 4$
$y = 4$

27. $-2x + 5y = -4$
$y = -\frac{5}{2}x + 6$

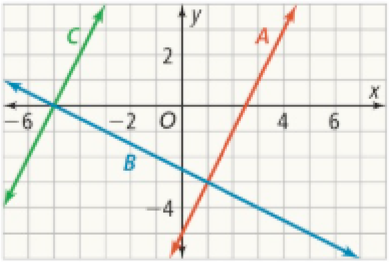
28. Copy and complete the table.

	Equation	Slope of a parallel line	Slope of a perpendicular line
a.	$y = \frac{1}{2}x + 6$		
b.	$x = -4.2$		
c.	$3x + 4y = 3$		
d.	$y = 3$		
e.	$y = x$		

PRACTICE & PROBLEM SOLVING

APPLY

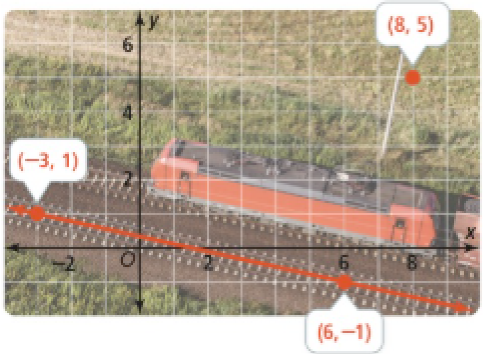
29. Represent and Connect An artist is drawing up plans for a mural. She wants to include a rectangle in her design.



a. What is an equation of Line *D* that will make the figure a rectangle?

b. Explain how the artist can use algebra to confirm that the figure is a rectangle.

30. Apply Math Models A construction crew will build a new railroad track, parallel to one modeled by the line, which passes through the point (8, 5). What equation models the path of the new track?



31. Analyze and Persevere Elijah and Aubrey have summer jobs. Elijah deposits the same amount of money in his account every week. The equation $y = 125x + 72$ represents his bank balance any given week of the summer. Aubrey also deposits the same amount into her account every week. At the end of the third week she has \$398. At the end of the sixth week she has \$773.

a. Write an equation to represent Aubrey's bank balance any given week of the summer.

b. Would the graph of the equation for Aubrey's balance be parallel to the graph of Elijah's balance? Explain.

c. What do the parallel graphs mean in terms of the situation?

ASSESSMENT PRACTICE

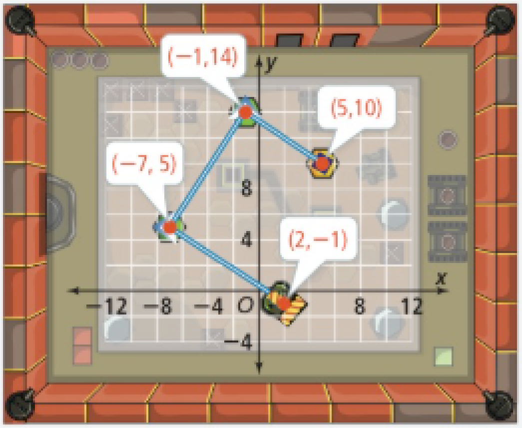
32. In a trail map, the equation $y = \frac{1}{2}x + 1$ represents the Tamiami Trail. Choose the equation for a perpendicular trail. AR.2.3

Ⓐ $y = \frac{1}{2}x + 5$
Ⓑ $y = -2x + 4$
Ⓒ $y = -\frac{1}{2}x + 5$
Ⓓ $y = 2x$

33. SAT/ACT A line passing through (6, *a*) and (9, −4) is parallel to $2x - 3y = 6$. What is the value of *a*?

Ⓐ −6
Ⓑ −3
Ⓒ −2
Ⓓ 3
Ⓔ 6

34. Performance Task A video game is designed to model the path of a laser. A laser is placed at (2, −1) and is aimed at Mirror 1. Other mirrors are placed as shown. Each mirror is placed so the light will reflect at a 90° angle.



Part A After reflecting off of all three mirrors, where will the light cross the *y*-axis?

Part B Write an equation to model the path of the light between the following:

a. Laser and Mirror 1

b. Mirror 1 and Mirror 2

c. Mirror 2 and Mirror 3

d. Mirror 3 and *y*-axis

Part C Change the placement of the mirrors so that the laser light hits a target in Quadrant IV. Give the coordinates of the mirrors and the equations of lines that the path of the light would follow.

TOPIC

2

Topic Review

TOPIC ESSENTIAL QUESTION

1. Why is it useful to have different forms of linear equations?

Vocabulary Review

Choose the correct term to complete each sentence.

2. The slopes of two perpendicular lines are negative ________.
3. The ________ of a linear equation is $Ax + By = C$, where A, B, and C are integers.
4. Nonvertical lines that are ________ have the same slope and different y-intercepts.
5. The ________ of a linear equation is $y = mx + b$.
6. You can write the equation of a line using any point (x_1, y_1) and the slope, m, in ________, $y - y_1 = m(x - x_1)$.

- parallel
- perpendicular
- point-slope form
- reciprocals
- slope-intercept form
- standard form
- y-intercept

Concepts & Skills Review

LESSON 2-1 Slope-Intercept Form

Quick Review

The **slope-intercept form** of a linear equation is $y = mx + b$, where m is the slope of the line and the y-intercept is b. The slope-intercept form is useful when the slope and the y-intercept of the line are known.

Example

Write the equation of the line in slope-intercept form that passes through (0, 4) and (2, 3).

$m = \frac{4-3}{0-2}$ Use the slope formula.

$= -\frac{1}{2}$

$b = 4$ The line intersects y-axis at (0, 4).

$y = mx + b$ Write the equation in slope-intercept form.

$y = -\frac{1}{2}x + 4$ Substitute $-\frac{1}{2}$ for m and 4 for b.

Practice & Problem Solving

Sketch the graph of each equation.

7. $y = 3x - 1$
8. $y = -1.5x + 3.5$

Write the equation of the line in slope-intercept form that passes through the given points.

9. (2, 0) and (4, 6)
10. (–1, 8) and (5, –2)

11. **Apply Math Models** Ricardo wants to buy a new tablet computer that costs \$1,150. He will make a down payment of \$250 and will make monthly payments of \$50. Write an equation in slope-intercept form that Ricardo can use to determine how much he will owe after x months.

LESSON 2-2 Point-Slope Form

Quick Review

The **point-slope form** of a linear equation is $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a specific point and (x, y) is any point on the line. The point-slope form is useful when you know the slope and a point that is not $(0, b)$.

Example

Write the equation of the line in point-slope form that passes through the points (2, 2) and (5, 1).

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Find the slope of the line.

$= \frac{1 - 2}{5 - 2}$ Substitute (5, 1) for (x_2, y_2) and (2, 2) for (x_1, y_1).

$= -\frac{1}{3}$

$y - y_1 = m(x - x_1)$ Write the equation in point-slope form.

$y - 2 = -\frac{1}{3}(x - 2)$ Substitute $-\frac{1}{3}$ for m and (2, 2) for (x_1, y_1).

Practice & Problem Solving

Write the equation in point-slope form of the line that passes through the given point with the given slope.

12. $(4, -2)$; $m = 0.5$

13. $(-2, 5)$; $m = -3$

Write an equation in point-slope form of the line that passes through the given points.

14. $(3, 1)$ and $(-5, -2)$

15. $(1.5, 4)$ and $(-2.5, 6)$

16. **Use Patterns and Structure** Jeffrey purchased a card for \$180 that gives him 20 visits to a new gym and includes a one-time fee for unlimited use of the sauna. After 5 visits, Jeff has \$123.75 left on the card, and after 11 visits, he has \$74.25 left on the card. Write an equation that Jeffrey can use to determine the cost of each visit and the fee for the sauna use.

LESSON 2-3 Standard Form

Quick Review

The **standard form** of a linear equation is $Ax + By = C$, where A, B, and C are integers. The standard form is useful for graphing vertical and horizontal lines, for finding the x- and y-intercepts, and for representing certain situations in terms of constraints.

Example

What are the x- and y-intercepts of the line $3x - 4y = 24$?

Substitute 0 for y and solve for x.

$3x - 4(0) = 24$

$x = 8$

Then substitute 0 for x and solve for y.

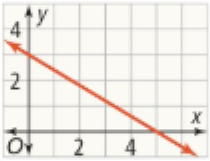
$3(0) - 4y = 24$

$y = -6$

The x-intercept is 8 and the y-intercept is –6.

Practice & Problem Solving

17. If $C = 15$, what values of A and B complete $Ax + By = C$ for the graph shown? Write the standard form of the equation.



Write each equation in standard form.

18. $y = 4x - 5$

19. $y - 3 = 5(4 - x)$

Determine the x- and y-intercepts of each line.

20. $5x - 3y = 30$

21. $x + 3y = 24$

22. **Apply Math Models** Jung-Soon has \$25 to spend on prizes for a game at the school fair. Lip balm costs \$1.25 each, and mini-notebooks cost \$1.50 each. Write a linear equation that can be used to determine how many of each prize she can buy.

LESSON 2-4 Parallel and Perpendicular Lines

Quick Review

Two nonvertical lines are **parallel** if they have the same slope, but different *y*-intercepts. Vertical lines are parallel if they have different *x*-intercepts. Two nonvertical lines are **perpendicular** if their slopes are negative reciprocals. A vertical line and a horizontal line are perpendicular.

Example

Are the graphs of the equations $4y = 2x - 5$ and $y = -2x + 7$ parallel, perpendicular, or neither?

Determine the slope of each line.

$$4y = 2x - 5 \qquad\qquad y = -2x + 7$$

$$\frac{4y}{4} = \frac{2x - 5}{4}$$

$$y = \frac{1}{2}x - \frac{5}{4}$$

The slopes of the lines are $\frac{1}{2}$ and -2, so the graphs of the equations are perpendicular lines.

Practice & Problem Solving

23. The graphs of $3x + 9y = 15$ and $y = mx - 4$ are parallel lines. What is the value of m?

Write an equation for the line that passes through the given point and is parallel to the given line.

24. $(2, 1)$; $y = -3x + 8$

25. $(-3, -1)$; $x - 2y = 5$

Write an equation for the line that passes through the given point and is perpendicular to the given line.

26. $(1, 7)$; $x - 4y = 8$

27. $(-2, 6)$; $y = 0.5x - 3$

Are the graphs of the given pairs of equations parallel, perpendicular, or neither?

28. $y = \frac{1}{4}x - 8$

$2x + y = 5$

29. $3y + 2x = 9$

$y = -\frac{2}{3}x - 4$

TOPIC

3

Linear and Absolute Value Functions

TOPIC ESSENTIAL QUESTION

How can linear and absolute value functions be used to model situations and solve problems?

Topic Overview

enVision® STEM Project:
Planning a Recycling Drive

3-1 Domain and Range of Functions
AR.2.5, MTR.2.1, MTR.3.1, MTR.5.1

3-2 Linear Functions
AR.2.4, AR.2.5, F.1.2, F.1.5, FL.3.2, FL.3.4, MTR.2.1, MTR.4.1, MTR.7.1

3-3 Transforming Linear Functions
F.1.5, F.2.1, MTR.2.1, MTR.4.1, MTR.5.1

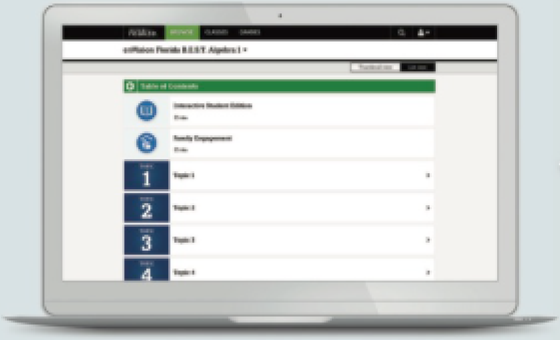
Mathematical Modeling in 3 Acts:
The Express Lane
AR.2.2, AR.2.5, MTR.7.1

3-4 Absolute Value Functions
AR.4.3, F.1.2, F.1.6, F.2.1, MTR.4.1, MTR.5.1, MTR.6.1

3-5 Transforming Absolute Value Functions
AR.4.3, F.2.1, MTR.1.1, MTR.5.1, MTR.7.1

Topic Vocabulary

- absolute value function
- axis of symmetry
- continuous
- discrete
- function notation
- linear function
- set-builder notation
- transformation
- translation
- vertex



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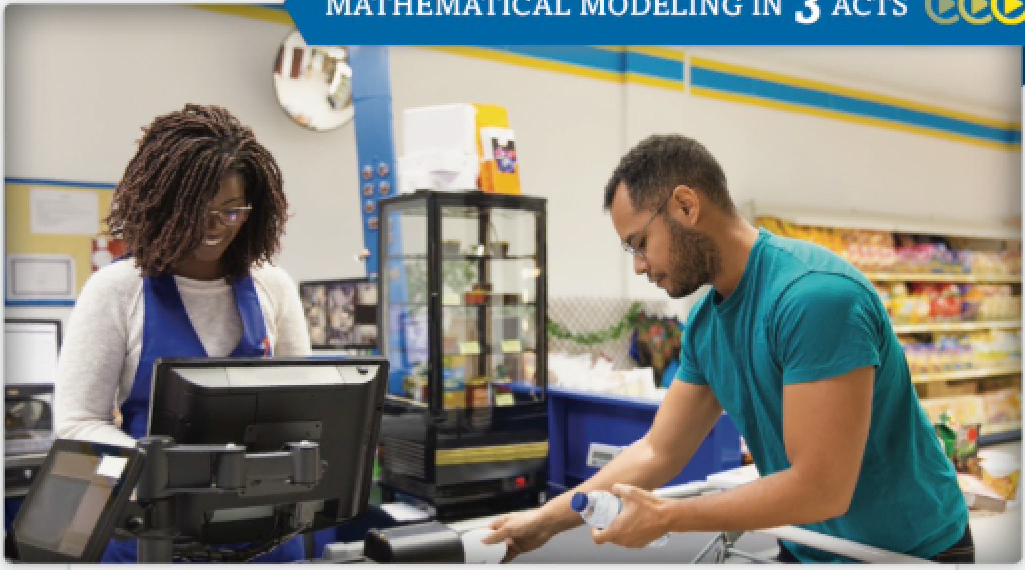
ACTIVITIES Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its.***

ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



The Express Lane

Some supermarkets have self-checkout lanes. Customers scan their items themselves and then pay with either cash or credit when they have finished scanning all of the items. Some customers think these lanes are faster than the checkout lanes with cashiers, but others don't like having to bag all of their purchases themselves.

What's your strategy for picking a checkout lane at the grocery store? Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

MATH TOOLS Explore math with digital tools and manipulatives.

DESMOS Use Anytime and as embedded Interactives in Lesson content.

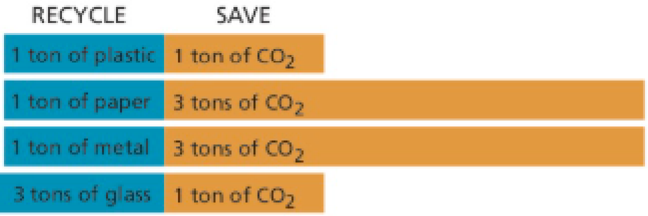
QR CODES Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

Did You Know?

Glass, aluminum, and other metals can be melted over and over again **without a loss in quality.** Paper can be recycled up to six times, with its quality decreasing each time.

Americans throw away **25,000,000 plastic bottles every hour.** If those bottles were recycled, they would offset the environmental impact of 625 round-trip flights between New York and London.

How **Recycling Offsets CO^2** Production



Your Task: Planning a Recycling Drive

About 75% of the trash Americans generate is recyclable, but only about 30% gets recycled. You and your classmates will plan a recycling drive at your school to increase the amount of trash that gets recycled.

3-1

Domain and Range of Functions

I CAN… describe constraints on the domain and range of a function.

VOCABULARY

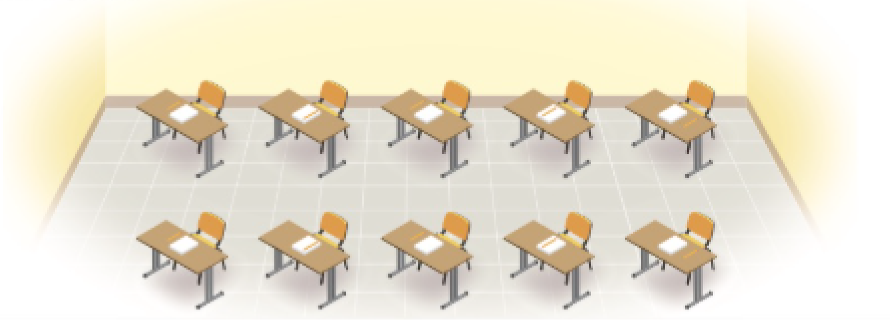
- continuous
- discrete
- set-builder notation

MA.912.AR.2.5–Solve and graph mathematical and real-world problems that are modeled with linear functions. Interpret key features and determine constraints in terms of the context.

MA.K12.MTR.2.1, MTR.3.1, MTR.5.1

EXPLORE & REASON

The desks in a study hall are arranged in rows like the horizontal ones in the picture.



A. What is a reasonable number of rows for the study hall? What is a reasonable number of desks?

B. **Use Patterns and Structure** What number of rows would be impossible? What number of desks would be impossible? Explain.

C. What do your answers to Parts A and B reveal about what the graph of rows to desks looks like?

ESSENTIAL QUESTION

Why is identifying the domain and range important in defining a function?

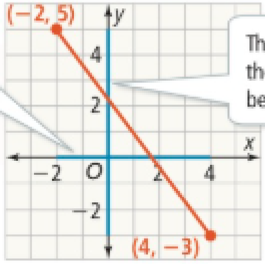
EXAMPLE 1 Use Notations to Describe Domain and Range

How can you use notation to describe the domain and the range of a function?

VOCABULARY

Recall that the *domain* of a function is the complete set of possible input values. The *range* of a function is the complete set of possible output values.

A. Use inequalities.



The domain of this function is the set of all the inputs, or the *x*-values, between −2 and 4.

The range is the set of all the outputs, or *y*-values, between −3 and 5.

The domain is $-2 \le x \le 4$.

The range is $-3 \le y \le 5$.

B. Use set-builder notation.

Domains and ranges can also be described using *set-builder notation*.

Set-Builder Notation is a form of set notation that specifies a variable and conditions.

"Variable" can be any variable.

$\{\text{variable} \mid \text{conditions}\}$

There are one or more conditions for the variable. They can be described verbally or algebraically.

The domain is $\{x \mid -2 \le x \le 4\}$.

The range is $\{y \mid -3 \le y \le 5\}$.

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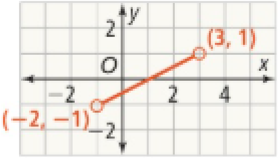
EXAMPLE 1 CONTINUED

Try It! **1.** Describe the domain and range of the function using each type of notation.

a. Inequality

b. Set-builder notation



CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Analyze Reasonable Domains and Ranges

A. A function can model each situation. What is a reasonable domain and range of each function?

A hose fills a 10,000-gallon swimming pool at a rate of 10 gallons per minute.

A reasonable domain is from 0 minutes to the time it takes to fill the pool. A reasonable range is from 0 to 10,000 gallons, the capacity of the pool.

A restaurant needs to order chairs for its tables. One table can accommodate four chairs.

A reasonable domain is from 1 table to the number of tables needed. A reasonable range is multiples of 4 from 4 to 4 times the number of tables needed.

B. Is the domain for each situation continuous or discrete?

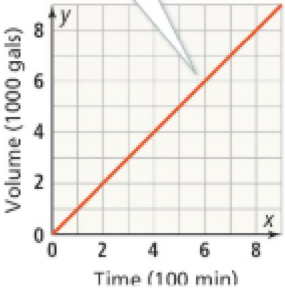
The domain of a function is **continuous** when it includes all real numbers over an interval. The graph of the function is a line or curve.

The domain of a function is **discrete** if it does not include all the real numbers in an interval, for example, the rational numbers or the integers.

Sketch a graph of each situation.

REPRESENT AND CONNECT
Why is the condition "x is an integer" needed, along with "$x > 0$", to represent the number of tables in the restaurant?

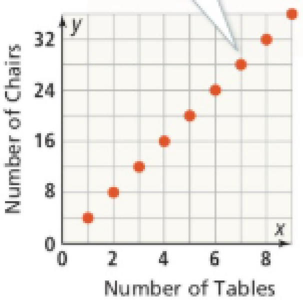
The volume of water in the pool can be determined at any point in time, for any value of x.



Domain: $0 \leq x \leq 10$
Range: $0 \leq y \leq 10,000$

The domain is continuous.

The number of tables and chairs must be whole numbers. There cannot be parts of tables or chairs.



Domain: $\{x \mid x \text{ is an integer and } x > 0\}$
Range: $\{y \mid y \text{ is a multiple of } 4\}$

The domain is discrete.

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EXAMPLE 2 CONTINUED

Try It! **2.** Analyze each situation. Identify and represent a reasonable domain and range for each situation. Explain.

a. A bowler pays \$2.75 per game.

b. A car can travel 25 miles per gallon of gas.

APPLICATION

EXAMPLE 3 Identify Constraints on the Domain

Alberto is making mofongo. His recipe for one batch of mofongo is shown below.

Mofongo Recipe Ingredients

4 green plantains

1 lb. crunchy pork skin

3 garlic cloves

4 teaspoons of olive oil

2 cups frying oil

A. Describe the number of batches of mofongo as a function of the number of whole plantains. What are the constraints on the domain?

Look for patterns between the number of whole plantains and the number of batches that can be made.

Number of Plantains, p	Number of Batches, b
1	$\frac{1}{4}$
2	$\frac{1}{2}$
3	$\frac{3}{4}$
4	1
8	2
12	3

The number of batches is the number plantains divided by 4.

The function can be described algebraically as $b = \frac{1}{4}p$. Since whole plantains are used, the domain is $\{p \mid p \geq 0, p \text{ is a whole number}\}$.

APPLY MATH MODELS
How would the constraints on the domain and range change if Alberto decided to make only whole batches?

B. What are the constraints on the range of the function?

The number of batches Alberto can make is $\frac{1}{4}$ the number of plantains used.

The range is $\left\{b \mid b = \frac{p}{4}, p \text{ is a whole number}\right\}$.

Try It! **3.** It costs Margaret between \$20 and \$25 to fill the gas tank of her car. She fills the tank once or twice a month. Margaret maps the number of times she fills the tank to the total she spends on gas. What are the constraints on the domain and the range? Use notation to represent the domain.

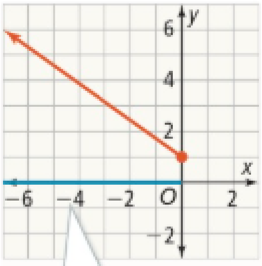
CONCEPT SUMMARY Domain and Range of a Function

WORDS

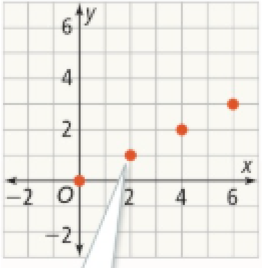
Use **inequalities** to describe the domain and range of a function if they are intervals defined over the real numbers.

Set-builder notation can include inequalities, verbal descriptions, or both to describe the domain and range of a function.

GRAPHS



The domain of the function includes all real numbers over an interval, so it is continuous on that interval.



The graph of the function is a series of points, so the domain is discrete.

ALGEBRA

Domain: $x \leq 0$

Range: $y \geq 1$

Domain: $\{x \mid x \text{ is even and } 0 \leq x \leq 6\}$

Range: $\{y \mid y \text{ is an integer and } 0 \leq y \leq 3\}$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** Why is identifying the domain and range important when defining a function?

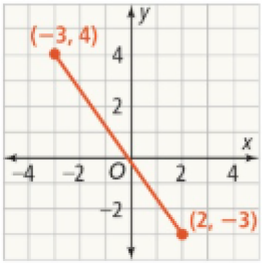
2. **Vocabulary** Maya is tracking the amount of rainfall during a storm. Describe the *domain* and *range* for this situation. Include *continuous* or *discrete* in your description.

3. **Choose Efficient Methods** If a function has a domain over a continuous interval can both inequality notation and set-builder notation be used to represent it? Which method is simpler? Explain.

4. **Error Analysis** Felipe states that he can use the inequality $1 \leq x \leq 4$ to describe the domain {1, 2, 3, 4} for a given function. Explain Felipe's error.

Do You KNOW HOW?

Use the graph for Exercises 5 and 6.



5. Use inequality notation to describe the domain and range.

6. Use set-builder notation to describe the domain and range.

7. Each day Jacob records the number of laps and the distance he walks, in miles, on a track. Graph the function and describe the domain and range.

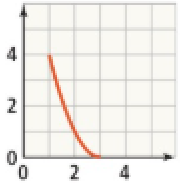
{(3, 0.75), (6, 1.5), (5, 1.25), (2, 0.5), (7, 1.75), (8, 4), (4, 1)}

PRACTICE & PROBLEM SOLVING

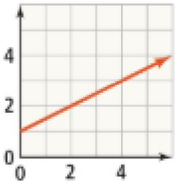
UNDERSTAND

8. **Use Patterns and Structure** Identify the domain and range of each function.

a.

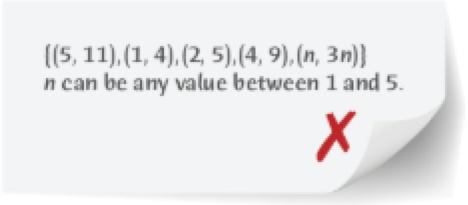


b.



9. **Represent and Connect** The domain of a function is all whole numbers between 2.5 and 7.5. Can you represent the domain using set-builder notation in more than one way? Explain.

10. **Error Analysis** The function has a domain $1 \leq x \leq 5$ and range of $3 \leq y \leq 12$. A student was asked to describe all possible values of n. Correct the error.



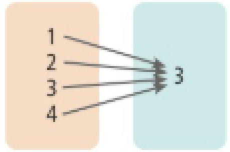
$\{(5, 11), (1, 4), (2, 5), (4, 9), (n, 3n)\}$

n can be any value between 1 and 5.

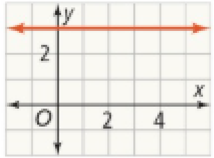
11. **Choose Efficient Methods** Previously you used set notation such as {7, 8, 9, 10}, to list the elements of a domain. What is a possible advantage of using set-builder notation in the form {Variable | Constraints} to describe discrete domains?

12. **Use Patterns and Structure** Describe the domain and range of each function.

a.



b.

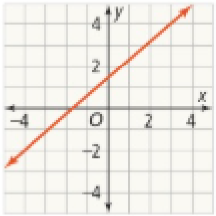


c. How are the functions the same? How are they different?

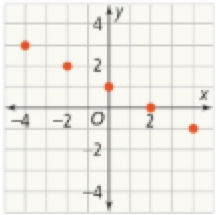
PRACTICE

Describe the domain and range of each function using inequality or set-builder notation. SEE EXAMPLE 1

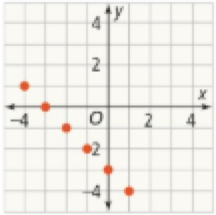
13.



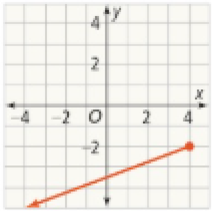
14.



15.



16.



Analyze each situation. Identify a reasonable domain and range. Is the domain continuous or discrete? SEE EXAMPLE 2

17. A car has a 15-gallon gas tank and gets a maximum average mileage of 24 miles per gallon.

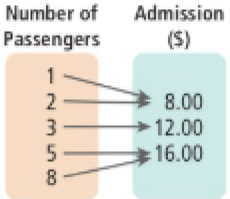
18. Kiyo invites 5 friends to a sporting event. Tickets cost $40. He knows at least one friend will choose to attend the event.

Identify any reasonable constraints on the domain. Describe the domain using inequality or set-builder notation. SEE EXAMPLE 3

19. Cameron earns an hourly wage at his job. He makes a table of the number of hours he works each week and the amount of money he earns.

20. Every day Isabel swims 10 to 20 laps in a 50-meter pool. She tracks the numbers of laps she swims and how long it takes her to complete the lap, in minutes.

21. The diagram shows parking fees for some vehicles at a local beach. Assume that the sample includes the minimum and maximum charges for parking. Describe any reasonable constraints on domain and range.



PRACTICE & PROBLEM SOLVING

APPLY

22. Apply Math Models The table shows the number of minutes Drew spends in each class for each of two weeks. Assume that Biology Lab is the one class Drew may not have on certain weeks and that durations of his classes will vary only as shown in the table for any given week.

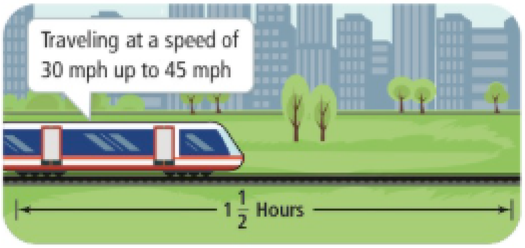
Class	Week 1	Week 2
	Time (min)	Time (min)
English	60	60
Math	90	60
History	45	45
Biology	45	45
Biology Lab	0	60

a. Describe the domain and range relating the number of classes to the total weekly class time.

b. Is the relation a function? Explain.

23. Analyze and Persevere Felix is slicing a tortilla Española (Spanish omelet) by cutting diameters through the center. He plans on cutting 8 diameters. The number of slices is a function of the number of diameters. Describe the domain and range of function.

24. Apply Math Models After a train has traveled for $\frac{1}{2}$ hour, it increases its speed and travels at a constant rate for $1\frac{1}{2}$ hours.



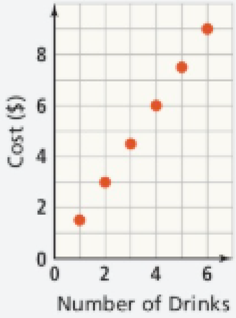
a. What is the domain? What is the range?

b. How can you represent the relationship between time traveled and speed?

c. Why did you choose this representation?

ASSESSMENT PRACTICE

25. A local store sells yogurt drinks for $1.50. The function that relates the number of drinks purchased to the total cost is shown below. Pilar can spend up to $10 on drinks. Select all of the statements about the domain or range of the function that are true. AR.2.5



☐ The domain is $1 \leq x \leq 6$.

☐ The domain is $\{x \mid x \text{ is an integer and } 1 \leq x \leq 6\}$.

☐ The domain is $\{x \mid x \text{ is an integer and } x \leq 10\}$.

☐ The range is $1 \leq y < 10$.

☐ The range is $\{y \mid y \text{ is a multiple of 1.5 and } 1.5 \leq y \leq 9\}$.

26. SAT/ACT The domain of a function is $\{x \mid x \text{ is an even number and } 3 \leq x \leq 17\}$. The range is $\{y \mid 1 < y < 8\}$. The points (6, 3), (10, 5), and (12, 6) are part of the function. Which point could not be part of the function?

Ⓐ (8, 4)

Ⓑ (16, 8)

Ⓒ (14, 7)

Ⓓ (4, 2)

27. Performance Task City Tours rents bicycles for $10 per hour or partial hour with a maximum daily fee of $100.

Part A Make a table that show the cost for renting a bicycle for 1, 3, 11, and 20 hours.

Part B Describe any constraints on the domain or the range. Explain.

Part C Represent the domain and range using inequality or set-builder notation.

3-2

Linear Functions

I CAN… identify, evaluate, graph, and write linear functions.

VOCABULARY

- function notation
- linear function

MA.912.AR.2.4–Given a table, equation or written description of a linear function, graph that function, and determine and interpret its key features. **Also AR.2.5, F.1.2, F.1.5, FL.3.2, FL.3.4**

MA.K12.MTR.2.1, MTR.4.1, MTR.7.1

EXPLORE & REASON

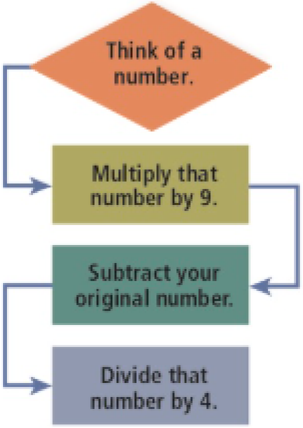
The flowchart shows the steps of a math puzzle.

A. Try the puzzle with 6 different integers.

B. Record each number you try and the result.

C. Make a prediction about what the final number will be for any starting number. Explain.

D. **Use Patterns and Structure** Would your prediction be true for all numbers? Explain.



ESSENTIAL QUESTION

How can you identify a linear function?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Evaluate Functions in Function Notation

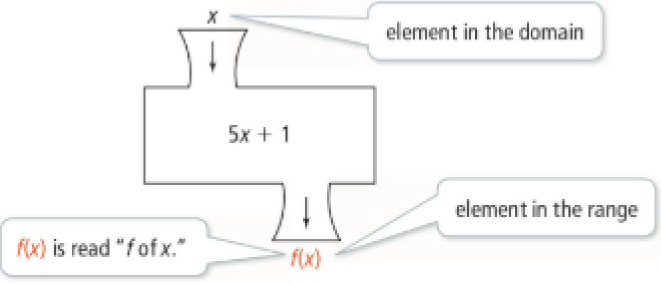
A. How can you represent a function rule?

Write the equation $y = 5x + 1$ using function notation.

Remember that a function is a rule that takes an input, or an element in the domain, and maps it to an output, or an element in the range.

Function notation is a method for writing variables as a function of other variables. The variable y becomes a function of x, meaning the variable x is used to find the value of y. Function notation helps distinguish between different functions. You can use the relationship between variables to solve problems and make predictions.

REPRESENT AND CONNECT
Function names are not restricted to f. What advantages are there to giving letter names to functions when modeling real-world situations?



The function f is defined in function notation by $f(x) = 5x + 1$.

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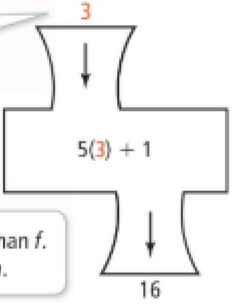
EXAMPLE 1 CONTINUED

B. What is the value of $g(x) = 5x + 1$ when $x = 3$?

HAVE A GROWTH MINDSET
In what ways can you be inquisitive and open to learning new things?

Evaluate $g(x) = 5x + 1$ for $x = 3$.

Substitute the input, 3, for every instance of x in the function.



If $g(x) = 5x + 1$, then $g(3) = 16$.

Function notation can use letters other than *f*. Other commonly used letters are *g* and *h*.

Try It! **1. Evaluate each function for $x = 4$.**

a. $g(x) = -2x - 3$

b. $h(x) = 7x + 15$

EXAMPLE 2 Write a Linear Function Rule

The cost to make 4 bracelets is shown in the table.

number of bracelets	1	2	3	4
cost	17	32	47	62

How can you determine the cost to make any number of bracelets?

Step 1 Examine the relationship between the values in the table.

+1 +1 +1

number of bracelets	1	2	3	4
cost ($)	17	32	47	62

+15 +15 +15

The rate of change is $\frac{15}{1}$ or $15 per bracelet.

The relationship is linear.

Step 2 Write a function using slope-intercept form for the rule.

$$f(x) = mx + b$$

$$f(x) = 15x + b$$

Step 3 Find the value of *b*.

Substitute any ordered pair from the table.

$$17 = 15(1) + b$$

$$2 = b$$

COMMON ERROR
You may think that the domain and range are all real numbers because the function $f(x) = 15x + 2$ has a domain and range of all real numbers. However, you need to consider the situation when determining the domain and range of a particular scenario.

You can use the function $f(x) = 15x + 2$ to determine the cost to make any number of bracelets.

The function $f(x) = 15x + 2$ is a linear function because the rule, $15x + 2$, is the same as the rule of the linear equation $y = 15x + 2$.

Try It! **2.** Write a linear function for the data in each table.

a.

x	1	2	3	4
y	6.5	13	19.5	26

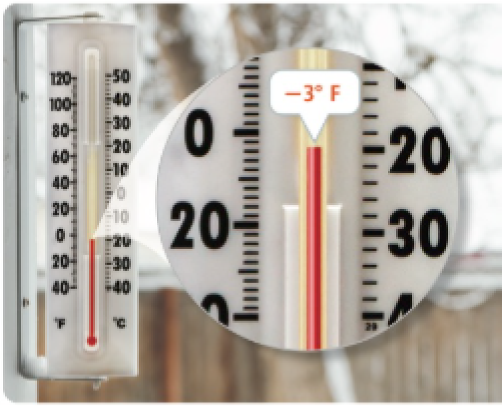
b.

x	1	2	3	4
y	1	4	7	10

APPLICATION

EXAMPLE 3 Graph a Linear Function

Tamika records the outside temperature at 6:00 a.m. The outside temperature increases by 2°F every hour for the next 8 hours.



A. Write and graph a function to model the situation.

Step 1 Write a function that models the situation.

Let $f(x)$ be the outdoor temperature x hours after 6:00 A.M.

The rate of change is the change in temperature divided by the change in time, or $\frac{2°\text{F}}{1\text{ h}}$.

The constant rate of change is 2° F per hour.

The initial temperature at time $x = 0$, or 6.00 A.M.

$$f(x) = 2x - 3$$

CHOOSE EFFICIENT METHODS
If you decide that time $x = 0$ corresponds to the initial time of recording, then you can use slope-intercept form to model the situation.

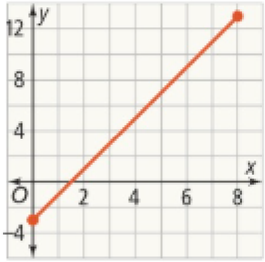
Step 2 Determine constraints on the domain.

The function models the outdoor temperature for 8 hours. A reasonable domain is $0 \leq x \leq 8$.

Step 3 Graph the function.

Use the slope and y-intercept to graph the function over the domain.

The graph of $f(x) = 2x - 3$ is a line. A **linear function** is a function whose graph is a line.



B. What will the temperature be at 1:30 P.M.?

Evaluate $f(x) = 2x - 3$ for $x = 7.5$.

$$f(7.5) = 2(7.5) - 3$$
$$= 12$$

Based on the linear model, the temperature will be 12°F at 1:30 P.M.

Try It! 3. The next day Tamika records the temperature of 0°F at 8:00 A.M. and 1°F at 10:00 A.M. Assume the temperature has been rising linearly since sunrise at 6:00 A.M.

a. Use point-slope form to write a linear function to model the situation. (Hint: Solve for y, then replace y with $f(x)$).

b. What will the temperature be at 11:00 A.M.?

EXAMPLE 4 Relate Simple Interest to Linear Growth

Lourdes invests $200 in a friend's startup company. She agrees to keep her money with them for the next 4 years, and they will pay Lourdes 7% interest annually on her principal, the original amount she invested.

A. How much is Lourdes' investment worth after 4 years?

Make a table to determine how much interest Lourdes earns and the total amount of her investment.

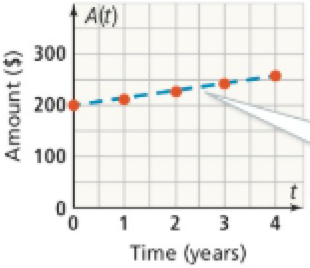
Year	Principal ($)	Interest ($)	Total ($)
1	200	$(0.07)200 = 14$	$200 + 14 = 214$
2	200	$(0.07)200 = 14$	$200 + 14 + 14 = 228$
3	200	$(0.07)200 = 14$	$200 + 14 + 14 + 14 = 242$
4	200	$(0.07)200 = 14$	$200 + 14 + 14 + 14 + 14 = 256$

VOCABULARY
Recall that interest earned or charged based on the principal only is called *simple interest*.

After 4 years Lourdes will have earned $56 in interest and her total investment will be worth $256.

B. How is simple interest related to linear growth?

Simple interest is paid on the original principal, so the investment increases, or grows, by 7% of $200 each year.



The graph shows that the growth of the investment is a linear relationship.

C. How can you write a linear function to model Lourdes' investment?

Let $A(t)$ be the total amount after t years. $A(t)$ grows linearly, so start by writing it in the form $A(t) = mt + b$.

$$\begin{aligned} A(t) &= mt + b \\ &= (Pr)t + P \\ &= P + Prt \\ &= P(1 + rt) \end{aligned}$$

The rate of change of A is Pr, and the initial amount when $t = 0$ is P.

Use the Distributive Property to rewrite the expression.

REPRESENT AND CONNECT
How does the 1 in the function $A(t) = P(1 + rt)$ connect to the simple interest situation?

Write the linear function for Lourdes' investment.

$$\begin{aligned} A(t) &= P(1 + rt) \\ &= 200(1 + 0.07t) \end{aligned}$$

Substitute 200 for P and 0.07 for r.

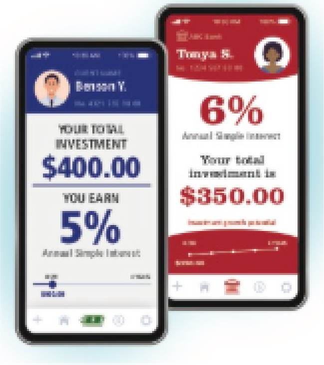
The linear function $A(t) = 200(1 + 0.07t)$ models Lourdes' investment.

Try It! **4.** What is the domain of the function that models Lourdes' investment?

APPLICATION

EXAMPLE 5 Solve Simple Interest Problems

Benson and Tonya invest their money into two different opportunities shown at the right.



A. After 4 years, who will have more money in their account?

Step 1 Write the linear function to represent each person's account.

Use the simple interest formula $A(t) = P(1 + rt)$ to write the linear functions.

Benson	Tonya
$f(t) = P(1 + rt)$	$g(t) = P(1 + rt)$
$f(t) = 400(1 + 0.05t)$	$g(t) = 350(1 + 0.06t)$

Step 2 Evaluate each function for $t = 4$ to determine how much money will be in Benson's and Tonya's accounts after 4 years.

$$f(4) = 400(1 + 0.05(4)) \qquad g(4) = 350(1 + 0.06(4))$$

$$f(4) = 480 \qquad g(4) = 434$$

After 4 years Benson will have $480 in his account and Tonya will have $434 in her account, so Benson will have more money in his account.

B. Compare the linear functions for Benson's and Tonya's accounts. What do the key features of the functions indicate? Who collects more interest over the 4 years?

ANALYZE AND PERSEVERE
How do the domains and ranges of the two functions compare?

Write the interest functions in slope-intercept form.

$$f(t) = 400(1 + 0.05t) \qquad g(t) = 350(1 + 0.06t)$$

$$= 400 + 400(0.05t) \qquad = 350 + 350(0.06t)$$

$$= 20t + 400 \qquad = 21t + 350$$

The y-intercept for Benson's function, 400, shows his principal was greater.

The slope for Tonya's function, 21, means that she collects more interest each year.

After 4 years, Tonya will collect more interest than Benson.

Try It! 5. Paulo invests $450 in an account that earns 4% annual simple interest for 6 years.

a. Write a linear function that models the situation.

b. Evaluate the function to determine how much money he will have after 6 years.

c. How much did Paulo earn in interest?

CONCEPT SUMMARY Linear Function Representations

WORDS Linear functions are represented by words, rules, tables, or graphs. Function notation tells us the name of a function and the input variable.

ALGEBRA $f(x) = -2x + 1$

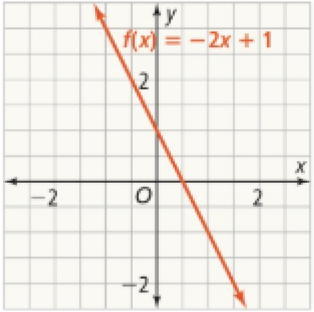
"f of x"

TABLE

x	−2	−1	0	1	2
$f(x)$	5	3	1	−1	−3

The table shows the domain and range of the function.

GRAPH The graph of the function $f(x) = -2x + 1$ is the graph of the linear equation $y = -2x + 1$.



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you identify a linear function?

2. **Communicate and Justify** Give a real-world example of a function that is linear and one that is not linear. Explain.

3. **Vocabulary** In the function rule for simple interest $A(t) = P(1 + rt)$, is P a variable? Explain.

4. **Error Analysis** The cost of using a game facility is \$1 for every 12 minutes. Talisa writes the function for the cost per hour as $f(x) = 12x$. Explain Talisa's error.

Do You KNOW HOW?

Evaluate each function for $x = 2$ and $x = 6$.

5. $f(x) = 4x - 3$

6. $f(x) = -(x - 2)$

7. Sketch the graph of $f(x) = \frac{1}{2}x + 5$.

8. What function models the height of the periscope lens at time t? If the periscope reaches its maximum height after ascending for 22 seconds, what is the maximum height in feet?

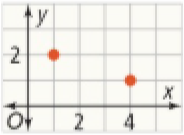

24 inches above the surface

ascends at 6 inches per second

PRACTICE & PROBLEM SOLVING

UNDERSTAND

9. **Use Patterns and Structure** The two points on the graph are given by the function f.



a. Use the two points to find the equation that represents the function f.

b. What is $f(6)$?

10. **Higher Order Thinking** Consider the functions $g(x) = 2x + 1$ and $h(x) = 2x + 2$ for the domain $0 < x < 5$.

a. Without evaluating or graphing the functions, how do the ranges compare?

b. Graph the two functions and describe each range over the given interval.

11. **Analyze and Persevere** Compare the key features of the linear functions f and g, given in the tables below.

x	0	1	2	3	4
$f(x)$	5	7	9	11	13

x	0	1	2	3	4
$g(x)$	4	7	10	13	16

a. What do the slopes of the graphs of functions f and g indicate?

b. Write rules for $f(x)$ and $g(x)$ in slope-intercept form.

c. If the domains of each function are all real numbers, compare their ranges.

12. **Error Analysis** Describe and correct the error a student made when finding the function rule for the data in the table.

x	1	2	3	4
y	10	19	28	37

When x increases by 1,
y increases by 9 each time.
When $x = 1$, $y = 10$.
So $y = 9x + 10$. ✗

PRACTICE

Find the value of $f(5)$ for each function.
EXAMPLE 1

13. $f(x) = 6 + 3x$

14. $f(x) = -2(x + 1)$

15. $f(a) = 3(a + 2) - 1$

16. $f(h) = -\frac{h}{10}$

For the data in each table, graph the linear function, determine the rate of change, and the y-intercept. Then write a linear function for the data. SEE EXAMPLES 2 AND 3

17.

x	0	1	2	3	4
y	−1	4	9	14	19

18.

x	0	1	2	3	4
y	4	1.5	−1	−3.5	−6

Sketch the graph of each linear function.
SEE EXAMPLE 3

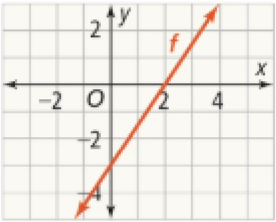
19. $f(x) = \frac{1}{2}x - 1$

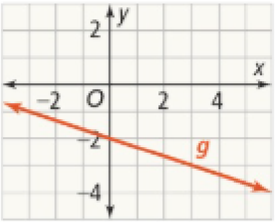
20. $f(x) = 0.75(10 - x) + 1$

21. Katrina buys a 64-ft roll of fencing to make a rectangular play area for her dogs. Use $2(l + w) = 64$ to write a function for the length, given the width. Graph the function. What is a reasonable domain for the situation? Explain. SEE EXAMPLE 3

22. Kyle invests \$600 which earns 3% annual simple interest for 5 years. Use $A(t) = P(1 + rt)$ to determine how much money he will have after 5 years. Interpret the rate of change in this situation. SEE EXAMPLES 4 AND 5

Compare the key features of the linear functions shown below.





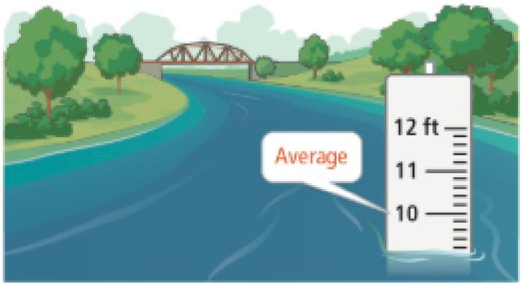
23. What do the slopes and the y-intercepts indicate about the linear functions f and g?

24. Compare the domains and ranges of functions f and g.

PRACTICE & PROBLEM SOLVING

APPLY

25. **Apply Math Models** A staff gauge measures the height of the water level in a river compared to the average water level. At one gauge the river is 1 ft below its average water level of 10 ft. It begins to rise by a constant rate of 1.5 ft per hour.



a. Graph the linear function to show the change in the water level over time.

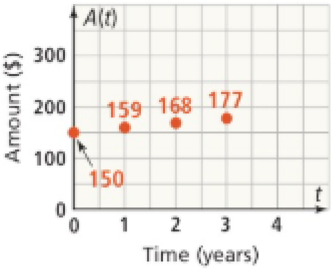
b. Will the river reach a level of 7 ft above normal after 5 hours? Explain.

26. **Apply Math Models** A drive-in movie theater charges $20 per car, regardless of the number of people in the vehicle.

a. Make a graph of the cost for a carload of moviegoers as function of the number of people in the car.

b. Describe the y-intercept, slope, domain, and range of the function.

27. **Represent and Connect** The graph shows Ramona's investment earning simple interest over 3 years. Her friend Becky invests $160 in a different account which earns 4% annual simple interest. How do the key features of the functions compare? Who will have more money after for 4 years? Explain.



ASSESSMENT PRACTICE

28. A veterinarian uses the function $f(t) = 3.7t + 21$ to estimate a puppy's future weight in pounds as a function of time in months.

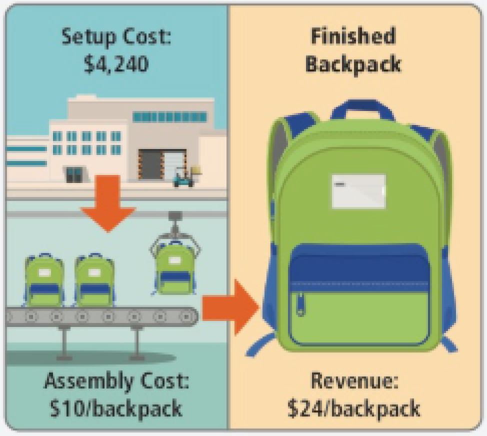
The key feature of the graph that models how much the puppy's weight is increasing every month is the ________ . AR.2.4

29. **SAT/ACT** Determine a linear function from the data in the table. Which point is not part of the function?

x	$f(x)$
0	180
1	174
2	168
3	162
4	156

Ⓐ (12, 108)
Ⓑ (30, 0)
Ⓒ (−15, 270)
Ⓓ (21, 54)
Ⓔ (9, 120)

30. **Performance Task** Manuel calculates the business costs and profits to produce n hiking backpacks. Manuel's profit is his revenue minus his total costs.



Part A Write a function to represent the profit Manuel makes selling n backpacks.

Part B Graph the profit function. What is a reasonable domain for this function for one year if his revenue is between $4,000 and $30,000? Is the function discrete or continuous? Explain.

Part C How much is his profit if he sells 43 backpacks? Explain.

3-3

Transforming Linear Functions

I CAN… transform linear functions.

VOCABULARY

- transformation
- translation

MA.912.F.2.1–Identify the effect on the graph or table of a given function after replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$ and $f(x+k)$ for specific values of k. Also F.1.5

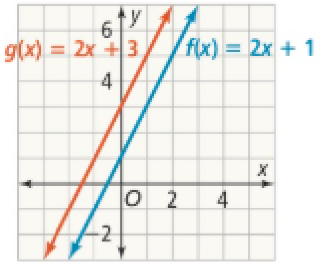
MA.K12.MTR.2.1, MTR.4.1, MTR.5.1

APPLICATION

LEARN TOGETHER
What are ways to stay positive and work toward goals?

CRITIQUE & EXPLAIN

Avery states that the graph of g is the same as the graph of f with every point shifted vertically. Cindy states that the graph of g is the same as the graph of f with every point shifted horizontally.



A. Give an argument to support Avery's statement.

B. Give an argument to support Cindy's statement.

C. **Represent and Connect** What do you know about linear equations that might support either of their statements?

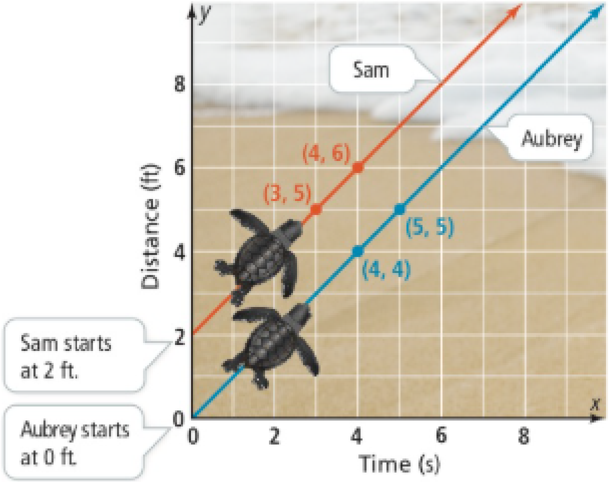
ESSENTIAL QUESTION

How does modifying the input or the output of a linear function rule transform its graph?

EXAMPLE 1 Vertical Translations of Linear Functions

The positions of 2 baby sea turtles making their way to the water after hatching from their eggs is recorded. They move at the same speed, with Sam starting 2 ft ahead of Aubrey's starting point.

A. What function represents each turtle's position as they make their way toward the shore?



Find the speed of each turtle by finding the slope of each line.

Aubrey: $m = \frac{5-0}{5-0} = 1$ Sam: $m = \frac{6-2}{4-0} = 1$

Both turtles are moving at 1 ft/s.

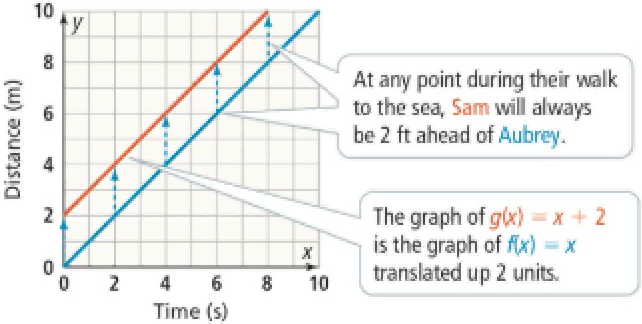
The function $f(x) = x$ represents Aubrey's distance from his starting point, and $g(x) = x + 2$ represents Sam's distance from his starting point.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

B. What happens to the graph of a function when you add a constant to its output?

Compare Sam's and Aubrey's graph.



At any point during their walk to the sea, Sam will always be 2 ft ahead of Aubrey.

The graph of $g(x) = x + 2$ is the graph of $f(x) = x$ translated up 2 units.

USE PATTERNS AND STRUCTURE
How do the key features of $g(x) = x + 2$ compare to key features of $f(x) = x$?

Adding a constant k to the output of a linear function *translates* the graph vertically by k units.

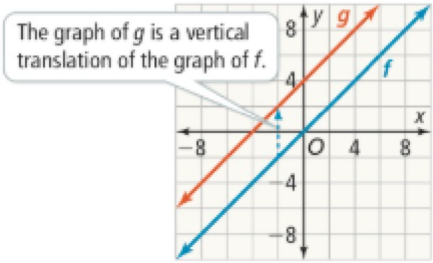
Try It! 1. Let $f(x) = -4x$.

a. How does the graph of $g(x) = -4x - 3$ compare with the graph of f?

b. How does the graph of $g(x) = -4x + 1.5$ compare with the graph of f?

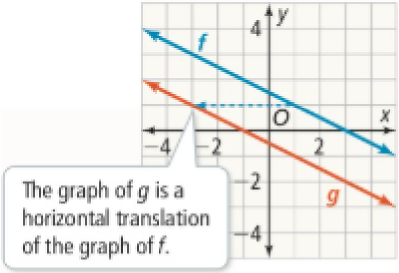
DEFINITION

A **transformation** of a function f maps each point of its graph to a new location. One type of transformation is a *translation*. A **translation** shifts each point of the graph of a function the same distance. A translation may be horizontal or vertical.



The graph of g is a vertical translation of the graph of f.

Vertical Translation



The graph of g is a horizontal translation of the graph of f.

Horizontal Translation

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Horizontal Translations of Linear Functions

How does subtracting a constant k from the input of a linear function affect its graph?

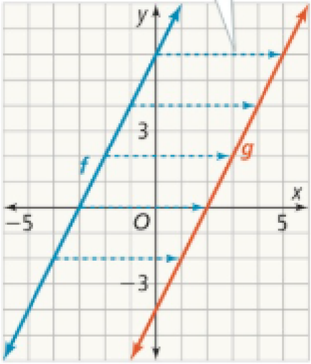
Consider the graphs of $f(x) = 2x + 6$ and $g(x) = 2(x - 5) + 6$.

Step 1 Make a table of values for $f(x) = 2x + 6$ and $g(x) = 2(x - 5) + 6$.

x	$f(x) = 2x + 6$	$x - 5$	$g(x) = 2(x - 5) + 6$
-4	-2	-9	$2(-9) + 6 = -12$
-3	0	-8	$2(-8) + 6 = -10$
-2	2	-7	$2(-7) + 6 = -8$
-1	4	-6	$2(-6) + 6 = -6$
0	6	-5	$2(-5) + 6 = -4$
1	8	-4	$2(-4) + 6 = -2$
2	10	-3	$2(-3) + 6 = 0$
3	12	-2	$2(-2) + 6 = 2$

Step 2 Graph the functions $f(x) = 2x + 6$ and $g(x) = 2(x - 5) + 6$.

The graph of g is the graph of f translated 5 units to the right.



COMMON ERROR
You may think that subtracting a positive value of k in $g(x) = 2(x - k) + 6$ shifts the graph in a negative direction, to the left. However, subtracting a positive value of k shifts the graph to the right.

Subtracting a constant k from the input of the function translates the graph horizontally by k units.

Translating the graph of f horizontally does not change the slope of the line, but it changes both the x- and y-intercepts.

Try It! **2.** Let $f(x) = 3x + 7$.

a. How does the graph of $g(x) = 3(x - 4) + 7$ compare with the graph of f?

b. How does the graph of $g(x) = 3(x + 9.5) + 7$ compare with the graph of f?

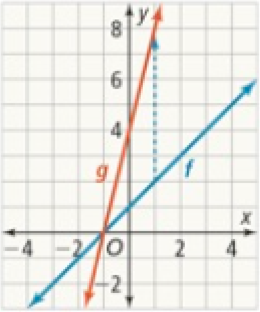
EXAMPLE 3 Stretches and Compressions of Linear Functions

A. How does multiplying the output of a linear function affect its graph?

Compare the graphs of $f(x) = x + 1$ and $g(x) = 4(x + 1)$.

x	$f(x) = x + 1$	$g(x) = 4(x + 1)$
−3	−2	−8
−2	−1	−4
−1	0	0
0	1	4
1	2	8

×4



The graph of g is vertical stretch of the graph of f, by a scale factor of 4. The slope and y-intercept are scaled by the same factor.

GENERALIZE
Is it always true that for a linear function f, the graphs of f and kf would have different y-intercepts?

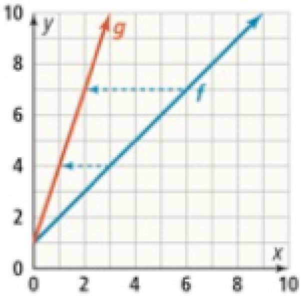
Multiplying the output of a linear function f by k scales its graph vertically. If $|k| > 1$, the transformed graph is a **vertical stretch**. If $0 < |k| < 1$ the transformed graph is a **vertical compression**.

B. How does multiplying the input of a linear function affect its graph?

Compare the graphs of $f(x) = x + 1$ and $g(x) = (3x) + 1$.

$\times\frac{1}{3}$

x	$f(x) = x + 1$	$g(x) = (3x) + 1$
0	1	1
1	2	4
2	3	7
3	4	10
6	7	19



The graph of g is horizontal compression of the graph of f, by a scale factor of $\frac{1}{3}$, the reciprocal of the coefficient of x. The slope is changed by the coefficient of x, but the y-intercept is unchanged.

Multiplying the input of a linear function f by k scales its graph horizontally. If $|k| > 1$, the transformed graph is a **horizontal compression**. If $0 < |k| < 1$ the transformed graph is a **horizontal stretch**.

Try It! 3. Let $f(x) = x - 2$. Compare the graph of each function with the graph of f. Describe the transformation and tell whether the slopes and intercepts of f and g are the same or different.

a. $g(x) = 0.25(x - 2)$

b. $g(x) = 0.5x - 2$

CONCEPT SUMMARY Transformations of Linear Functions

	Translations	Stretches and Compressions
WORDS	Translations shift each point of the graph the same distance horizontally or vertically.	Stretches and compressions scale the graph either horizontally or vertically.

ALGEBRA

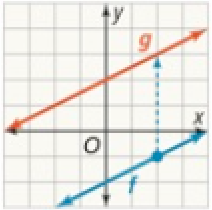
Vertical by k units: The graph of $g(x) = \left(\frac{1}{2}x - 2\right) + k$ is a vertical translation of $f(x) = \frac{1}{2}x - 2$.

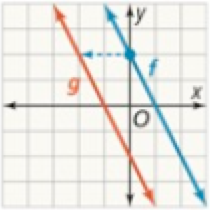
Horizontal by k units: The graph of $g(x) = -2(x - k) + 2$ is a horizontal translation of $f(x) = -2x + 2$.

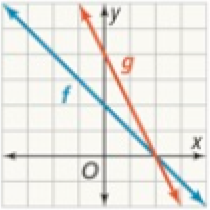
Vertical by scale factor k: The graph of $g(x) = k(-x + 2)$ is a stretch of $f(x) = -x + 2$ when $|k| > 1$.

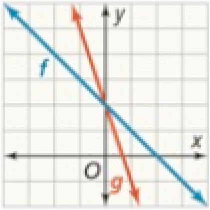
Horizontal by scale factor k: The graph of $g(x) = -(kx) + 2$ is a compression of $f(x) = -x + 2$ when $|k| > 1$.

GRAPHS









Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does modifying the input or the output of a linear function rule transform its graph?

2. **Vocabulary** Why is the addition or subtraction of k to the output of a function considered a *translation*?

3. **Error Analysis** The addition or subtraction of a number to a linear function always moves the line up or down. Describe the error with this reasoning.

4. **Use Patterns and Structure** Why does multiplying the input of a linear function change only the slope while multiplying the output changes both the slope and the y-intercept?

Do You KNOW HOW?

Given $f(x) = 4x + 1$, describe how the graph of g compares with the graph of f.

5. $g(x) = 4(x + 3) + 1$

6. $g(x) = (4x + 1) + 3$

Given $f(x) = x + 2$, describe how setting $k = 4$ affects the slope and y-intercept of the graph of g compared to the graph of f.

7. $g(x) = 4(x + 2)$

8. $g(x) = (4x) + 2$

9. The minimum wage for employees of a company is modeled by the function $f(x) = 13x$. The company decided to offer a signing bonus of $75. How does adding this amount affect a graph of an employee's earnings?

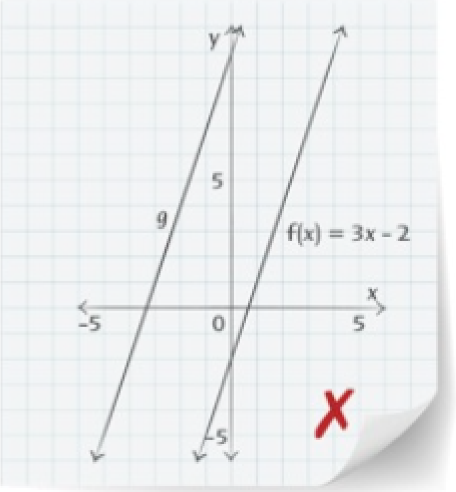
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Analyze and Persevere** Describe the transformation of the function $f(x) = \frac{1}{2}x - 2$ that makes the slope 2 and the y-intercept –8.

11. **Use Patterns and Structure** Why do translations produce parallel lines?

12. **Error Analysis** A student graphs $f(x) = 3x - 2$. On the same grid they graph the function g which is a transformation of f made by subtracting 4 from the input of f. Describe and correct the error they made when graphing g.



13. **Use Patterns and Structure** Let $f(x) = \frac{1}{2}x - 3$. Suppose you subtract 6 from the input of f to create a new function g, then multiply the input of function g by 4 to create a function h. What equation represents h?

14. **Represent and Connect** The graph of g is a transformation of the graph of f. Describe the transformation.

x	$f(x)$	$g(x)$
0	4	2
1	6	3
2	8	4
3	10	5
4	12	6

PRACTICE

Given $f(x) = 3x + 5$, describe how the graph of g compares with the graph of f.
SEE EXAMPLES 1, 2, AND 3

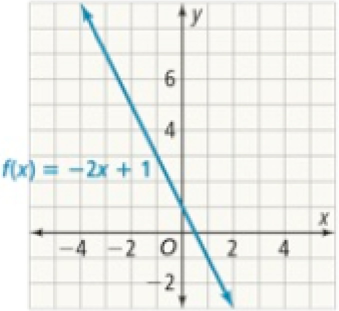
15. $g(x) = (3x + 5) + 8$
16. $g(x) = (3x + 5) - 4$
17. $g(x) = 3(x + 10) + 5$
18. $g(x) = 3(x - 1) + 5$
19. $g(x) = 3(0.1x) + 5$
20. $g(x) = 5(3x + 5)$
21. $g(x) = 3(2x) + 5$
22. $g(x) = 8(3x + 5)$

Compare the key features (slope and y-intercept) of each linear function g with those of the graph of $f(x) = 2x + 3$. Describe how the value of k affects the key features. SEE EXAMPLES 1, 2, AND 3

23. $g(x) = 3(2x + 3)$
24. $g(x) = 2(0.5x) + 3$
25. $g(x) = \frac{1}{6}(2x + 3)$
26. $g(x) = 2\left(\frac{1}{8}\right)x + 3$
27. $g(x) = (2x + 3) - 3$
28. $g(x) = 2(x - 0.5) + 3$

Key features of $f(x) = -2x + 1$ are shown in the table shown below. For each transformation, write the function rule and compare the key features of the function g to the key features of function f.
SEE EXAMPLES 2 AND 3

$f(x) = -2x + 1$	
Domain	{x | x is a real number}
Range	{y | y is a real number}
Slope	–2
y-intercept	1
x-intercept	0.5



29. The graph of g is the graph of f translated up 3 units.

30. The graph of g is a vertical stretch of the graph of f, by a scale factor of 4.

PRACTICE & PROBLEM SOLVING

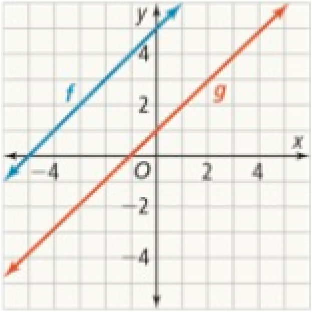
APPLY

31. **Mathematical Connections** The cost of renting a landscaping tractor is a \$100 security deposit plus the hourly rate.

a. The function f represents the cost of renting the tractor. The function g represents the cost if the hourly rate were doubled. Write each function.

b. How would the slope and y-intercept of the graph g compare to the slope and y-intercept of the graph of f?

32. **Communicate and Justify** Veronica said the graph of g below represents a vertical translation of the function $f(x) = x + 1$ by 4 units. Dawn argued that the graph of g represents a horizontal translation of f by 4 units. Who is correct? Explain.



33. **Higher Order Thinking** The graph of a linear function f has a negative slope. Describe the effect on the graph of the function if the transformation has a value of $k < 0$.

a. adding k to the outputs of f

b. adding k to the inputs of f

c. multiplying the outputs of f by k

d. multiplying the inputs of f by k

ASSESSMENT PRACTICE

34. The table shows the values for the function $f(x) = 2x$.

x	−2	−1	0	1	2
y	−4	−2	0	2	4

Copy and complete the table for the function $g(x) = 2x + 6$. F.2.1

x	−2	−1	0	1	2
y					

35. **SAT/ACT** Which of the following describes the differences between the graph of f and the graph of the output of f multiplied by 3?

Ⓐ The slope changes by a factor of 3; the y-intercept does not change.

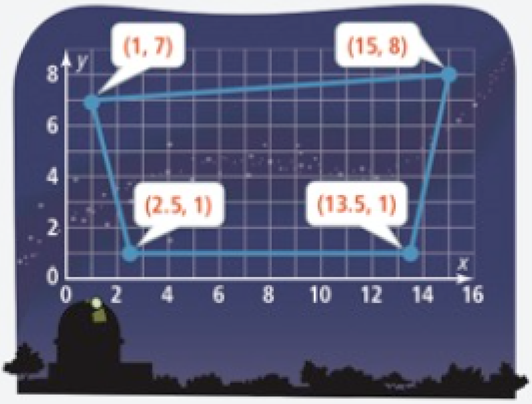
Ⓑ Both the slope and y-intercept change by a factor of 3.

Ⓒ The slope does not change; the y-intercept changes by a factor of 3.

Ⓓ Neither the slope nor y-intercept change.

36. **Performance Task** The science club members are using transformations on coordinate grids to track the movement of constellations in the sky.

Choose one side of the constellation depicted below and describe a series of transformations to move the side.



Copy and complete the table to record the motion.

Transformation	Function

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.2.5–Solve and graph mathematical and real-world problems that are modeled with linear functions. Interpret key features and determine constraints in terms of the context **Also AR.2.2.**

MA.K12.MTR.7.1

The Express Lane

Some supermarkets have self checkout lanes. Customers scan their items themselves and then pay with either cash or credit when they have finished scanning all of the items. Some customers think these lanes are faster than the checkout lanes with cashiers, but others don't like having to bag all of their purchases themselves.

What's your strategy for picking a checkout lane at the grocery store? Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

3-4

Absolute Value Functions

I CAN… analyze functions that include absolute value expressions.

VOCABULARY

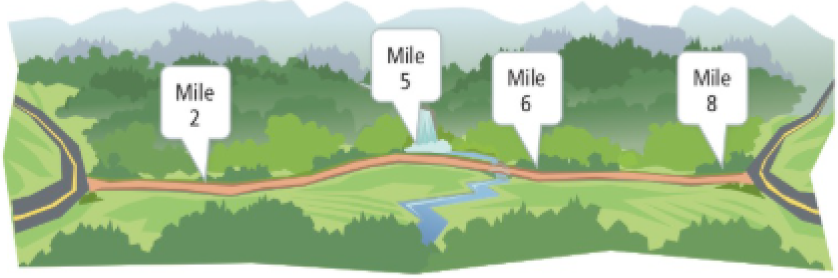
- absolute value function
- axis of symmetry
- vertex

MA.912.AR.4.3–Given a table, equation or written description of an absolute value function, graph that function and determine its key features. **Also F.1.2, F.1.6, F.2.1**

MA.K12.MTR.4.1, MTR.5.1, MTR.6.1

EXPLORE & REASON

Groups of students are hiking from mile markers 2, 6 and 8 to meet at the waterfall located at mile marker 5.



A. How can you use the mile marker to determine the number of miles each group of students needs to hike to the waterfall?

B. **Apply Math Models** Make a graph that relates the position of each group on the trail to their distance from the waterfall.

C. How would the points in your graph from part B change as the groups of students approach the waterfall?

ESSENTIAL QUESTION

What are the key features of the graph of the absolute value function?

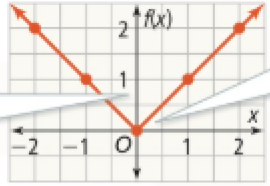
EXAMPLE 1 Graph the Absolute Value Function

What are the features of the graph of $f(x) = |x|$?

Make a table of values and graph the **absolute value function** $f(x) = |x|$.

x	$f(x) = \|x\|$	$(x, f(x))$
−2	2	(−2, 2)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)

The *vertex*, (0, 0), is the turning point of the graph, where the function changes from decreasing to increasing.



The *axis of symmetry* is the line $x = 0$.

The graph has a **vertex**, where the axis of symmetry intersects the graph. It represents the minimum value in the range.

USE PATTERNS AND STRUCTURE How does the nature of absolute value contribute to the symmetric appearance of the graph?

The graph has an **axis of symmetry**, which intersects the vertex and divides the graph into two sections, or pieces, that are images of each other under a reflection.

Try It! **1.** What are the domain and range of $f(x) = |x|$?

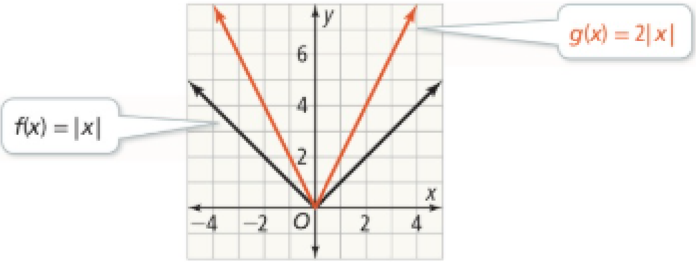
CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Key Features of Absolute Value Functions

Powered By desmos

A. How do the domain and range of $g(x) = 2|x|$ compare with the domain and range of $f(x) = |x|$?

Compare the graphs of g and f.



The domain of f and the domain of g is $\{x \mid x \text{ is a real number}\}$.

Because the absolute value expression produces only nonnegative values, the range of f is $\{y \mid y \text{ is a nonnegative real number}\}$. Multiplying $|x|$ by a positive factor, in this case, 2, yields nonnegative outputs, so the range of g is also $\{y \mid y \text{ is a nonnegative real number}\}$.

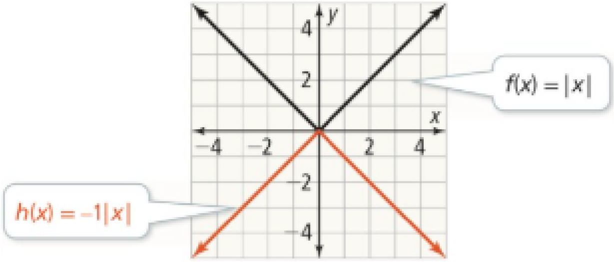
The domain and range of the function g are the same as those of function f.

VOCABULARY

A *reflection* across the x-axis maps (x, y) to $(x, -y)$.

B. The graph of function h is a reflection of the graph of $f(x) = |x|$ over the x-axis. Over what intervals is $h(x)$ positive? Over what intervals is $h(x)$ negative? How does function h compare with positive and negative intervals of $f(x) = |x|$?

Compare the graphs of h and f.



Multiplying $|x|$ by a negative factor, -1, yields nonpositive outputs, so h is never positive.

The function h is negative for values $\{x \mid x \neq 0\}$. This is opposite of f, which is never negative and positive for values $\{x \mid x \neq 0\}$.

Try It! **2.** For each function, describe the domain, range, and the intervals over which they are positive or negative.

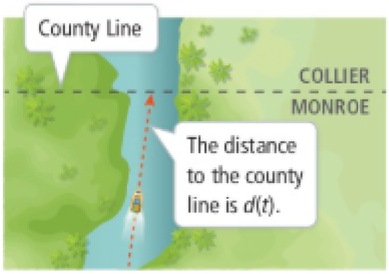
a. The graph of function h is a reflection of the graph $f(x) = 2|x|$ over the x-axis.

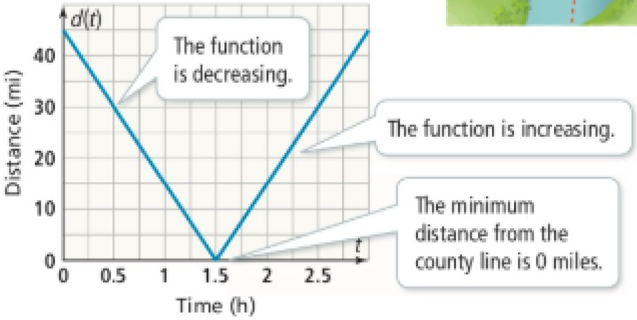
b. $g(x) = \frac{1}{2}|x|$

APPLICATION

EXAMPLE 3 Interpret the Graph of a Function

Jay rides in a boat from Monroe County to Collier County. The graph of the function $d(t) = 30|t - 1.5|$ shows the distance of the boat in miles from the county line at t hours. Assume the graph shows Jay's entire trip.





COMMON ERROR
Remember that a graph representing the motion of an object may not be a picture of its path.

A. How far does Jay travel to visit his friend?

Jay began his trip 45 mi from the county line, traveled towards the county line, which he crossed after an hour and a half. He then traveled away from the county line and was 45 mi from the county line after 3 h. He traveled a total of 90 mi.

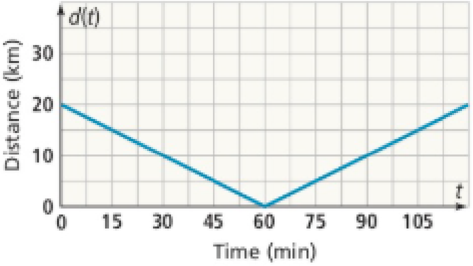
B. How does the graph relate to the domain and range of the function?

Since Jay's entire trip is 3 h, the domain of the function is $0 \leq t \leq 3$.

For the section of the domain $0 < t < 1.5$, his distance to the border is decreasing. For $1.5 < t < 3$ his distance from the border is increasing.

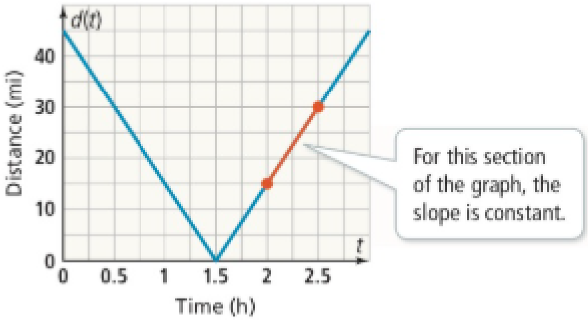
The maximum and minimum values on the graph are 45 and 0, so the range of the function is $0 \leq d(t) \leq 45$.

Try It! **3.** A cyclist competing in a race rides past a water station. The graph of the function $d(t) = \frac{1}{3}|t - 60|$ shows her distance from the water station at t minutes. Assume the graph represents the entire race. What does the graph tell you about her race?



EXAMPLE 4 Determine Rate of Change

The graph shows Jay's boat ride across the state line from Example 3. What is the rate of change over the interval $2 \leq t \leq 2.5$? What does it mean in terms of the situation?



For this section of the graph, the slope is constant.

Use the slope formula to determine the rate of change from $t = 2$ to $t = 2.5$.

Use the points from (2, 15) and (2.5, 30).

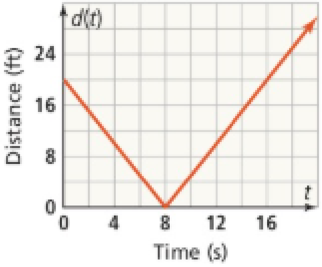
$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{30 - 15}{2.5 - 2} \\ &= \frac{15}{0.5} \\ &= 30 \end{aligned}$$

CHECK FOR REASONABLENESS
The rate of change is a different constant value for each section of the graph. How does this relate to the situation?

The rate of change over this interval is 30.

The rate of change represents the speed of the boat in miles per hour. Since the rate of change is positive, Jay's distance from the border is increasing. The boat is traveling at 30 mi/h away from the border.

Try It! 4. Kata gets on a moving walkway at the airport. Then, 8 s after she gets on, she taps Lisa, who is standing alongside the walkway. The graph shows Kata's distance from Lisa over time. Calculate the rate of change in her distance from Lisa from 6 s to 8 s, and then from 8 s to 12 s. What do the rates of change mean in terms of Kata's movement?

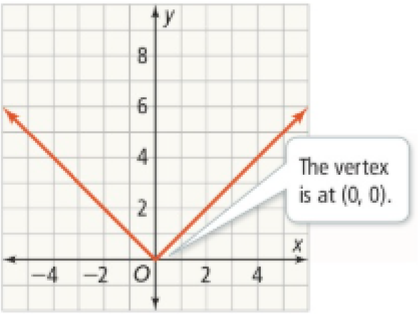


CONCEPT SUMMARY The Absolute Value Function

WORDS The graph of the absolute value function has a vertex, which represents the minimum value of the function. The axis of symmetry intersects the vertex and divides the graph into two sections that are images of each other under a reflection. The absolute value function changes from decreasing to increasing at the vertex.

ALGEBRA $f(x) = |x|$

GRAPH



The vertex is at (0, 0).

The domain is $\{x \mid x \text{ is a real number}\}$.
The range is $\{y \mid y \geq 0\}$.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are the key features of the graph of the absolute value function?

2. **Communicate and Justify** How do the domain and range of $g(x) = a|x|$ compare to the domain and range of $f(x) = |x|$ when $0 < a < 1$? Explain.

3. **Analyze and Persevere** The graph of the function $g(x) = a|x|$ includes the point (1, 16). What is another point on the graph of the function? What is the value of a?

4. **Error Analysis** Janiece says that the vertex of the graph of $g(x) = a|x|$ always represents the minimum value of the function g. Explain her error.

Do You KNOW HOW?

Find the domain and range of each function.

5. $g(x) = 5|x|$

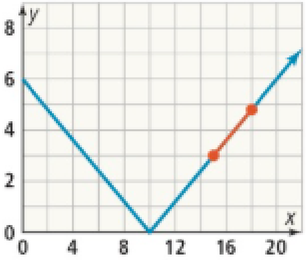
6. $h(x) = -2|x|$

Graph each function.

7. $g(x) = 1.5|x|$

8. $h(x) = -0.8|x|$

9. What is the rate of change over the interval $15 \leq x \leq 18$?



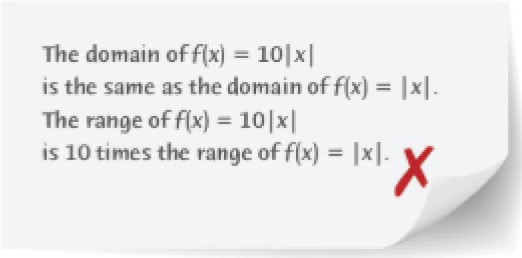
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Generalize** How does changing the sign of the constant a from positive to negative affect the domain and range of $f(x) = a|x|$?

11. **Communicate and Justify** Compare and contrast the graph of $f(x) = |x|$ and the graph of $f(x) = x$. How are they alike? How do they differ?

12. **Error Analysis** Describe and correct the error a student made in determining the relationship between the domain and range of $f(x) = 10|x|$ and $f(x) = |x|$.



The domain of $f(x) = 10|x|$ is the same as the domain of $f(x) = |x|$. The range of $f(x) = 10|x|$ is 10 times the range of $f(x) = |x|$. ✗

13. **Higher Order Thinking** For which values of a would the graph of $f(x) = a|x|$ form a right angle at the vertex? Explain.

14. **Use Patterns and Structure** Four values of an absolute value function h are shown below. The vertex is located at (0, 0).

a. Graph the function.

b. Describe the domain, range, and the intervals over which it is increasing or decreasing.

x	$h(x)$
–6	1.5
–4	1
–2	0.5
0	0

15. **Use Patterns and Structure** Consider the function $f(x) = 2|x|$.

a. Graph f over the domain $\{x \mid -4 \le x \le 4\}$.

b. What is the rate of change over the interval $0 \le x \le 4$?

c. How is the rate of change over this interval related to the form of the function?

PRACTICE

Tell whether each point is on the graph of $f(x) = |x|$. If it is, give the coordinates of another point with the same y value. SEE EXAMPLE 1

16. (11, 11)

17. (–2.3, –2.3)

18. (0, 1)

19. (15, –15)

Describe the key features of each absolute value function. Identify the vertex, axis of symmetry, and the domain and range. SEE EXAMPLES 1 AND 2

20.

x	$g(x)$
–2	–6
–1	–3
0	0
1	–3
2	–6

21.

x	$h(x)$
0	1
1	0
2	1
3	2
4	3

Graph each function. What is the domain and range of each function? Describe the intervals over which each function is positive and over which it is negative. SEE EXAMPLE 2

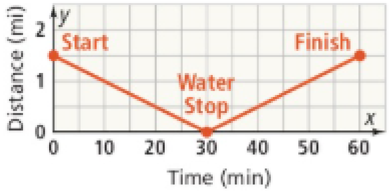
22. $g(x) = -\frac{1}{4}|x|$

23. $h(x) = 3.5|x|$

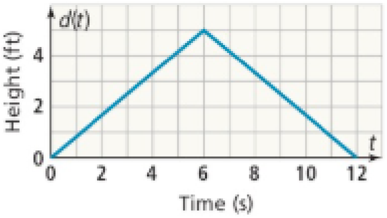
24. $p(x) = -5|x|$

25. $d(x) = \frac{1}{3}|x|$

26. Oscar participates in a charity walk. The graph shows his distance in miles from the water stop as a function of time. How many miles did Oscar walk? Explain your answer. SEE EXAMPLE 3



For the graph shown, find the rate of change over the interval. SEE EXAMPLE 4



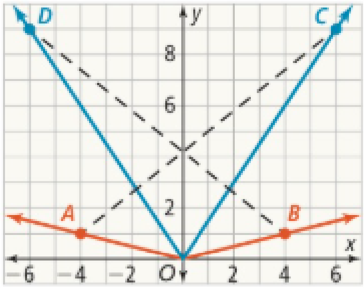
27. $3 \le t \le 6$

28. $7 \le t \le 10$

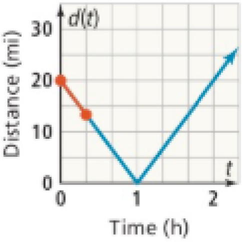
PRACTICE & PROBLEM SOLVING

APPLY

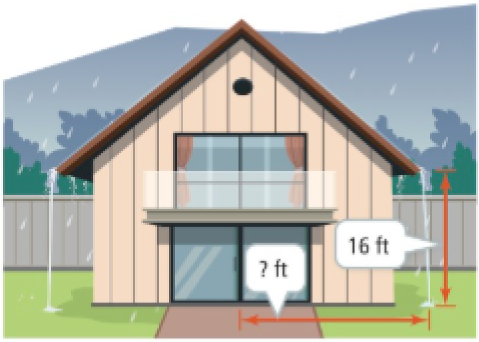
29. **Apply Math Models** A game designer is looking for two functions to model the solid lines in the figure she constructed. What functions represent the solid lines?



30. **Analyze and Persevere** The graph shows the distance between a bicyclist and a sandwich shop along her route. Estimate the rate of change over the highlighted interval. What does the rate mean in terms of the situation?



31. **Apply Math Models** The function $h(x) = -|x| + 34$ models the height of the roof of a house, where x is the horizontal distance from the center of the house. If a raindrop falls from the end of the roof, how far from the center of the base does it land? Explain your solution.



ASSESSMENT PRACTICE

32. Robert walks 4 miles along a single, straight road to his friend's house at a rate of 3 miles per hour. After 1 mile, he passes his school. Graph a function that models Robert's distance from the school over time. Label the x- and y-intercepts, and explain what they represent.

AR.4.3

33. **SAT/ACT** For what domain is the range of $y = -x$ and $y = -|x|$ the same?

Ⓐ $\{x \mid x < 0\}$

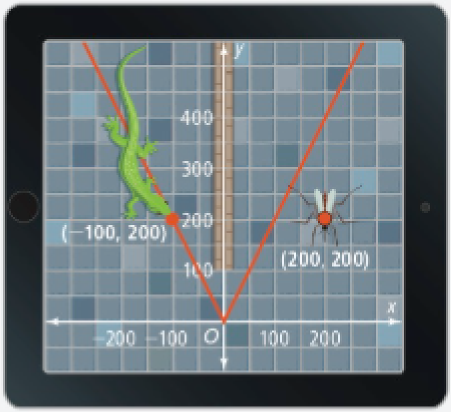
Ⓑ $\{x \mid x \leq 0\}$

Ⓒ $\{x \mid x > 0\}$

Ⓓ $\{x \mid x \geq 0\}$

Ⓔ $\{x \mid x \text{ is a real number}\}$

34. **Performance Task** The position of a lizard in a video game is modeled on a coordinate plane. The lizard follows the path shown.



Part A Write a function that includes an absolute value expression for the position of the lizard.

Part B Interpret the graph. Find the vertex and determine the intervals in which the function is increasing, decreasing; and any maximum or minimum values.

Part C Where would the function need to intersect the x-axis so that the lizard can eat the mosquito?

Part D Write a function for which the new vertex that you found in Part C is a solution to the function, and allows the lizard to eat the mosquito.

3-5

Transforming Absolute Value Functions

I CAN… graph and analyze transformations of absolute value functions.

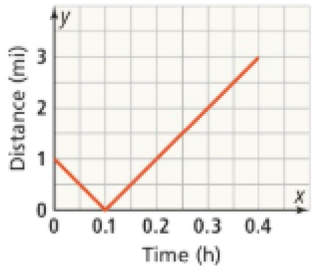
MA.912.F.2.1–Identify the effect on the graph or table of a given function after replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$ and $f(x + k)$ for specific values of k. **Also AR.4.3**

MA.K12.MTR.1.1, MTR.5.1, MTR.7.1

MODEL & DISCUSS

Cleo rides her bike at a steady 10 mi/h to Danielle's house, 4 miles away. She passes the high school, 1 mile from her house. The graph shows her distance from the high school, as a function of time.

Cleo's Distance From High School



A. The mall is 2 miles from Cleo's house, also on the way to Danielle's house. Sketch a new graph of the function that shows Cleo's distance from the mall, as a function of time, as she rides to Danielle's house.

B. Describe the domain and range of each function.

C. **Represent and Connect** How is the graph of Cleo's distance from the mall related to the graph of her distance from the high school? Explain.

ESSENTIAL QUESTION

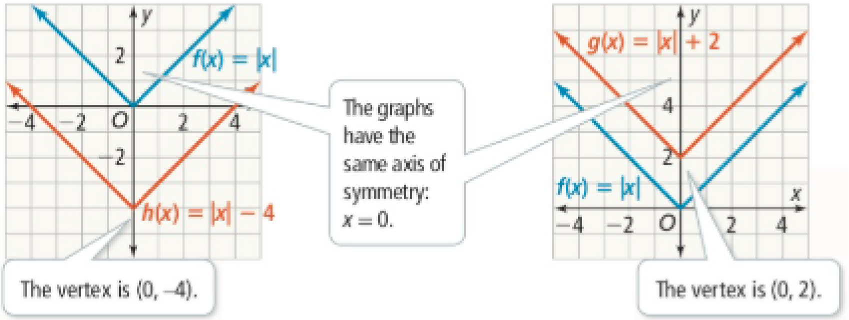
How do the constants affect the graphs of absolute value functions?

EXAMPLE 1 Vertical Translations of the Absolute Value Function

How does adding a constant to the output affect the graph of $f(x) = |x|$?

Compare the graphs of $h(x) = |x| - 4$ and $g(x) = |x| + 2$ with the graph of $f(x) = |x|$.

STUDY TIP
Recall that the vertex is the point where the axis of symmetry intersects the graph.



The graphs have the same axis of symmetry: $x = 0$.

The vertex is (0, –4).

The vertex is (0, 2).

Adding a constant, k, outside of the absolute value bars changes the value of $f(x)$, or the output. It does not change the input. The value of k, in $g(x) = |x| + k$, translates the graph of $f(x) = |x|$ vertically by k units. The axis of symmetry does not change.

Try It! 1. For each function, identify the vertex and the axis of symmetry.

a. $p(x) = |x| + 3$

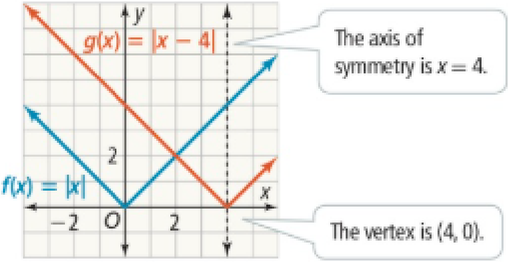
b.

x	–2	–1	0	1	2
$g(x)$	–1	–2	–3	–2	–1

EXAMPLE 2 Horizontal Translations of the Absolute Value Function

How does adding a constant to the input affect the graph of $f(x) = |x|$?

Compare the graph of $g(x) = |x - 4|$ with the graph of $f(x) = |x|$.



The axis of symmetry is $x = 4$.

The vertex is (4, 0).

COMMON ERROR

You may think that the expression $x - 4$ in the function $g(x) = |x - 4|$ shifts the graph of $f(x) = |x|$ horizontally 4 units to the left, in the *negative* direction. But the function will shift 4 units to the right because (4, 0) is a solution to the function when $h = 4$.

Adding a constant, h, inside the absolute value bars changes the value of x, the input, as well as the value of $f(x)$, the output.

The value of h, in $g(x) = |x - h|$ translates the graph of $f(x) = |x|$ horizontally by h units. If $h > 0$, the translation is to the right. If $h < 0$, the translation is to the left. Because the input is changed, the translation is horizontal, and the axis of symmetry also shifts.

Try It! 2. For each function, identify the vertex and the axis of symmetry.

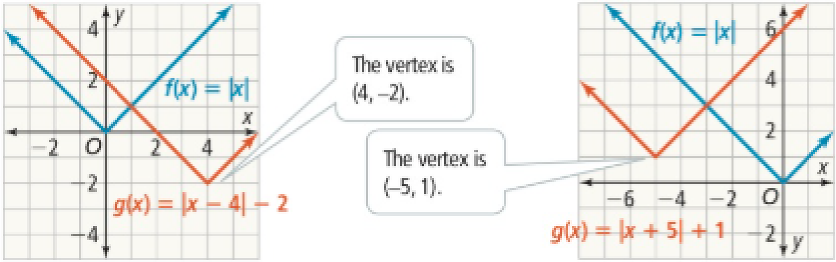
a. $g(x) = |x - 3|$ **b.** $p(x) = |x + 5|$

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Understand Vertical and Horizontal Translations

What information do the constants h and k provide about the graph of $g(x) = |x - h| + k$?

Compare the graphs of $g(x) = |x - 4| - 2$ and $g(x) = |x + 5| + 1$ with the graph of $f(x) = |x|$.



The vertex is (4, −2).

The vertex is (−5, 1).

USE PATTERNS AND STRUCTURE

Does a vertical translation of the graph of $f(x) = |x - 4|$ by −2 result in the same graph as horizontal translation of the graph of $f(x) = |x| - 2$ to the right by 4 units?

The value of h translates the graph horizontally and the value of k translates it vertically. The vertex of the graph $g(x) = |x - h| + k$ is at (h, k).

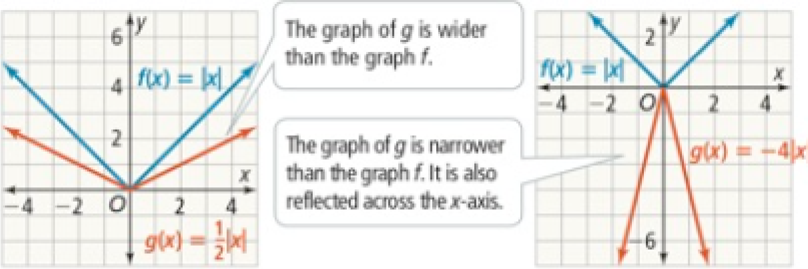
Try It! 3. Find the vertex of the graph of each function.

a. $g(x) = |x - 1| - 3$ **b.** $g(x) = |x + 2| + 6$

EXAMPLE 4 Understand Vertical Stretches and Compressions

How does the constant a affect the graph of $g(x) = a|x|$?

Compare the graphs of $g(x) = \frac{1}{2}|x|$ and $g(x) = -4|x|$ with the graph of $f(x) = |x|$.



In $g(x) = a|x|$, the constant a multiplies the output of the function $f(x) = |x|$ by a.

- When $0 < |a| < 1$ the graph of $g(x) = a|x|$ is a vertical compression towards the x-axis of the graph of $f(x) = |x|$.
- When $|a| > 1$, the graph of $g(x) = a|x|$ is a vertical stretch away from the x-axis of the graph of $f(x) = |x|$.
- When $a < 0$, the graph of g is reflected across the x-axis.

The value of a stretches or compresses the graph vertically.

Try It! 4. Compare the graph of each function with the graph of $f(x) = |x|$.

a. $g(x) = 3|x|$

b. $g(x) = -\frac{1}{3}|x|$

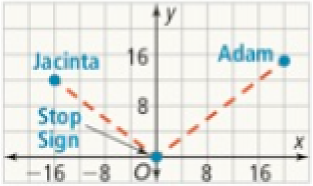
APPLICATION

EXAMPLE 5 Apply Transformations of the Absolute Value Function

Jacinta wants to signal Adam by reflecting a light off of the stop sign at the end of their street. The path of light beam can be modeled by an absolute value function, with the point of reflection being the vertex. Their positions are $J(-16, 12)$ and $A(20, 15)$. The stop sign is located at the origin.

REPRESENT AND CONNECT
The graph of the function passes through the points (–16, 12) and (16, 12). How does this help you determine that it is vertical compression of $f(x) = |x|$?

A. **Graph the function that represents the situation. How is the function related to $f(x) = |x|$?**



The graph of the function is a vertical compression of the graph of $f(x) = |x|$.

B. **Describe the domain, range, and symmetry of the function.**

The domain of the function is $-16 \leq x \leq 20$ and the range is $0 \leq y \leq 15$.

The graph is symmetric for the values $-16 \leq x \leq 16$, about the axis of symmetry $x = 0$.

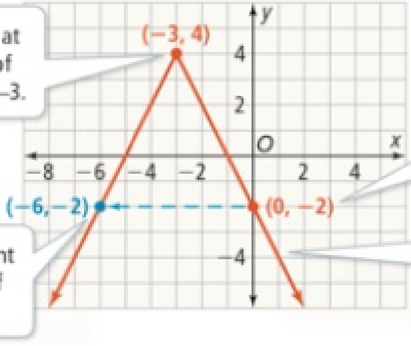
Try It! 5. If Adam's position was (16, 10) would Jacinta be able to signal him using this method? Explain.

EXAMPLE 6 Understand Transformations of the Absolute Value Function

A. How do the constants a, h, and k affect the graph of $g(x) = a|x - h| + k$?

Graph $g(x) = -2|x + 3| + 4$.

The values of h and k determine the location of the vertex and the axis of symmetry. The value of a determines the direction of the graph and whether it is a vertical stretch or compression of the graph of $f(x) = |x|$.



1 Plot the vertex at (−3, 4). The axis of symmetry is $x = -3$.

2 Evaluate the function to find another point. $g(0) = -2$.

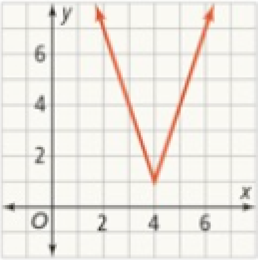
3 Reflect the point across the axis of symmetry.

4 Sketch the graph through the points.

GENERALIZE
Does the graph of every function of the form $g(x) = a|x - h| + k$ have a y-intercept?

Since $|a| > 1$ and a is negative the graph is a vertical stretch of the graph of $f(x) = |x|$ that is reflected across the x-axis.

B. How can you use the constants a, h, and k to write a function given its graph?



Step 1 Identify the vertex of the graph.

The vertex is (4, 1), so $h = 4$ and $k = 1$.

The function has the form $f(x) = a|x - 4| + 1$.

Step 2 Find the value of a. Select another point on the graph, $(x, f(x))$, and solve for a.

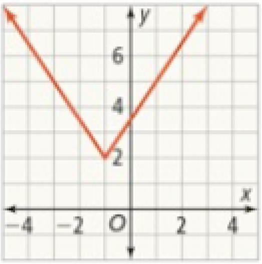
$$f(x) = a|x - 4| + 1$$
$$4 = a|5 - 4| + 1$$
$$a = 3$$

Substitute 5 for x and 4 for $f(x)$.

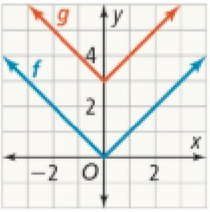
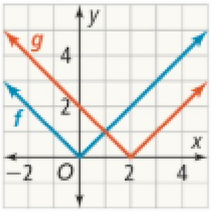
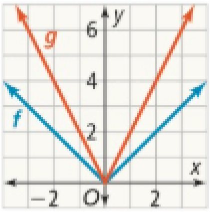
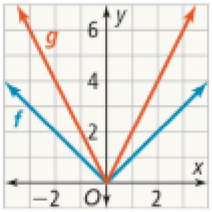
The graph represents the function $f(x) = 3|x - 4| + 1$.

Try It! 6. a. Write a function for the graph shown.

b. Write the function of the graph after a translation 1 unit right and 4 units up.



CONCEPT SUMMARY Transformations of the Absolute Value Function

	Translations		Stretches and Compressions	
WORDS	Translations shift each point of the graph the same distance horizontally or vertically.		Stretches and compressions scale the graph either vertically or horizontally.	
ALGEBRA	**Vertical by *k* units:** The graph of $g(x) = \lvert x \rvert + k$ is vertical translation of $f(x) = \lvert x \rvert$.	**Horizontal by *h* units:** The graph of $g(x) = \lvert x - h \rvert$ is a horizontal translation of $f(x) = \lvert x \rvert$.	**Vertical by scale factor *a*:** The graph of $g(x) = a\lvert x \rvert$ is a stretch of $f(x) = \lvert x \rvert$ when $\lvert a \rvert > 1$ and a compression of $f(x) = \lvert x \rvert$ when $0 < \lvert a \rvert < 1$.	**Horizontal by scale factor *a*:** The graph of $g(x) = \lvert ax \rvert$ is a compression of $f(x) = \lvert x \rvert$ when $\lvert a \rvert > 1$ and a stretch of $f(x) = \lvert x \rvert$ when $0 < \lvert a \rvert < 1$.
GRAPHS				

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do the constants affect the graphs of absolute value functions?

2. **Generalize** How does k affect the domain and range of $g(x) = |x| + k$?

3. **Error Analysis** Jacy says that $f(x) = 4|x - 1|$ and $f(x) = |4x - 1|$ have the same graph. Is Jacy correct? Explain.

4. **Use Patterns and Structure** How does the graph of $g(x) = a|x|$ compare with the graph of $h(x) = |ax|$ if $a > 0$? If $a < 0$? Explain.

Do You KNOW HOW?

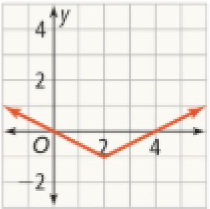
Find the vertex and graph each function.

5. $f(x) = |x| + 2.5$
6. $f(x) = |x + 2.5|$
7. $f(x) = |x - 2| + 4$
8. $f(x) = -3|x + 1| - 5$

9. Identify the vertex and the axis of symmetry of the function.

x	−5	−4	−3	−2	−1
g(x)	1	0	1	2	3

10. Write a function for the graph.



PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. **Analyze and Persevere** Give two examples of functions that include an absolute value expression and have a vertex of (−1, 3).

12. **Represent and Connect** Consider the functions and $f(x) = |x| - 2$ and $g(x) = f(2x)$.

    a. Copy and complete the table

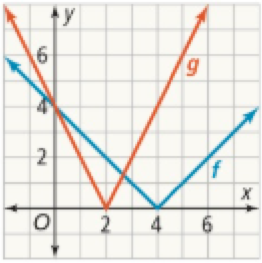
x	$f(x) = \|x\| - 2$	$g(x) = 2f(x)$
−2	0	■
−1	−1	■
0	−2	■
1	−1	■
2	0	■

   b. Describe the graph of g as a transformation of the graph of f. How does each point on the graph of f map to the corresponding point on g?

13. **Error Analysis** Describe and correct the errors a student made in describing the graph of the function $f(x) = -0.5|x + 1| + 3$.

> The graph of $y = -0.5|x + 1| + 3$ compresses the graph of $y = |x|$ vertically toward the x-axis, and moves the vertex to (1, 3). ✗

14. **Use Patterns and Structure** The graph of $g(x) = f(2x)$ is a transformation of the graph of $f(x) = |x - 4|$.



   a. The points (0, 4), (2, 2), and (4, 0) lie on the graph of function f. What are the corresponding points on the graph of function g?

   b. How does each point on the graph of f map to the corresponding point on g? Use the y-axis as a reference.

   c. Describe the graph of g as a transformation of the graph of f.

PRACTICE

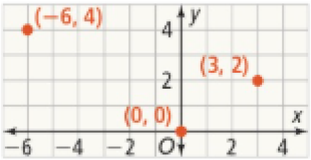
Find the vertex and graph each function. SEE EXAMPLES 1, 2, AND 3

15. $f(x) = |x| - 2$
16. $f(x) = |x| + 1$
17. $f(x) = |x + 0.5|$
18. $f(x) = |x - 1|$
19. $f(x) = |x + 7| - 2$
20. $f(x) = |x - 0.5| + 0.5$

Compare the graph of each function with the graph of $f(x) = |x|$. Describe the transformation, then graph the function. SEE EXAMPLES 4 AND 6

21. $g(x) = \frac{1}{5}|x|$
22. $g(x) = -3|x|$
23. $g(x) = |-3x|$
24. $g(x) = |3x|$
25. $g(x) = \frac{1}{3}|x + 6| - 1$
26. $g(x) = -4|x - 2| - 1$

27. The graph of an absolute function passes through the points shown below.
SEE EXAMPLES 4 AND 5



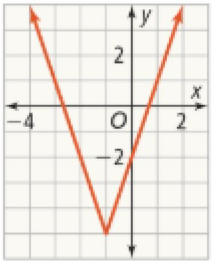
   a. Describe the function as a transformation of $f(x) = |x|$.

   b. Write a rule for the function.

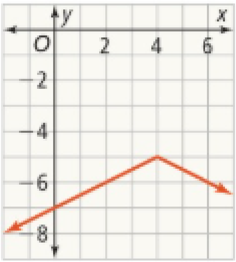
   c. Does the graph of the function pass through the point $(-1, \frac{2}{3})$?

Write a function for each graph. SEE EXAMPLE 6

28.



29.



What function g describes the graph of f after the given transformations?

30. $f(x) = |x| + 1$; translated 2 units left

31. $f(x) = |x + 3|$; translated down 6 units

32. $f(x) = 4|x|$; translated 5 units right

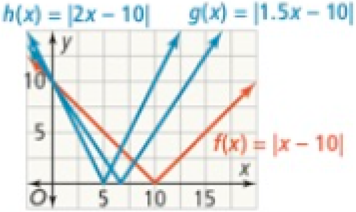
PRACTICE & PROBLEM SOLVING

APPLY

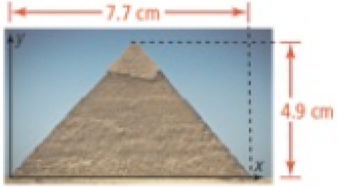
33. **Apply Math Models** Carolina is studying the reflection of light. She models the path of a light beam passing through a point at (0, 10) reflecting off of a mirror. The function $f(x) = |x - 10|$ models light hitting the mirror at a 45° angle. The functions $g(x) = |1.5x - 10|$ and $h(x) = |2x - 10|$ model the light beam hitting the mirror at steeper angles.

    a. Describe the graphs of g and h as transformations of the graph of f.

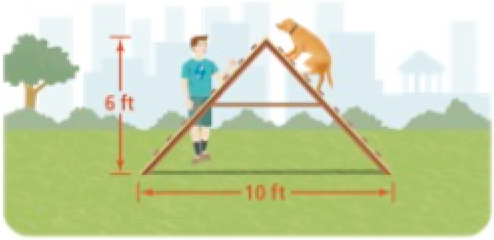
    b. Compare the domains, ranges, and intercepts of functions g and h with the domain, range and intercepts of f.



34. **Apply Math Models** Emma wants to model the sides of a pyramid by using a function that includes an absolute value expression. Emma will place the pyramid on a coordinate grid as shown. What function should she use? For what domain?



35. **Analyze and Persevere** One part of a dog agility course is an obstacle called an A-frame. Assume that the left corner of the A-frame corresponds to the point (0, 0). What function that includes an absolute value expression could you use to model the obstacle? What is the domain of the function? Explain your reasoning.



ASSESSMENT PRACTICE

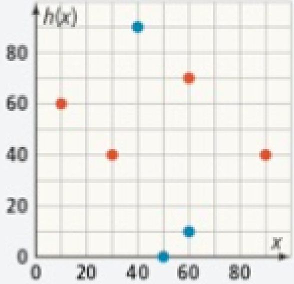
36. The function g is the transformation of $f(x) = |x|$ given by $g(x) = -f(x + 5)$. Copy and complete the table below. F.2.1

x	$f(x)$	$g(x)$
−15	____	____
−11	____	____
3	____	____

37. **SAT/ACT** Which the graph has the same axis of symmetry as the graph of the $f(x) = |x - 4|$ translated down 2 units?

    Ⓐ $f(x) = |x + 4|$ translated up 2 units
    Ⓑ $f(x) = |x + 2|$ translated 2 units right
    Ⓒ $f(x) = |x - 3|$ translated up 1 unit
    Ⓓ $f(x) = |x - 7|$ translated 3 units left
    Ⓔ none of these

38. **Performance Task** You are playing a ship trapping game. There are 4 of your opponent's red ships on the screen. You can send out 3 strikes from your blue ships through the red ships' positions to capture them. Each strike sends two lasers that resemble the graph of a function with an absolute value expression.



Part A How can symmetry help you find a path to capture two ships?

Part B Write three functions that represent strike paths to capture the ships. Show how each ship is captured by a function.

Part C For your function that captures two ships, can you write a different function from one of your other ships that represent strikes paths to capture these two ships? Explain.

TOPIC

3

Topic Review

TOPIC ESSENTIAL QUESTION

1. How can linear and absolute value functions be used to model situations and solve problems?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A _________ domain does not include all the real numbers in an interval.

3. The _________ intersects the vertex and divides the graph into two congruent halves that are images of each other under a reflection.

4. A _________ maps each point of a function to a new location.

5. The graph of a _________ is a straight line or a set of values that lie on a straight line.

6. A _________ domain includes an interval of real numbers.

- linear function
- transformation
- *y*-intercept
- continuous
- axis of symmetry
- discrete

Concepts & Skills Review

LESSON 3-1 Domain and Range of Functions

Quick Review

The domain of a function can be described using inequalities when it includes all real numbers over a given interval. Set-builder notation is useful in describing domains the require more conditions, such as sets of integers. It can include inequalities and verbal descriptions.

Example

Use set-builder notation to describe the domain and range of the function.

x	0	1	2	3	4
y	5	10	15	20	25

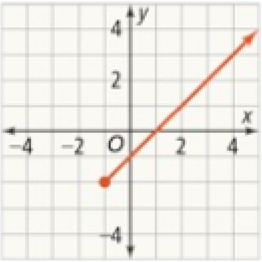
The domain is $\{x \mid x$ is integer and $1 \le x \le 4\}$. The range is $\{y \mid y$ is a multiple of 5 and $5 \le y \le 25\}$.

Practice & Problem Solving

Identify the domain and range of each function.

7. {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5)}

8.



Communicate and Justify What constraints, if any, are there on the domain? Explain.

9. An airplane ascends to a cruising altitude at the rate of 1,000 ft/min for *m* minutes.

10. The value of an automobile in *d* dollars decreases by about 10% each year.

LESSON 3-2 Linear Functions

Quick Review

A **linear function** is a function whose graph is a straight line. It represents a linear relationship between two variables. A linear function written in **function notation** is $f(x) = mx + b$ and $f(x)$ is read "*f* of *x*."

Example

A taxi company charges \$3.50 plus \$0.85 per mile. What linear function can be used to determine the cost of a taxi ride of *x* miles? How much would a 3.5-mile taxi ride cost?

Let d = distance of the taxi ride.

Cost of taxi ride = cost × distance + fee

$$f(d) = 0.85d + 3.5$$

Use the function to determine the cost of a 3.5-mile ride.

$$f(3.5) = (0.85)(3.5) + 3.5$$
$$= 6.475$$

The cost of a 3.5-mile taxi ride is \$6.48.

Practice and Problem Solving

11. Use the simple interest formula $A(t) = P(1 + rt)$. Evaluate the function for $t = 7$ to determine how much money Oliva will have if she invests \$850, which earns 4% annual simple interest for 7 years.

12. **Analyze and Persevere** Melissa runs a graphic design business. She charges by the page, and has a setup fee. The table shows her earnings for the last few projects. What is her per-page rate, and what is her setup fee?

Cost (\$)	185	335	485	635
Page totals	2	4	6	8

13. **Use Patterns and Structure** Tia's Computer Repair Shop charges the labor rates shown for computer repairs. What linear function can she use to determine the cost of a repair that takes 5.5 hours and includes \$180 in parts?

Hours	1	1.5	2	2.5
Labor (\$)	85	127.5	170	212.5

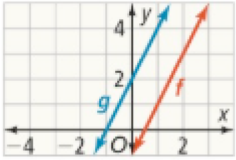
LESSON 3-3 Transforming Linear Functions

Quick Review

A **transformation** of a function *f* maps each point of its graph to a distinct location. A **translation** shifts each point of the graph of a function the same distance horizontally, vertically, or both. **Stretches** and **compressions** scale each point of a graph either horizontally or vertically.

Example

Let $f(x) = 2x - 1$. If $g(x) = (2x - 1) + 3$, how does the graph of *g* compare to the graph of *f*?



The graph of *g* is the translation of the graph of *f* three units up.

Practice & Problem Solving

Given the function $f(x) = x$, how does the addition or subtraction of a constant to the output affect the graph?

14. $f(x) = x - 2$

15. $f(x) = x + 5$

Given $f(x) = 4x - 5$, describe how the graph of *g* compares with the graph of *f*.

16. $g(x) = 4(x - 3) - 5$

17. $g(x) = 2(4x - 5)$

18. **Generalize** Given $f(x) = -3x + 9$, how does multiplying the output of *f* by 2 affect the slope and *y*-intercept of the graph?

19. **Apply Math Models** A hotel business center charges \$40 per hour to rent a computer plus a \$65 security deposit. The total rental charge is represented by $f(x) = 40x + 65$. How would the equation change if the business center increased the security deposit by \$15?

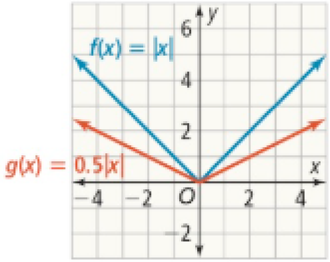
LESSON 3-4 Absolute Value Functions

Quick Review

The graph of the absolute value function, $f(x) = |x|$ has a **vertex** at (0, 0) and an **axis of symmetry** $x = 0$.

Example

How do the domain and range of $g(x) = 0.5|x|$ compare with the domain and range of $f(x) = |x|$?



The domain of f and the domain of g is $\{x \mid x \text{ is a real number}\}$. The range of both functions is $\{y \mid y \geq 0\}$.

Practice & Problem Solving

Tell whether each point is on the graph of $f(x) = |x|$. If it is, give the coordinates of another point with the same y-coordinate.

20. (8, 8) **21.** (−5, 5) **22.** (−3.5, −3.5)

Graph each function. What is the domain and range of each function?

23. $f(x) = -2.5|x|$

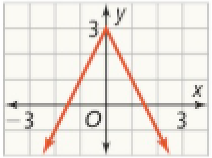
24. $g(x) = \frac{1}{3}|x|$

For each function, find the vertex and tell whether it represents a maximum or minimum value of the function.

25. $g(x) = -6.3|x|$

26. $g(x) = 7|x|$

27. **Represent and Connect** Find the domain, range, and vertex of the graphed function.



LESSON 3-5 Transforming Absolute Value Functions

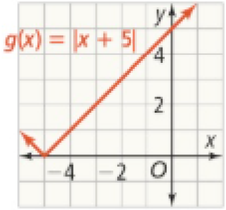
Quick Review

The graph of $g(x) = |x| + k$ is vertical translation of the graph of $f(x) = |x|$, when $k \neq 0$. The graph of $g(x) = |x - h|$ is a horizontal translation of the graph of f when $h \neq 0$.

The graph of $g(x) = a|x|$ is a vertical stretch of the graph of f when $|a| > 1$ and a compression of the graph of f when $0 < |a| < 1$.

Example

For the function $g(x) = |x + 5|$, graph the function. Describe the graph of g as a transformation of the graph of $f(x) = |x|$.



The value of h in $g(x) = |x - h|$ translates the graph of $f(x) = |x|$ horizontally.

$g(x) = |x + 5|$

$g(x) = |x - (-5)|$

So $h = -5$. The graph of g is the graph of f translated 5 units to the left.

Practice & Problem Solving

Find the vertex and graph each function.

28. $g(x) = |x| + 4$

29. $g(x) = |x - 2|$

30. $g(x) = |x + 1| - 2$

31. $g(x) = |x - 3| + 1$

Compare each function with $f(x) = |x|$. Describe the graph of g as transformation of the graph of f.

32. $g(x) = |-6x|$

33. $g(x) = |6x|$

34. $g(x) = 2|x + 6| - 1$

35. $g(x) = -|x - 2| - 1$

What function g describes the graph of f after the given transformations?

36. $f(x) = |x| + 3$; translated 4 units right

37. $f(x) = |x - 1|$; translated 2 units down

38. **Analyze and Persevere** A traffic cone is 18 in. tall and 12 in. wide. You want to sketch an image of the traffic cone on a coordinate grid with one edge at (0, 0). What function that includes an absolute value expression could represent the traffic cone? What would be the domain of the function? Explain.

TOPIC

4

Systems of Linear Equations and Inequalities

TOPIC ESSENTIAL QUESTION

How do you use systems of linear equations and inequalities to model situations and solve problems?

Topic Overview

enVision® STEM Project:
Growing Grain

4-1 Solving Systems of Equations by Graphing
AR.9.1, MTR.5.1, MTR.6.1, MTR.7.1

4-2 Solving Systems of Equations by Substitution
AR.9.1, MTR.3.1, MTR.6.1, MTR.7.1

4-3 Solving Systems of Equations by Elimination
AR.9.1, AR.9.6, MTR.2.1, MTR.4.1, MTR.5.1

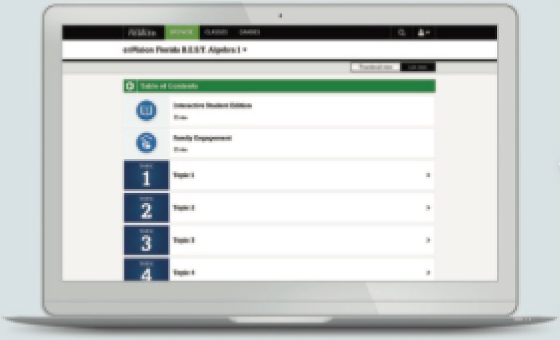
4-4 Linear Inequalities in Two Variables
AR.2.7, AR.2.8, MTR.1.1, MTR.5.1, MTR.7.1

Mathematical Modeling in 3 Acts:
Get Up There!
AR.9.1, MTR.7.1

4-5 Systems of Linear Inequalities
AR.9.4, AR.9.6, MTR.1.1, MTR.2.1, MTR.6.1

Topic Vocabulary

- linear inequality in two variables
- solution of an inequality in two variables
- solution of a system of linear inequalities
- system of linear inequalities



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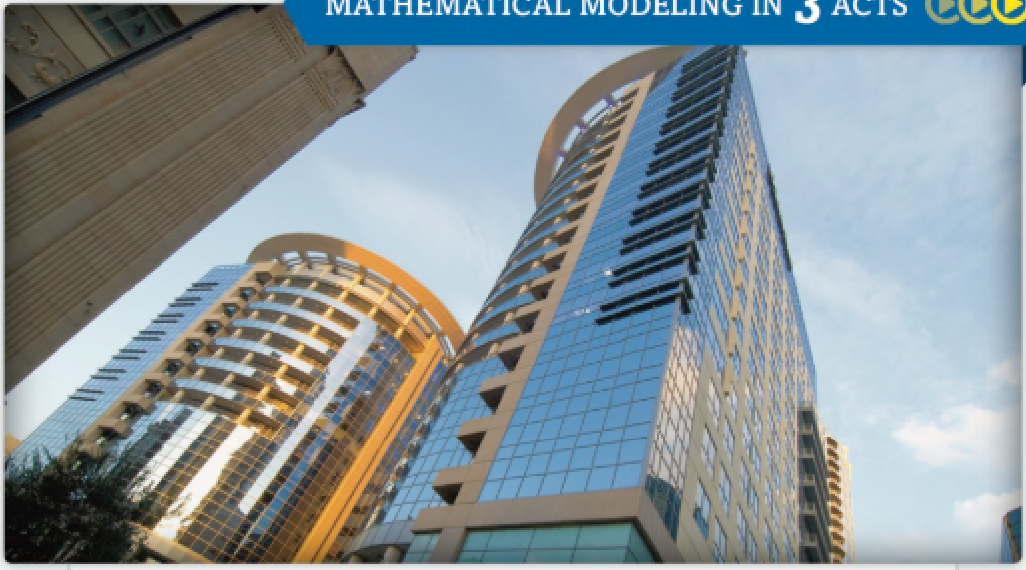
ACTIVITIES Complete ***Explore & Reason***, ***Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its***.

ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Get Up There!

Have you ever been to the top of a skyscraper? If so, you probably didn't take the stairs. You probably took an elevator. How long did it take you to get to the top? Did you take an express elevator?

Express elevators travel more quickly because they do not stop at every floor. How much more quickly can you get to the top in an express elevator? Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision® *STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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Did You Know?

American farmers produce enough meat and grain for the United States plus extra to export to other countries. Farms use about **53.5 billion gallons of groundwater** each day for irrigation.

Peak Water Demands

Corn	July
Wheat	May and June
Soybeans	August

Corn is **easier to grow than wheat** and hardier in northern climates.

In 2012, soybeans became the biggest crop in the United States, with four times the acreage as in 1992. **More soybeans are grown in the United States** than in any other country.

1 million individual kernels of wheat

42 pounds of white flour

60 pounds of whole-wheat flour

42 one and a half-pound loaves of white bread

Your Task: Growing Grain

You and your classmates will make decisions about growing crops on a farm. How much of each crop will you plant, and why?

4-1

Solving Systems of Equations by Graphing

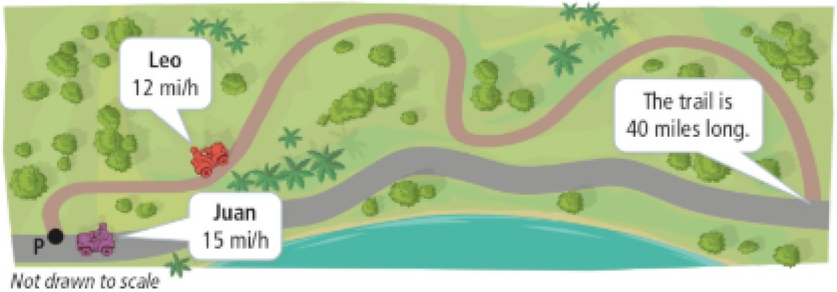
I CAN… use graphs to find approximate solutions to systems of equations.

MA.912.AR.9.1–Given a mathematical or real-world context, write and solve a system of two-variable linear equations algebraically or graphically.

MA.K12.MTR.5.1, MTR.6.1, MTR.7.1

EXPLORE & REASON

Juan and Leo were supposed to meet and drive ATVs on a trail together. Juan is late so Leo started without him.



Not drawn to scale

A. Write an equation for Leo's distance from the starting point after riding for x hours. Write an equation for Juan's distance from the starting point if he starts h hours after Leo.

B. **Apply Math Models** Suppose $h = 1$. How can you use graphs of the two equations to determine who finishes the trail first?

C. How much of a head start must Leo have to finish the trail at the same time as Juan?

ESSENTIAL QUESTION How can you use a graph to illustrate the solution to a system of linear equations?

CONCEPTUAL UNDERSTANDING

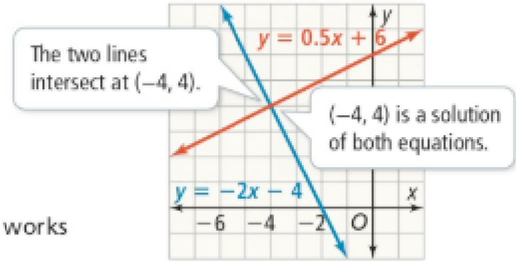
EXAMPLE 1 Solve a System of Equations by Graphing

What is the solution of the system of equations? $y = -2x - 4$
$y = 0.5x + 6$

Use a graph to solve this system of equations.

Step 1 Graph both equations.

Step 2 Find the point of intersection. Since the point of intersection lies on both lines, it is a solution to both equations.



Step 3 Check that the solution works for both equations.

$y = -2x - 4$	$y = 0.5x + 6$
$4 \overset{?}{=} -2(-4) - 4$	$4 \overset{?}{=} 0.5(-4) + 6$
$4 \overset{?}{=} 8 - 4$	$4 \overset{?}{=} -2 + 6$
$4 = 4$ ✓	$4 = 4$ ✓

Since there are no other points of intersection the system of equations has exactly one solution, (–4, 4).

STUDY TIP
Substitute the x-coordinate for x and the y-coordinate for y.

Try It! **1.** Use a graph to solve each system of equations.

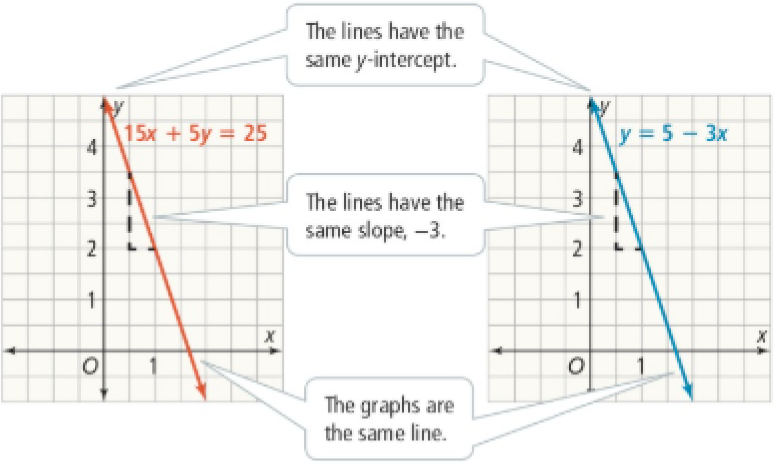
a. $y = \frac{1}{2}x - 2$
$y = 3x - 7$

b. $y = 2x + 10$
$y = -\frac{1}{4}x + 1$

EXAMPLE 2 Graph Systems of Equations With Infinitely Many Solutions or No Solution

What is the solution of each system of equations? Use a graph to explain your answer.

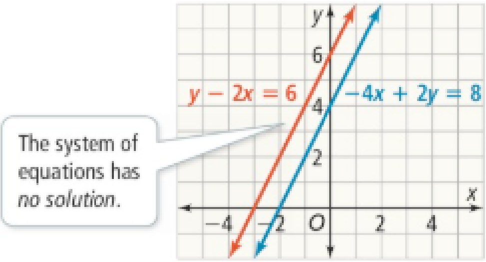
A. $15x + 5y = 25$
$y = 5 - 3x$



COMMON ERROR
You may think that the lines cannot be the same since the equations are different. However, if you write both equations in the same form and divide out the common factor, the equations will be identical.

All ordered pairs on the line are solutions of both equations, so all points on the line are solutions to the system of equations. There are infinitely many solutions to the system.

B. $y - 2x = 6$
$-4x + 2y = 8$



The lines are parallel, so there is no point of intersection. Therefore, there is no solution to this system of equations.

Try It! **2.** Use a graph to solve each system of equations.

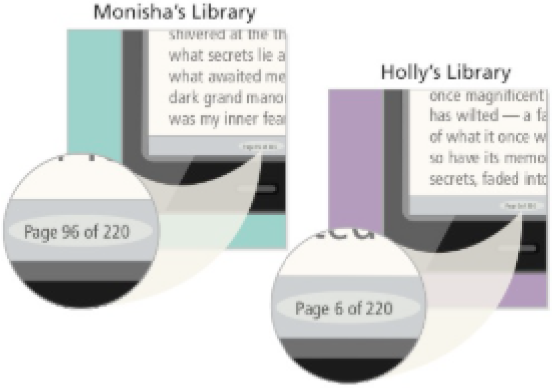
a. $y = \frac{1}{2}x + 7$
$4x - 8y = 12$

b. $3x + 2y = 9$
$\frac{2}{3}y = 3 - x$

APPLICATION

EXAMPLE 3 Write a System of Equations

Monisha and Holly have 14 more days to finish reading the same novel for class. Monisha plans to read 9 pages each day, while Holly plans to read 20 pages each day. Assuming Holly and Monisha both maintain their reading plan, when will Holly catch up with Monisha? Who will finish reading the novel first?



Formulate ◀ Write a system of equations to represent Holly's and Monisha's reading paces.

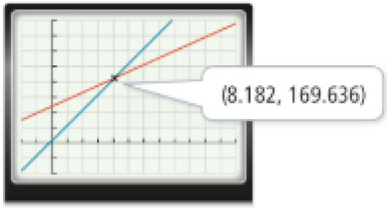
Monisha: Total pages = 96 pages already read + 9 pages/day • *x* days

$y = 96 + 9x$

Holly: Total pages = 6 pages already read + 20 pages/day • *x* days

$y = 6 + 20x$

Compute ◀ Graph the system of equations. Find the point where the graphs intersect.



Interpret ◀ After a little more than 8 days of reading, Monisha will have read $96 + 9(8.182) \approx 170$ pages.

Holly will have read $6 + 20(8.182) \approx 170$ pages. So, Holly will catch up with Monisha in a little over 8 days.

Since the book is 220 pages long, and Holly is reading at a faster rate, she will finish reading the novel before Monisha.

Try It! **3.** Suppose Monisha reads 10 pages each day instead.

a. How will that change the length of time it takes for Holly to catch up with Monisha?

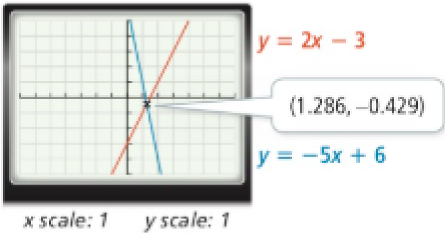
b. Will Holly still finish the novel first? Explain.

EXAMPLE 4 Solve a System of Equations Approximately

What is the solution of the system of equations? $y = 2x - 3$

$y = -5x + 6$

Step 1 Use a graphing utility to graph both equations. Find the point of intersection.



Step 2 Check the values of x and y in each equation to see if they satisfy both equations.

$$y = 2x - 3$$
$$-0.429 \stackrel{?}{=} 2(1.286) - 3$$
$$-0.429 \stackrel{?}{=} 2.572 - 3$$
$$-0.429 \neq -0.428$$

The solutions obtained by graphing are close to, but not equal to, the actual solutions.

$$y = -5x + 6$$
$$-0.429 \stackrel{?}{=} -5(1.286) + 6$$
$$-0.429 \stackrel{?}{=} -6.43 + 6$$
$$-0.429 \neq -0.43$$

CHECK FOR REASONABLENESS
What would happen if you substituted the exact solution into the system of equations?

The solution obtained by graphing, (1.286, –0.429) is correct to three decimal places, but it is not an exact solution. What is the exact solution? Consider which x-value gives you the same y-value in each equation.

Set the expressions for y equal to each other and solve for x.

$$2x - 3 = -5x + 6$$
$$2x + 5x = 6 + 3$$
$$7x = 9$$
$$x = \frac{9}{7}$$
$$= 1.\overline{285714}$$

The y-values in both equations are equal when $x = \frac{9}{7}$.

Now substitute for x in either equation to find y.

$$y = 2x - 3$$
$$= 2\left(\frac{9}{7}\right) - 3$$
$$= \frac{18}{7} - 3$$
$$= \frac{-3}{7}$$
$$= -0.\overline{428571}$$

$$y = -5x + 6$$
$$= -5\left(\frac{9}{7}\right) + 6$$
$$= \frac{-45}{7} + 6$$
$$= \frac{-3}{7}$$
$$= -0.\overline{428571}$$

The exact solution is $\left(\frac{9}{7}, -\frac{3}{7}\right)$.

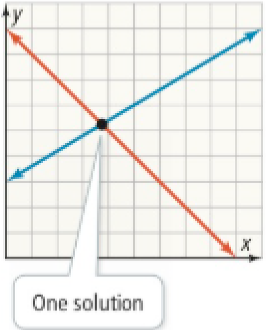
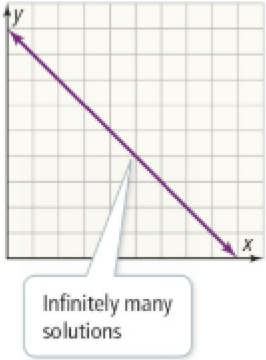
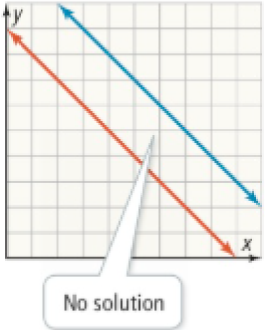
You will see more methods for finding exact solutions in later lessons.

Try It! **4.** What solution do you obtain for the system of equations by graphing? What is the exact solution?

$y = 5x - 4$

$y = -6x + 14$

CONCEPT SUMMARY Graphing to Solve Systems of Equations

WORDS	A system of linear equations may have one solution . . .	. . . infinitely many solutions . . .	. . . or no solution.
ALGEBRA	$y = -x + 9$ $y = \frac{3}{5}x + 3$	$y = -x + 9$ $2y = -2x + 18$	$y = -x + 9$ $y = -x + 12$
GRAPHS			

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you use a graph to illustrate the solution to a system of linear equations?

2. **Represent and Connect** How does the graph of a system of equations with one solution differ from the graph of a system of equations with infinitely many solutions or no solution?

3. **Communicate and Justify** How does the point of intersection of the two graphs represent the solution to a system of linear equations?

4. **Error Analysis** Reese states that the system of equations has no solution because the slopes are the same. Describe Reese's error.

$$y = -3x - 1$$
$$3x + y = -1$$

Do You KNOW HOW?

Solve each system of equations by graphing.

5. $y = 2x + 5$
$y = -\frac{1}{2}x$

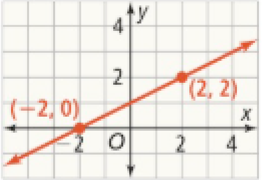
6. $y = -\frac{2}{3}x + 2$
$2x + 3y = 6$

7. Juanita is painting her house. She can either buy Brand A paint and a paint roller tray or Brand B paint and a grid for the paint roller. For how many gallons of paint would the price for both options be the same? If Juanita needs 15 gallons of paint, which is the better option for the money?

PRACTICE & PROBLEM SOLVING

UNDERSTAND

8. **Use Patterns and Structure** Describe the solution set for the system of equations that includes the equation of the line shown and each equation below.



a. $y = \frac{1}{2}x - 3$

b. $2x + y = 6$

c. $x - 2y = -2$

9. **Analyze and Persevere** Write an equation in slope-intercept form that would have infinitely many solutions in a system of equations with $5x - 2y = 8$.

10. **Generalize** Copy and complete the table by writing the word *same* or *different* to show how the slope and *y*-intercept of each equation relate to the number of solutions in a system of two linear equations.

Number of solutions	Slopes	*y*-intercepts
One solution	■	■
Infinitely many solutions	■	■
No solution	■	■

11. **Error Analysis** Describe and correct the error a student made in finding the solution of the system of equations.

$y + 3x = 9$
$y = 3x + 9$

There are an infinite number of solutions since the coefficients of the variables and the constants are the same. ✗

12. **Higher Order Thinking** The solution of a system of equations is (3, 2). One of the equations in the system is $2x + 3y = 12$. Write an equation in slope-intercept form that could be the second equation in the system.

PRACTICE

Solve each system of equations by graphing. SEE EXAMPLE 1

13. $y = -2x - 2$
$y = 3x - 7$

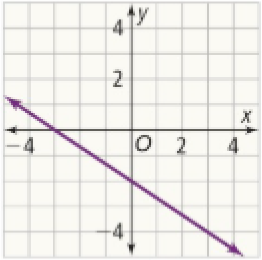
14. $y = x$
$y = 2x$

15. $x + y = -5$
$y = \frac{1}{2}x - 2$

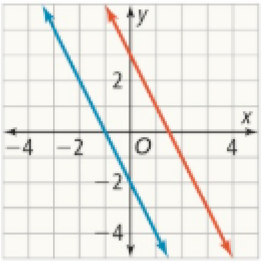
16. $3x + 2y = -3$
$2x - 3y = -15$

Determine whether each system of equations shown in the graph has *no solution* or *infinitely many solutions*. SEE EXAMPLE 2

17.



18.



Write and solve a system of equations for the given situation. SEE EXAMPLE 3

19. Roshaun has saved \$150 and continues to add \$10 each week. Keegan starts with \$0 and saves \$25 each week.

a. In how many weeks will they have the same amount of money?

b. What amount of money will they each have saved?

Solve each system of equations by graphing. Round your answers to the thousandths, if necessary. SEE EXAMPLE 4

20. $y = 5x + 1$
$y = 2x + 6$

21. $y = -6x + 5$
$y = 4x + 3$

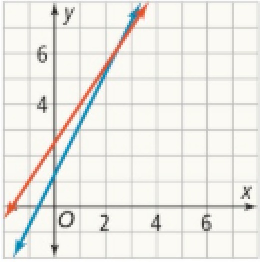
22. $y = 9x + 2$
$y = -3x - 4$

23. $y = \frac{1}{3}x + 9$
$y = -\frac{3}{4}x + 4$

PRACTICE & PROBLEM SOLVING

APPLY

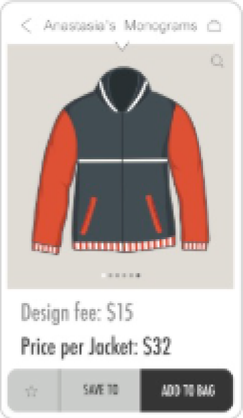
24. Use the graph to determine the solution for the system of equations.



a. **Check for Reasonableness** How does the graph show that the solution of the system of equations has an x-value between 2 and 3?

b. What is the approximate solution of the system of equations?

25. **Apply Math Models** Gabriela considers buying fleece jackets from Anastasia's Monograms or Monograms Unlimited. Anastasia's charges a one time design fee and a price per jacket. Monograms Unlimited only charges a price per jacket.



Design fee: $15

Price per Jacket: $32

Price per Jacket: $35

Design fee: $0

a. Write and solve a system of equations to represent the cost for a jacket from each company.

b. What does the solution mean?

c. Gabriela needs to buy 10 jackets. Which company should she choose? How does the graph help her decide? Explain.

26. **Generalize** How do you know when the solution to a system of equations is a precise answer and when it is an approximate answer?

ASSESSMENT PRACTICE

27. Consider the system of equations. AR.9.1

$y = \frac{3}{4}x + 2$

$3x + 4y = 8$

How many lines does the graph of this system have? What is the solution of the system?

28. **SAT/ACT** Select which is the solution of the system of equations.

$y = -3x - 3$

$y = -0.5x + 2$

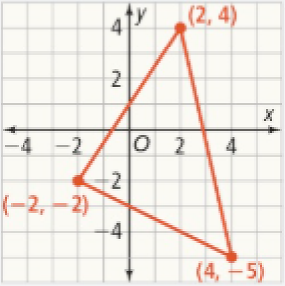
Ⓐ (0, 2)

Ⓑ (−1, 0)

Ⓒ (−1, 2)

Ⓓ (−2, 3)

29. **Performance Task** The lines that form the three sides of the triangle can be grouped into three different systems of two linear equations.



Part A Describe the system of equations that has each solution.

a. (2, 4)

b. (−2, −2)

c. (4, −5)

Part B Replace the solution (4, −5) to make an acute triangle. What are the coordinates of the new triangle?

Part C Describe the system of equations that will produce each of the new coordinates.

4-2

Solving Systems of Equations by Substitution

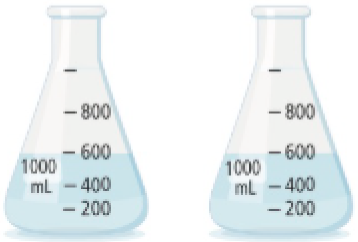
I CAN… solve a system of equations using the substitution method.

MA.912.AR.9.1–Given a mathematical or real-world context, write and solve a system of two-variable linear equations algebraically or graphically.

MA.K12.MTR.3.1, MTR.6.1, MTR.7.1

MODEL & DISCUSS

Rochelle is conducting an experiment on cells of Elodea, a kind of water plant. To induce plasmolysis at the correct rate, she needs to use an 8% saline solution, but she has only the solutions shown.



Solution A
10% saline

Solution B
5% saline

A. If Rochelle mixes the two solutions to get 1,000 mL of an 8% saline solution, which will she use more of? Explain.

B. How can Rochelle determine the amount of each solution she needs to make the 8% saline solution?

C. **Choose Efficient Methods** Are there any methods for solving this problem other than the one you used in part B? Explain.

ESSENTIAL QUESTION **How do you use substitution to solve a system of linear equations?**

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Solve Systems of Equations Using Substitution

What is the solution of the system of equations?

$$y = 6x + 7$$
$$3x - 8y = 4$$

COMMON ERROR
The value of the variable can be an expression, so be sure to use parentheses when substituting an expression for the value of a variable.

Step 1 The first equation is already solved for y, so substitute $6x + 7$ for y in the second equation.

$$3x - 8y = 4$$
$$3x - 8(6x + 7) = 4$$

Substitute $6x + 7$ for y.

Step 2 Solve for x.

$$3x - 8(6x + 7) = 4$$
$$3x - 48x - 56 = 4$$
$$-45x - 56 = 4$$
$$-45x = 60$$
$$x = -\frac{4}{3}$$

Step 3 Substitute $-\frac{4}{3}$ for x in one of the equations and solve for y.

$$y = 6x + 7$$
$$y = 6\left(-\frac{4}{3}\right) + 7$$
$$y = -8 + 7$$
$$y = -1$$

This system of equations has exactly one solution, $\left(-\frac{4}{3}, -1\right)$.

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EXAMPLE 1 CONTINUED

Step 4 Check by substituting the values into each of the original equations.

$y = 6x + 7$	$3x - 8y = 4$
$(-1) \stackrel{?}{=} 6\left(-\frac{4}{3}\right) + 7$	$3\left(-\frac{4}{3}\right) - 8(-1) \stackrel{?}{=} 4$
$-1 \stackrel{?}{=} -8 + 7$	$-4 + 8 \stackrel{?}{=} 4$
$-1 = -1$ ✓	$4 = 4$ ✓

LEARN TOGETHER
How can you provide constructive feedback while being aware of the feelings and reactions of others?

Try It! **1.** Use substitution to solve each system of equations.

a. $x = y + 6$
$x + y = 10$

b. $y = 2x - 1$
$2x + 3y = -7$

EXAMPLE 2 Compare Graphing and Substitution Methods

A vacation resort offers surfing lessons and parasailing. If a person takes a surfing lesson and goes parasailing, she will pay a total of \$175. On Friday, the resort collects a total of \$3,101 for activities shown in the photo. How much does each activity cost?


16 people go parasailing.

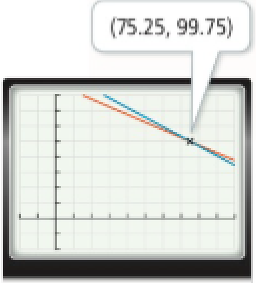
20 people take surfing lessons.

$x + y = 175 \qquad 20x + 16y = 3{,}101$

Let x be the price of a surfing lesson per person. Let y be the price of parasailing per person.

Method 1

Solve the system of equations by graphing.



x scale: 10 *y scale: 20*

Check your answer.

$$x + y = 175$$
$$75.25 + 99.75 \stackrel{?}{=} 175$$
$$175 = 175 \checkmark$$

$$20x + 16y = 3{,}101$$
$$20(75.25) + 16(99.75) \stackrel{?}{=} 3{,}101$$
$$1{,}505 + 1{,}596 \stackrel{?}{=} 3{,}101$$
$$3{,}101 = 3{,}101 \checkmark$$

CHECK FOR REASONABLENESS
How could you show that (75, 100) is a good estimate for the solution of the system?

Surfing lessons cost \$75.25 and parasailing lessons cost \$99.75.

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

Method 2

Solve the system of equations by substitution.

Step 1 Solve one of the equations for either x or y.

$$x + y = 175$$
$$x + y - y = 175 - y$$
$$x = 175 - y$$

STUDY TIP
Remember that after solving one equation for a variable, you need to substitute the solution into the other equation in the system of equations.

Step 2 Substitute for x and solve for y.

$$20x + 16y = 3{,}101$$
$$20(175 - y) + 16y = 3{,}101$$

Substitute $175 - y$ for x.

$$-4y = -399$$
$$y = 99.75$$

Step 3 Substitute 99.75 for y in one of the equations and solve for x.

$$x + y = 175$$
$$x + 99.75 = 175$$
$$x = 75.25$$

Step 4 Check by substituting the values for x and y into each of the original equations.

$$x + y = 175$$
$$75.25 + 99.75 \stackrel{?}{=} 175$$
$$175 = 175\ \checkmark$$

$$20x + 16y = 3{,}101$$
$$20(75.25) + 16(99.75) \stackrel{?}{=} 3{,}101$$
$$1{,}505 + 1{,}596 \stackrel{?}{=} 3{,}101$$
$$3{,}101 = 3{,}101\ \checkmark$$

It costs \$75.25 to take a surfing lesson and \$99.75 to go parasailing.

Try It! **2.** On Saturday, the vacation resort offers a discount on both activities. To take a surfing lesson and go parasailing costs \$130. That day, 25 people take surfing lessons, and 30 people go parasailing. A total of \$3,650 is collected. What is the discounted price of each activity?

EXAMPLE 3 Systems With Infinitely Many Solutions or No Solution

What is the solution of each system of equations?

A. $\mathbf{y = 3x + 1}$
$\mathbf{6x - 2y = -2}$

$$6x - 2y = -2$$
$$6x - 2(3x + 1) = -2$$

Substitute $3x + 1$ for y and then simplify.

$$6x - 6x - 2 = -2$$
$$-2 = -2$$

The statement $-2 = -2$ is an identity, so the system of equations has infinitely many solutions. Both equations represent the same line. All points on the line are solutions to the system of equations.

CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

B. $5x - y = -4$
$y = 5x - 4$

$$5x - y = -4$$
$$5x - (5x - 4) = -4$$
$$5x - 5x + 4 = -4$$
$$4 = -4$$

Substitute $5x - 4$ for y and then simplify.

The statement $4 = -4$ is false, so the system of equations has no solution.

CHECK FOR REASONABLENESS
When the result of solving a system of equations is a false statement, there are no values of x and y that satisfy both equations.

Try It! **3.** Solve each system of equations.

a. $x + y = -4$
$y = -x + 5$

b. $y = -2x + 5$
$2x + y = 5$

APPLICATION

EXAMPLE 4 Model Using Systems of Equations

Nate starts a lawn-mowing business. In his business he has expenses and revenue. Nate's expenses are the cost of the lawn mower and gas, and his revenue is $25 per lawn. At what point will Nate's revenue exceed his expenses?

Formulate Write a system of linear equations to model Nate's expenses and revenue.

In both equations, let y represent the dollar amount, either of expenses or revenue. Let x represent the number of lawns Nate mows.

Nate's expenses: $y = 2x + 200$

Nate's revenue: $y = 25x$

Nate has an initial expense of $200 and then an additional expense of $2 per lawn.

Compute Substitute for y in one of the equations.

$$y = 2x + 200$$
$$25x = 2x + 200$$
$$23x = 200$$
$$x \approx 8.7$$

Interpret Since x is the number of lawns Nate mows, he needs to mow 8.7 lawns before his expenses and revenue are equal. However, Nate is hired to mow whole lawns and not partial lawns, so 8.7 is not a viable solution to the problem. He will need to mow 9 lawns.

Nate will need to mow 9 lawns before his revenue exceeds his expenses.

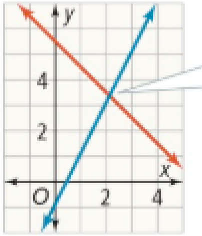
Try It! **4.** Funtime Amusement Park charges $12.50 for admission and then $0.75 per ride. River's Edge Park charges $18.50 for admission and then $0.50 per ride. For what number of rides is the cost the same at both parks?

CONCEPT SUMMARY Solve by Graphing and by Substitution

GRAPHING

$x + y = 5.5$

$8x - 4y = 3.5$



The point of intersection is about (2.1, 3.4).

SUBSTITUTION

$x + y = 5.5$

$8x - 4y = 3.5$

Solve for one variable.	Substitute for x.	Substitute for y.
$x + y = 5.5$	$8x - 4y = 3.5$	$x + y = 5.5$
$x = 5.5 - y$	$8(5.5 - y) - 4y = 3.5$	$x + 3.375 = 5.5$
	$44 - 8y - 4y = 3.5$	$x = 2.125$
	$-12y = -40.5$	
	$y = 3.375$	

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do you use substitution to solve a system of linear equations?

2. **Choose Efficient Methods** When is using a graph to solve a system of equations more useful than the substitution method?

3. **Error Analysis** Simon solves a system of equations, in x and y, by substitution and gets an answer of $5 = 5$. He states that the solution to the system is all of the points (x, y) where x and y are real numbers. Describe Simon's error.

4. **Choose Efficient Methods** When solving a system of equations using substitution, how can you determine whether the system has one solution, no solution, or infinitely many solutions?

Do You KNOW HOW?

Use substitution to solve each system of equations.

5. $y = 6 - x$
   $4x - 3y = -4$

6. $x = -y + 3$
   $3x - 2y = -1$

7. $-3x - y = 7$
   $x + 2y = 6$

8. $6x - 3y = -6$
   $y = 2x + 2$

9. A sports store sells a total of 70 soccer balls in one month and collects a total of $2,400. Write and solve a system of equations to determine how many of each type of soccer ball were sold.

Limited Edition soccer ball
$65.00

Pro NSL soccer ball
$15.00

PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Choose Efficient Methods** When given a system of equations in slope-intercept form, which is the most efficient method to solve: graphing or substitution? Explain.

11. **Use Patterns and Structure** After solving a system of equations using substitution, you end up with the equation $3 = 2$. What is true about the slope and y-intercepts of the lines in the system of equations?

12. **Error Analysis** Describe and correct the error a student made in finding the number of solutions of the system of equations.

$x - 2y = -4$

$5x - 3y = 1$

$x = 2y - 4$

$5x - 3y = 1$

$2y - 4 - 2y = -4$

$-4 = -4$

Infinitely many solutions ✗

13. **Generalize** When using substitution to solve systems of equations that have no solution or infinitely many solutions, the variables are the same on both sides. How is the solution determined by the constants in the equations?

14. **Apply Math Models** The perimeter of a rectangle is 124 cm. The length is six more than three times the width. What are the dimensions of the rectangle?

15. **Mathematical Connections** Two angles are complementary. One angle is six more than twice the other. What is the measure of each angle?

16. **Higher Order Thinking** One equation in a system of equations is $5x - 2y = -4$.

a. Write a second equation in the system of equations that would produce a graph with parallel lines.

b. Write a second equation in the system of equations that would produce a graph with one line.

PRACTICE

Use substitution to solve each system of equations.
SEE EXAMPLE 1

17. $y = 2x - 4$
$3x - 2y = 1$

18. $y = 3x - 8$
$y = 13 - 4x$

19. $y = 2x - 7$
$9x + y = 15$

20. $y = -\frac{1}{2}x$
$2x + 2y = 5$

21. $x = 3y - 4$
$2x - 3y = -2$

22. $x + 2y = -10$
$y = -\frac{1}{2}x + 2$

Consider the system of equations. SEE EXAMPLE 2

$\mathbf{x + y = 5}$

$\mathbf{2x - y = -2}$

23. Solve the system of equations by graphing.

24. Solve the system of equations using the substitution method.

25. Which method do you prefer in this instance? Explain.

Identify whether each system of equations has infinitely many solutions or no solution.
SEE EXAMPLE 3

26. $4x + 8y = -8$
$x = -2y + 1$

27. $2x - 3y = 6$
$y = \frac{2}{3}x - 2$

28. $2x + 2y = 6$
$4x + 4y = 4$

29. $2x + 5y = -5$
$y = -\frac{2}{5}x - 1$

Write and solve a system of equations for the situation. SEE EXAMPLE 4

30. At a hot air balloon festival, Mohamed's balloon is at an altitude of 40 m and rises 10 m/min. Dana's balloon is at an altitude of 165 m and descends 15 m/min.

a. In how many minutes will both balloons be at the same altitude?

b. What will be the altitude?

31. Richard and Teo have a combined age of 31. Richard is 4 years older than twice Teo's age. How old are Richard and Teo?

PRACTICE & PROBLEM SOLVING

APPLY

32. **Use Patterns and Structure** The sum of two numbers is 4. The larger number is 12 more than three times the smaller number. What are the numbers?

33. **Apply Math Models** In a basketball game, the Bulldogs make a total of 21 shots. Some of the shots are 2-point shots while others are 3-point shots. The Bulldogs score a total of 50 points. How many 2-point and 3-point shots did they make?

34. **Analyze and Persevere** Stay Fit gym charges a membership fee of \$75. They offer karate classes for an additional fee.

a. How many classes could members and non-members take before they pay the same amount?

b. How much would they pay?

35. **Apply Math Models** Abby uses two social media sites. She has 52 more followers on Site A than on Site B. How many followers does she have on each site?


Social Media Site A

@abbystudioone.com my new photo and art studio is now open!

700 total followers for both social media sites.

Social Media Site B

ASSESSMENT PRACTICE

36. What are the x- and y-coordinates of the solution for the system of equations? AR.9.1

$$x = -y + 4$$
$$2x + 3y = 4$$

37. **SAT/ACT** Describe the solution of the system of equations.

$$2x - 5y = -5$$
$$y = \frac{2}{5}x - 2$$

Ⓐ No solution

Ⓑ Infinitely many solutions

Ⓒ (10, 5)

Ⓓ (5, 3)

38. **Performance Task** Each side of a triangle lies along a line in a coordinate plane. The three lines that contain these sides are represented by the given equations.

Equation 1: $x - 2y = -4$

Equation 2: $2x + y = -3$

Equation 3: $7x - 4y = 12$

Part A Write three systems of equations that can be used to determine the vertices of the triangle.

Part B What are the coordinates of the vertices?

Part C Is this a right triangle? Explain.

4-3

Solving Systems of Equations by Elimination

I CAN… solve systems of linear equations using the elimination method.

MA.912.AR.9.1–Given a mathematical or real-world context, write and solve a system of two-variable linear equations algebraically or graphically.
Also AR.9.6

MA.K12.MTR.2.1, MTR.4.1, MTR.5.1

CRITIQUE & EXPLAIN

Sadie and Micah used different methods to solve the system of equations.

$$y = 2x + 3$$
$$4x - y = 5$$

Sadie's work

$$4x - (2x + 3) = 5$$
$$4x - 2x - 3 = 5$$
$$2x - 3 = 5$$
$$2x = 8$$
$$x = 4$$
$$y = 2(4) + 3 = 11$$

The solution is (4, 11).

Micah's work

$$y = 2x + 3 \text{ and } y = 4x - 5$$
$$2x + 3 = 4x - 5$$
$$8 = 2x$$
$$x = 4$$
$$y = 2(4) + 3$$
$$y = 11$$

The solution is (4, 11).

A. In what ways are Sadie's and Micah's approaches similar? In what ways are they different?

B. Are both Sadie's and Micah's approaches valid solution methods? Explain.

C. **Choose Efficient Methods** Which method of solving systems of equations do you prefer when solving, Sadie's method, or Micah's method? Explain.

ESSENTIAL QUESTION

Why does the elimination method work when solving a system of equations?

EXAMPLE 1 Solve a System of Equations by Adding

What is the solution to the system of equations?

$$x + y = 7$$
$$2x - y = 2$$

You can add equations to get a new equation that is easier to solve.

Match like terms.

$$\begin{array}{r} x + y = 7 \\ + \; 2x - y = 2 \\ \hline 3x + 0 = 9 \end{array}$$

Write the sums of like terms below.

This method works because of the Addition Property of Equality.

$$x + y = 7$$
$$x + y + (2x - y) = 7 + 2$$
$$(x + 2x) + (y - y) = 7 + 2$$
$$3x + 0 = 9$$

Since $2x - y$ and 2 are equal, you can add $2x - y$ to the left side and 2 to the right side.

Look carefully at the last two steps. They are the same as adding like terms in the original system of equations.

$$\begin{array}{r} x + y = 7 \\ +2x - y = 2 \\ \hline 3x + 0 = 9 \end{array}$$

$$(x + 2x) + (y - y) = 7 + 2$$
$$3x + 0 = 9$$

COMMON ERROR
You may think that you need to add the same expression to each side of the equation. However, since $2x - y = 2$, adding one of the expressions to each side of the equation is still adding an equivalent value to each side of the equation.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

So $3x = 9$, or $x = 3$.

Now substitute 3 for x in either of the two equations in the system of equations.

$$\begin{aligned} x + y &= 7 \\ 3 + y &= 7 \\ y &= 4 \end{aligned}$$

The solution to the system of equations is (3, 4).

Try It! **1.** Solve each system of equations.

a. $2x - 4y = 2$
$-x + 4y = 3$

b. $2x + 3y = 1$
$-2x + 2y = -6$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Understand Equivalent Systems of Equations

What is the solution to the system of equations?

$$\begin{aligned} \mathbf{x + 3y} &\mathbf{= 7} \\ \mathbf{2x + 2y} &\mathbf{= 6} \end{aligned}$$

Before adding equations, multiply each side of one of the equations by a constant that makes either the x or y terms opposites.

$$\begin{array}{lcl} x + 3y = 7 & \xrightarrow{\text{Multiply by } -2.} & -2(x + 3y) = -2 \cdot 7 \\ 2x + 2y = 6 & & 2x + 2y = 6 \end{array}$$

The result is an *equivalent system* that has the same solution as the original system. This is because the first equation has the same solution after multiplying each side by the same nonzero value.

REPRESENT AND CONNECT
The two equations have the same solution because of the Multiplication Property of Equality.

Now solve by adding the equations.

$$\begin{array}{r} -2x - 6y = -14 \\ + 2x + 2y = 6 \\ \hline 0 - 4y = -8 \end{array}$$

Distribute the -2 on each side before adding.

So $-4y = -8$, or $y = 2$. Now substitute 2 for y in either of the two equations in the system.

$$\begin{aligned} x + 3y &= 7 \\ x + 3(2) &= 7 \\ x &= 1 \end{aligned}$$

The solution to the system is (1, 2).

Try It! **2.** Solve each system of equations.

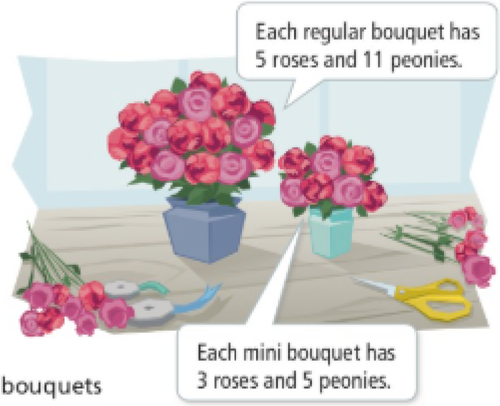
a. $x + 2y = 4$
$2x - 5y = -1$

b. $2x + y = 2$
$x - 2y = -5$

APPLICATION

EXAMPLE 3 Apply Elimination

A florist is making regular bouquets and mini bouquets. The florist has 118 roses and 226 peonies to use in the bouquets. How many of each type of bouquet can the florist make?



Each regular bouquet has 5 roses and 11 peonies.

Each mini bouquet has 3 roses and 5 peonies.

Formulate Let x be the number of regular bouquets and y be the number of mini bouquets.

Roses: $5x + 3y = 118$

Peonies: $11x + 5y = 226$

Compute Multiply each equation by constants to eliminate one variable.

$$\begin{aligned} 5x + 3y &= 118 \quad \xrightarrow{\text{Multiply by 5.}} \quad & 25x + 15y &= 590 \\ 11x + 5y &= 226 \quad \xrightarrow{\text{Multiply by } -3.} \quad & -33x - 15y &= -678 \\ & & \hline -8x + 0 &= -88 \\ & & x &= 11 \end{aligned}$$

There are many ways to do this. In this case, 15 is the LCM of 3 and 5.

Add the equations to eliminate y.

Solve for y.

$$\begin{aligned} 5(11) + 3y &= 118 \\ 55 + 3y &= 118 \\ 3y &= 63 \\ y &= 21 \end{aligned}$$

Substitute 11 for x and solve for y.

The solution is (11, 21).

Interpret Only whole number values for x and y are viable options because the florist would only make complete regular bouquets or mini bouquets. So (11, 21) is a viable solution to the problem.

The florist has enough roses and peonies to make 11 regular and 21 mini bouquets.

Try It! **3.** Before the florist has a chance to finish the bouquets, a large order is placed. After the order, only 85 roses and 163 peonies remain. How many regular bouquets and mini bouquets can the florist make now?

EXAMPLE 4 Choose a Method of Solving

What is the solution of the system of equations?

A. $y = x + 13$

$2x + 7y = 10$

Since the first equation is already solved for one of the variables, you can easily substitute $x + 13$ for y.

$$\begin{aligned} 2x + 7(x + 13) &= 10 \\ 9x + 91 &= 10 \\ 9x &= -81 \\ x &= -9 \end{aligned} \qquad \begin{aligned} y &= -9 + 13 \\ y &= 4 \end{aligned}$$

The solution is (−9, 4).

B. $8x - 2y = -8$

$5x - 4y = 17$

STUDY TIP

Since the coefficients of the y-terms have the same sign, multiply by a negative number.

The coefficient of y in the second equation is an integer multiple of the coefficient of y in the first equation. This makes it easy to eliminate the y variable.

$$\begin{aligned} 8x - 2y &= -8 \\ 5x - 4y &= 17 \end{aligned} \xrightarrow{\text{Multiply by } -2.} \begin{aligned} -16x + 4y &= 16 \\ 5x - 4y &= 17 \\ \hline -11x + 0 &= 33 \\ x &= -3 \end{aligned}$$

Add the equations.

Solve for x.

Now solve for y.

$$\begin{aligned} 5(-3) - 4y &= 17 \\ -15 - 4y &= 17 \\ -4y &= 32 \\ y &= -8 \end{aligned}$$

To use the substitution method, you would have to solve for one of the variables first. Because of the structure of the equations, elimination is an easier method.

The solution is (−3, −8).

Try It! **4.** What is the solution of each system of equations? Explain your choice of solution method.

a. $6x + 12y = -6$
$3x - 2y = -27$

b. $3x - 2y = 38$
$x = 6 - y$

CONCEPT SUMMARY Elimination vs. Substitution

	Substitution	Elimination
WORDS	When one equation is already solved for one variable, or if it is easy to solve for one variable, use substitution.	When you can multiply one or both equations by a constant to get like coefficients that are opposite, use elimination.
ALGEBRA	$3x + y = 8$ $x = 2y - 2$ $3(2y - 2) + y = 8$ $6y - 6 + y = 8$ $7y = 14$ $y = 2$ $x = 2(2) - 2$ $= 2$	$3x - 7y = 16 \xrightarrow{\text{Multiply by 5.}} 15x - 35y = 80$ $5x - 4y = 19 \xrightarrow{\text{Multiply by } -3.} -15x + 12y = -57$ $0 - 23y = 23$ $y = -1$ $3x - 7(-1) = 16$ $3x = 9$ $x = 3$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** Why does the elimination method work when solving a system of equations?

2. **Error Analysis** Esteban tries to solve the following system.

   $7x - 4y = -12$
   $x - 2y = 4$

   His first step is to multiply the second equation by 3.

   $7x - 4y = -12$
   $3x - 6y = 12$

   Then he adds the equations to eliminate a term. What is Esteban's error?

3. **Communicate and Justify** How can you determine whether two systems of equations are equivalent?

4. **Mathematical Connections** The sum of 5 times the width of a rectangle and twice its length is 26 units. The difference of 15 times the width and three times the length is 6 units. Write and solve a system of equations to find the length and width of the rectangle.

Do You KNOW HOW?

Solve each system of equations.

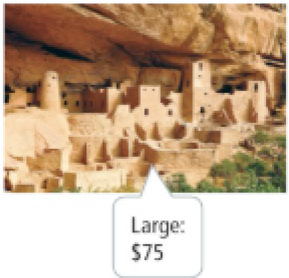
5. $4x - 2y = -2$
   $3x + 2y = -12$

6. $3x + 2y = 4$
   $3x + 6y = -24$

7. $4x - 3y = -9$
   $3x + 2y = -11$

8. $x - 3y = -4$
   $2x - 6y = 6$

9. Catori is a landscape photographer. One weekend at her gallery she sells a total of 52 prints for a total of $2,975. How many of each size print did Catori sell?



PRACTICE & PROBLEM SOLVING

UNDERSTAND

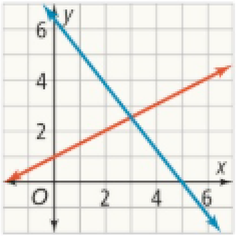
10. **Communicate and Justify** How does the structure of a system of equations help you choose which solution method to use?

11. **Generalize** Consider the system of equations.

$Ax + By = C$
$Px + Qy = R$

If the system has infinitely many solutions, how are the coefficients A, B, C, P, Q, and R related? If the system has no solution, how are the coefficients related?

12. **Represent and Connect** Write and solve a system of equations for the graph shown.



13. **Error Analysis** Describe and correct the error a student made in finding the solution to the system of equations.

$2x - y = -1$
$x - y = -4$

$2x - y = -1$
$-1(x - y) = -4$

$2x - y = -1$
$-x + y = -4$
$x = -5$

$2(-5) - y = -1$
$-10 - y = -1$
$-y = 9$
The solution is $(-5, -9)$. ✗

14. **Use Patterns and Structure** Explain the advantages of using substitution to solve the system of equations instead of elimination.

$x = 6 + y$
$48 = 2x + 2y$

PRACTICE

Solve each system of equations. SEE EXAMPLES 1 AND 3

15. $x - y = 4$
$2x + y = 5$

16. $x - 2y = -2$
$3x + 2y = 30$

17. $3x + 2y = 8$
$x + 4y = -4$

18. $x - 2y = 1$
$2x + 3y = -12$

19. $7x - 4y = -12$
$x - 2y = 4$

20. $5x + 6y = -6$
$7x - 3y = -54$

21. $2x + 5y = -20$
$3x - 2y = -11$

22. $4x - 3y = 17$
$2x - 5y = 5$

Is each pair of systems of equations equivalent? Explain. SEE EXAMPLE 2

23. $3x - 9y = 5$ $\quad$ $6x - 9y = 10$
$6x + 2y = 18$ $\quad$ $6x + 2y = 18$

24. $4y + 2x = -7$ $\quad$ $4y + 2x = -7$
$2y - 6x = 8$ $\quad$ $4y - 12x = 16$

25. $5x + 3y = 19$ $\quad$ $10x + 6y = 38$
$2x + 4y = 20$ $\quad$ $10x + 20y = 100$

Write and solve a system of equations to model each situation. SEE EXAMPLE 3

26. Two pizzas and four sandwiches cost \$62. Four pizzas and ten sandwiches cost \$140. How much does each pizza and sandwich cost?

27. At a clothing store, 3 shirts and 8 hats cost \$65. The cost for 2 shirts and 2 hats is \$30. How much does each shirt and hat cost?

Solve each system. Explain your choice of solution method. SEE EXAMPLE 4

28. $6x - 5y = -1$
$6x + 4y = -10$

29. $8x - 4y = -4$
$x = y - 4$

30. $5x - 2y = -6$
$3x - 4y = -26$

31. $2x - 3y = 14$
$5x + 4y = 12$

PRACTICE & PROBLEM SOLVING

APPLY

32. Communicate and Justify DeShawn and Chris are solving the following system of equations.

$$x - 4y = -8$$
$$3x + 4y = 0$$

DeShawn says that the first step should be to add the two equations to eliminate *y*. Chris says that the first step should be to multiply the first equation by −3 so you can eliminate the *x*-terms.

Who is correct? Explain.

33. Choose Efficient Methods Describe a system of equations where each solution method would be the most efficient to use.

a. Graphing

b. Substitution

c. Elimination

34. Apply Math Models Two groups of friends go to a baseball game. Each group plans to share the snacks shown. What is the price of one drink and one order of churros?

3 drinks and
2 orders of churros:
$16.00

5 drinks and
5 orders of churros:
$31.25

35. Higher Order Thinking Determine the value of *n* that makes a system of equations with a solution that has a *y*-value of 2.

$$5x + 6y = 32$$
$$2x + ny = 18$$

36. A group of 30 students from the senior class charters a bus to an amusement park. The total amount they spend on the bus and admission to the park for each student is \$1,770.

A group of 50 students from the junior class also go to the amusement park, but they require two buses. If the group from the junior class spent \$3,190 in total, how much does it cost to charter one bus?

ASSESSMENT PRACTICE

37. Julia is solving a system of equations using elimination. The first two steps of her solution are shown below. List a property of equality to justify each of her steps. Then, find the solution of the system of equations. AR.9.1

$$\begin{array}{lcr} 4x + 3y = 6 & & 4x + 3y = 6 \\ 2x - 5y = 16 & \xrightarrow{\text{Multiply by } -2.} & \underline{-4x + 10y = -32} \\ & & 0 + 13y = -26 \end{array}$$

38. SAT/ACT A rental company can set up 3 small tents and 1 large tent in 115 min. They can set up 2 small tents and 2 large tents in 130 min. How much time is required to set up a small tent?

Ⓐ 15 min

Ⓑ 25 min

Ⓒ 35 min

Ⓓ 40 min

39. Performance Task At Concessions Unlimited, four granola bars and three drinks cost \$12.50. Two granola bars and five drinks cost \$15.00.

At Snacks To Go, three granola bars and three drinks cost \$10.50. Four granola bars and two drinks cost \$10.00.

Part A Write a system of equations for each concession stand that models the price of its items.

Part B Solve each system of equations. What do the solutions represent?

Part C You decide to open a new concessions stand and sell granola bars and drinks. Determine a price for each item that differs from the prices at Snacks To Go. Then write a system of equations to model the prices at your snack bar.

4-4

Linear Inequalities in Two Variables

I CAN… graph solutions to linear inequalities in two variables.

VOCABULARY

- linear inequality in two variables
- solution of a linear inequality in two variables

MA.912.AR.2.7–Write two-variable linear inequalities to represent relationships between quantities from a graph or a written description within a mathematical or real-world context. Also AR.2.8

MA.K12.MTR.1.1, MTR.5.1, MTR.7.1

CONCEPTUAL UNDERSTANDING

STUDY TIP

You used substitution to test whether an ordered pair is a solution of an equation. You can do the same to test whether an ordered pair is a solution of an inequality.

MODEL & DISCUSS

A flatbed trailer carrying a load can have a maximum total height of 13 feet, 6 inches. The photograph shows the height of the trailer before a load is placed on top. What are the possible heights of loads that could be carried on the trailer?

A. What type of model could represent this situation? Explain.

B. Will the type of model you chose show all the possible heights of the loads without going over the maximum height? Explain.

C. **Represent and Connect** Interpret the solutions of the model. How many solutions are there? Explain.

ESSENTIAL QUESTION

How does the graph of a linear inequality in two variables help you identify the solutions of the inequality?

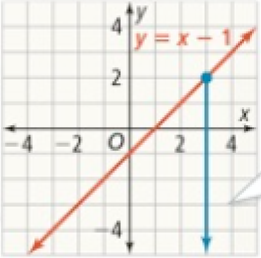
EXAMPLE 1 Understand an Inequality in Two Variables

A. What is the solution of the inequality $y \leq x - 1$?

The inequality $y \leq x - 1$ is an example of a **linear inequality in two variables**. It has the same form as a linear equation but uses an inequality symbol. The **solution of a linear inequality in two variables** is all ordered pairs (x, y) that make the inequality true.

You can plot these ordered pairs on a coordinate plane to understand the solution of the inequality.

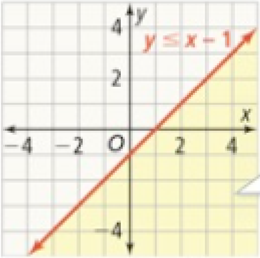
When $y = x - 1$, the solutions are all points on the red line.



The y-value of every point on the blue ray is less than or equal to $x - 1$ for each x.

Now imagine drawing the blue ray for *every* x-value.

The solution of the inequality is all points on the line (called a *boundary line*) and in the shaded region.



All the blue rays together form the shaded region.

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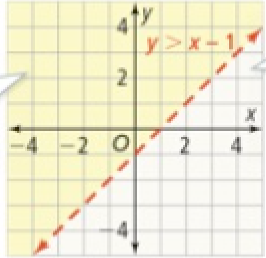
EXAMPLE 1 CONTINUED

B. What is the solution of the inequality $y > x - 1$?

COMMON ERROR
Be sure to use a dotted line when graphing linear inequalities with $<$ or $>$.

The process for finding the solution of $y > x - 1$ is similar to finding the solution of $y \leq x - 1$. For each x, find all values of y that are greater than $x - 1$.



Since the inequality uses *greater than*, shade *above* the line.

The boundary line is a dashed line to indicate that points on it are *not* part of the solution.

The solution of the inequality is all points in the shaded region.

Try It! **1.** Describe the graph of the solutions of each inequality.

a. $y < -3x + 5$

b. $y \geq -3x + 5$

APPLICATION

EXAMPLE 2 Rewrite an Inequality to Graph It

The Science Club sells T-shirts and key chains to raise money. How many T-shirts and key chains could they sell to meet or exceed their goal?

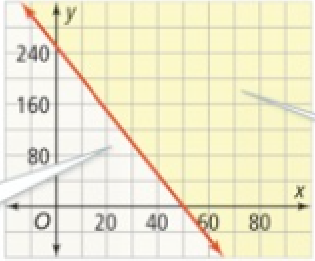
Formulate ◀ Let x represent the number of T-shirts sold and y represent the number of key chains sold. The total amount of money they make must equal or exceed $500.

$$10x + 2y \geq 500$$

Compute ◀ Solve the inequality for y.

$$10x + 2y \geq 500$$
$$2y \geq -10x + 500$$
$$y \geq -5x + 250$$

Graph the inequality.



Draw a solid line since the inequality is greater than or equal to.

Shade above the line since the slope-intercept form of the inequality uses $\geq$.

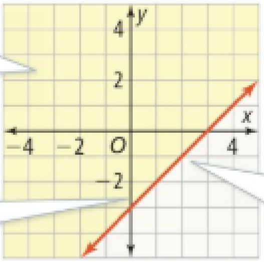
Interpret ◀ Any point in the shaded region or on the boundary line is a solution of the inequality. However, since it is not possible to sell a negative number of T-shirts or key chains, you must exclude negative values for each.

Try It! **2.** Will the Science Club meet their goal if they sell 30 T-shirts and 90 key chains? Explain in terms of the graph of the inequality.

EXAMPLE 3 Write an Inequality From a Graph

What inequality does the graph represent?

Determine the equation of the boundary line.



The graph is shaded above the line.

The boundary line is solid.

The equation of the boundary line is $y = x - 3$.

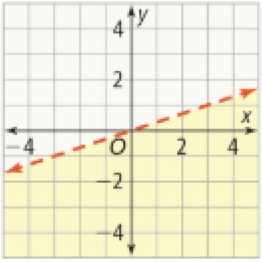
USE PATTERNS AND STRUCTURE
The graph gives you information about the inequality. What does the solid line tell you about the inequality?

The graph is shaded above the boundary line and the boundary line is solid, so the inequality symbol is $\geq$.

The inequality shown by the graph is $y \geq x - 3$.

Try It! **3.** What inequality does each graph represent?

a.



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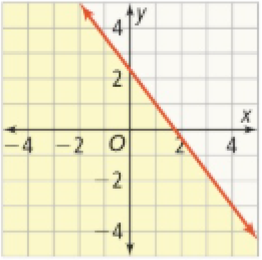
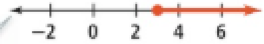


EXAMPLE 4 Inequalities in One Variable in the Coordinate Plane

What is the graph of the inequality in the coordinate plane?

A. $x \geq 3$?

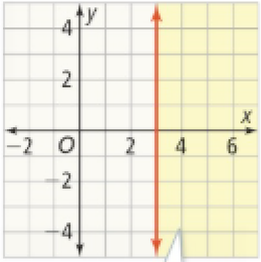
You have graphed the solution of a one-variable inequality on a number line.



solution of $x \geq 3$ on a number line

You can write $x \geq 3$ as $x + 0 \cdot y \geq 3$. The inequality is true for all y, whenever $x \geq 3$.

Imagine stacking copies of the solution on the number line on top of each other, one for each y-value. The combined solutions graphed on the number line make up the shaded region on the coordinate plane.



solution of $x \geq 3$ on a coordinate plane

HAVE A GROWTH MINDSET
When it takes time to learn something new, how do you stick with it?

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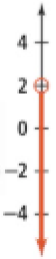
EXAMPLE 4 CONTINUED

B. $y < 2$?

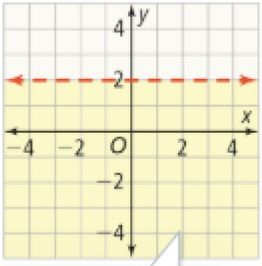
USE PATTERNS AND STRUCTURE
How are the open circle and the dashed line similar?

You can graph the solution of the inequality on a vertical number line.



solution of $y < 2$ on a vertical number line

Notice that the solution on the number line matches the shaded area for any vertical line on the coordinate grid. This is because x can be any number, and the inequality will still be $y < 2$.



solution of $y < 2$ on a coordinate plane

Try It! **4.** Graph each inequality in the coordinate plane.

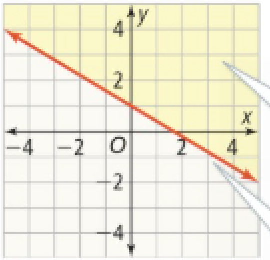
a. $y > -2$

b. $x \leq 1$

CONCEPT SUMMARY Linear Inequalities in Two Variables

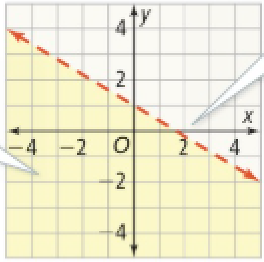
ALGEBRA $y \geq -\frac{3}{5}x + 1$ $y < -\frac{3}{5}x + 1$

GRAPH



Solutions are points in the shaded region.

Solutions are also points on the solid line.



No solutions are on the dashed line.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does the graph of a linear inequality in two variables help you identify the solutions of the inequality?

2. **Communicate and Justify** How many solutions does a linear inequality in two variables have?

3. **Vocabulary** In what form do you write one of the *solutions of an inequality in two variables*?

4. **Error Analysis** A student claims that the inequality $y < 1$ cannot be graphed on a coordinate grid since it has only one variable. Explain the error the student made.

Do You KNOW HOW?

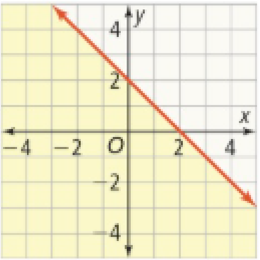
Tell whether each ordered pair is a solution of the inequality $y > x + 1$.

5. (0, 1) **6.** (3, 5)

Graph each inequality in the coordinate plane.

7. $y \geq 2x$ **8.** $y < x - 2$

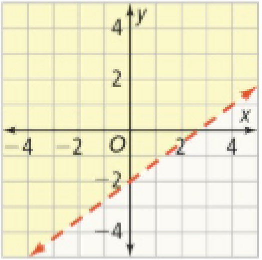
9. What inequality is shown by the graph?



PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Analyze and Persevere** Which inequality, $y-1>\frac{3}{4}(x-4)$ or $3x-4y<8$, is shown by the graph? Explain.



11. **Error Analysis** Describe and correct the error a student made in determining whether the ordered pair (1, 1) is a solution of the inequality $y \leq -4x+5$.

$y \leq -4x+5$
$1 \leq -4(1)+5$
$1 \leq -4+5$
$1 \leq 1$

Since 1 is not less than 1, the inequality is not true. So, (1, 1) is not a solution of the inequality. ✗

12. **Higher Order Thinking** What is the graph of the inequality $x<y+3$? How is this graph different from the graph of the inequality $y<x+3$?

13. **Analyze and Persevere** Write an inequality in two variables for which (3, 7) and (−2, 3) are solutions.

14. **Mathematical Connections** Compare the graph of a linear inequality $x<4$ on a number line with its graph on a coordinate plane. How are they similar?

15. **Generalize** Explain why you can immediately determine which side of the line to shade when an inequality in two variables is solved for *y*.

PRACTICE

Graph each inequality in the coordinate plane.
SEE EXAMPLES 1, 2, AND 4

16. $y \geq -2x+3$

17. $y<x-6$

18. $y \leq \frac{2}{3}x-1$

19. $y>x-2$

20. $y<-0.5x+2$

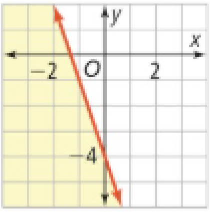
21. $y \geq 1.5x-4$

22. $2x>12$

23. $-2y \leq 6$

What inequality is shown by each graph?
SEE EXAMPLE 3

24.



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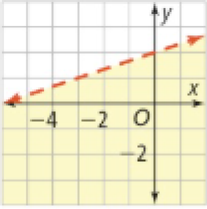

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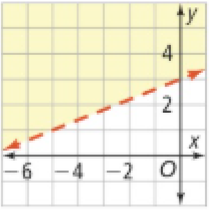

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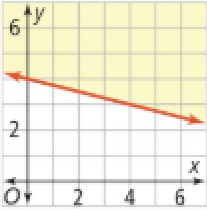
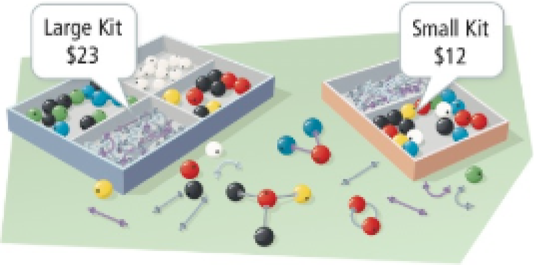


PRACTICE & PROBLEM SOLVING

APPLY

28. **Apply Math Models** A school has $600 to buy molecular sets for students to build models.

a. Write and graph an inequality that represents the number of each type of molecular set the school can buy.



b. Suppose the school decides to buy 20 of the large kits. How many of the small kits can the school now afford?

29. **Apply Math Models** A freight elevator can hold a maximum weight of 2,500 pounds. A 180-pound person has a load of boxes to deliver. Some of the boxes weigh 25 pounds each and some weigh 60 pounds each.

a. Write and graph an inequality that represents the number of boxes the elevator can hold in one trip if the person is not in the elevator.

b. Write and graph an inequality that represents the number of boxes the elevator can hold in one trip if the person rides in the elevator.

c. Compare the graphs of the two inequalities.

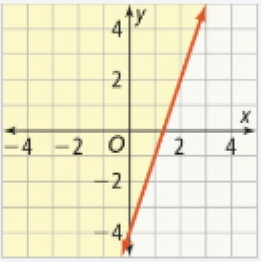
30. **Analyze and Persevere** A soccer team holds a banquet at the end of the season. The team needs to seat at least 100 people and plans to use two different-sized tables. A small table can seat 6 people, and a large table can seat 8 people. Write a linear inequality that represents the numbers of each size table the team needs. Graph the inequality. If the school has 5 small tables and 9 large tables, will this be enough for the banquet?

ASSESSMENT PRACTICE

31. Select all the ordered pairs that are solutions of the inequality $y > 7x - 3$. AR.2.7

☐ **A.** (2, 15)
☐ **B.** (−3, −15)
☐ **C.** (0, −3)
☐ **D.** (1, 5)
☐ **E.** (−2, −18)

32. **SAT/ACT** What inequality is shown by the graph?



Ⓐ $y > 3x - 4$
Ⓒ $y \geq 3x - 4$
Ⓑ $y > 4x - 3$
Ⓓ $y \geq 4x - 3$

33. **Performance Task** A phone has a certain amount of storage space remaining. There are 1000 mb in 1 GB. The average photo uses 3.6 MB of space and the average song uses 4 MB of space.

Part A Write a linear inequality to represent how many additional photos x and songs y the phone can store.

Part B Graph the inequality. Describe how the number of photos that are stored affects the number of songs that can be stored.

Part C Does the graph make sense outside of the first quadrant? Explain.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.9.1–Given a mathematical or real-world context, write and solve a system of two-variable linear equations algebraically or graphically.

MA.K12.MTR.7.1

Get Up There!

Have you ever been to the top of a skyscraper? If so, you probably didn't take the stairs. You probably took an elevator. How long did it take you to get to the top? Did you take an express elevator?

Express elevators travel more quickly because they do not stop at every floor. How much more quickly can you get to the top in an express elevator? Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

4-5

Systems of Linear Inequalities

I CAN… graph and solve a system of linear inequalities.

VOCABULARY

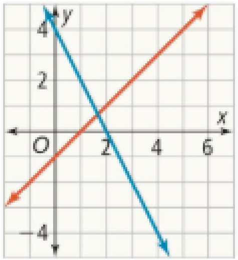
- solution of a system of linear inequalities
- system of linear inequalities

MA.912.AR.9.4–Graph the solution set of a system of two-variable linear inequalities. Also AR.9.6

MA.K12.MTR.1.1, MTR.2.1, MTR.6.1

EXPLORE & REASON

The graph shows the equations $y = x - 1$ and $y = -2x + 4$.



A. Choose some points above and below the line $y = x - 1$. Which of them are solutions to $y > x - 1$? Which are solutions to $y < x - 1$?

B. Choose some points above and below the line $y = -2x + 4$. Which of them are solutions to $y > -2x + 4$? Which are solutions to $y < -2x + 4$?

C. **Use Patterns and Structure** The two lines divide the plane into four regions. How can you describe each region in terms of the inequalities in parts A and B?

ESSENTIAL QUESTION

How is the graph of a system of linear inequalities related to the solutions of the system of inequalities?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Graph a System of Inequalities

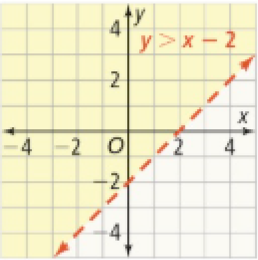
What are the solutions to the system of linear inequalities?

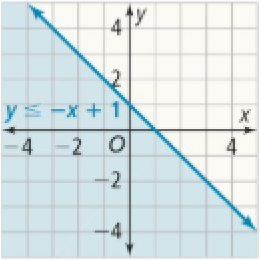
A. $y > x - 2$

$y \leq -x + 1$

A **system of linear inequalities** is made up of two or more linear inequalities. **Solutions of a system of linear inequalities** are ordered pairs that make *all* of the inequalities true.

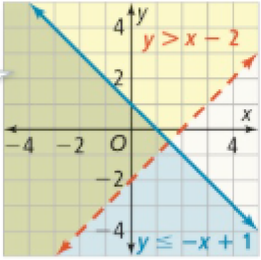
Look at the solutions of each inequality separately.





COMMON ERROR
Points on the solid boundary line for $y \leq -x + 1$ are solutions of the system only when they are also in the region representing the solutions to $y > x - 2$.

Now find points that are solutions to *both* inequalities.



Points in the overlapping region are in *both* shaded regions.

The solutions of the system of linear inequalities are the ordered pairs where the regions overlap.

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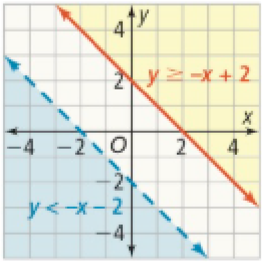
B. $y \geq -x + 2$
$y < -x - 2$

Graph each inequality.



Since the slopes of the boundary lines are equal and their y-intercepts are different, they are parallel, and do not intersect.

The graphs do not overlap, so there is no solution to this system of inequalities.

Try It! **1.** Graph each system of inequalities.

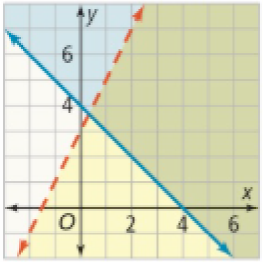
a. $y < 2x$
$y > -3$

b. $y \geq -2x + 1$
$y > x + 2$

EXAMPLE 2 Write a System of Inequalities From a Graph

REPRESENT AND CONNECT
What information can you determine from the graphs of the lines? What information can you determine from the shaded region of the graph?

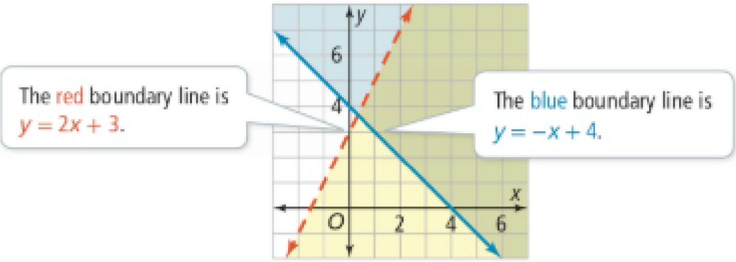
What system of inequalities is shown by the graph?



Determine the equation of each line using the slope and y-intercept.

The slope of the red boundary line is 2 and it has a y-intercept of 3.

The slope of the blue boundary line is −1 and it has a y-intercept of 4.



The red boundary line is $y = 2x + 3$.

The blue boundary line is $y = -x + 4$.

The solutions to the system are below the dashed red line, so one inequality is $y < 2x + 3$.

The solutions to the system are above the solid blue line, so the other inequality is $y \geq -x + 4$.

The graph shows the system of inequalities, $y < 2x + 3$ and $y \geq -x + 4$.

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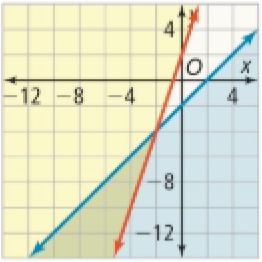
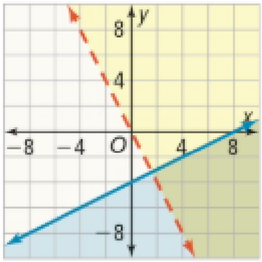
EXAMPLE 2 CONTINUED

Try It! **2.** What system of inequalities is shown by each graph?

a.



b.



APPLICATION

EXAMPLE 3 Use a System of Inequalities

Malia has $500 to purchase water bottles and pairs of socks for a fundraiser for her school's cross-country team. She needs to buy a total of at least 200 items. What graph shows the possible numbers of water bottles and pairs of socks that Malia could buy?

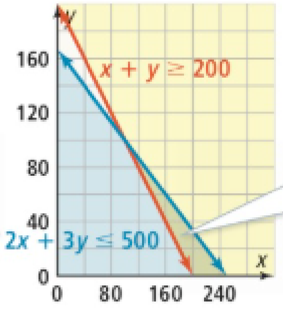
Formulate ◀ Let x represent the number of water bottles and y represent the number of pairs of socks that Malia buys.

Write a system of inequalities with the constraints $x \geq 0$ and $y \geq 0$.

$x + y \geq 200$ .... Malia needs to buy *at least* 200 items.

$2x + 3y \leq 500$ .... Malia can spend *at most* $500.

Compute ◀ Graph the system of inequalities.



2x and 3y represent the total cost for each item.

All the points where both shaded regions overlap are solutions of the system of inequalities.

Interpret ◀ Every point in the overlapping region is a solution of the system. However every solution of the system is not necessarily a viable solution of the real-world situation. Since Malia can only buy whole numbers of water bottles or pairs of socks, only points with whole number coordinates are viable solutions.

Try It! **3.** Use the graph in Example 3 to determine if Malia can buy 75 water bottles and 100 pairs of socks. Explain.

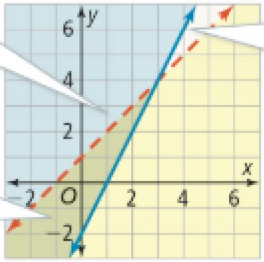
CONCEPT SUMMARY Systems of Linear Inequalities

ALGEBRA

$y < x + 1$ ………… All points below the dashed line

$y \geq 2x - 2$ ………… All points above the solid line

GRAPH



The line is dashed so the points on the line are not included in the solution.

The line is solid so the points on the line may be included in the solution.

The solution of the system of linear inequalities is the shaded region.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is the graph of a system of linear inequalities related to the solutions of the system of inequalities?

2. **Error Analysis** A student says that (0, 1) is a solution to the following system of inequalities.

   $y > x$

   $y > 2x + 1$

   She says that (0, 1) is a solution because it is a solution of $y > x$. Explain the error that the student made.

3. **Vocabulary** How many inequalities are in a *system of inequalities*?

4. **Choose Efficient Methods** Is it easier to describe the solution of a system of linear inequalities in words or to show it using a graph? Explain.

Do You KNOW HOW?

Identify the boundary lines for each system of inequalities.

5. $y > -3x + 4$

   $y \leq 8x + 1$

6. $y < -6x$

   $y \geq 10x - 3$

Graph each system of inequalities.

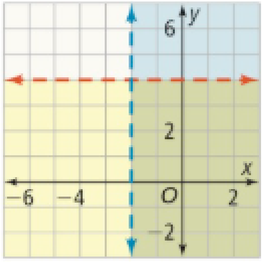
7. $y \leq -3x$

   $y < 2$

8. $y \geq x - 4$

   $y < -x$

9. What system of inequalities is shown by the graph?

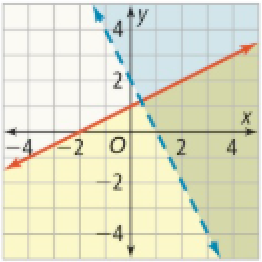


PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Generalize** How does a real-world situation that is best described by a system of linear inequalities differ from a real-world situation that is best described by a single linear inequality?

11. **Error Analysis** Describe and correct the error a student made in writing the system of inequalities represented by the graph shown below.



The red boundary line is $y = 0.5x + 1$.
Since the line is solid, use $\leq$ or $\geq$.
The blue boundary line is $y = -2x + 2$.
Since the line is dashed, use $<$ or $>$.

$y \leq 0.5x + 1$
$y < -2x + 2$

✗

12. **Mathematical Connections** How is a system of two linear inequalities in two variables similar to a system of two linear equations in two variables? How is it different?

13. **Check for Reasonableness** Suppose you use a system of inequalities to model a real-world situation. After graphing the system, you see that point (a, b) lies in the solution region of the coordinate plane. Is it possible that (a, b) could represent a non-viable solution to the real-world situation? Explain.

14. **Higher Order Thinking** Can you write a system of three inequalities that has no solutions? Explain.

15. **Represent and Connect** Could the solutions of a system of inequalities be a rectangular region? If so, give an example.

PRACTICE

Graph each system of inequalities.
SEE EXAMPLES 1 AND 3

16. $y < 2x + 1$
$y \leq -x - 4$

17. $y \leq 3x - 2$
$y > x - 2$

18. $y \geq -\frac{1}{2}x + 1$
$y > x + 3$

19. $y < \frac{1}{3}x$
$y \geq -4x + 1$

20. $2x + 3y < 5$
$y \geq 2x - 3$

21. $x + 4y > 3$
$x - y \leq 2$

22. $y > 0.3x + 2$
$y < -0.2x + 1$

23. $y \leq 0.25x - 4$
$y \geq -x - 3$

24. $y < -2x - 5$
$4x - y < 3$

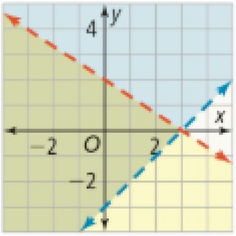
25. $-6x + 4y \geq 8$
$y < -x - 1$

26. $x > 1$
$y < 2x - 3$
$y > x$

27. $y \leq -3x$
$y > -x - 2$
$y > 2$

What system of inequalities is shown by each graph? SEE EXAMPLES 2 AND 3

28.



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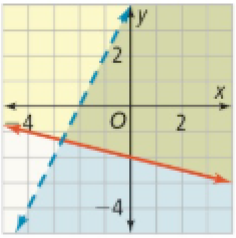

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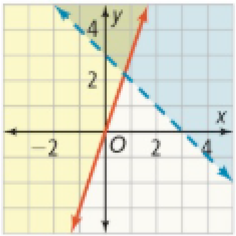

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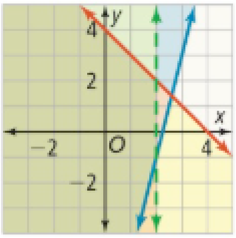


PRACTICE & PROBLEM SOLVING

APPLY

32. **Analyze and Persevere** A group of at most 10 people wants to purchase a combination of seats in Section A and Section B, but does not want to spend more than $450. Graph the system of inequalities that represents the possible ticket combinations they could buy. List three possible combinations they could buy.

STADIUM SEATING

Section A	$50
Section B	$40
Section C	$25
Section D	$15

33. **Apply Math Models** Kendra earns $10 per hour babysitting and $15 per hour providing tech support. Her goal is to save at least $1,000 by the end of the month while not working more than 80 hours. Write and graph a system of inequalities that shows how many hours Kendra could work at each job to meet her goal. What is the fewest number of hours she could work and still meet her goal?

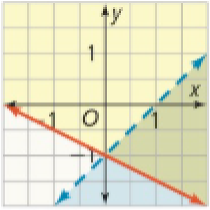
34. **Analyze and Persevere** Alex knits hats and scarves to sell at an art fair. He can make at most 20 hats and 30 scarves, but no more than 40 items altogether, in time for the art fair. Write and graph a system of inequalities that shows the possible numbers of hats and scarves Alex can bring to the art fair if he wants to bring at least 25 items. How do the solutions change if he wants to make more hats than scarves? Explain.

35. **Communicate and Justify** Shannon and Dyani graph the following system of inequalities.

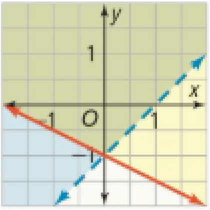
$$y \geq \frac{1}{2}x - 1$$
$$x - y > 1$$

Which graph is correct? Explain.

Shannon's graph

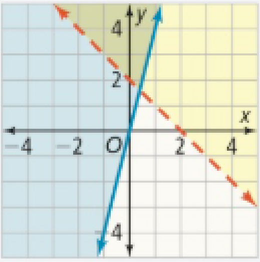


Dyani's graph



ASSESSMENT PRACTICE

Use the graph to answer Exercises 36 and 37.



36. Which of the following is a solution of the system of inequalities shown in the graph? AR.9.4

Ⓐ (3, 2)
Ⓑ (−3, 2)
Ⓒ (−1, 4)
Ⓓ (1, −4)

37. **SAT/ACT** Which of the following is the system of inequalities shown by the graph?

Ⓐ $y > -x + 2$
$y > 4x$

Ⓑ $y < -x + 2$
$y \geq 4x$

Ⓒ $y \geq -x + 2$
$y > 4x$

Ⓓ $y > -x + 2$
$y \geq 4x$

38. **Performance Task** A person is planning a weekly workout schedule of cardio and yoga. He has at most 12 hours per week to work out. The amounts of time he wants to spend on cardio and yoga are shown.

hours of yoga per week

hours of cardio per week

total hours of training

Part A Write a system of linear inequalities to represent this situation.

Part B Graph the system of inequalities. Is there a minimum number of hours the person will be doing cardio? Explain.

TOPIC

4

Topic Review

TOPIC ESSENTIAL QUESTION

1. How do you use systems of linear equations and inequalities to model situations and solve problems?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A(n) __________ is made up of two or more linear inequalities.
3. A(n) __________ is an inequality that is in the same form as a linear equation in two variables, but with an inequality symbol instead of an equal sign.
4. A(n) __________ is an ordered pair that makes all of the inequalities in the system true.
5. The __________ is the set of all ordered pairs that satisfy the inequality.

- linear inequality in two variables
- solution of an inequality in two variables
- solution of a system of linear inequalities
- system of linear inequalities

Concepts & Skills Review

LESSON 4-1 Solving Systems of Equations by Graphing

Quick Review

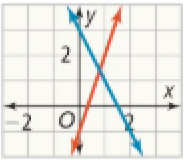
Systems of linear equations can have one solution, infinitely many solutions, or no solution. Graphing a system of linear equations can result in either an approximate solution or an exact solution.

Example

What is the solution of the system of equations? Use a graph.

$y = 3x - 1$

$y = -2x + 3$



The graph intersects at one point, so the system of linear equations has one solution. Find the point of intersection. The graph intersects at (0.8, 1.4).

Check that the solution works for both equations.

$y = 3x - 1$ $\qquad$ $y = -2x + 3$

$1.4 \overset{?}{=} 3(0.8) - 1$ $\qquad$ $1.4 \overset{?}{=} -2(0.8) + 3$

$1.4 = 1.4$ ✓ $\qquad$ $1.4 = 1.4$ ✓

The system of equations has one solution at (0.8, 1.4).

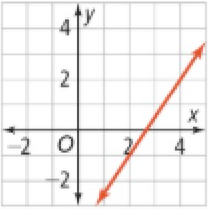
Practice & Problem Solving

Approximate the solution of each system of equations by graphing.

6. $y = 5x + 4$
   $y = -3x - 8$

7. $y = -3x - 7$
   $y = 1.5x + 4$

7. **Use Patterns and Structure** Describe the solution set of the system of equations made by the equation $y = 1.5x + 4.5$ and the graphed line.



8. **Apply Math Models** Kiyo is considering two catering companies for a party. A+ Food charges $35 per person and $75 to setup. Super Cater charges $38 per person with no setup fee. Write and solve a system of equations to represent the charges for catering by each company. Which company should Kiyo use if she expects 28 guests?

LESSON 4-2 Solving Systems of Equations by Substitution

Quick Review

To use substitution to solve a system of equations, isolate the variable in one equation and substitute its value into the other equation. Solve for the variable. Then use that value to solve for the other variable.

Example

Solve the system of equations. $\begin{aligned} y &= 3x - 5 \\ 4x - 2y &= 8 \end{aligned}$

Substitute $3x - 5$ for y and solve for x.

$$\begin{aligned} 4x - 2y &= 8 \\ 4x - 2(3x - 5) &= 8 \\ 4x - 6x + 10 &= 8 \\ -2x &= -2 \\ x &= 1 \end{aligned}$$

Substitute 1 for x in either equation and solve for y.

$$\begin{aligned} y &= 3x - 5 \\ &= 3(1) - 5 \\ &= -2 \end{aligned}$$

The solution of the system of equations is $(1, -2)$.

Practice & Problem Solving

Use substitution to solve each system of equations.

10. $y = 5x - 2$
    $3x - 5y = 4$

11. $y = 2x - 3$
    $y = 8 - 2x$

12. $x = 4y - 8$
    $3x - 6y = 12$

13. $y = 2.5x - 8$
    $3x + 5y = 12$

Identify whether each system of equations has infinitely many solutions or no solution.

14. $3y = 3x - 9$
    $y - 2 = x$

15. $3x - 4y = 12$
    $\frac{3}{4}x = y + 3$

16. **Mathematical Connections** A room has a perimeter of 40 feet. The length is 4 less than 2 times the width. What are the dimensions of the room?

17. **Apply Math Models** Benson has 58 more boxed action figures than collector pins. In total he has 246 collectible items. How many of each type of collectible item does Benson own?

LESSON 4-3 Solving Systems of Equations by Elimination

Quick Review

To use elimination to solve a system of equations, multiply one or both equations by a number so that the coefficient of one variable in both equations is the same or opposite. Then add or subtract to eliminate one variable, and solve for the remaining variable.

Example

Solve the system of equations. $\begin{aligned} 4x - 3y &= 12 \\ 5x - 6y &= 18 \end{aligned}$

Multiply the first equation by -2 and add the two equations to eliminate y and solve for x.

$$\begin{aligned} 4x - 3y &= 12 \quad \xrightarrow{\text{Multiply by } -2.} & -8x + 6y &= -24 \\ 5x - 6y &= 18 & \underline{5x - 6y} &\underline{= 18} \\ & & -3x &= -6 \\ & & x &= 2 \end{aligned}$$

Substitute 2 for x into either equation and solve for y.

$$\begin{aligned} 5(2) - 6y &= 18 \\ y &= -\frac{4}{3} \end{aligned}$$

The solution of the system of equations is $\left(2, -\frac{4}{3}\right)$.

Practice & Problem Solving

Solve each system of equations.

18. $2x - y = -2$
    $3x - 2y = 4$

19. $5x - 2y = 10$
    $4x + 3y = -6$

Is each pair of systems equivalent? Explain.

20. $2x - 3y = 14$ $\quad$ $4x - 6y = 28$
    $5x - 2y = 8$ $\quad$ $-15x + 6y = -24$

21. $3x - 4y = -6$ $\quad$ $6x - 8y = 12$
    $2x + 5y = 1$ $\quad$ $6x + 15y = 3$

22. **Generalize** Do you always have to multiply one or both equations to use elimination? Explain.

23. **Apply Math Models** Carmen and Alicia go to the office supply store to purchase packs of pens and paper. Carmen bought 5 packs of paper and 3 packs of pens for $36.60. Alicia bought 6 packs of paper and 6 packs of pens for $53.40. What is the price of one pack of paper and one pack of pens?

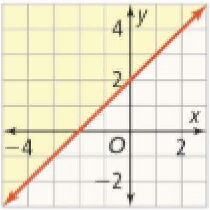
LESSON 4-4 Linear Inequalities in Two Variables

Quick Review

A **linear inequality in two variables** is an inequality that is in the same form as a linear equation in two variables but with an inequality symbol instead of an equal sign. A **solution of a linear inequality in two variables** is an ordered pair that satisfies the inequality.

Example

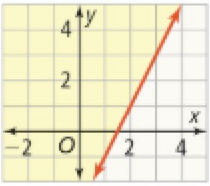
What inequality is shown by the graph?



The slope of the line is 1 and its y-intercept is 2. Therefore, the equation of the line is $y = x + 2$. The boundary line is solid, and all values of x and y that make the inequality true lie on the line or above the line. The inequality shown by the graph is $y \geq x + 2$.

Practice & Problem Solving

Use the graph to tell whether each ordered pair is a solution of the inequality $y \geq 2x - 3$.



24. (2, 5)

25. (3, −1)

26. (−2, 4)

Graph the inequality in the coordinate plane.

27. $y > 4x - 9$

28. $y \leq 1.5x + 4$

29. **Generalize** Write an inequality in two variables for which (2, 5) and (−3, −1) are solutions.

30. **Analyze and Persevere** Renaldo has a budget of $500 to buy gift boxes for a party. Large boxes cost $65 and small boxes cost $35. Write and graph an inequality that represents the number of each type of gift box that Renaldo can buy. If Renaldo buys 6 small gift boxes, how many large gift boxes can he afford to buy?

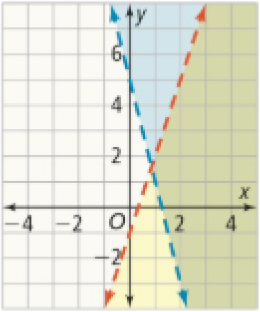
LESSON 4-5 Systems of Linear Inequalities

Quick Review

A **system of linear inequalities** is made up of two or more inequalities. The **solutions of a system of linear inequalities** is the set of all ordered pairs that satisfy the inequalities in the system.

Example

What system of inequalities is shown by the graph? Describe the solutions of the system of inequalities.



Determine the equation of each line using the slope and y-intercept.

Points below the red dashed line satisfy the inequality $y < 3x - 1$. Points above the blue dashed line satisfy the inequality $y > -4x + 5$.

The solutions to the system lie in the region where the graphs overlap.

Practice & Problem Solving

Graph each system of inequalities.

31. $y < 2x + 3$
$y \leq -3x + 1$

32. $y \geq 4x$
$y < -x - 5$

33. **Generalize** What two inequalities can you add to any system of inequalities to indicate that negative values of the variables could not be meaningful answers in the real-world situation.

34. **Apply Math Models** Olivia makes and sells bracelets and necklaces. She can make up to 60 pieces per week, but she can only make up to 40 bracelets and 40 necklaces. Write and graph a system of inequalities that shows the combination of bracelets and necklaces that she can make if she wants to sell at least 30 items per week. If necklaces sell for $80 each and bracelets sell for $5 each, what is the most money she can make in a week? Explain.

TOPIC

5

Exponents and Exponential Functions

TOPIC ESSENTIAL QUESTION

How do you use exponential functions to model situations and solve problems?

Topic Overview

enVision® STEM Project:
Predict the Future Using Moore's Law

5-1 Rational Exponents and Properties of Exponents
NSO.1.1, NSO.1.2, MTR.3.1, MTR.4.1, MTR.6.1

5-2 Radical Expressions
NSO.1.4, MTR.3.1, MTR.4.1, MTR.5.1

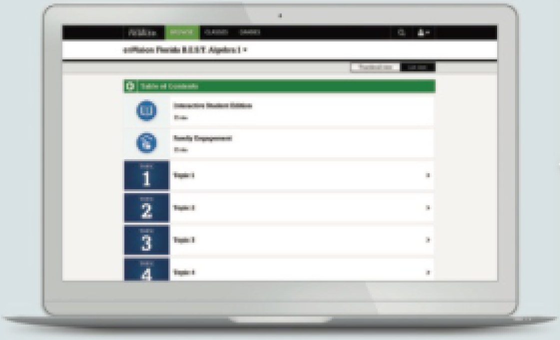
5-3 Exponential Functions
AR.5.4, AR.5.6, F.1.1, F.1.2, F.1.8, MTR.1.1, MTR.5.1, MTR.7.1

5-4 Exponential Growth and Decay
NSO.1.2, AR.1.1, AR.5.3, AR.5.4, F.1.2, F.1.3, FL.3.2, FL.3.4, MTR.2.1, MTR.6.1, MTR.7.1

Mathematical Modeling in 3 Acts:
Big Time Pay Back
AR.5.4, MTR.7.1

Topic Vocabulary

- asymptote
- compound interest
- constant ratio
- decay factor
- exponential decay
- exponential function
- exponential growth
- growth factor
- Product Property of Cube Roots
- Product Property of Square Roots
- rational exponent
- simplest form of a radical expression



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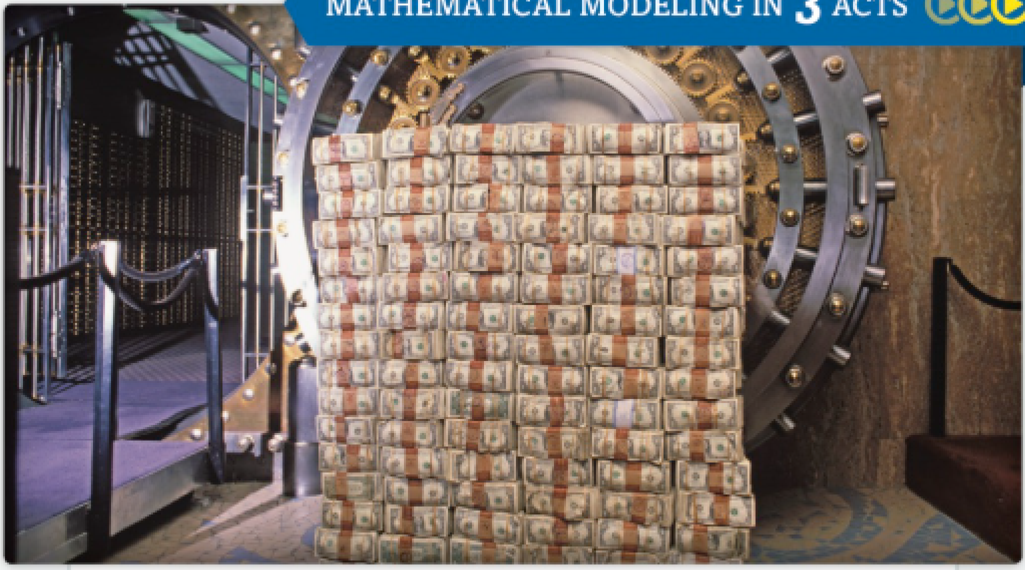
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Big Time Pay Back

Most people agree that investing your money is a good idea. Some people might advise you to put money into a bank savings account. Other people might say that you should invest in the stock market. Still others think that buying bonds is the best investment option.

Is a bank savings account a good way to let your money grow? Just how much money can you make from a savings account? In the Mathematical Modeling in 3 Acts lesson, you'll see an intriguing situation about an investment option.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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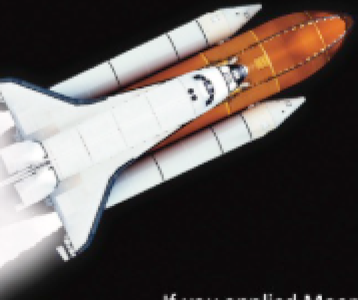
TOPIC 5 **PROJECT** enVision® STEM

Did You Know?

Moore's Law predicts advancements in many digital electronics, stating that **growth is exponential.**



Moore's Law, 1965 (projected for 10 years):
The number of **transistors in a chip will double approximately every 12 months.**

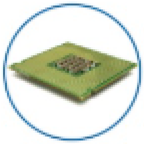
Moore's Law, amended 1975 (projected for 10 years):
The number of **transistors in a chip will double approximately every 24 months.**

If you applied Moore's Law to space travel, a trip to the moon would take one minute.

Corollary to Moore's Law: For a chip of fixed size, the transistors will **decrease in size by 50% every 24 months.**

Transistor

Integrated Circuit



Microprocessor

If cars and transistors shrank at the same rate, **today's cars would be the size of ants.**

Your Task: Predict the Future Using Moore's Law

You and your classmates will predict the features of a cellular phone released 3 years from now, and decide whether or not Moore's Law is sustainable for the next 20 years.

5-1

Rational Exponents and Properties of Exponents

I CAN… use properties of exponents to generate equivalent expressions with rational exponents.

VOCABULARY

- rational exponent

MA.912.NSO.1.1–Extend previous understanding of the Laws of Exponents to include rational exponents. Apply the Laws of Exponents to evaluate numerical expressions and generate equivalent numerical expressions involving rational exponents.
Also NSO.1.2

MA.K12.MTR.3.1, MTR.4.1, MTR.6.1

CRITIQUE & EXPLAIN

Students are asked to write an equivalent expression for 3^{-3}.

Casey and Jacinta each write an expression on the board.

Casey	Jacinta
$3^{-3} = -27$	$3^{-3} = \frac{1}{27}$

A. Who is correct, Casey or Jacinta? Explain.

B. **Communicate and Justify** What is the most likely error that was made?

ESSENTIAL QUESTION

What are the properties of rational exponents and how are they used to solve problems?

EXAMPLE 1 Understand Rational Exponents

What does $3^{\frac{1}{2}}$ equal?

You can think of exponentiation as repeated multiplication.

$3^2 = 3 \cdot 3$ Multiply 3 by itself 2 times.

$3^3 = 3 \cdot 3 \cdot 3$ Multiply 3 by itself 3 times.

$3^4 = 3 \cdot 3 \cdot 3 \cdot 3$ Multiply 3 by itself 4 times.

etc.

CONCEPTUAL UNDERSTANDING

But what does $3^{\frac{1}{2}}$ mean? You cannot multiply 3 by itself $\frac{1}{2}$ times. Since interpreting exponents as repeated multiplication does not work in this case, you have to *define* a new meaning for expressions like $3^{\frac{1}{2}}$.

Whatever the new definition is, you want it to obey the same rules of exponents that you know for integers, such as the Power of a Power Property.

GENERALIZE
The Power of a Power Property says that $(a^m)^n = a^{mn}$ for all integers m and n.

$$\left(3^{\frac{1}{2}}\right)^2 = 3^{\frac{1}{2} \cdot 2} = 3^1 = 3$$

When you square $3^{\frac{1}{2}}$ …

… the result is 3.

You know that a number whose square is 3 is $\sqrt{3}$. So in order to define raising a number to the $\frac{1}{2}$ power in a way that makes sense, define $3^{\frac{1}{2}}$ to be $\sqrt{3}$.

You can define the meaning of other rational exponents in a similar way.

If the nth root of a is a real number and m is an integer, then, $a^{\frac{1}{n}} = \sqrt[n]{a}$, and $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

Try It! 1. What does $2^{\frac{1}{3}}$ equal? Explain.

EXAMPLE 2 Use the Properties of Exponents

A. Use the properties of exponents to generate an expression equivalent to $\left(\frac{1}{125}\right)^{-\frac{1}{3}}$.

$$\begin{aligned}\left(\frac{1}{125}\right)^{-\frac{1}{3}} &= \left(\frac{1}{125}\right)^{(-1)\left(\frac{1}{3}\right)}\\ &= \left(\left(\frac{1}{125}\right)^{(-1)}\right)^{\frac{1}{3}}\\ &= 125^{\frac{1}{3}}\\ &= \sqrt[3]{125}\\ &= 5\end{aligned}$$

Apply the Power of a Power Property and the Negative Exponent Property.

The expression $\left(\frac{1}{125}\right)^{-\frac{1}{3}}$ is equivalent to 5.

B. Use the properties of exponents to show that $8^{\frac{1}{2}}$ and $16^{\frac{3}{8}}$ are equivalent.

Rewrite each expression so they both have the same base.

$$\begin{aligned}8^{\frac{1}{2}} &= 16^{\frac{3}{8}}\\ ((2)^3)^{\frac{1}{2}} &= ((2)^4)^{\frac{3}{8}}\\ 2^{3\left(\frac{1}{2}\right)} &= 2^{4\left(\frac{3}{8}\right)}\\ 2^{\frac{3}{2}} &= 2^{\frac{3}{2}}\end{aligned}$$

Write 8 as 2^3 and 16 as 2^4.

Apply the Power of a Power Property.

The expressions can be written using same base with equal exponents so $8^{\frac{1}{2}}$ and $16^{\frac{3}{8}}$ are equivalent.

STUDY TIP

Finding the prime factorizations of 8 and 16 can help you determine a common base to use to rewrite each number, if one exists. $8 = 2 \cdot 2 \cdot 2$ and $16 = 2 \cdot 2 \cdot 2 \cdot 2$. You can write 16 as 2^4, and 8 as 2^3. So 2 is a common base.

C. Use the properties of exponents to generate an expression with only positive exponents, equivalent to $\left(\frac{a^2}{b^{-3}}\right)^{-4}$.

$$\begin{aligned}\left(\frac{a^2}{b^{-3}}\right)^{-4} &= \left(\frac{b^{-3}}{a^2}\right)^{4}\\ &= \frac{(b^{-3})^4}{(a^2)^4}\\ &= \frac{b^{-12}}{a^8}\\ &= \frac{1}{a^8b^{12}}\end{aligned}$$

Apply the Negative Exponent Property.

Apply the Power of a Quotient Property.

Write the expression using positive exponents.

The expression $\left(\frac{a^2}{b^{-3}}\right)^{-4}$ is equivalent to $\frac{1}{a^8b^{12}}$.

HAVE A GROWTH MINDSET

What other strategies can you try when you get stuck?

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

D. Use the properties of exponents to generate an algebraic expression equivalent to $(-8)^{(3t+2)}$.

$$\begin{aligned}(-8)^{(3t+2)} &= (-8)^{3t}(-8)^{2}\\ &= (-8)^{3t}(64)\\ &= 64(-8)^{3t}\\ &= 64((-8)^{3})^{t}\\ &= 64(-512)^{t}\end{aligned}$$

Apply the Product of Powers Property.

Apply the Power of a Power Property.

The expression $(-8)^{(3t+2)}$ is equivalent to $64(-512)^{t}$.

Try It! **2.** Use the properties of exponents to generate an expression equivalent to each of the following expressions.

a. $9^{\frac{3}{4}}$ **b.** $49^{-\frac{3}{2}}$ **c.** $(a^{-2}b^{5})^{3}$ **d.** $5^{(2t+3)}$

EXAMPLE 3 Evaluate Expressions Using the Properties of Exponents

A. Evaluate the product $\left(3^{\frac{27}{5}}\right)\left(3^{\frac{18}{5}}\right)$.

$$\begin{aligned}\left(3^{\frac{27}{5}}\right)\left(3^{\frac{18}{5}}\right) &= 3^{\frac{27}{5}+\frac{18}{5}}\\ &= 3^{\frac{45}{5}}\\ &= 3^{9}\end{aligned}$$

The product $\left(3^{\frac{27}{5}}\right)\left(3^{\frac{18}{5}}\right)$ is equal to 3^9, or 19,683.

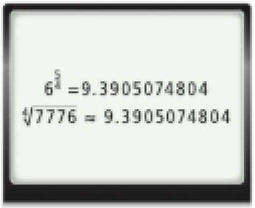
B. Evaluate the quotient $\frac{6^{\frac{27}{4}}}{6^{\frac{11}{2}}}$.

$$\begin{aligned}\frac{6^{\frac{27}{4}}}{6^{\frac{11}{2}}} &= 6^{\frac{27}{4}-\frac{11}{2}}\\ &= 6^{\frac{27}{4}-\frac{22}{4}}\\ &= 6^{\frac{5}{4}}\end{aligned}$$

Apply the Quotient of Powers Property.

The quotient $\frac{6^{\frac{27}{4}}}{6^{\frac{11}{2}}}$ is equivalent to $6^{\frac{5}{4}}$, which cannot be written as a rational number.

In radical form, $6^{\frac{5}{4}} = \sqrt[4]{6^5} = \sqrt[4]{7776}$.
To evaluate the quotient,
enter $6^{\frac{5}{4}}$ or $\sqrt[4]{7776}$ in a calculator.



$6^{\frac{5}{4}} = 9.3905074804$
$\sqrt[4]{7776} = 9.3905074804$

The quotient $\frac{6^{\frac{27}{4}}}{6^{\frac{11}{2}}}$ is approximately 9.39.

CHECK FOR REASONABLENESS
How can you use the powers 6^1 and 6^2 to check that the answer 9.39 is a reasonable approximation of $6^{\frac{5}{4}}$?

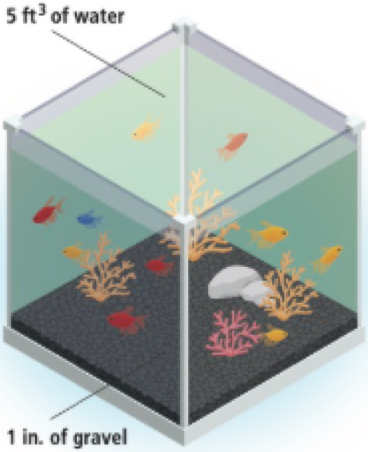
Try It! **3 a.** Evaluate the product $\left(2^{\frac{9}{5}}\right)\left(2^{\frac{6}{5}}\right)$.

b. Evaluate the quotient $\frac{125^{\frac{12}{7}}}{125^{\frac{12}{7}}}$.

APPLICATION

EXAMPLE 4 Solve Problems Using Rational Exponents

How much gravel is needed to cover the floor of the cube-shaped fish tank to a depth of 1 inch?



The amount of gravel needed to cover the floor 1 in. deep is $A \cdot 1$ in.^{3}, where A is the area in in.^{2} of the floor. Let s be the side length of the tank. Then, $A = s^2$.

The volume V of the tank is equal to the cube of the side length, or $V = s^3$. So, $s = V^{\frac{1}{3}}$.

COMMUNICATE AND JUSTIFY
When is it more appropriate to give an answer in approximate form instead of exponent form?

$$\begin{aligned} A &= s^2 \\ &= \left(V^{\frac{1}{3}}\right)^2 = \left(5^{\frac{1}{3}}\right)^2 \\ &= 5^{\frac{2}{3}} \\ &\approx 2.9240 \text{ ft}^2 \end{aligned}$$

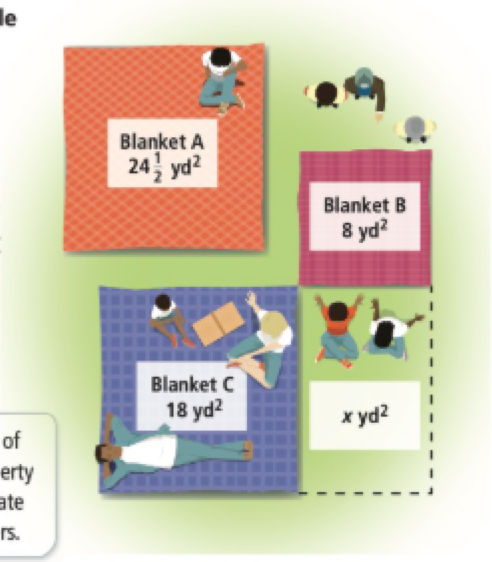
1 square foot is 144 square inches, so the area of the floor of the tank is $144 \cdot 2.9240 \approx 421$ in.^{2}. To a depth of 1 inch, 421 cubic inches of gravel are needed for the floor of the tank.

Try It! 4. A second cube-shaped fish tank holds 10 cubic feet of water. How much gravel is needed to cover the floor of the second tank to the same depth?

APPLICATION

EXAMPLE 5 Use the Properties of Exponents to Solve Problems

What is the area of the grass rectangle next to square blankets B and C?



Formulate

Area	=	length	•	width
x	=	$8^{\frac{1}{2}}$	•	$18^{\frac{1}{2}}$
Area of the grass rectangle		Side length of Blanket B		Side length of Blanket C

Compute

Simplify the expression using the Power of a Product Property.

$$\begin{aligned} x &= \left(8^{\frac{1}{2}}\right)\left(18^{\frac{1}{2}}\right) \\ &= (8 \cdot 18)^{\frac{1}{2}} \\ &= (2^3 \cdot 2 \cdot 3^2)^{\frac{1}{2}} \\ &= (2^4)^{\frac{1}{2}} (3^2)^{\frac{1}{2}} \\ &= 2^2 \cdot 3 \\ &= 12 \end{aligned}$$

Use the Power of a Product Property again to separate the even powers.

CONTINUED ON THE NEXT PAGE

EXAMPLE 5 CONTINUED

Interpret ◀ The area of the grass rectangle next to blankets B and C is 12 square yards.

Check Use a calculator to check that the product is 12.

$$\begin{aligned}\left(8^{\frac{1}{2}}\right)\left(18^{\frac{1}{2}}\right) &\approx (2.8284)(4.2426)\\ &\approx 11.9998\\ &\approx 12\end{aligned}$$

Try It! 5. Blanket A is also square. If Blanket B and Blanket A traded places, what would the area of the grass rectangle next to Blankets A and C be?

CONCEPT SUMMARY Rational Exponents and Properties of Exponents

WORDS If the nth root of a is a real number and m is an integer, then

$a^{\frac{1}{n}} = \sqrt[n]{a}$ $\qquad$ $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

	Power of a Power	Power of a Product	Product of Powers	Quotient of Powers
ALGEBRA	$(a^m)^n = a^{mn}$	$(ab)^m = a^m \cdot b^m$	$a^m \cdot a^n = a^{m+n}$	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
NUMBERS	$\left(256^{\frac{1}{4}}\right)^{\frac{1}{2}} = 256^{\frac{1}{4}\cdot\frac{1}{2}}$ $= 256^{\frac{1}{8}}$ $= 2$	$(4 \times 9)^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$ $= 2 \cdot 3$ $= 6$	$16^{\frac{1}{4}} \times 16^{\frac{1}{4}} = 16^{\frac{1}{4}+\frac{1}{4}}$ $= 16^{\frac{2}{4}}$ $= 4$	$\frac{8^{\frac{2}{3}}}{8^{\frac{1}{3}}} = 8^{\frac{2}{3}-\frac{1}{3}}$ $= 8^{\frac{1}{3}}$ $= 2$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are the properties of rational exponents and how are they used to solve problems?

2. **Represent and Connect** A square has an area of 15 ft^2. What are two ways of expressing its side length?

3. **Error Analysis** Corey wrote $\sqrt[3]{4^2}$ as $4^{\frac{3}{2}}$. What error did Corey make?

4. **Choose Efficient Methods** When is it useful to have rational exponents instead of radicals?

5. **Vocabulary** How are *rational exponents* different from whole number exponents? How are they the same?

Do You KNOW HOW?

Write each radical using rational exponents.

6. $\sqrt{7}$

7. $\sqrt{15}$

8. $\sqrt[3]{6^4}$

9. $\sqrt[5]{2^3}$

Use the properties of exponents to determine whether the expressions are equivalent.

10. $81^{\frac{1}{3}}$ and $9^{\frac{2}{3}}$

11. $(a^3b^4)^2$ and $a^6(b^2)^4$

Use the properties of exponents to generate an expression equivalent to each given expression.

12. $\left(\frac{m^2}{n}\right)^{-1}$

13. $\frac{25^{\frac{11}{4}}}{25^{\frac{9}{4}}}$

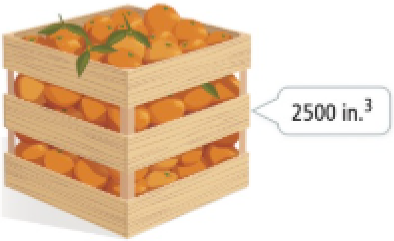
14. $\left(4^{\frac{1}{3}}\right)^9$

15. $(p^{-1}q^2)^3$

PRACTICE & PROBLEM SOLVING

UNDERSTAND

16. **Analyze and Persevere** Describe two ways to express the edge length of a cube with a volume shown.



17. **Communicate and Justify** Explain why $5^{\frac{4}{3}}$ must be equal to $\sqrt[3]{5^4}$ if the Power of a Power Property holds for rational exponents.

18. **Error Analysis** Describe and correct the error a student made when simplifying the product $\left(16^{\frac{1}{2}}\right)\left(16^{\frac{3}{2}}\right)$.

$$
\begin{aligned}
(16^{\frac{1}{2}})(16^{\frac{3}{2}}) &= (16)^{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)} \\
&= (16)^{\frac{3}{4}} \\
&= (16^{\frac{1}{4}})^3 \\
&= 2^3 \\
&= 8
\end{aligned}
$$

✗

19. **Communicate and Justify** The Power of a Quotient rule is $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $b \neq 0$. Will this rule work with rational exponents if $\frac{a}{b}$ is a positive number? Give an example to support your argument.

20. **Higher Order Thinking** The Zero Exponent Property is $a^0 = 1$, $a \neq 0$.
    a. How could you use properties of exponents to explain why $a^0 = 1$?
    b. How could the Zero Exponent Property be applied when solving equations with rational exponents?

21. **Communicate and Justify** Explain how you can use Properties of Exponents to show that the expression $9(3)^t$ is equivalent to 3^{2+t}.

PRACTICE

Write each radical using rational exponents.
SEE EXAMPLE 1

22. $\sqrt{3}$

23. $\sqrt[3]{7}$

24. $\sqrt[5]{3^2}$

25. $\sqrt[4]{2^{-5}}$

Use the properties of exponents to determine if each pair of expressions are equivalent.
SEE EXAMPLE 2

26. $8^{\frac{4}{5}}$ and $16^{\frac{3}{5}}$

27. $(x^5y^3)^2$ and x^7y^5

Use the properties of exponents to generate an expression equivalent to each given expression.
SEE EXAMPLE 2

28. $\left(\frac{1}{9}\right)^{-\frac{1}{2}}$

29. $\left(\frac{c^{-3}}{d^4}\right)^{-3}$

30. $6^{(2t+3)}$

31. $\left(11^{\frac{1}{3}}\right)^4$

32. $(pq^5)^6$

33. $\left(49^{-\frac{1}{4}}\right)^2$

Use the properties of exponents to evaluate each expression. If the expression cannot be written as a rational number, use technology to evaluate the result to the nearest hundredth. SEE EXAMPLES 3 AND 5

34. $\left(7^{\frac{12}{5}}\right)\left(7^{-\frac{2}{5}}\right)$

35. $\frac{5^{\frac{5}{3}}}{5^{\frac{4}{3}}}$

36. $\left(\frac{2}{11}\right)^{\frac{1}{2}}\left(\frac{2}{11}\right)^{\frac{3}{2}}$

37. $\frac{10^{\frac{9}{4}}}{10^{-\frac{3}{4}}}$

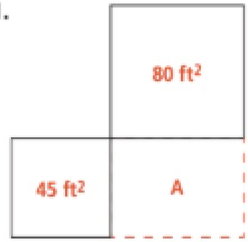
38. $(2)^{\frac{1}{2}}(32)^{\frac{1}{2}}$

39. $(12)^{\frac{1}{3}}(18)^{\frac{1}{3}}$

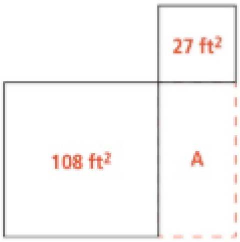
40. A cube has a volume of 150 in.^{3}. What is the area of the base of the cube? Round to the nearest square inch. SEE EXAMPLE 4

For each figure, find the area of rectangle A.
SEE EXAMPLE 5

41.



42.



PRACTICE & PROBLEM SOLVING

APPLY

43. Apply Math Models The formula for the volume V of a sphere is $\frac{4}{3}\pi r^3$. What is the radius of the basketball shown?

44. Analyze and Persevere A singing contest eliminates contestants after each round. To find the number of contestants in the next round, raise the number of contestants in the current round to the power of $\frac{6-n}{7-n}$, where n is the number of the current round. The number of contestants in Round 2 is 243. How many contestants will be in Round 5?

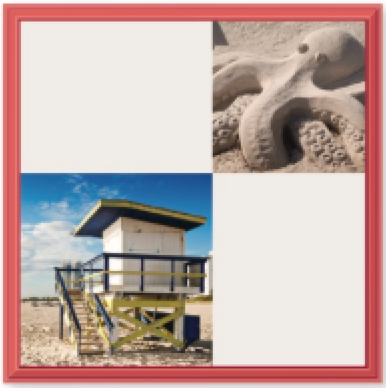
45. Apply Math Models Photos A and B are square photos. Damian wants to place them within a square frame as shown below. What is the total area of the empty spaces within the large square frame?

Photo A

Area = 128 cm^2

Photo B

Area = 72 cm^2



ASSESSMENT PRACTICE

46. Rewrite each radical using rational exponents.
NSO.1.1

a. $\sqrt[4]{2^5}$

b. $\sqrt{5}$

c. $\sqrt[5]{2^4}$

d. $\sqrt[5]{2}$

47. Which expression does not equal 4?

Ⓐ $\left(8^{\frac{5}{3}}\right)(8)^{-1}$

Ⓑ $\frac{16^{\frac{1}{6}}}{16^{-\frac{1}{3}}}$

Ⓒ $\left(4^{\frac{2}{5}}\right)\left(4^{\frac{5}{2}}\right)$

Ⓓ $\left(2^{\frac{5}{7}}\right)\left(2^{\frac{3}{7}}\right)\left(2^{\frac{6}{7}}\right)$

Ⓔ $\frac{\left(\frac{1}{64}\right)^{\frac{10}{3}}}{\left(\frac{1}{64}\right)^{\frac{11}{3}}}$

48. Performance Task It is possible to write any positive integer as the sum of powers of 2 with whole number exponents. For example, you can write 75 in the following manner.

$$2^0 + 2^1 + 2^3 + 2^6 = 75$$

Part A Use the equation above to write 75 as the sum of powers of 8, using rational exponents. What are possible values for a, b, c and d?

$$8^a + 8^b + 8^c + 8^d = 75$$

Part B How can you modify the equation you wrote in part A to express 75 as sum of powers of 16?

$$16^a + 16^b + 16^c + 16^d = 75$$

Part C Given that a, b, c, and d are rational numbers, for what types of integer values of x is the following equation true? Explain your answer.

$$x^a + x^b + x^c + x^d = 75$$

5-2

Radical Expressions

I CAN… write equivalent radical expressions.

VOCABULARY

- Product Property of Cube Roots
- Product Property of Square Roots
- simplest form of a radical expression

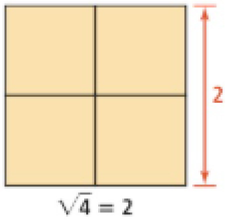
MA.912.NSO.1.4–Apply previous understanding of operations with rational numbers to add, subtract, multiply, and divide numerical radicals.

MA.K12.MTR.3.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

The table shows the relationship between the area of a square, the side length of the square, and the square root of the area. A square with an area of 4 and a side length of 2 is shown at the right.

Area of Square (Square units)	$s = \sqrt{\text{area}}$	Side Length, s (units)
1	$s = \sqrt{1}$	1
4	$s = \sqrt{4}$	2
9	$s = \sqrt{9}$	3
16	$s = \sqrt{16}$	4
25	$s = \sqrt{25}$	5



$\sqrt{4} = 2$

A. What is the side length of a square with an area of 49 square units?

B. **Use Patterns and Structure** Between what two consecutive integers is $\sqrt{20}$? How do you know?

C. Think of three squares that have a side length between 3 and 4. What is the area of each square?

ESSENTIAL QUESTION

How does rewriting radicals in different forms help you communicate your answers?

EXAMPLE 1 Write Equivalent Radical Expressions

A. What is an equivalent expression for $\sqrt{63}$?

$$\begin{aligned}\sqrt{63} &= \sqrt{9 \cdot 7}\\ &= (9 \cdot 7)^{\frac{1}{2}}\\ &= 9^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}\\ &= \sqrt{9} \cdot \sqrt{7}\\ &= 3\sqrt{7}\end{aligned}$$

The equivalence $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ when a and b are both greater than or equal to 0 is known as the **Product Property of Square Roots**.

The expression $\sqrt{63}$ is equivalent to $3\sqrt{7}$.

B. What is an equivalent expression for $\sqrt{2^7 \cdot 5^2}$?

$$\begin{aligned}\sqrt{2^7 \cdot 5^2} &= \sqrt{2^6 \cdot 2 \cdot 5^2}\\ &= \sqrt{2^6} \cdot \sqrt{5^2} \cdot \sqrt{2}\\ &= 2^{\frac{6}{2}} \cdot 5^{\frac{2}{2}} \cdot \sqrt{2}\\ &= 2^3 \cdot 5 \cdot \sqrt{2}\\ &= 40\sqrt{2}\end{aligned}$$

Separate out factors of the radicand that have even powers.

Try It! **1.** Rewrite each expression to remove perfect square factors other than 1 in the radicand.

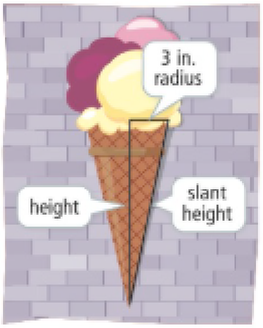
a. $\sqrt{44}$

b. $3\sqrt{27}$

APPLICATION

EXAMPLE 2 Write a Radical Expression

A light fixture in an ice cream shop is in the shape of an ice cream cone and the cone has a height that is 7 times the radius. The expression for the slant height is $\sqrt{r^2 + h^2}$, where r is the radius and h is the height of the cone. What is the slant height of the cone?



Formulate Let $r = 3$ and $h = 7 \cdot 3$.

$$\sqrt{r^2 + h^2} = \sqrt{3^2 + (7 \cdot 3)^2}$$

Compute

$$\begin{aligned}\sqrt{3^2 + (7 \cdot 3)^2} &= \sqrt{3^2 + 7^2 \cdot 3^2}\\ &= \sqrt{(1 + 7^2) \cdot 3^2}\\ &= 3\sqrt{50}\\ &= 3\sqrt{5^2 \cdot 2}\\ &= 15\sqrt{2}\end{aligned}$$

When there is a common factor in the radicand, use the Distributive Property to make it easier to factor later.

Interpret The slant height of the cone is $15\sqrt{2} \approx 21.2$ inches.

Try It! 2. Another cone has a slant height s equal to $5r$. Simplify the expression for h if $r = 4$.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Multiply and Divide Radicals

A. Multiply $\sqrt[3]{45} \cdot \sqrt[3]{-75}$.

$$\begin{aligned}\sqrt[3]{45} \cdot \sqrt[3]{-75} &= \sqrt[3]{45(-75)}\\ &= \sqrt[3]{3 \cdot 3 \cdot 5 \cdot (-1) \cdot 3 \cdot 5 \cdot 5}\\ &= \sqrt[3]{-1} \cdot \sqrt[3]{3^3} \cdot \sqrt[3]{5^3}\\ &= -1 \cdot 3 \cdot 5\\ &= -15\end{aligned}$$

The **Product Property of Cube Roots** does not require that the radicands be greater than or equal to zero.

Gather the like radicands and apply the Product Property.

USE PATTERNS AND STRUCTURE
Why is the cube root of −1 equal to −1?

The expression $\sqrt[3]{45} \cdot \sqrt[3]{-75}$ is equivalent to −15.

B. Divide $\frac{\sqrt[3]{15}}{\sqrt[3]{405}}$.

$$\frac{\sqrt[3]{15}}{\sqrt[3]{405}} = \sqrt[3]{\frac{15}{405}}$$

The product properties of square and cube roots also apply to quotients.

$$= \sqrt[3]{\frac{3 \cdot 5}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 5}}$$

Factor the numerator and denominator.

$$= \sqrt[3]{\frac{1}{3^3}}$$

Simplify the fraction. Rewrite the denominator using an exponent.

$$= \frac{1}{\sqrt[3]{3^3}}$$

Take the cube root of the numerator.

$$= \frac{1}{3}$$

Try It! 3. Multiply or divide each expression.

a. $\sqrt[3]{21} \cdot \sqrt[3]{147}$

b. $\frac{\sqrt[3]{225}}{\sqrt[3]{75}}$

EXAMPLE 4 Multiply Radical Expressions

How can you write an expression for the product of $3\sqrt{10} \cdot 2\sqrt{120}$ without any perfect square factors in the radicand?

Find the product.

CHOOSE EFFICIENT METHODS
When rewriting the product under the radicand, it is easier to find perfect squares by replacing each number with its prime factors than by using other factors.

$$\begin{aligned}
&3\sqrt{10} \cdot 2\sqrt{120}\\
&= 3 \cdot 2\sqrt{10 \cdot 120}\\
&= 6\sqrt{2 \cdot 5 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5}\\
&= 6 \cdot \sqrt{5^2} \cdot \sqrt{2^4} \cdot \sqrt{3}\\
&= 6 \cdot 5 \cdot 2^2 \cdot \sqrt{3}\\
&= 120\sqrt{3}
\end{aligned}$$

Use the Product Property of Square Roots to multiply the radicands.

Rewrite to show the perfect-square factors.

The expression $3\sqrt{10} \cdot 2\sqrt{120}$ is equivalent to $120\sqrt{3}$.

Try It! **4.** Write an expression for each product without perfect square factors other than 1 in the radicand.

a. $\frac{1}{2}\sqrt{168} \cdot 4\sqrt{28}$

b. $2\sqrt{3072} \cdot \sqrt{576}$

EXAMPLE 5 Write Quotients of Radical Expressions in Simplest Form

How can you write the quotient $\frac{2 \cdot \sqrt[3]{72}}{\sqrt[3]{3888}}$ in simplest form?

Find the quotient.

$$\begin{aligned}
\frac{2 \cdot \sqrt[3]{72}}{\sqrt[3]{3888}} &= 2 \cdot \sqrt[3]{\frac{72}{3888}}\\
&= 2 \cdot \sqrt[3]{\frac{2^3 \cdot 3^2}{2^4 \cdot 3^5}}\\
&= 2 \cdot \sqrt[3]{\frac{1}{2 \cdot 3^3}}\\
&= \frac{2}{3\sqrt[3]{2}}
\end{aligned}$$

Write the prime factorizations of the numerator and the denominator.

Take the cube root of the numerator and the denominator.

LEARN TOGETHER
Do you seek help when needed? Do you offer help and support others?

In the **simplest form of a radical expression**, each radicand is greater than 1 and has no other perfect square factors under a square root or perfect cube factors under a cube root. Additionally, there should be no radicals in the denominator of any fraction.

$$\begin{aligned}
\frac{2}{3 \cdot \sqrt[3]{2}} &= \frac{2}{3 \cdot \sqrt[3]{2}} \cdot \frac{\sqrt[3]{2} \cdot \sqrt[3]{2}}{\sqrt[3]{2} \cdot \sqrt[3]{2}}\\
&= \frac{2 \cdot \sqrt[3]{4}}{3 \cdot 2}\\
&= \frac{\sqrt[3]{4}}{3}
\end{aligned}$$

Multiply numerator and denominator by the cube root until there is a product of three cube roots in the denominator.

The simplest form of $\frac{2\sqrt[3]{72}}{\sqrt[3]{3888}}$ is $\frac{\sqrt[3]{4}}{3}$.

Try It! **5.** Write $\sqrt{\frac{63}{98}}$ in simplest form.

EXAMPLE 6 Add and Subtract Radical Expressions

A. What is the sum of $12\sqrt{7}$ and $6\sqrt{7}$?

The Distributive Property states that $ax + bx = (a + b)x$.

Let $a = 12$, $b = 6$, and $x = \sqrt{7}$.

$$\begin{aligned} 12\sqrt{7} + 6\sqrt{7} &= (12 + 6)\sqrt{7} \\ &= 18\sqrt{7} \end{aligned}$$

The sum of $12\sqrt{7}$ and $6\sqrt{7}$ is $18\sqrt{7}$.

B. What is the difference of $10\sqrt{3}$ and $\frac{6}{\sqrt{3}}$?

Write both radicals in simplest form.

$10\sqrt{3}$ is already in simplest form, but $\frac{6}{\sqrt{3}}$ has a radical in the denominator of a fraction.

$$\begin{aligned} \frac{6}{\sqrt{3}} &= \frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{6\sqrt{3}}{\sqrt{9}} \\ &= \frac{6\sqrt{3}}{3} \\ &= 2\sqrt{3} \end{aligned}$$

To rationalize the denominator, multiply the numerator and denominator by $\sqrt{3}$ to create a perfect square.

The radical expressions have a common radicand, so you can use the Distributive Property to find the difference.

$$\begin{aligned} 10\sqrt{3} - \frac{6}{\sqrt{3}} &= 10\sqrt{3} - 2\sqrt{3} \\ &= (10 - 2)\sqrt{3} \\ &= 8\sqrt{3} \end{aligned}$$

The difference of $10\sqrt{3}$ and $\frac{6}{\sqrt{3}}$ is $8\sqrt{3}$.

C. What is the sum of $2\sqrt{5}$ and $8 \cdot \sqrt[3]{5}$?

You cannot combine square and cube roots, so the sum is $2\sqrt{5} + 8 \cdot \sqrt[3]{5}$.

COMMUNICATE AND JUSTIFY
How could you show that unlike radicands cannot be combined?

Try It! **6.** Simplify each sum or difference.

a. $\frac{1}{2}\sqrt{28} + 4\sqrt{28}$

b. $\sqrt{3072} - 2\sqrt{1728}$

c. $31\sqrt{6} - 31\sqrt[3]{6}$

CONCEPT SUMMARY Rewriting Radical Expressions

WORDS A radical expression is written in the simplest form when all of the following are true:

- There are no perfect square factors other than 1 in the radicand of a square root.
- There are no perfect cube factors other than 1 in the radicand of a cube root.
- Each radicand is greater than 1.
- There are no radicals in the denominator of a fraction.

ALGEBRA Product Property of Square Roots

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ when $a \geq 0$ and $b \geq 0$

Product Property of Cube Roots

$\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b}$

NUMBERS

$$\begin{aligned} \sqrt{63} &= \sqrt{9 \cdot 7} \\ &= \sqrt{3 \cdot 3 \cdot 7} \\ &= 3\sqrt{7} \end{aligned}$$

$$\begin{aligned} 4\sqrt{3} \cdot 6\sqrt{6} &= 4 \cdot 6\sqrt{3 \cdot 6} \\ &= 24\sqrt{3 \cdot 3 \cdot 2} \\ &= 24 \cdot 3\sqrt{2} \\ &= 72\sqrt{2} \end{aligned}$$

$$\begin{aligned} \sqrt[3]{40} &= \sqrt[3]{8 \cdot 5} \\ &= \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 5} \\ &= 2\sqrt[3]{5} \end{aligned}$$

$$\begin{aligned} 2\sqrt[3]{9} \cdot 4\sqrt[3]{15} &= 2 \cdot 4\sqrt[3]{9 \cdot 15} \\ &= 8\sqrt[3]{3 \cdot 3 \cdot 3 \cdot 5} \\ &= 8 \cdot 3\sqrt[3]{5} \\ &= 24\sqrt[3]{5} \end{aligned}$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does rewriting radicals in different forms help you communicate your answer?

2. **Vocabulary** State the *Product Property of Square Roots* in your own words.

3. **Communicate and Justify** Write an expression for $\sqrt{32}$ without any perfect square factors other than 1 in the radicand. Explain your steps.

4. **Error Analysis** Rikki says that the product $\sqrt{3 \cdot 5^3} \cdot \sqrt{5}$ is $3 \cdot 5^2$ or 75. Explain Rikki's error and write the correct product.

5. **Communicate and Justify** Is $\sqrt{45}$ in simplest form? Explain.

6. **Analyze and Persevere** Describe how you would simplify an expression so that there are no perfect square factors in the radicand.

Do You KNOW HOW?

Write each expression in simplest form.

7. $\sqrt{80}$

8. $\sqrt{200}$

9. $2\sqrt{50}$

10. $\sqrt[3]{24}$

Write each product or quotient in simplest form.

11. $\sqrt{14} \cdot \sqrt{21}$

12. $\frac{\sqrt{54}}{\sqrt{3}}$

13. $\frac{2\sqrt[3]{500}}{\sqrt[3]{4}}$

14. $\sqrt{28} \cdot \sqrt{27}$

Write each sum or difference in simplest form.

15. $6\sqrt{11} + 3\sqrt{11}$

16. $9\sqrt{13} - 5\sqrt{13}$

17 $\sqrt{18} + \sqrt{8}$

18. $12\sqrt{5} - 3\sqrt{20}$

19. $\frac{14}{\sqrt{7}} + \sqrt{7}$

20. $\sqrt{8} - \frac{1}{\sqrt{2}}$

PRACTICE & PROBLEM SOLVING

UNDERSTAND

21. **Generalize** For $\sqrt{x^n}$, consider rewriting this expression without a perfect square factor in the radicand for even and odd values of n, where n is a positive integer.

    a. What is the expression when n is even?

    b. What is the expression when n is odd?

22. **Error Analysis** Describe and correct the error a student made in multiplying $\sqrt[3]{24}$ and $\sqrt[3]{45}$

$$\begin{aligned}\sqrt[3]{24}\cdot\sqrt[3]{45} &= \sqrt[3]{24\cdot 45}\\ &= \sqrt[3]{2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 5}\\ &= 2\cdot 3\sqrt[3]{2\cdot 3\cdot 5}\\ &= 6\sqrt[3]{30}\end{aligned}$$

✗

23. **Use Patterns and Structure** Suppose n is an integer greater than 1. Write an expression equivalent to $\sqrt{753^{2n}}$.

24. **Communicate and Justify** Why do the multiplication properties of exponents apply to radicals? Explain.

25. **Analyze and Persevere** How many perfect squares other than 1 are under each radical?

Radical	Perfect squares
$\sqrt{8}$	
$\sqrt{18}$	
$\sqrt{32x^6}$	
$\sqrt{50x}$	
$\sqrt{72}$	

26. **Choose Efficient Methods** Determine if the following sums can be written as a single term without simplifying the radicands first. Explain.

    a. $7\sqrt{72} + 3\sqrt{72}$

    b. $3\sqrt{48} + 9\sqrt{75}$

PRACTICE

Write each expression in simplest form.
SEE EXAMPLE 1

27. $\sqrt{180}$

28. $\sqrt{120}$

29. $2\sqrt{294}$

30. $\sqrt{192}$

31. $\sqrt{3^2 \cdot 11^4}$

32. $\sqrt{2^5 \cdot 13^3}$

Write each product or quotient in simplest form. SEE EXAMPLES 3, 4, AND 5

33. $\sqrt{72}\cdot\sqrt{63}$

34. $2\sqrt{30}\cdot 3\sqrt{70}$

35. $\dfrac{\sqrt{640}}{\sqrt{5}}$

36. $\sqrt[3]{16}\cdot\sqrt[3]{-48}$

37. $\sqrt[3]{81}\cdot\sqrt[3]{45}$

38. $\sqrt{\dfrac{845}{5}}$

39. $\dfrac{6\sqrt[3]{1620}}{\sqrt[3]{-20}}$

40. $\sqrt{28}\cdot\sqrt{27}$

Write each sum or difference in simplest form.
SEE EXAMPLE 6

41. $5\sqrt{7} + 8\sqrt{7}$

42. $2\sqrt{12} - \sqrt{3}$

43. $8\sqrt{3} + \sqrt{5}$

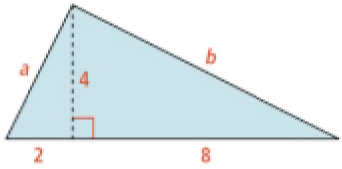
44. $\sqrt{2} + \dfrac{6}{\sqrt{2}}$

45. $4\sqrt{112} - 2\sqrt{175}$

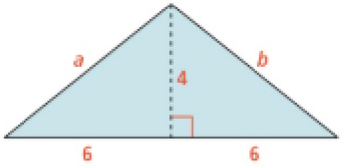
46. $\dfrac{9}{\sqrt{5}} - \sqrt{20}$

Write the expression for the sum of the missing sides a and b in simplest form. Then write an expression for the perimeter of each triangle.
SEE EXAMPLES 2 AND 6

47.



48.



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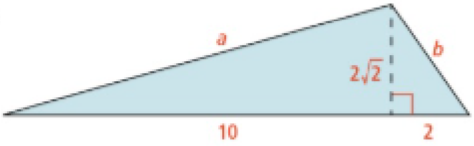
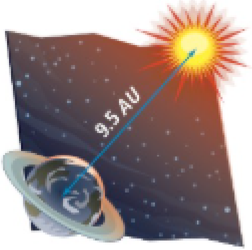


PRACTICE & PROBLEM SOLVING

APPLY

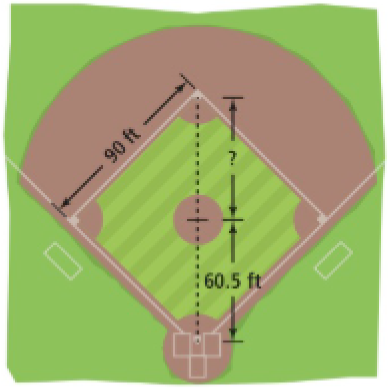


50. **Use Patterns and Structure** The time it takes a planet to revolve around the sun in Earth years can be modeled by $t = \sqrt{d^3}$, where d is the average distance from the sun in astronomical units (AU).

a. Write an equivalent equation for the function.

b. How long does it take Saturn, pictured above, to orbit the sun? Show that both expressions give the same value.

51. **Apply Math Models** A baseball "diamond" is a square that measures 90 ft on each side.



a. Write an expression for the distance from 2^{nd} base to home plate in feet. What is this distance to the nearest tenth?

b. The pitcher standing on the pitcher's mound is about to throw to home plate but turns around and throws to 2^{nd} base. How much farther is the throw? Explain.

52. **Apply Math Models** A television has a ratio of width to height of about 1.732 : 1.

a. For a television with a height of h inches, what is an equivalent expression for the length of the diagonal? Justify your answer.

b. Write an expression for the perimeter.

ASSESSMENT PRACTICE

53. Which pair of values for a and b makes the following statement true? NS0.1.4

$$\sqrt[3]{(-2)^a \cdot (-11)^b} = 22\sqrt[3]{22}$$

Ⓐ $a = -4$, $b = -4$

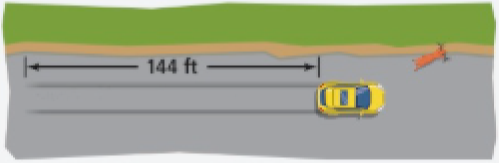
Ⓑ $a = 1$, $b = -1$

Ⓒ $a = 1$, $b = 1$

Ⓓ $a = 2$, $b = 2$

Ⓔ $a = 4$, $b = 4$

54. **SAT/ACT** A car skidded s ft when traveling on a damp paved road. The expression $r = \sqrt{18s}$ is an estimate of the car's rate of speed in ft/s.



Which expression represents the speed of the car in feet per second?

Ⓐ $24\sqrt{6}$

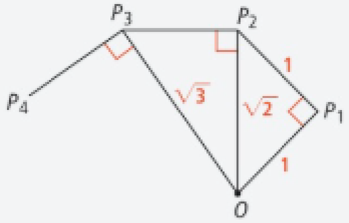
Ⓑ $12\sqrt{6}$

Ⓒ $36\sqrt{2}$

Ⓓ $24\sqrt{3}$

Ⓔ $48\sqrt{2}$

55. **Performance Task** Copy the figure. Center it on a large piece of paper so you can expand it.



Part A Use the pattern to complete the triangle on the left. Label the side lengths.

Part B Continue using the pattern to add triangles while labeling side lengths.

Part C Are equivalent expressions of the square roots appropriate? Explain your reasoning.

5-3

Exponential Functions

I CAN… describe and graph exponential functions.

VOCABULARY

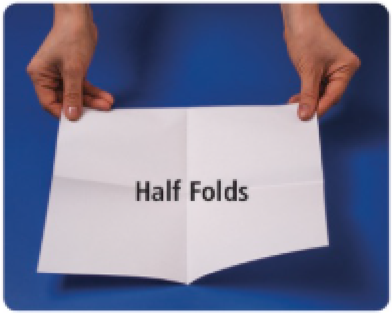
- asymptote
- constant ratio
- exponential function

MA.912.AR.5.6–Given a table, equation or written description of an exponential function, graph that function and determine its key features. **Also AR.5.4, F.1.1, F.1.2, F.1.8**

MA.K12.MTR.1.1, MTR.5.1, MTR.7.1

EXPLORE & REASON

Use two pieces of $8\frac{1}{2}$ in.-by-11 in. paper. Fold one of the pieces of paper accordion-style for five folds. Fold the other in half for five folds. After each fold, unfold each piece of paper and count the total number of rectangular sections.



A. Find the pattern relating the number of folds to the number of sections for each folding style. What do you notice?

B. **Analyze and Persevere** Explain why the two different folded styles of paper produce different results.

ESSENTIAL QUESTION

What are the characteristics of exponential functions?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Key Features of $f(x) = 2^x$

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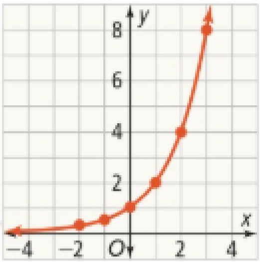
A. What does the graph of $f(x) = 2^x$ look like?

The table and graph show $f(x) = 2^x$.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

As x-values approach $-\infty$, y-values approach 0.

The y-intercept is 1.



COMMON ERROR

The graph of $f(x) = 2^x$ gets infinitely close to the x-axis, but will never reach it. So it does not have an x-intercept.

B. What are the key features of the graph of $f(x) = 2^x$?

The graph of $f(x) = 2^x$ is continuous between and beyond the x-values shown, so the domain is $\{x \mid x \text{ is a real number}\}$. The function is increasing over its entire domain. Because it is defined by powers of a positive number, the function is always positive.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

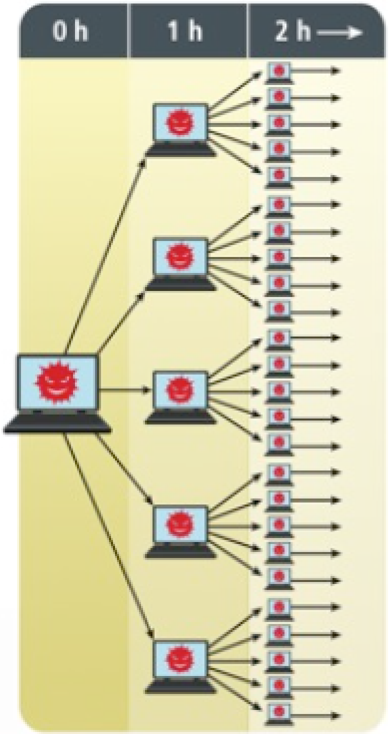
The function gets closer and closer to the x-axis, but never quite reaches it. When a function approaches a line in this manner, the line is called an **asymptote.** The asymptote of f is $y = 0$. The range is $\{y \mid y > 0\}$.

Try It! 1. Identify the following key features of the function $f(x) = b^x$ for $b = 2$ and $b = \frac{1}{2}$; domain, range, intercepts, increasing and decreasing intervals, positive and negative intervals, and asymptotes.

APPLICATION

EXAMPLE 2 Graph Exponential Functions

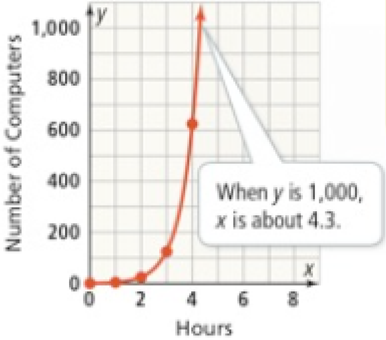
A network administrator uses the function $f(x) = 5^x$ to model the number of computers a virus spreads to after x hours. If there are 1,000 computers on the network, about how many hours will it take for the virus to spread to the entire network?



Step 1 Make a table.

x	$f(x) = 5^x$
0	1
1	5
2	25
3	125
4	625
5	3,125

Step 2 Graph the function.



STUDY TIP
When graphing an exponential function, choose a scale for the vertical axis so that the relevant domain is shown.

Step 3 Use a calculator to check if your answer, $5^{4.3} = 1{,}012.91$, is close to 1,000. ✓

$5^{4.3}$

1012.910373

It will take the virus about 4.3 hours to spread to the entire network.

Try It! 2. How long will it take for the virus to spread to 50,000 computers?

CONCEPT Exponential Function

An **exponential function** is the product of an initial amount and a **constant ratio** raised to a power. Exponential functions are modeled using $f(x) = a \cdot b^x$, where a is a nonzero constant, $b > 0$, and $b \neq 1$.

$$f(x) = a \cdot b^x$$

a is the initial amount.

b is the constant ratio.

EXAMPLE 3 Write Exponential Functions

STUDY TIP
Note that exponential functions have constant ratios rather than constant differences.

A. What is the written form of the function represented by the table?

x	$f(x)$
0	4
1	12
2	36
3	108
4	324

The initial amount is 4.

$12 \div 4 = 3$

$36 \div 12 = 3$

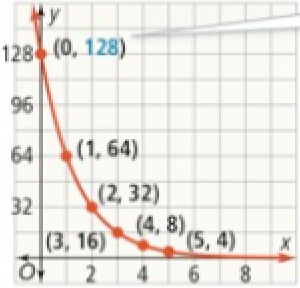
$108 \div 36 = 3$

$324 \div 108 = 3$

The constant ratio is 3.

In $f(x) = a \cdot b^x$, substitute 4 for a and 3 for b.
The function is $f(x) = 4(3)^x$.

B. What is the written form of the function represented by the graph?



The initial amount is 128.

$64 \div 128 = \frac{1}{2}$

$32 \div 64 = \frac{1}{2}$

$16 \div 32 = \frac{1}{2}$

$8 \div 16 = \frac{1}{2}$

$4 \div 8 = \frac{1}{2}$

The constant ratio is $\frac{1}{2}$.

In $f(x) = a \cdot b^x$, substitute 128 for a and $\frac{1}{2}$ for b.
The function is $f(x) = 128\left(\frac{1}{2}\right)^x$.

Try It! **3.** Write an exponential function for each set of points.

a. (0, 3), (1, 12), (2, 48), (3, 192), and (4, 768)

b. (0, 2,187), (1, 729), (2, 243), (3, 81), and (4, 27)

APPLICATION

EXAMPLE 4 Compare Linear and Exponential Functions

Talisha is offered two pledge options for donating to a charity. Which option will increase the pledge amount faster over time?

Option A: \$100 for the first week, and each week after that the amount increases by \$25

Week	Payment (\$)
0	100
1	125
2	150
3	175
4	200
5	225

Initial value

+25
+25
+25
+25
+25

Constant increase

Option B: \$1 for the first week, and each week after that the amount triples

Week	Payment (\$)
0	1
1	3
2	9
3	27
4	81
5	243

Initial value

×3
×3
×3
×3
×3

Constant ratio

Option A is a linear function and increases at a constant rate.

Since the ratio of consecutive terms in Option B is constant, the exponential function will increase faster over time.

REPRESENT AND CONNECT
Will an exponential model with a base greater than 1 always have a greater rate of change over time? Explain.

Try It! 4. Identify each function as linear or exponential. Explain.

a. $f(x)$ equals the number of branches at level x in a tree diagram, where at each level each branch extends into 4 branches.

b. $f(x)$ equals the number of boxes in row x of a stack in which each row increases by 2 boxes.

CONCEPT SUMMARY Exponential Functions

Exponential Functions are modeled using $f(x) = a \cdot b^x$, where a is the initial amount and b is the constant ratio.

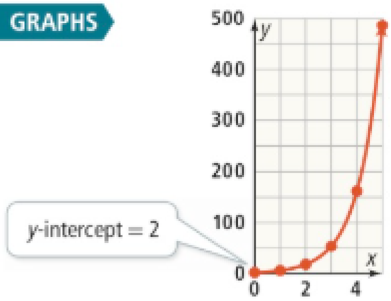
	$b > 1$	$0 < b < 1$
ALGEBRA	$f(x) = 2(3)^x$	$f(x) = 81\left(\frac{2}{3}\right)^x$

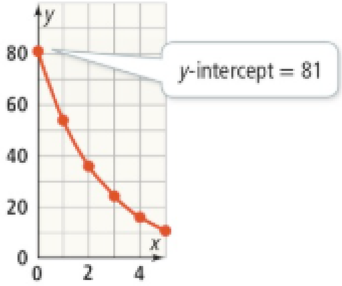
TABLES

x	f(x)	
0	2	
1	6	× 3
2	18	× 3
3	54	× 3
4	162	× 3
5	486	× 3

x	f(x)	
0	81	
1	54	$\times \frac{2}{3}$
2	36	$\times \frac{2}{3}$
3	24	$\times \frac{2}{3}$
4	16	$\times \frac{2}{3}$
5	10.7	$\times \frac{2}{3}$

GRAPHS





Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are the characteristics of exponential functions?

2. **Generalize** How can you tell whether the graph of a function of form $f(x) = ab^x$ where $a > 0$ will increase or decrease from left to right?

3. **Analyze and Persevere** Why is $b \neq 1$ a condition for $f(x) = ab^x$ to be considered an exponential function?

4. **Error Analysis** Martin says that $f(x) = 2(4)^x$ starts at 4 and has constant ratio of 2. What error did Martin make? Explain.

Do You KNOW HOW?

Graph each function.

5. $f(x) = 3^x$

6. $f(x) = \left(\frac{1}{4}\right)^x$

Write each exponential function.

7.

x	f(x)
0	4
1	2
2	1
3	$\frac{1}{2}$
4	$\frac{1}{4}$

8.

x	f(x)
0	3
1	6
2	12
3	24
4	48

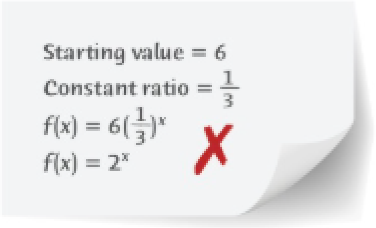
PRACTICE & PROBLEM SOLVING

UNDERSTAND

9. **Analyze and Persevere** An exponential function of form $f(x) = b^x$ includes the points (2, 16), (3, 64), and (4, 256). What is the value of b?

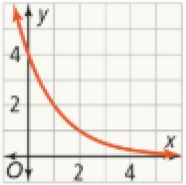
10. **Communicate and Justify** Is $y = 0$ the asymptote of all functions of the form $f(x) = ab^x$? Explain your reasoning.

11. **Error Analysis** Describe and correct the error a student made in writing an exponential function.



Starting value = 6
Constant ratio = $\frac{1}{3}$
$f(x) = 6(\frac{1}{3})^x$
$f(x) = 2^x$

12. **Use Patterns and Structure** The function $f(x) = a\left(\frac{1}{2}\right)^x$ is graphed below for $a = 4$. Describe how the graph would change for $a > 4$ and $1 < a < 4$.



13. **Higher Order Thinking** The exponential function $f(x) = 2^x$ increases as x increases. Do all exponential functions behave this way? Use algebraic reasoning to support your answer.

14. **Use Patterns and Structure** How do the following key features of $f(x) = ab^x$ change if $a < 0$, compared to when $a > 0$.

a. domain and range

b. intercepts

c. increasing or decreasing intervals

d. positive or negative intervals

e. asymptotes

PRACTICE

For each exponential function, identify the domain, range, intercepts, intervals over which the function is increasing, decreasing, positive or negative, and asymptotes. SEE EXAMPLE 1

15. $f(x) = 4^x$

16. $f(x) = \left(\frac{1}{3}\right)^x$

Graph each exponential function. SEE EXAMPLE 2

17. $f(x) = 0.5^x$

18. $f(x) = 6^x$

19. $f(x) = 2(3)^x$

20. $f(x) = 4\left(\frac{1}{2}\right)^x$

Write each exponential function. SEE EXAMPLE 3

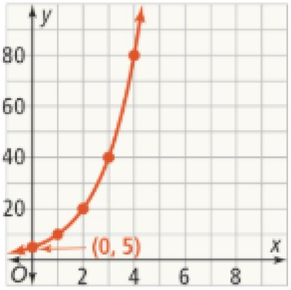
21.

x	$f(x)$
0	2
1	8
2	32
3	128
4	512

22.

x	$f(x)$
0	4
1	$\frac{4}{3}$
2	$\frac{4}{9}$
3	$\frac{4}{27}$
4	$\frac{4}{81}$

23.



Tell whether each function is linear or exponential. Explain your reasoning. SEE EXAMPLE 4

24.

x	$f(x)$
0	5
1	9
2	13
3	17
4	21

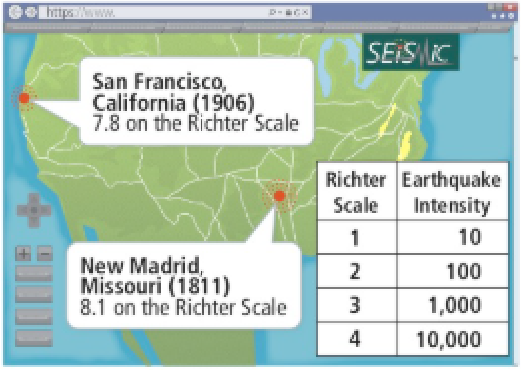
25.

x	$f(x)$
0	216
1	36
2	6
3	1
4	$\frac{1}{6}$

PRACTICE & PROBLEM SOLVING

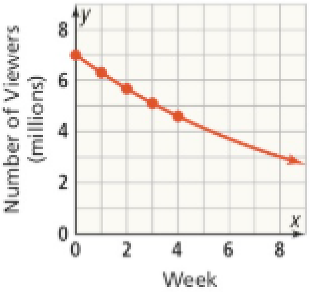
APPLY

26. **Analyze and Persevere** Write an exponential function to model earthquake intensity as a function of a Richter Scale number. How can you use your function to compare the intensity of the 1811 New Madrid and 1906 San Francisco earthquakes?



Richter Scale	Earthquake Intensity
1	10
2	100
3	1,000
4	10,000

27. **Apply Math Models** A television show will be canceled if the estimated number of viewers falls below 2.5 million by Week 10. Use the graph to write an exponential function to model the situation. If this pattern continues, will the show be canceled?



28. **Apply Math Models** The table shows the number of algae cells in pool water samples. A pool will turn green when there are 24 million algae cells or more. Write and graph an exponential function to model the expected number of algae cells as a function of the number of days. If the pattern continues, in how many days will the water turn green?

Day	Number of Algae Cells
0	2000
1	10,000
2	50,000
3	250,000
4	1,250,000

ASSESSMENT PRACTICE

29. A school uses a phone tree to communicate updates to families. First, the 3 school administrators decide when an announcement is needed. They each call 5 families in the first round of calls. Then each of those families calls 5 families in the second round. In every round, each family that received a call in the previous round makes calls to 5 new families. Let x be the number of rounds, and let y be the number of calls made during each round. Write and graph a function that models this situation. Label the y-intercept, and explain what it represents. AR.5.4

30. **SAT/ACT** What is the y-intercept of $f(x) = 8\left(\frac{1}{2}\right)^x$?

Ⓐ 0

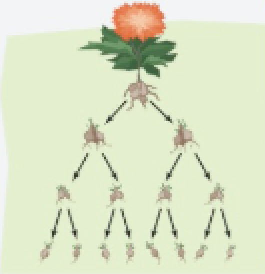
Ⓑ $\frac{1}{2}$

Ⓒ 1

Ⓓ 2

Ⓔ 8

31. **Performance Task** A gardener can increase the number of dahlia plants in an annual garden by either buying new bulbs each year or dividing the existing bulbs to create new plants. The table shows the expected number of bulbs for each method.



Year	Buy New Bulbs	Divide Existing Bulbs
0	6	6
1	56	12
2	106	24
3	156	48
4	206	96

Part A For each method, write a function to model the expected number of plants for each year.

Part B Use your functions to find the expected number of plants in 10 years for each method.

Part C How does the expected number of plants in five years compare to the expected number of plants in 15 years? Explain how these patterns could affect the method the gardener decides to use.

5-4

Exponential Growth and Decay

I CAN… use exponential functions to model situations and make predictions.

VOCABULARY

- compound interest
- decay factor
- exponential decay
- exponential growth
- growth factor

MA.912.AR.5.4–Write an exponential function to represent a relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. **Also NSO.1.2, AR.1.1, AR.5.3, F.1.2, F.1.3, FL.3.2, FL.3.4**

MA.K12.MTR.2.1, MTR.6.1, MTR.7.1

CONCEPTUAL UNDERSTANDING

MODEL & DISCUSS

Cindy is buying a new car and wants to learn how the value of her car will change over time. Insurance actuaries predict the future value of cars using depreciation functions. One such function is applied to the car whose declining value is shown at the right.

Years After Purchase	Value
0 yr	 $10,000
1 yr	 $8,520
2 yr	 $7,213
3 yr	 $6,100
4 yr	 $5,210

A. Describe how the value of the car decreases from year to year.

B. **Represent and Connect** What kind of function would explain this type of pattern?

C. Given your answer to Part B, what is needed to find the function the actuary is using? Explain.

ESSENTIAL QUESTION

What kinds of situations can be modeled with exponential growth or exponential decay functions?

EXAMPLE 1 Exponential Growth

The population of Hillville grows at an annual rate of 15%. What will the estimated population of Hillville be in five years?

You can model **exponential growth** with a function $f(x) = a \cdot b^x$, where $a > 0$ and $b > 1$.

$$f(x) = a \cdot b^x$$

$$f(x) = a(1 + r)^x$$

An exponential growth function has a **growth factor** that is equal to 1 plus the growth rate.

VOCABULARY

You can also refer to the annual growth rate of 15% as the *constant percent rate of change.* It is expressed as a decimal when used in the exponential growth function.

Step 1 Write the exponential growth function that models the expected population growth.

Let x = time in years, a = initial amount, and r = growth rate.

$$\begin{aligned} f(x) &= a(1 + r)^x \\ &= 5{,}000(1 + 0.15)^x \\ &= 5{,}000(1.15)^x \end{aligned}$$

The function is $f(x) = 5{,}000(1.15)^x$.

CONTINUED ON THE NEXT PAGE

COMMON ERROR
You may incorrectly record a decimal number when you are finding "how many" people. Remember that the number of people should be a whole number.

EXAMPLE 1 CONTINUED

Step 2 Find the expected population in 5 years.

$$f(5) = 5{,}000(1.15)^5$$
$$\approx 10{,}056.79$$

In 5 years, the population is expected to be about 10,057.

Try It! **1.** The population of Valleytown is also 5,000, with an annual increase of 1,000. Can the expected population for Valleytown be modeled with an exponential growth function? Explain.

CONCEPT Interest

Interest is calculated in two ways: simple interest and compound interest. Recall that simple interest is interest paid only on the principal.

Compound interest is interest that is paid both on the principal and on the interest that has already been paid.

The table shows \$500 compounded annually at a rate of 4% for 3 years.

Year	Principal and Interest (\$)	Total (\$), A
1	500	$500(1 + 0.04)$
2	$500(1 + 0.04)$	$500(1 + 0.04)^2$
3	$500(1 + 0.04)^2$	$500(1 + 0.04)^3$

The total grows by a constant percent rate. The exponential function $A = 500(1 + 0.04)^t$ models the total as a function of time in years.

The compound interest formula is an exponential growth function and can be written in the general form shown below.

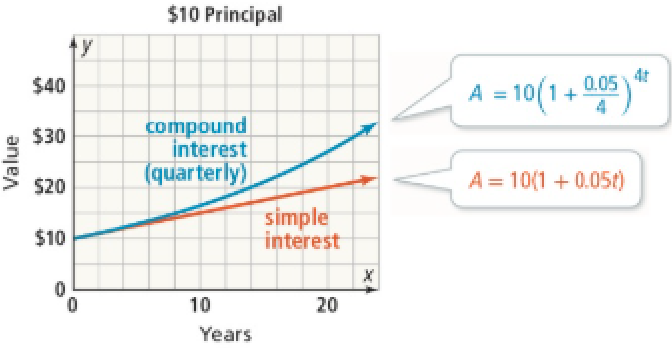
REPRESENT AND CONNECT
How is the compound interest formula related to $f(x) = a(1 + r)^x$? How is it related to $f(x) = ab^x$?

Compound Interest Formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A = amount paid
P = principal amount
r = rate of interest
n = number of times per year the interest is compounded
t = time in years

The graph below shows \$10 at 5% simple interest and at 5% interest compounded quarterly.



APPLICATION

EXAMPLE 2 Exponential Models of Growth

Kimberly's family invested in a Certificate of Deposit (CD) for her when she was born. The interest is compounded quarterly.

CERTIFICATE
of Deposit

Amount Deposited: $3,000.00
Annual Interest: 8%
Interest compounded quarterly

A. What is the value of the CD at the end of five years?

Use the compound interest formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$= 3{,}000\left(1 + \frac{0.08}{4}\right)^{4t}$$

The principal amount is 3,000. The rate of interest is 8%, or 0.08. The number of times per year the interest is calculated is 4.

$$= 3{,}000(1 + 0.02)^{4t}$$

The 8% interest is paid over 4 periods, so 2% interest is paid each period.

$$= 3{,}000(1.02)^{4t}$$

$$= 3{,}000(1.02)^{4(5)}$$

$$= 4{,}457.84$$

At the end of five years, the value of the CD will be $4,457.84.

CHECK FOR REASONABLENESS
Use the order of operations when entering your numbers on a calculator. How can the omission of parentheses change your answer?

B. Will Kimberly's CD earn more money if her interest is compounded annually rather than quarterly?

Use the laws of exponents to compare $A = P\left(1 + \frac{r}{1}\right)^{1t}$ with $A = P\left(1 + \frac{r}{4}\right)^{4t}$, with $r = 0.08$.

Annually

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$= P\left(1 + \frac{0.08}{1}\right)^{1t}$$

$$= P(1 + 0.08)^{t}$$

$$= P(1.08)^{t}$$

Quarterly

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$= P\left(1 + \frac{0.08}{4}\right)^{4t}$$

$$= P(1 + 0.02)^{4t}$$

$$= P(1.02)^{4t}$$

$$= P((1.02)^{4})^{t}$$

Use the Power of a Power Property to write 1.02^{4t} as $((1.02)^{4})^{t}$.

$$= P(1.0824316)^{t}$$

CHECK FOR REASONABLENESS
Why does it make sense that there is a relatively small difference in the two interest rates?

Compounding an 8% annual rate quarterly is roughly equivalent to an 8.24% annual interest rate. Kimberly will earn more money if her interest is compounded quarterly.

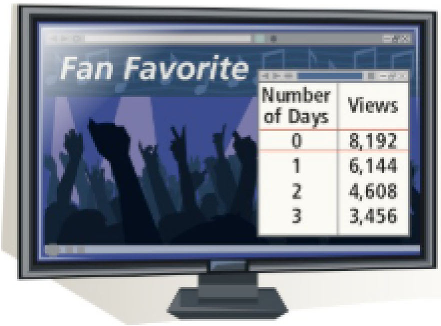
Try It! **2. a.** What will be the difference after 15 years if the interest is compounded semiannually rather than quarterly?

b. A CD earns 4% annual interest, compounded monthly. What is the equivalent annual interest rate?

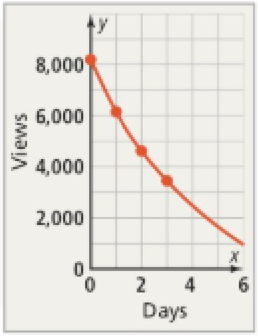
APPLICATION

EXAMPLE 3 Exponential Decay

A video is labeled a fan favorite if it receives at least 1,000 views per day. Amelia posts a video that gets 8,192 views on the first day. The number of views decreases by 25% each day after that. In how many days total will the video stop being a fan favorite?



Number of Days	Views
0	8,192
1	6,144
2	4,608
3	3,456



GENERALIZE
How does the exponential decay function differ from the exponential growth function?

Because the number of views decrease rather increase, by 25% each day, the situation is modeled by *exponential decay*. You can model **exponential decay** with a function of the form $f(x) = a \cdot b^x$, where $a > 0$ and $0 < b < 1$.

$$f(x) = a \cdot b^x$$

$$f(x) = a(1 - r)^x$$

r is the decay rate.

The **decay factor** is 1 minus the decay rate.

Step 1 Model the situation.

Let x = time in years, a = initial amount, and r = decay rate.

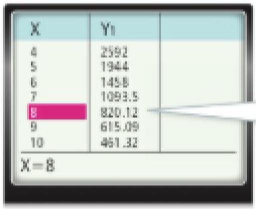
$$\begin{aligned} f(x) &= a(1 - r)^x \\ &= 8{,}192(1 - 0.25)^x \\ &= 8{,}192(0.75)^x \end{aligned}$$

The decay factor is written as a decimal less than 1.

Step 2 Write an equation to find x if $f(x) = 1{,}000$.

$$1{,}000 = 8{,}192(0.75)^x$$

Step 3 Estimate the solution of the equation using a graphing calculator.



X	Y_1	
4	2592	
5	1944	
6	1458	
7	1093.5	
8	820.12	
9	615.09	
10	461.32	

X=8

Use the table feature to find the value of x when $y < 1{,}000$.

The video will stop being a fan favorite in 8 days.

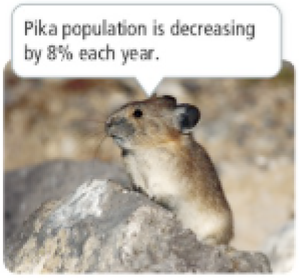
Try It! **3. a.** What is the constant percent rate of change in Example 3?

b. Suppose the number of views increases by 10% per day. Is the situation modeled by exponential growth or decay?

APPLICATION

EXAMPLE 4 Exponential Models of Decay

The number of pikas in a region is decreasing. How does the decrease in the pika population for years 1 to 5 compare to the population for years 6 to 10?



Initial population: 144 pikas

Write an exponential decay function to model the situation.

$$f(x) = a(1 - r)^x$$
$$= 144(1 - 0.08)^x$$
$$f(x) = 144(0.92)^x$$

Find the average rate of change for each interval.

Year 1 to Year 5: $1 \leq x \leq 5$

$f(1) \approx 132$ $f(5) \approx 95$

$$\frac{95 - 132}{5 - 1} = \frac{-37}{4} = -9.25$$

Year 6 to Year 10: $6 \leq x \leq 10$

$f(6) \approx 87$ $f(10) \approx 63$

$$\frac{63 - 87}{10 - 6} = \frac{-24}{4} = -6$$

STUDY TIP
Use the slope formula to find the average rate of change between two points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

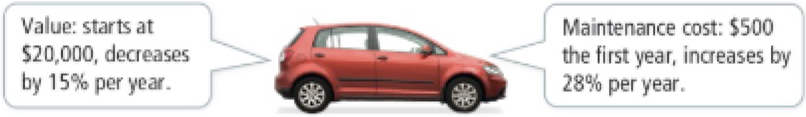
For years 1 to 5, the pika population decreases by an average of 9.25 pikas per year. For years 6 to 10 the pika population decreases by an average of 6 pikas per year. The average rate of change for the pika population decreases as years increase.

Try It! **4.** How would the average rate of change over the same intervals be affected if the population increased at a rate of 8%?

APPLICATION

EXAMPLE 5 Exponential Growth and Decay

Rich is comparing the cost of maintaining his car with the depreciating value of the car. When will the cost and value be the same?



Formulate ◀ Write the exponential functions.

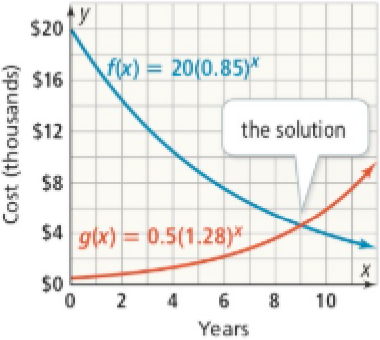
Value of the car: $f(x) = 20(0.85)^x$

Cost of maintenance: $g(x) = 0.5(1.28)^x$

Compute ◀ Solve by graphing.

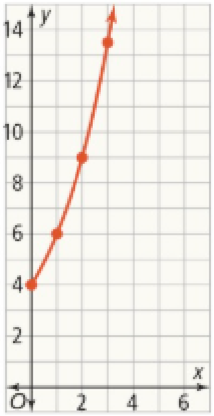
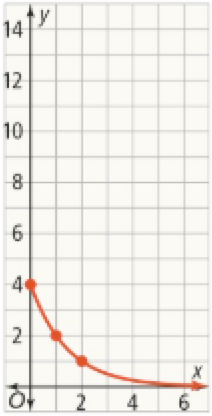
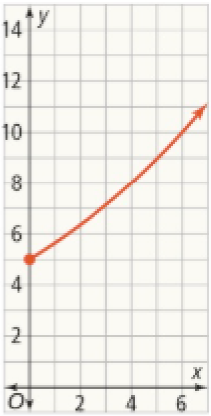
Find the point of intersection: (9, 4.5).

Interpret ◀ The value of the car and the cost of maintenance are both about $4,500 at 9 years.



Try It! **5.** Explain how to use tables on a graphing calculator to answer this question.

CONCEPT SUMMARY Exponential Growth and Decay

	Exponential Growth	Exponential Decay	Compound Interest
ALGEBRA	$f(x) = a \cdot b^x$ $f(x) = a(1 + r)^x$	$f(x) = a \cdot b^x$ $f(x) = a(1 - r)^x$	$f(x) = a \cdot b^x$ $A = P\left(1 + \frac{r}{n}\right)^{nt}$
NUMBERS	$f(x) = 4(1 + 0.5)^x$	$f(x) = 4(1 - 0.5)^x$	$A = 5\left(1 + \frac{0.12}{2}\right)^{2t}$
WORDS	initial value: 4 growth rate: 50% growth factor: 1.5	initial value: 4 decay rate: 50% decay factor: 1 – 0.5 = 0.5	principal: 5 annual interest rate: 12% periods per year: 2
GRAPHS			

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What kinds of situations can be modeled with exponential growth or exponential decay functions?

2. **Vocabulary** What is the difference between simple interest and *compound interest*?

3. **Error Analysis** LaTanya says that the growth factor of $f(x) = 100(1.25)^x$ is 25%. What mistake did LaTanya make? Explain.

4. **Use Patterns and Structure** Why is the growth factor $1 + r$ for an exponential growth function?

Do You KNOW HOW?

Write an exponential growth or decay function for each situation.

5. initial value of 100 increasing at a rate of 5%

6. initial value of 1,250 increasing at a rate of 25%

7. initial value of 512 decreasing at a rate of 50%

8. initial value of 10,000 decreasing at a rate of 12%

9. What is the difference in the value after 10 years of an initial investment of \$2,000 at 5% annual interest when the interest is compounded quarterly rather than annually?

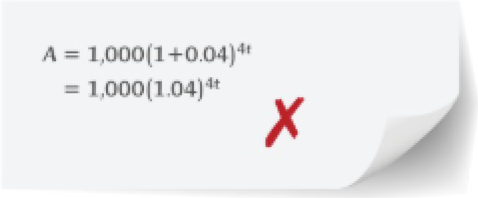
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Use Patterns and Structure** How is an exponential growth function of the form $f(x) = a(1 + r)^x$ related to an exponential function of the form $f(x) = a \cdot b^x$?

11. **Generalize** What is the asymptote of the graph of an exponential growth or exponential decay function? Explain your reasoning.

12. **Error Analysis** Describe and correct the error a student made in writing an equation to find the annual value of an investment of $1,000 at 4% annual interest compounded semiannually.



$$A = 1{,}000(1+0.04)^{4t}$$
$$= 1{,}000(1.04)^{4t}$$

13. **Choose Efficient Methods** Describe how you could use a graphing calculator to estimate the value of x when $f(x) = 8(1.25)^x$ equals 15.

14. **Represent and Connect** The population of a town over several years is shown in the table below.

Year	Population	Percent Change
0	2400	
1	2352	■
2	2305	■
3	2259	■
4	2214	■

a. Copy and complete the table. Round the percent changes to nearest percent.

b. Why can the population be modeled by an exponential function? Would it be a growth or decay function? Explain.

c. Write the function.

15. **Communicate and Justify** Explain how to use the laws of exponents to determine if an investment earning 5% annual interest would earn more compounded monthly or compounded quarterly.

PRACTICE

Write an exponential growth function to model each situation. SEE EXAMPLE 1

16. initial value: 20
growth factor: 1.25

17. initial value: 100
growth factor: 1.05

Compare each investment to an investment of the same principal at the same rate compounded annually. SEE EXAMPLE 2

18. principal: $8,000
annual interest: 6%
interest periods: 4
number of years: 20

19. principal: P
annual interest: 3.5%
interest periods: 2
number of years: t

Write an exponential decay function to model each situation. Then estimate the value of x for the given value of $f(x)$. SEE EXAMPLE 3

20. initial value: 100
decay factor: 0.95
$f(x) = 60$

21. initial value: 5,000
decay factor: 0.7
$f(x) = 100$

Write an exponential decay function to model each situation. Compare the average rates of change over the given intervals. SEE EXAMPLE 4

22. initial value: 50
decay factor: 0.9
$1 \leq x \leq 4$ and
$5 \leq x \leq 8$

23. initial value: 25
decay factor: 0.8
$2 \leq x \leq 4$ and
$6 \leq x \leq 8$

Write an exponential function to model the data in each table. Classify the function as representing growth or decay. SEE EXAMPLES 1–4

24.

x	$f(x)$
0	16
1	8
2	4
3	2
4	1

25.

x	$f(x)$
0	100
1	110
2	121
3	133.1
4	146.41

Model each pair of situations with exponential functions f and g. Find the approximate value of x that makes $f(x) = g(x)$. SEE EXAMPLE 5

26. f: initial value of 100 decreasing at a rate of 5%
g: initial value of 20 increasing at a rate of 5%

27. f: initial value of 40 increasing at a rate of 25%
g: initial value of 10,000 decreasing at a rate of 16%

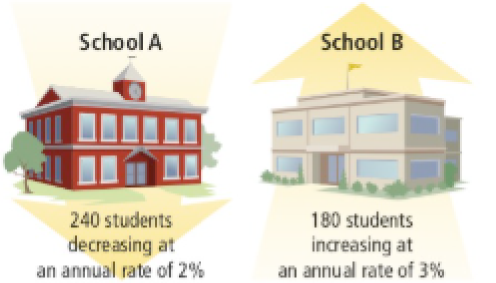
PRACTICE & PROBLEM SOLVING

APPLY

28. **Apply Math Models** A plant will become invasive when the number of plants reaches 10,000. Model the situation with an exponential growth function. How many years will it take for the plant to become invasive? Explain how you found the solution.

29. **Apply Math Models** Joshua invests \$500 at the interest rate shown. Felix invests \$1,000 in an account with the same compounding, but at 6% interest rate. Model each investment with an exponential growth function. Whose money will double first? Explain.

30. **Analyze and Persevere** Write and graph exponential functions to model the number of students at School A and at School B as a function of number of years. In about how many years will the number of students at both schools be approximately the same? Explain how you can use a graph to determine the answer.



ASSESSMENT PRACTICE

31. Select all the functions that are exponential growth functions. AR.5.3

☐ **A.** $f(x) = 2(1.02)^x$

☐ **B.** $f(x) = 5000(3)^x$

☐ **C.** $f(x) = 7500(0.91)^x$

☐ **D.** $f(x) = 189(1 - 0.25)^x$

☐ **E.** $f(x) = 2485(1 + 0.25)^x$

32. **SAT/ACT** Which function models the value in *x* years of an investment at 3% annual interest compounded quarterly?

Ⓐ $150(1 - 0.03)^{4x}$

Ⓑ $150(1 + 0.03)^{4x}$

Ⓒ $150(1 - 0.03)^x$

Ⓓ $150(1 + 0.0075)^{4x}$

Ⓔ $150(1 - 0.0075)^x$

33. **Performance Task** Isabel has \$10,000 to invest. She is choosing between the three investment opportunities shown.

Investment	Annual Interest	Number of Interest Periods
A	4%	1
B	4%	4
C	4.2%	1

Part A Write a function for each investment to model its value in *x* years.

Part B Suppose Isabel only wants to invest her money for five years. Which investment will have the greatest value in five years?

Part C Which investment will make Isabel a millionaire first?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.5.4–Write an exponential function to represent a relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.

MA.K12.MTR.7.1

Big Time Pay Back

Most people agree that investing your money is a good idea. Some people might advise you to put money into a bank savings account. Other people might say that you should invest in the stock market. Still others think that buying bonds is the best investment option.

Is a bank savings account a good way to let your money grow? Just how much money can you make from a savings account? In the Mathematical Modeling in 3 Acts lesson, you'll see an intriguing situation about an investment option.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. Write a number that you know is too small.
6. Write a number that you know is too large.
7. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

8. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

9. Is your refined conjecture between the highs and lows you set up earlier?
10. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC
5

Topic Review

TOPIC ESSENTIAL QUESTION

1. How do you use exponential functions to model situations and solve problems?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A population's growth can be modeled by a(n) __________ function of the form $f(x) = a \cdot b^x$, where $a > 0$ and $b > 1$.

3. An exponential function repeatedly multiplies an initial amount by the same positive number, called the __________.

4. In the exponential function $f(x) = a\,(1 - r)^x$, the term $(1 - r)$ is called the __________.

5. __________ is interest that is paid both on the principal and on the interest that has already been paid.

6. As x or y gets larger in absolute value, the graph of the exponential function gets closer to the line called a(n) __________.

- constant ratio
- simple interest
- decay factor
- compound interest
- exponential decay
- exponential growth
- exponential function
- asymptote
- growth factor

Concepts & Skills Review

LESSON 5-1 Rational Exponents and Properties of Exponents

Quick Review

If the nth root of a is a real number and m is an integer, then $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.

Power of a Power: $(a^m)^n = a^{mn}$

Power of a Product: $(a \cdot b)^m = a^m b^m$

Product of Powers: $a^m \cdot a^n = a^{m+n}$

Quotient of Powers: $\frac{a^m}{a^n} = a^{m-n}$, $a \neq 0$

Example

Find the product of and $9^{\frac{1}{3}}$ and $9^{\frac{8}{3}}$.

Use the Product of Powers Property. Simplify the rational exponent, and then the power.

$$\begin{aligned}\left(9^{\frac{1}{3}}\right)\left(9^{\frac{8}{3}}\right) &= (9)^{\frac{1}{3}+\frac{8}{3}}\\ &= 9^{\frac{9}{3}}\\ &= 9^3\\ &= 729\end{aligned}$$

The product of $9^{\frac{1}{3}}$ and $9^{\frac{8}{3}}$ is 729.

Practice & Problem Solving

Write each radical using rational exponents.

7. $\sqrt{8}$

8. $\sqrt[3]{12}$

Use the laws of exponents to simplify each expression.

9. $\left(6^{\frac{13}{5}}\right)\left(6^{-\frac{3}{5}}\right)$

10. $\frac{27^{\frac{7}{3}}}{27^{\frac{5}{3}}}$

11. **Represent and Connect** Describe two ways to express the edge length of a cube with a volume of 64 cm^3.

12. **Apply Math Models** Use rational exponents to express the relationship between the dollar values of two prizes in a contest.

Prize	Value
Bicycle	$256
Luxury vehicle	$65, 536

LESSON 5-2 Radical Expressions

Quick Review

The Product Property of Square Roots states that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, when $a \geq 0$ and $b \geq 0$. The Product Property of Cube Roots states that $\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b}$, with no restrictions on the signs of a or b.

Example

Write an expression for $5\sqrt{3x} \cdot 2\sqrt{12x^3}$ without any perfect squares in the radicand.

$5\sqrt{3x} \cdot 2\sqrt{12x^3}$ .... Multiply the constants, and use the Product Property of Square Roots to multiply the radicands.

$= 5 \cdot 2\sqrt{3x \cdot 12x^3}$

$= 10\sqrt{36x^4}$ .... Simplify.

$= 10 \cdot 6 \cdot x^2$ .... Simplify.

$= 60x^2$

The expression $5\sqrt{3x} \cdot 2\sqrt{12x^3}$ is equivalent to $60x^2$.

Practice & Problem Solving

Write each expression in simplest form.

13. $\sqrt{420}$

14. $4\sqrt{84}$

15. $\sqrt[3]{54}$

16. $11\sqrt[3]{-16}$

Write each product or quotient in simplest form.

17. $\sqrt{8} \cdot \sqrt{20}$

18. $\frac{\sqrt{108}}{\sqrt{12}}$

19. $\frac{6\sqrt[3]{16}}{\sqrt[3]{32}}$

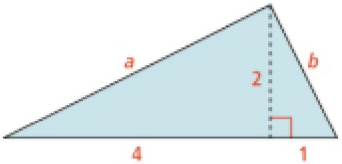
20. $2\sqrt{35} \cdot \sqrt{45}$

Write each sum or difference in simplest form.

21. $3\sqrt{17} + 2\sqrt{17}$

22. $5\sqrt{80} - 2\sqrt{96}$

23. **Analyze and Persevere** Write the sum of a and b in simplest form. Then write an expression for the perimeter of the triangle.



LESSON 5-3 Exponential Functions

Quick Review

An exponential function is the product of an initial amount and a **constant ratio** raised to a power. Exponential functions are expressed using $f(x) = a \cdot b^x$, where a is a nonzero constant, $b > 0$, and $b \neq 1$.

Example

Find the initial amount and the constant ratio of the exponential function represented by the table.

x	$f(x)$
0	3
1	12
2	48
3	192
4	768

The initial amount is 3.

$12 \div 3 = 4$

$48 \div 12 = 4$

$192 \div 48 = 4$

$768 \div 192 = 4$

The constant ratio is 4.

In $f(x) = a \cdot b^x$, substitute 3 for a and 4 for b.

The function is $f(x) = 3(4)^x$.

Practice & Problem Solving

Describe the domain, range, intercepts, and constant ratio for each exponential function.

24. $f(x) = -4(6)^x$

25. $f(x) = 8\left(\frac{1}{7}\right)^x$

Graph each exponential function.

26. $f(x) = 2.5^x$

27. $f(x) = 5(2)^x$

28. Write the exponential function for values shown in the table.

x	0	1	2	3
$f(x)$	0.5	1	2	4

Tell whether each function is linear or exponential. Explain your reasoning.

29.

x	$f(x)$
0	5
1	15
2	45
3	135
4	405

30.

x	$f(x)$
0	5
1	15
2	25
3	35
4	45

LESSON 5-4 Exponential Growth and Decay

Quick Review

An **exponential growth function** can be written as $f(x) = ab^x$, where $a > 0$ and $b > 1$. You can express the **growth factor**, b as $(1 + r)$, with r being the **growth rate**. The resulting function is $f(x) = a(1 + r)^x$.

If $a > 0$ and $0 < b < 1$ then $f(x) = ab^x$ is an **exponential decay function**. The **decay factor** b can be written as $(1 - r)$, with r being the **decay rate**. The resulting function can be written as $f(x) = a(1 - r)^x$.

Compound interest is special application of exponential growth. Compound interest is paid both on the principal and on the interest that has already been paid.

Example

Chapter City has a population of 18,000 and grows at an annual rate of 8%. What is the estimated population of Chapter City in 6 years?

Let x = time in years, a = initial amount, and r = growth rate.

$$\begin{aligned} f(x) &= a(1 + r)^x \\ &= 18{,}000(1 + 0.08)^x \end{aligned}$$

The function is $f(x) = 18{,}000(1.08)^x$.

Find the expected population in 6 years.

$$\begin{aligned} f(6) &= 18{,}000(1.08)^6 \\ &\approx 28{,}563.74 \end{aligned}$$

After 6 years, the population is expected to be about 28,564.

Practice & Problem Solving

31. **Analyze and Persevere** An exponential function of the form $f(x) = b^x$ includes the points (2, 36), (3, 216), and (4, 1,296). What is the value of b?

Write an exponential growth or decay function to model each situation.

32. initial value: 50, growth factor: 1.15

33. initial value: 200, decay factor: 0.85

Write an exponential decay function to model each situation. Compare the average rates of change over the given intervals.

34. initial value: 20
decay factor: 0.8
$1 \leq x \leq 4$ and
$5 \leq x \leq 8$

35. initial value: 24
decay factor: 0.7
$2 \leq x \leq 4$ and
$6 \leq x \leq 8$

Write an exponential function to model the data in each table. Classify the function as representing growth or decay.

36.

x	$f(x)$
0	128
1	160
2	200
3	250
4	312.5

37.

x	$g(x)$
0	256
1	64
2	16
3	4
4	1

Apply Math Models **Compare each investment to an investment of the same principal at the same rate compounded annually.**

38. principal: \$12,000
annual interest: 5%
interest periods: 2
number of years: 10

39. principal: P
annual interest: 2.5%
interest periods: 4
number of years: t

TOPIC 5 REVIEW

TOPIC

6

Polynomials and Factoring

TOPIC ESSENTIAL QUESTION

How do you work with polynomials to rewrite expressions and solve problems?

Topic Overview

enVision® STEM Project:
Make Business Decisions

6-1 Adding and Subtracting Polynomials
AR.1.3, MTR.1.1, MTR.4.1, MTR.5.1

6-2 Multiplying Polynomials
AR.1.3, MTR.1.1, MTR.2.1, MTR.5.1

6-3 Multiplying Special Cases
AR.1.3, MTR.3.1, MTR.4.1, MTR.5.1

6-4 Factoring Polynomials
AR.1.4, AR.1.7, MTR.1.1, MTR.5.1, MTR.7.1

6-5 Factoring $x^2 + bx + c$
AR.1.7, MTR.1.1, MTR.3.1, MTR.5.1

Mathematical Modeling in 3 Acts:
Who's Right?
AR.1.7, MTR.7.1

6-6 Factoring $ax^2 + bx + c$
AR.1.7, MTR.1.1, MTR.4.1, MTR.5.1

6-7 Factoring Special Cases
AR.1.1, AR.1.7, MTR.4.1, MTR.5.1, MTR.6.1

Topic Vocabulary

- Closure Property
- degree of a monomial
- degree of a polynomial
- difference of two squares
- monomial
- perfect-square trinomial
- polynomial
- standard form of a polynomial

Digital Experience

INTERACTIVE STUDENT EDITION Access online or offline.

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ACTIVITIES Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its***.

ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

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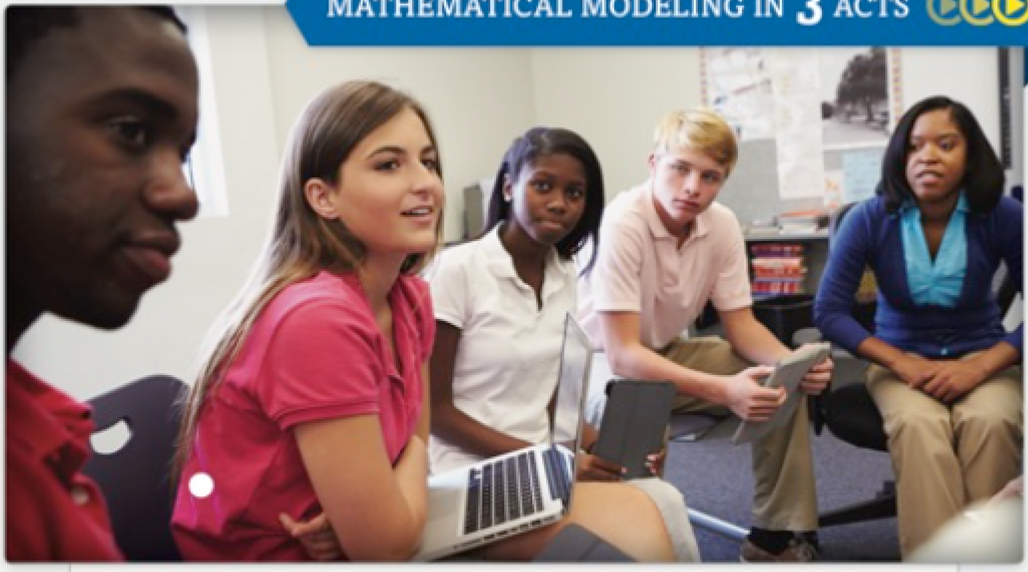
MATHEMATICAL MODELING IN 3 ACTS



Who's Right?

People often approach a problem in different ways. Sometimes their solutions are the same, but other times different approaches lead to very different, but still valid, solutions.

Suppose you had to solve a system of linear equations. You might solve it by graphing while a classmate might use substitution. Is one way of solving a problem always better than another? Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision® *STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you.*

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from *Virtual Nerd*, right when you need it.

MATH TOOLS Explore math with digital tools and manipulatives.

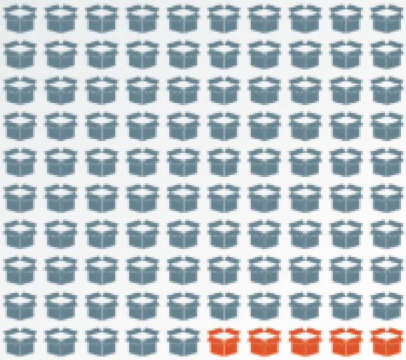
DESMOS Use Anytime and as embedded Interactives in Lesson content.

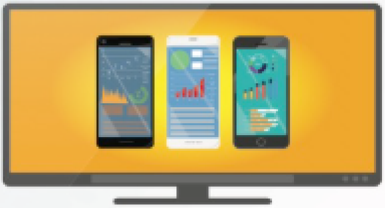
QR CODES Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

Did You Know?

Businesses can use functions to estimate their revenue and expenses, and then use that information to set sales targets and prices.

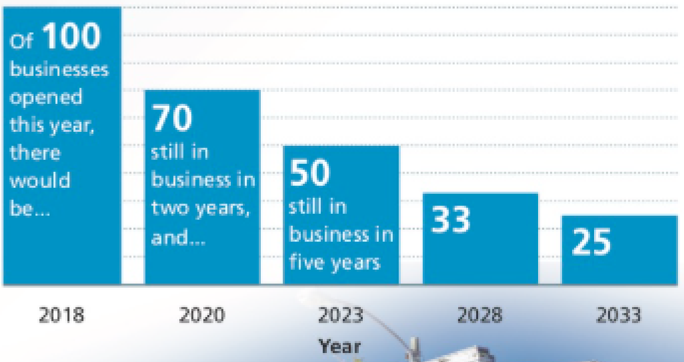
For every **100 new products** introduced each year, only **5 succeed**.





The **biggest advertiser** on TV
COMMUNICATION COMPANIES

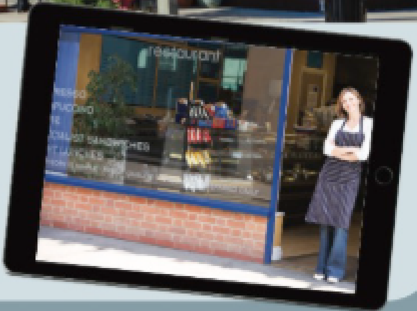
The second biggest
CARS



About **543,000 new businesses** are started every month in the United States.

Your Task: Make Business Decisions

You and your classmates will choose a business to model. You will suggest and defend choices for the number of items to make and the price(s) at which to sell them. Then, you will research ways that your decisions could change based on market factors.



6-1

Adding and Subtracting Polynomials

I CAN… combine like terms to simplify polynomials.

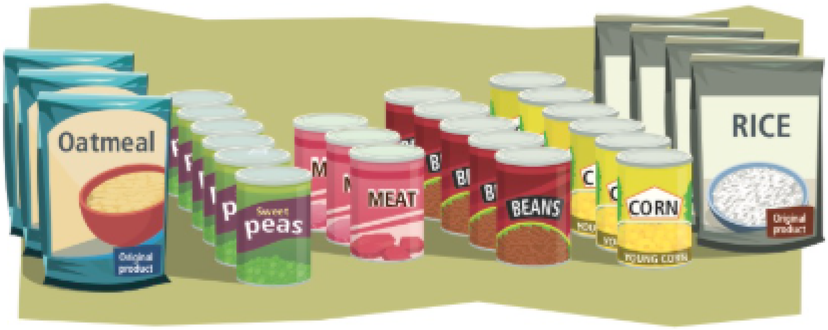
VOCABULARY

- Closure Property
- degree of a monomial
- degree of a polynomial
- monomial
- polynomial
- standard form of a polynomial

MA.912.AR.1.3–Add, subtract and multiply polynomial expressions with rational number coefficients.
MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

Each year the Student Council conducts a food drive. At the end of the drive, the members report on the items collected.



A. Describe two different ways that the students can sort the items that were collected.

B. **Apply Math Models** Write two expressions to represent the number and type of items collected.

C. Share your expression with classmates. How are the expressions similar? How are they different? Why are they different?

ESSENTIAL QUESTION

How does adding or subtracting polynomials compare to adding or subtracting integers?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Understand Polynomials

A. Why does a constant have a degree of 0?

A **monomial** is a real number, a variable, or the product of a real number and one or more variables with whole number exponents.

The **degree of a monomial** is the sum of the exponents of the variables of a monomial.

STUDY TIP
Remember to find the sum of the exponents of the variables in each term, not just the greatest exponent of any variable.

The variable has an exponent of 2, so the degree is 2.

$15x^2$

$\frac{1}{2}x^5y^2$

The sum of the variable exponents is 7, so the degree is 7.

A constant has no variable. However, any number to the 0 power is 1, so you can multiply a constant by x^0 without changing the value of the constant.

$$7 \longrightarrow 7x^0$$
$$= 7(1)$$
$$= 7$$

The exponent of the variable is 0, so the degree of this monomial is 0.

So, the degree of a constant is 0.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

B. Why is $5x^3 - 4$ called a cubic binomial?

A **polynomial** is a monomial or the sum or difference of two or more monomials, called terms.

Polynomials are named according to their degree. The **degree of a polynomial** is the greatest degree of any term of the polynomial.

Polynomial	Degree	Name Based on Degree
7	0	Constant
$4x$	1	Linear
$3x^2 + 2x + 1$	2	Quadratic
$-5x^2y$	3	Cubic
$4x^2y^2 + 5x - 2y + 6$	4	Fourth Degree

Polynomials with a degree greater than 3 can be named by their degree—fourth degree, fifth degree, and so on.

Polynomials are also named according to how many terms they have.

Polynomial	Number of Terms	Name Based on Number of Terms
$4x$	1	Monomial
$x + 7$	2	Binomial
$-5x^2y + x^2 + x$	3	Trinomial
$4x^2y^2 + 5x - 2y + 6$	4	Polynomial

Polynomials with more than 3 terms do not have special names.

$5x^3 - 4$ is called a cubic binomial because it has a degree of 3 and two terms.

Try It! **1.** Name each polynomial based on its degree and number of terms.

a. $-2xy^2$ **b.** $6xy - 3x + y$

EXAMPLE 2 Write Polynomials in Standard Form

What is the standard form of the polynomial $7x - 5 - x^3 + 6x^4 - 3x^2$?

The **standard form of a polynomial** is the form of a polynomial in which the terms are written in descending order according to their degree.

CHOOSE EFFICIENT METHODS
Why is writing polynomials in standard form important? How will it be useful?

Rewrite the polynomial in standard form.

$$7x - 5 - x^3 + 6x^4 - 3x^2$$

$$6x^4 - x^3 - 3x^2 + 7x - 5$$

$6x^4$ has the greatest degree, 4, so write it first.

The standard form of the polynomial is $6x^4 - x^3 - 3x^2 + 7x - 5$.

Try It! **2.** Write each polynomial in standard form.

a. $7 - 3x^3 + 6x^2$ **b.** $2y - 3 - 8y^2$

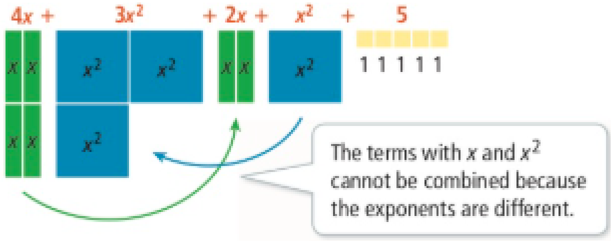
EXAMPLE 3 Add and Subtract Monomials

VOCABULARY
Remember, *like terms* are terms with exactly the same variable factors.

USE PATTERNS AND STRUCTURE
Which terms in a polynomial are like terms? How does combining them change the polynomial?

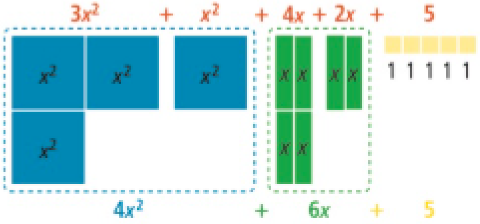
How can you use the properties of operations to combine like terms and write the expression $4x + 3x^2 + 2x + x^2 + 5$ in standard form?

Use algebra tiles to model the expression.



The terms with x and x^2 cannot be combined because the exponents are different.

Rearrange the tiles to group like terms together.



The expression $4x + 3x^2 + 2x + x^2 + 5$ written in standard form is $4x^2 + 6x + 5$.

You can also rewrite the expression in standard form using the properties of operations.

$$\begin{aligned} &4x + 3x^2 + 2x + x^2 + 5 \\ &= (3x^2 + x^2) + (4x + 2x) + 5 \\ &= 4x^2 + 6x + 5 \end{aligned}$$

You can apply the same properties of operations for real numbers to operations with monomials.

Try It! **3.** Combine like terms and write each expression in standard form.

a. $4x^2 - 3x - x^2 + 3x$

b. $7y^3 - 3y + 5y^3 - 2y + 7$

EXAMPLE 4 Add Polynomials

A. How is adding polynomials like adding numbers?

Consider the expressions $123 + 405$ and $(x^2 + 2x + 3) + (4x^2 + 5)$.

$$\begin{array}{r} 123 \\ +\,405 \\ \hline 528 \end{array}$$

Only like place values can be added.

Only like terms can be added.

$$\begin{array}{rrr} x^2 & + 2x & + 3 \\ +\,4x^2 & & + 5 \\ \hline 5x^2 & + 2x & + 8 \end{array}$$

Before you add polynomials, align like terms. This is similar to how, before adding numbers, the numbers must be aligned according to their place value.

CONTINUED ON THE NEXT PAGE

EXAMPLE 4 CONTINUED

B. What is the sum of $\left(\frac{1}{5}x^2 + 2x - 3\right)$ and $\left(\frac{2}{5}x^2 + 6\right)$?

To add two polynomials, combine like terms.

COMMON ERROR
Be careful to align like terms when adding polynomials vertically.

Method 1: Add vertically.

$$\begin{array}{r} \frac{1}{5}x^2 + 2x - 3 \\ +\frac{2}{5}x^2 \qquad + 6 \\ \hline \frac{3}{5}x^2 + 2x + 3 \end{array}$$

Align like terms.

Method 2: Add horizontally.

$$\left(\frac{1}{5}x^2 + 2x - 3\right) + \left(\frac{2}{5}x^2 + 6\right)$$

$$= \left(\frac{1}{5}x^2 + \frac{2}{5}x^2\right) + (2x) + (-3 + 6)$$

$$= \frac{3}{5}x^2 + 2x + 3$$

Use the Commutative and Associative Properties to group like terms.

The sum of $\left(\frac{1}{5}x^2 + 2x - 3\right)$ and $\left(\frac{2}{5}x^2 + 6\right)$ is $\frac{3}{5}x^2 + 2x + 3$.
The sum of these two polynomials is a polynomial.

Try It! **4.** Simplify. Write each answer in standard form.

a. $(3x^2 + 2x) + (-x + 9)$ **b.** $(-2x^2 + 5.6x - 7) + (3.8x + 7)$

EXAMPLE 5 Subtract Polynomials

What is the difference $(6x^2 + 3x - 2) - (3x^2 + 5x - 8)$?

To subtract two polynomials subtract like terms.

COMMON ERROR
Remember that subtraction is adding the opposite. Be sure to find the opposite of every term in the second polynomial by distributing –1 to each term, before adding.

Method 1: Subtract vertically by lining up like terms.

Line up like terms. Then subtract.

$$\begin{array}{r} 6x^2 + 3x - 2 \\ \hline -(3x^2 + 5x - 8) \end{array} \xrightarrow[\text{to each term.}]{\text{Distribute } -1} \begin{array}{r} 6x^2 + 3x - 2 \\ -3x^2 - 5x + 8 \\ \hline 3x^2 - 2x + 6 \end{array}$$

Method 2: Subtract horizontally.

$$(6x^2 + 3x - 2) - (3x^2 + 5x - 8)$$

$$= 6x^2 + 3x - 2 - 3x^2 - 5x + 8$$

Distribute –1 to each term in the subtracted expression.

$$= (6x^2 - 3x^2) + (3x - 5x) + (-2 + 8)$$

$$= 3x^2 - 2x + 6$$

Use the Commutative and Associative Properties to combine like terms. Then simplify.

The difference of $(6x^2 + 3x - 2)$ and $(3x^2 + 5x - 8)$ is $3x^2 - 2x + 6$.
The difference of these two polynomials is also a polynomial.

In Examples 4 and 5, the result of adding or subtracting two polynomials is another polynomial. The **Closure Property** indicates that polynomials are closed under addition or subtraction because the result of these operations is another polynomial.

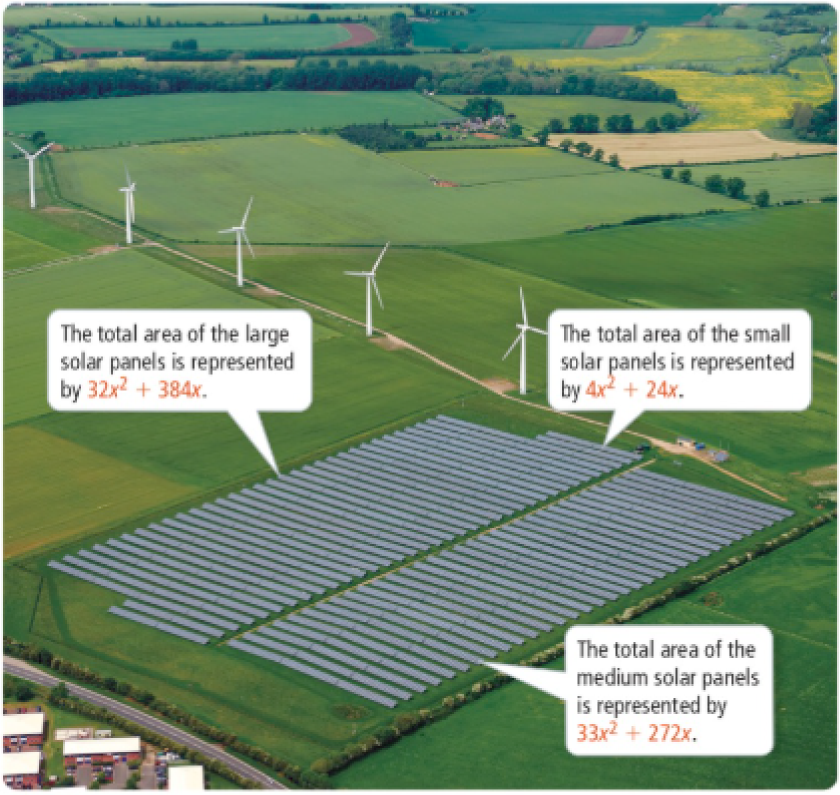
Try It! **5.** Simplify. Write each answer in standard form.

a. $(3x^2 + 4x + 2) - (-x + 4)$ **b.** $(-5x - 6) - (4x^2 + 6)$

APPLICATION

EXAMPLE 6 Apply Polynomials

An engineer is reviewing the layout of a solar farm. The solar farm shown has 4 small panels, 33 medium panels, and 32 large panels. What is the total area of the farm's solar panels?



The total area of the large solar panels is represented by $32x^2 + 384x$.

The total area of the small solar panels is represented by $4x^2 + 24x$.

The total area of the medium solar panels is represented by $33x^2 + 272x$.

USE PATTERNS AND STRUCTURE
How could you use what you know about polynomial addition to find an expression for the area of each small solar panel?

Write an expression to represent the total area of the solar panels.

$$\begin{aligned}\text{Total area} &= \text{Total area of large panels} + \text{Total area of medium panels} + \text{Total area of small panels}\\ &= (32x^2 + 384x) + (33x^2 + 272x) + (4x^2 + 24x)\\ &= (32x^2 + 33x^2 + 4x^2) + (384x + 272x + 24x)\\ &= 69x^2 + 680x\end{aligned}$$

Use the Commutative and Associative Properties to group like terms. Then add.

The total area of the solar panels is modeled by the expression $69x^2 + 680x$, where x is the width, in meters, of each solar panel.

Try It! **6.** What expression models the difference between the total area of the large solar panels and the total area of the small solar panels?

CONCEPT SUMMARY Adding and Subtracting Polynomials

STANDARD FORM **Standard Form of a Polynomial:** $3x^4 - 3x^2 + 4x - 2$

In standard form the monomial terms are written in descending order according to their degree.

NAMING POLYNOMIALS Polynomials can be named according to the number of terms and their degree.

$12x^3 + 6xy - 5$

There are 3 terms, so it is a trinomial.

The highest degree is 3, so it is cubic.

So $12x^3 + 6xy - 5$ is a cubic trinomial.

POLYNOMIAL OPERATIONS **Adding Polynomials**

$(-2x^3 + 4x^2 - 5) + (4x^3 + 2x^2 + 8)$
$= (-2x^3 + 4x^3) + (4x^2 + 2x^2) + (-5 + 8)$
$= 2x^3 + 6x^2 + 3$

Subtracting Polynomials

$(3x^2 - 2x + 4) - (-3x^2 - x + 6)$
$= 3x^2 - 2x + 4 + 3x^2 + x - 6$
$= (3x^2 + 3x^2) + (-2x + x) + (4 - 6)$
$= 6x^2 - x - 2$

Add or subtract like terms just like you add or subtract digits with the same place value.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does adding or subtracting polynomials compare to adding or subtracting integers?

2. **Represent and Connect** How does the definition of the prefixes *mono-*, *bi-*, and *tri-* help when naming polynomials?

3. **Vocabulary** Describe the relationship between the *degree of a monomial* and *the standard form of a polynomial.*

4. **Communicate and Justify** Explain why the sum $x + x$ is equal to $2x$ instead of x^2.

5. **Error Analysis** Rebecca says that all monomials with the same degree are like terms. Explain Rebecca's error.

Do You KNOW HOW?

Name each polynomial based on its degree and number of terms.

6. $\frac{x}{4} + 2$

7. $7x^3 + xy - 4$

Write each polynomial in standard form.

8. $2y - 3 - y^2$

9. $3x^2 - 2x + x^3 + 6$

Simplify each expression.

10. $(x^2 + 2.3x - 4) + (2x^2 - 5.4x - 3)$

11. $(3x^2 - 5x - 8) - (-4x^2 - 2x - 1)$

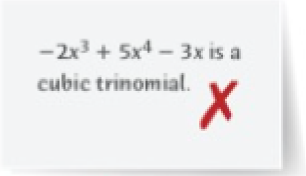
12. A square prism has square sides each with area $x^2 + 8x + 16$ and rectangular sides with each area $2x^2 + 15x + 28$. What expression represents the surface area of the square prism?

PRACTICE & PROBLEM SOLVING

UNDERSTAND

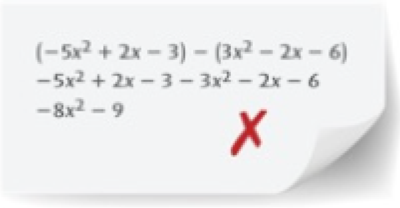
13. **Analyze and Persevere** How is it possible that the sum of two quadratic trinomials is a linear binomial?

14. **Error Analysis** Describe and correct the error a student made when naming the polynomial.



$-2x^3 + 5x^4 - 3x$ is a cubic trinomial. ✗

15. **Error Analysis** Describe and correct the error a student made when subtracting the polynomials.



$(-5x^2 + 2x - 3) - (3x^2 - 2x - 6)$

$-5x^2 + 2x - 3 - 3x^2 - 2x - 6$

$-8x^2 - 9$ ✗

16. **Use Patterns and Structure** What is the missing term in the equation?

a. $(\_\_\_ + 7) + (2x - 6) = -4x + 1$

b. $(a^2 + \_\_\_ + 1) - (\_\_\_ + 5a + \_\_\_) = 4a^2 - 2a + 7$

17. **Higher Order Thinking** Describe each statement as *always, sometimes,* or *never* true.

a. A linear binomial has a degree of 0.

b. A trinomial has a degree of 2.

c. A constant has a degree of 1.

d. A cubic monomial has a degree of 3.

18. **Communicate and Justify** Consider the set of linear binomials $ax + b$, where a and b are positive integers, $a > 0$ and $b > 0$.

a. Does the set have closure for addition? Explain.

b. Does the set have closure for subtraction? Explain.

PRACTICE

Find the degree of each monomial. SEE EXAMPLE 1

19. $\frac{x}{4}$

20. $-7xy$

21. 21

22. $4x^2y$

Name each polynomial based on its degree and number of terms. SEE EXAMPLE 1

23. $17yx^2 + xy - 5$

24. $5x^3 + 2x - 8$

25. $100x^2 + 3$

26. $-9x^4 + 8x^3 - 7x + 1$

Simplify each expression. Write the answer in standard form. SEE EXAMPLES 2 AND 3

27. $3x + 2x^2 - 4x + 3x^2 - 5x$

28. $5 + 8y^2 - 12y^2 + 3y$

29. $3z - 7z^2 - 5z + 5z^2 + 2z^2$

30. $7 - 2x + 3 + 5x + 4x^2$

Add or subtract. Write each answer in standard form. SEE EXAMPLES 4 AND 5

31. $\left(\frac{1}{6}b - 8\right) + \left(\frac{1}{6}b + 4\right)$

32. $(2x^2 - 7x^3 + 8x) + (-8x^3 - 3x^2 + 4)$

33. $(5y^2 - 2y + 1) - (y^2 + y + 3)$

34. $(-7a^4 - a + 4.6a^2) - (-7.4a^2 + a - 7a^4)$

35. $(4m^2 - 2m + 4) + (2m^2 + 2m - 5)$

Write an expression to represent each situation. SEE EXAMPLE 6

36. Find the perimeter of the rectangle.

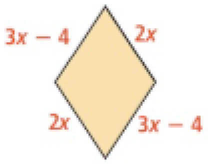
37. A cube has square sides with area $x^2 + 24x + 144$. What expression represents the surface area of the cube?

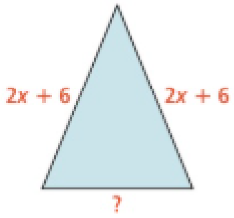
38. A rectangle has a length of $5x + 2$ in. and a width of $4x + 6$ in. What is the perimeter of the rectangle?

PRACTICE & PROBLEM SOLVING

APPLY

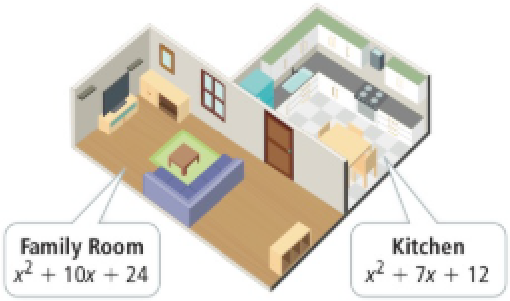
39. Mathematical Connections The perimeters of the two figures are equal.





What expression represents the missing side length?

40. Analyze and Persevere The owners of a house want to knock down the wall between the kitchen and family room.



What expression represents the area of the new combined open space?

41. Generalize Polynomial A has degree 2; Polynomial B has degree 4. What can you determine about the name and degree of the sum of the polynomials and the difference of the polynomials if

a. Polynomial A is a binomial and Polynomial B is a monomial?

b. Both Polynomial A and Polynomial B are binomials?

42. Apply Math Models A large indoor market is set up with 4 rows of booths. There are large booths with an area of x^2 sq. units, medium booths with an area of x sq. units, and small booths with an area of 1 sq. unit. In the marketplace, two of the rows contain 7 large booths, 6 medium booths, and 5 small booths each. The other two rows each contain 3 large booths, 5 medium booths, and 10 small booths. What is the total area of the booths in the marketplace?

ASSESSMENT PRACTICE

43. Which expression is equivalent to $(x^2 + 3.8x - 5) - (4x^2 + 3.8x - 6)$? AR.1.3

Ⓐ $5x^2 + 7.6x - 11$

Ⓑ $-3x^4 + 7.6x^2 + 1$

Ⓒ $-3x^2 + 1$

Ⓓ $-3x^2 + 7.6x - 11$

44. SAT/ACT What is the sum of $-2x^2 + 3x - 4$ and $3x^2 - 4x + 6$?

Ⓐ $x^4 - x^2 + 2$

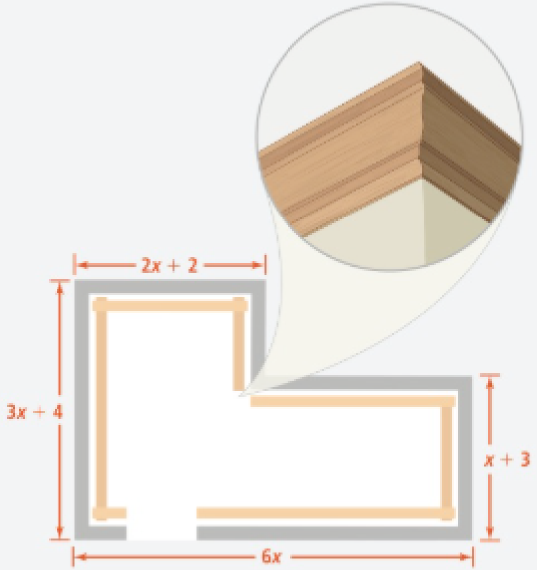
Ⓑ $5x^4 + 7x^2 + 10$

Ⓒ 2

Ⓓ $x^2 - x + 2$

Ⓔ $2x^6$

45. Performance Task A room has the dimensions shown below. Molding was installed around the edge of the ceiling.



Part A Write an expression to represent the amount of molding needed.

Part B Sam used 80 feet of molding to finish the room. What is the measurement of each edge of the ceiling?

6-2

Multiplying Polynomials

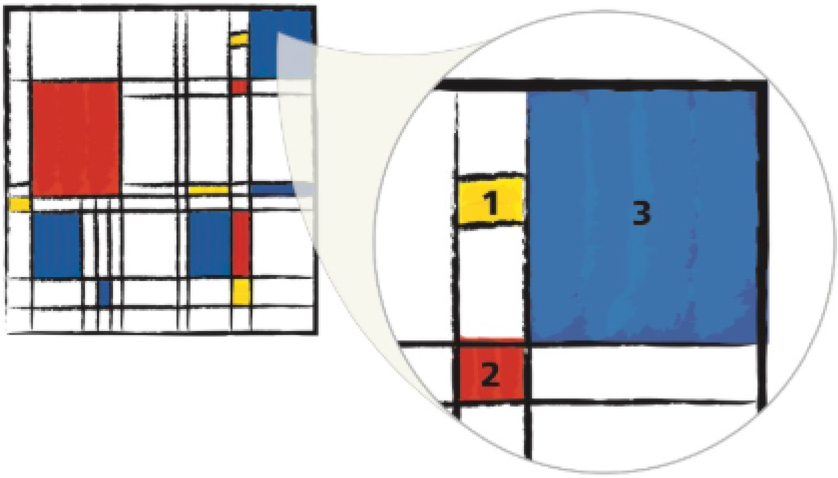
I CAN… multiply two polynomials.

MA.912.AR.1.3–Add, subtract and multiply polynomial expressions with rational number coefficients.

MA.K12.MTR.1.1, MTR.2.1, MTR.5.1

MODEL & DISCUSS

Samantha makes the abstract painting shown using vertical and horizontal lines and four colors.



A. How can you use mathematics to describe the areas of Rectangle 1 and Rectangle 2?

B. **Represent and Connect** How can you use mathematics to describe the area of Rectangle 3?

ESSENTIAL QUESTION

How does multiplying polynomials compare to multiplying integers?

EXAMPLE 1 Multiply a Monomial and a Trinomial

What is the product of $-4x^3$ and $(x^2 + 3x - 4)$?

Use the Distributive Property. The Distributive Property works for polynomials in the same way that it works for real numbers.

$$-4x^3(x^2 + 3x - 4) = -4x^3(x^2) + -4x^3(3x) + -4x^3(-4)$$
$$= -4x^5 - 12x^4 + 16x^3$$

Distribute $-4x^3$ to each term of the trinomial.

The product is $-4x^5 - 12x^4 + 16x^3$.

Notice that the product of these two polynomials is a polynomial.

COMMON ERROR
You may incorrectly state that $-4x^3(x^2)$ is $-4x^6$. Recall that when multiplying terms with exponents, you add the exponents of like bases.

Try It! **1.** Find each product.

a. $-2x^2(x^2 + 3x + 4)$

b. $-4x(2x^2 - 3x + 5)$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Use a Table to Find the Product of Polynomials

A. How is multiplying binomials like multiplying two-digit numbers?

Multiply the expressions 15 • 18 and $(x + 5)(x + 8)$ using a table.

15 • 18

	10	8
10	100	80
5	50	40

You can write 15 • 18 in expanded form as (10 + 5)(10 + 8).

$(x + 5)(x + 8)$

	x	8
x	x^2	$8x$
5	$5x$	40

STUDY TIP
Once the two-digit numbers are written in expanded form, you can also use the Distributive Property to find the product.

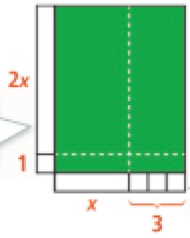
$$15 \cdot 18 = 100 + 80 + 50 + 40$$
$$= 270$$

$$(x + 5)(x + 8) = x^2 + 8x + 5x + 40$$
$$= x^2 + 13x + 40$$

You can multiply both binomials and two-digit numbers in expanded form using the Distributive Property.

B. What is the area of the green rectangle?

The area of the green rectangle is represented by the expression $(2x + 1)(x + 3)$.



The height of the rectangle is represented by $(2x + 1)$.

The width of the rectangle is represented by $(x + 3)$.

Use a table to find the area of each section of the rectangle.

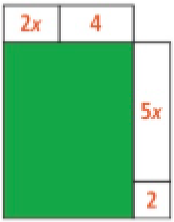
$2x$	$2x^2$	$6x$
1	x	3
	x	3

Combine like terms.

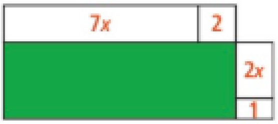
The area of the green rectangle is $2x^2 + 7x + 3$. Again, the product of these two polynomials is a polynomial.

Try It! 2. Find the area of each green rectangle.

a.



b.



EXAMPLE 3 Multiply Binomials

How can you use the Distributive Property to rewrite $(2x + 4)(x - 5)$ as a polynomial?

Distribute each term in the first binomial to each term in the second binomial.

$(2x + 4)(x - 5) = 2x(x - 5) + 4(x - 5)$ Distribute 2x and 4 to the second binomial.

$= 2x(x) + 2x(-5) + 4(x) + 4(-5)$ Distribute 2x and 4 to each term in the second binomial.

$= 2x^2 - 10x + 4x - 20$ Multiply.

$= 2x^2 - 6x - 20$ Combine like terms.

The product of $(2x + 4)$ and $(x - 5)$ is $2x^2 - 6x - 20$.

Again, the product of these two polynomials is a polynomial.

GENERALIZE
Compare the factors and the final product. What generalizations can you make from this example?

Try It! **3.** Find each product.

a. $(5x - 4)(2x + 1)$

b. $(3x - 5)(2x + 4)$

EXAMPLE 4 Multiply a Trinomial and a Binomial

A. How can you use a table to find the product of $(x^2 + 2x - 1)$ and $(3x + 4)$?

Write the terms for each polynomial in the first row and column of the table. Multiply to find each product.

$3x + 4$

$x^2 + 2x - 1$

	x^2	$2x$	-1
$3x$	$3x^3$	$6x^2$	$-3x$
4	$4x^2$	$8x$	-4

Combine the like terms.

	x^2	$2x$	-1
$3x$	$3x^3$	$6x^2$	$-3x$
4	$4x^2$	$8x$	-4

$(x^2 + 2x - 1)(3x + 4) = 3x^3 + 6x^2 + 4x^2 + 8x + (-3x) + (-4)$

$= 3x^3 + 10x^2 + 5x - 4$

So $(x^2 + 2x - 1)(3x + 4) = 3x^3 + 10x^2 + 5x - 4$. When you multiply a trinomial by a binomial, the result is six individual products. Using a table is one method you can use to help organize these products.

CONTINUED ON THE NEXT PAGE

EXAMPLE 4 CONTINUED

B. How is multiplying a trinomial by a binomial like multiplying a three-digit number by a two-digit number?

Consider the products 312 • 24 and $(3x^2 + x + 2)(2x + 4)$.

Multiply each place of the three-digit number by 4 ones and 2 tens. Then find the sum.

$$\begin{array}{r} 3\ \ 1\ \ 2 \\ \times \qquad 2\ \ 4 \\ \hline 1,\ 2\ \ 4\ \ 8 \\ +\ \ 6,\ 2\ \ 4\ \ 0 \\ \hline 7,\ 4\ \ 8\ \ 8 \end{array}$$

Multiply each term of the trinomial by +4 and $2x$. Then combine like terms.

$$\begin{array}{r} 3x^2 + x + 2 \\ \times \qquad 2x + 4 \\ \hline 12x^2 + 4x + 8 \\ +\ 6x^3 + 2x^2 + 4x \qquad \\ \hline 6x^3 + 14x^2 + 8x + 8 \end{array}$$

STUDY TIP
As you multiply, remember to line up like terms so combining them in the last step will be easier.

When multiplying a trinomial by a binomial, you multiply each term of the trinomial by each term of the binomial. This is similar to how, when multiplying a three-digit number by a two-digit number, you multiply by each place value of the two-digit number.

Try It! 4. Find each product.

a. $(2x - 5)(-3x^2 + 4x - 7)$
b. $(-3x^2 + 1)(2x^2 + 3x - 4)$

EXAMPLE 5 Closure and Multiplication

Why is the operation of multiplication closed over the set of polynomials?

For each example in this lesson, you have found the product of two polynomials. In each case the product has also been a polynomial.

Consider the first example: $-4x^3(x^2 + 3x - 4)$.

$$\begin{aligned} -4x^3(x^2 + 3x - 4) &= -4x^3(x^2) + -4x^3(3x) + -4x^3(-4) \\ &= -4x^5 - 12x^4 + 16x^3 \end{aligned}$$

x^3, x^2, and x have whole number exponents, so the product will also have whole number exponents.

LEARN TOGETHER
How do you value other perspectives and points of view respectfully?

When you multiply two polynomials, the result is the sum or difference of terms. Each term is a real number coefficient multiplied by a variable raised to a whole number exponent. Each term is a monomial and the sum or difference of monomials is a polynomial. So polynomials are closed under multiplication.

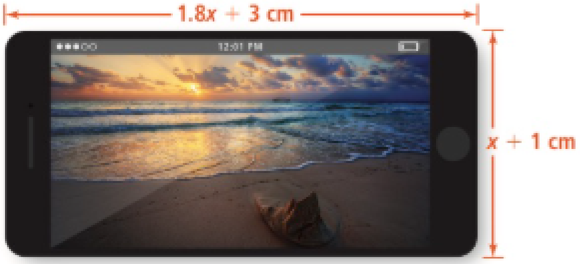
Try It! 5. Why is it important that the product of two polynomials have only whole number exponents?

APPLICATION

EXAMPLE 6 Apply Multiplication of Binomials

A smartphone has a screen that has a width of x and a height that is 1.8 times the width. The outer dimensions of the phone are shown.

Write an expression for the portion of the phone that is not occupied by the screen. Assume that the phone is rectangular.



Formulate Write expressions to represent the area of the screen and the area of the phone.

$$\text{Area of screen} = x(1.8x)$$
$$\text{Area of phone} = (x + 1)(1.8x + 3)$$

Compute Express each area in standard form.

$$\text{Area of screen} = x(1.8x) = 1.8x^2$$

$$\begin{aligned}\text{Area of phone} &= (x + 1)(1.8x + 3)\\ &= x(1.8x + 3) + 1(1.8x + 3)\\ &= 1.8x^2 + 3x + 1.8x + 3\\ &= 1.8x^2 + 4.8x + 3\end{aligned}$$

Subtract the area of the screen from the area of the phone.

$$\begin{aligned}\text{Non-screen Area} &= \text{Area of Phone} - \text{Area of Screen}\\ &= (1.8x^2 + 4.8x + 3) - 1.8x^2\\ &= 4.8x + 3\end{aligned}$$

Interpret The expression $4.8x + 3$ represents the portion of the phone's surface not occupied by the screen.

Try It! 6. Suppose the height of the phone in Example 6 were 1.9 times the width but all of the other conditions were the same. What expression would represent the area of the phone's surface not occupied by the screen?

CONCEPT SUMMARY Multiplying Polynomials

There are different methods that can be used to multiply polynomials. The methods used for multiplying polynomials are similar to the methods used for multiplying multi-digit numbers.

Binomial × Binomial

ALGEBRA

Multiply Horizontally

$(x + 3)(x - 2)$

$= x(x - 2) + 3(x - 2)$

$= x^2 - 2x + 3x - 6$

$= x^2 + x - 6$

Multiply Vertically

$$\begin{array}{rr} & x - 2 \\ \times & x + 3 \\ \hline & 3x - 6 \\ + & x^2 - 2x \\ \hline & x^2 + x - 6 \end{array}$$

DIAGRAMS

	x	3
x	x^2	$3x$
-2	$-2x$	-6

$x^2 + x - 6$

Binomial × Trinomial

Multiply Horizontally

$(x + 3)(x^2 + 4x - 2)$

$= x(x^2 + 4x - 2) + 3(x^2 + 4x - 2)$

$= x^3 + 4x^2 - 2x + 3x^2 + 12x - 6$

$= x^3 + 7x^2 + 10x - 6$

Multiply Vertically

$$\begin{array}{rr} & x^2 + 4x - 2 \\ \times & x + 3 \\ \hline & 3x^2 + 12x - 6 \\ + & x^3 + 4x^2 - 2x \\ \hline & x^3 + 7x^2 + 10x - 6 \end{array}$$

	x	3
x	x^2	$3x^2$
$4x$	$4x^2$	$12x$
-2	$-2x$	-6

$x^3 + 7x^2 + 10x - 6$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does multiplying polynomials compare to multiplying integers?

2. **Choose Efficient Methods** When multiplying two variables, how is using the Distributive Property similar to using a table?

3. **Error Analysis** Mercedes states that when multiplying $4x^3(x^3 + 2x^2 - 3)$ the product is $4x^9 + 8x^6 - 12x^3$. What was Mercedes's error?

4. **Use Patterns and Structure** When multiplying polynomials, why can the degree of the product different be from the degree of the factors?

Do You KNOW HOW?

Find each product.

5. $-2x^3(3x^2 - 4x + 7)$

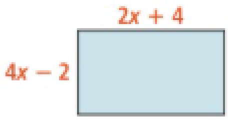
6. $(2x + 6)(x - 4)$

7. $(x - 2)(3x + 4)$

8. $(5y - 2)(4y^2 + 3y - 1)$

9. $(3x^2 + 2x -5)(2x - 3)$

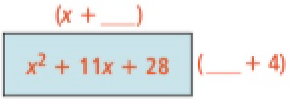
10. Find the area of the rectangle.



PRACTICE & PROBLEM SOLVING

UNDERSTAND

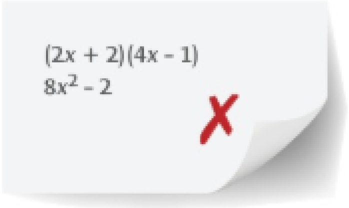
11. **Analyze and Persevere** The area of a rectangle is given. Identify the missing terms in the length and width.



12. **Use Patterns and Structure** The table shows the product when multiplying two binomials. What is the relationship between the numbers in the factors and the terms in the product?

Binomials	Products
$(x + 3)(x + 4)$	$x^2 + 7x + 12$
$(x + 2)(x - 5)$	$x^2 - 3x - 10$
$(x - 3)(x - 5)$	$x^2 - 8x + 15$

13. **Error Analysis** Describe and correct the error a student made when multiplying two binomials.



$(2x + 2)(4x - 1)$

$8x^2 - 2$

14. **Represent and Connect** Use a table to find the product of $(3x + 4)(x^2 + 3x - 2)$. How are the like terms in a table arranged?

15. **Higher Order Thinking** Is it possible for the product of a monomial and trinomial to be a binomial? Explain.

16. **Mathematical Connections** A triangle has a height of $2x + 6$ and a base length of $x + 4$. What is the area of the triangle?

17. **Communicate and Justify** Explain how to find the combined volume of the two rectangular prisms described. One has side lengths of $3x$, $2x + 1$, and $x + 3$. The other has side lengths of $5x - 2$, $x + 9$, and 8.

PRACTICE

Find each product. SEE EXAMPLE 1

18. $6x(x^2 - 4x - 3)$

19. $-y(-3y^2 + 2y - 7)$

20. $3x^2(-x^2 + 2x - 4)$

21. $-5x^3(2x^3 - 4x^2 + 2)$

Use a table to find each product. SEE EXAMPLE 2

22. $(x - 6)(3x + 4)$

23. $(2x + 1)(4x + 1)$

Use the Distributive Property to find each product. SEE EXAMPLE 3

24. $(x - 6)(x + 3)$

25. $(3x - 4)(2x + 5)$

26. $(5.2x - 8)(2x + 3)$

Find each product. SEE EXAMPLE 4

27. $(y + 3)(2y^2 - 3y + 4)$

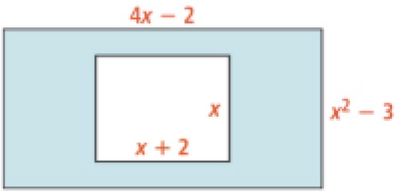
28. $(2x - 7)(3x^2 - 4x + 1)$

29. $(2x^2 - 3x)(-3x^2 + 4x - 2)$

30. $(-2x^2 + 1)(2x^2 - 3x - 7)$

31. $(x^2 + 3x)(3x^2 - 2x + 4)$

32. Find the area of the shaded region. SEE EXAMPLE 6

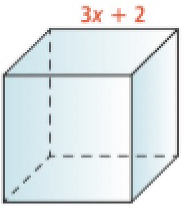


33. A rectangular park is $6x + 2$ ft long and $3x + 7$ ft wide. In the middle of the park is a square turtle pond that is 8 ft wide. What expression represents the area of the park not occupied by the turtle pond? SEE EXAMPLE 6

PRACTICE & PROBLEM SOLVING

APPLY

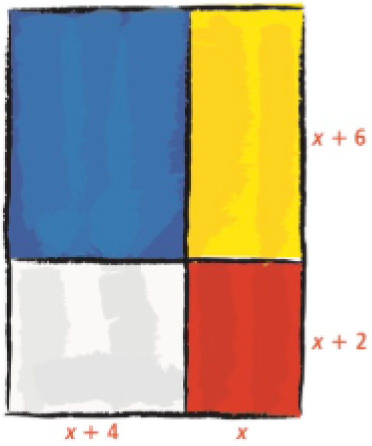
34. Apply Math Models The volume of a cube is calculated by multiplying the length, width, and height. What is the volume of this cube?



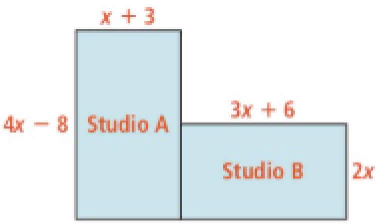
35. Analyze and Persevere The product of the binomial and the trinomial shown is a polynomial with four terms. Change one of the terms of the binomial or the trinomial so the product is also a trinomial.

$(2x + 2)(x^2 + 2x - 4) = 2x^3 + 7x^2 - 2x - 12$

36. Analyze and Persevere What is the area of the painting shown?



37. Analyze and Persevere A dance teacher wants to expand her studio to fit more classes. What is the combined area of Studio A and Studio B?



ASSESSMENT PRACTICE

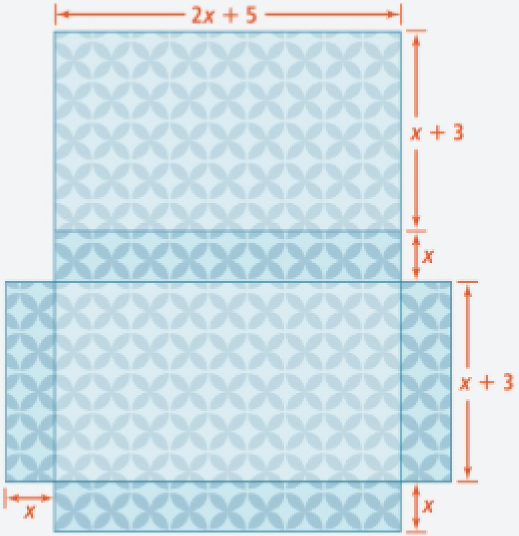
38. Rewrite the expression as the sum of monomials.

$$(x + 4)\left(\frac{3}{4}x + 1\right) - [(x - 5)(x + 3)] + 3x^2$$ AR.1.3

39. SAT/ACT What is the product $(-2x + 2)(x - 5)$?

Ⓐ $-2x^2 - 10$
Ⓑ $-2x^2 + 12x - 10$
Ⓒ $-x - 3$
Ⓓ $-2x^2 - 12x - 10$

40. Performance Task The net of a rectangular box and its dimensions are shown.



Part A Write an expression for the surface area of the box in terms of x.

Part B Evaluate the polynomial expression you found in Part A. What integer value of x would give the prism a surface area of about 600 cm^2?

6-3

Multiplying Special Cases

I CAN… use patterns to multiply binomials.

VOCABULARY

- difference of two squares

MA.912.AR.1.3–Add, subtract and multiply polynomial expressions with rational number coefficients.

MA.K12.MTR.3.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

The table gives values for *x* and *y* and different expressions.

x	y	$(x - y)(x + y)$	x^2	y^2	$(x^2 - y^2)$
7	4				
6	2				
3	9				

A. Copy and complete the table.

B. Describe any patterns you notice.

C. **Use Patterns and Structure** Try substituting variable expressions of the form $7p$ and $4q$ for x and y. Does the pattern still hold? Explain.

ESSENTIAL QUESTION

What patterns are there in the product of the square of a binomial and the product of a sum and a difference?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Determine the Square of a Binomial

A. Why is $(a + b)^2$ considered a special case when multiplying polynomials?

Use the Distributive Property.

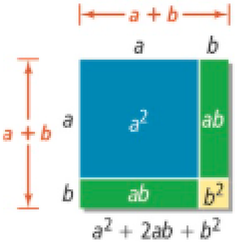
$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a(a + b) + b(a + b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

first term squared

twice the product of the first and last terms

last term squared

Use a visual model.



GENERALIZE
When squaring a binomial, think about how you can use the terms in the binomial to quickly determine the product. What generalizations about terms can you make?

The square of a binomial follows the pattern $(a + b)^2 = a^2 + 2ab + b^2$.

B. What is the product $(5x - 3)^2$?

Use the pattern you found in Part A to find the square of a difference.

$(5x - 3)^2 = [5x + (-3)]^2$ ········ Rewrite the difference as a sum.

$= (5x)^2 + 2(5x)(-3) + (-3)^2$ ········ Substitute $5x$ and -3 into $a^2 + 2ab + b^2$.

$= 25x^2 - 30x + 9$ ········ Simplify.

You can write the product $(5x - 3)^2$ as $25x^2 - 30x + 9$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

C. How can you use the square of a binomial to find the product 29^2?

Rewrite the product as a difference of two values whose squares you know, such as $(30 - 1)^2$. Then use the pattern for the square of a binomial to find its square.

$$\begin{aligned}(30 - 1)^2 &= (30)^2 + 2(30)(-1) + (-1)^2\\ &= 900 - 60 + 1\\ &= 841\end{aligned}$$

$(30 - 1)$ is the same as 29. So, $(30 - 1)^2$ is the same as 29^2.

So, $29^2 = 841$. In general, you can use the square of a binomial to find the square of a large number by rewriting the number as the sum or difference of two numbers with known squares.

Try It! **1.** Find each product.

a. $(3x - 4)^2$

b. 71^2

EXAMPLE 2 Find the Product of a Sum and a Difference

A. What is the product $(a + b)(a - b)$?

Use the Distributive Property to find the product.

$$\begin{aligned}(a + b)(a - b) &= a(a - b) + b(a - b)\\ &= a^2 - ab + ba - b^2\\ &= a^2 - b^2\end{aligned}$$

The middle terms add to 0 because they are opposites.

first term squared

last term squared

The product of two binomials in the form $(a + b)(a - b)$ is $a^2 - b^2$. The product of the sum and difference of the same two values results in the **difference of two squares**.

B. What is the product $(5x + 7)(5x - 7)$?

Use the pattern you found in Part A.

$$(5x + 7)(5x - 7) = (5x)^2 - (7)^2$$ Substitute $5x$ and 7 into $a^2 - b^2$.

$$= 25x^2 - 49$$ Simplify.

The product of $(5x + 7)(5x - 7)$ is $25x^2 - 49$. It is the difference of two squares, $(5x)^2 - 7^2$.

COMMON ERROR
Remember that the last terms of each binomial are opposites. So, the product of the last terms will always be negative.

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

C. How can you use the difference of two squares to find the product of 43 and 37?

Rewrite the product as the sum and difference of the same two numbers *a* and *b*.

43 and 37 are each 3 units from 40.

$$(40 + 3)(40 - 3) = (40)^2 - (3)^2$$
$$= 1{,}600 - 9$$
$$= 1{,}591$$

$(40 + 3)(40 - 3)$ is of the form $(a + b)(a - b)$, so it is equivalent to the difference of two squares.

You can use the difference of two squares to mentally find the product of large numbers when the numbers are the same distance from a known square.

CHOOSE EFFICIENT METHODS
What types of practical limitations are there on using the product of a sum and difference to find the product of two numbers?

Try It! 2. Find each product.

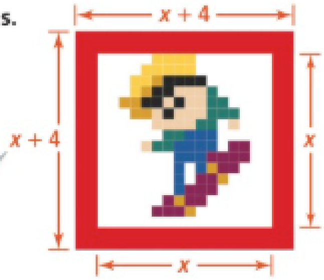
a. $(2x - 4)(2x + 4)$

b. $56 \cdot 44$

APPLICATION

EXAMPLE 3 Apply the Square of a Binomial

A graphic designer is developing images for icons. The square pixelated image is placed inside a border that is 2 pixels wide on all sides. If the area of the border of the image is 176 square pixels, what is the area of the image?



Let x represent the length and width of the image.

Formulate The area of the image and the border is represented by the expression $(x + 4)^2$.

Total area − Area of Image = Area of Border
$(x + 4)^2 - x^2 = 176$

Compute

$$(x + 4)^2 - x^2 = 176$$
$$x^2 + 8x + 16 - x^2 = 176$$
$$8x + 16 = 176$$
$$8x = 160$$
$$x = 20$$

Find the product of the squared binomial first.

Interpret The image will be 20 pixels by 20 pixels. The area of the image is $20 \cdot 20$, or 400 square pixels.

Try It! 3. What is the area of the square image if the area of the border is 704 square pixels and the border is 4 pixels wide?

CONCEPT SUMMARY Multiplying Special Cases

	Square of a Binomial	Product of a Sum and Difference
WORDS	The square of a binomial, $(a + b)^2$, always follows the same pattern: the square of the first term, plus twice the product of the first and last term, plus the square of the last term.	The product of two binomials in the form $(a + b)(a - b)$ results in the difference of two squares.
ALGEBRA	$(a + b)^2 = a^2 + 2ab + b^2$ or $(a - b)^2 = a^2 - 2ab + b^2$	$(a + b)(a - b) = a^2 - b^2$ difference of two squares
NUMBERS	$(x + 4)^2 = x^2 + 2(4)x + 4^2$ $= x^2 + 8x + 16$ or $(11 - 3)^2 = 11^2 - 2(11)(3) + 3^2$ $= 121 - 66 + 9$ $= 64$	$(x - 7)(x + 7) = x^2 - 7^2$ $= x^2 - 49$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What patterns are there in the product of the square of a binomial and the product of a sum and a difference?

2. **Error Analysis** Kennedy multiplies $(x - 3)(x + 3)$ and gets an answer of $x^2 - 6x - 9$. Describe and correct Kennedy's error.

3. **Vocabulary** The product $(x + 6)(x - 6)$ is equivalent to an expression that is called the *difference of two squares*. Explain why the term *difference of two squares* is appropriate.

4. **Use Patterns and Structure** Explain why the product of two binomials in the form $(a + b)$ $(a - b)$ is a binomial instead of a trinomial.

Do You KNOW HOW?

Write each product in standard form.

5. $(x - 7)^2$

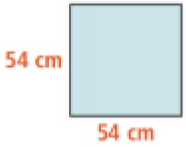
6. $(2x + 5)^2$

7. $(x + 4)(x - 4)$

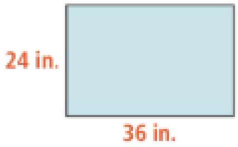
8. $(3y - 5)(3y + 5)$

Use either the square of a binomial or the difference of two squares to find the area of each rectangle.

9.



10.



PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. **Generalize** Find each product.

- $(x + 9)(x + 9)$
- $(x - 7)(x - 7)$
- $(2x - 1)^2$

a. What do all products of the square of a binomial have in common?

b. Will the third term of the square of a binomial always be positive? Explain.

c. What is the relationship between the sign of the second term of the binomial and the sign of the second term in the product?

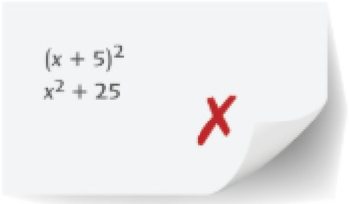
d. What is true about the exponents representing perfect square variables?

12. **Use Patterns and Structure** Find a value for m or n to make a true statement.

a. $mx^2 - 36 = (3x + 6)(3x - 6)$

b. $(mx + ny)^2 = 4x^2 + 12xy + 9y^2$

13. **Error Analysis** Describe and correct the error a student made when squaring $(x + 5)$.



$(x + 5)^2$

$x^2 + 25$

14. **Choose Efficient Methods** The expression $96^2 - 95^2$ is a difference of two squares. How can you use the factors $(96 - 95)(96 + 95)$ to make it easier to simplify this expression?

15. **Communicate and Justify** Jacob makes the following conjectures. Is each conjecture correct? Provide arguments to support your answers.

a. The product of any two consecutive even numbers is 1 less than a perfect square.

b. The product of any two consecutive odd numbers is 1 less than a perfect square.

PRACTICE

Write each product in standard form. SEE EXAMPLE 1

16. $(y + 9)(y + 9)$

17. $(5x - 3)(5x - 3)$

18. $(a + 11)(a + 11)$

19. $(x - 13)(x - 13)$

20. $(p + 15)^2$

21. $(3k + 8)^2$

22. $(x - 4y)^2$

23. $(2a + 3b)^2$

24. $\left(\frac{2}{5}x + \frac{1}{5}\right)^2$

25. $(0.4x + 1.2)^2$

Use the square of a binomial to find each product. SEE EXAMPLE 1

26. 56^2

27. 72^2

Write each product in standard form. SEE EXAMPLE 2

28. $(x - 12)(x + 12)$

29. $(2x + 5)(2x - 5)$

30. $(3a - 4b)(3a + 4b)$

31. $(x^2 - 2y)(x^2 + 2y)$

32. $\left(\frac{1}{4}x - \frac{2}{3}\right)\left(\frac{1}{4}x + \frac{2}{3}\right)$

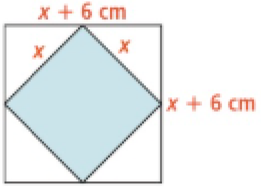
33. $(x + 2.5)(x - 2.5)$

Use the product of sum and difference to find each product. SEE EXAMPLE 2

34. $32 \cdot 28$

35. $83 \cdot 97$

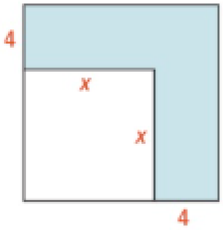
36. Consider the figure shown. SEE EXAMPLE 3



a. What expression represents the total area of the four white triangles?

b. If the length of each side of the shaded square is 12 cm, what is the total area of the four white triangles?

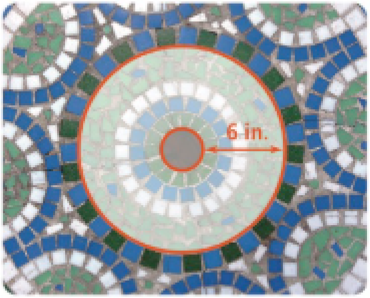
37. What is the area of the shaded region? SEE EXAMPLE 3



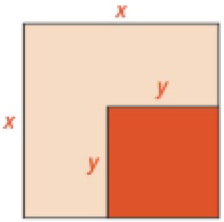
PRACTICE & PROBLEM SOLVING

APPLY

38. **Mathematical Connections** The radius of the inner circle of a tile pattern shown is x inches. Write a polynomial in standard form to represent the area of the space between the inner and outer circle.

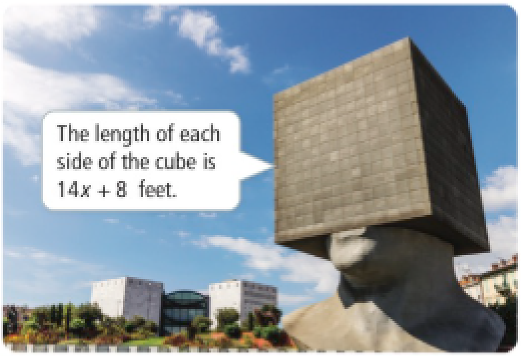


39. **Represent and Connect** In the figure shown, the darker square is removed.



- **a.** Divide the remaining figure into two rectangles. What are the dimensions of each rectangle?
- **b.** What is the area of each rectangle?
- **c.** What is the total area of the remaining figure? How does this figure represent the difference of two squares?

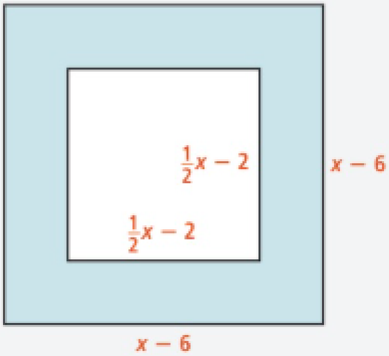
40. **Higher Order Thinking** The sculpture shown contains a large cube.



- **a.** Write a polynomial in standard form to represent the surface area of the cube.
- **b.** Write a polynomial in standard form to represent the volume of the cube.

ASSESSMENT PRACTICE

41. Select all of the expressions that could represent the shaded region in the figure below. AR.1.7



☐ **A.** $(x - 6)^2 + \left(\frac{1}{2}x - 2\right)^2$
☐ **B.** $(x^2 - 12x + 36) - \left(\frac{1}{2}x - 2\right)^2$
☐ **C.** $(x - 6)^2 - \left(\frac{1}{4}x^2 - 2x + 4\right)$
☐ **D.** $\frac{3}{4}x^2 - 10x + 32$
☐ **E.** $(x - 6)^2 - \left(\left(\frac{1}{2}x\right)^2 - 2^2\right)$

42. **SAT/ACT** What is the product of $(3x^2 - 4y)(3x^2 + 4y)$?

Ⓐ $9x^4 - 24x^2y - 16y^2$
Ⓑ $3x^2 - 4y^2$
Ⓒ $9x^4 - 16y^2$
Ⓓ $3x^2 + 14x^2y - 4y$

43. **Performance Task** Consider the difference of squares $a^2 - b^2$, for integer values of a and b.

Part A Make a table of the difference of squares using consecutive integers for a and b. What pattern do you notice?

Part B Use the pattern from Part A to find pair of consecutive integers that generates a difference of squares of –45.

Part C Make a table of the difference of squares using consecutive even integers for a and b. What pattern do you notice?

Part D Use the pattern from Part C to find a pair of consecutive even integers that generates a difference of squares of –100.

6-4

Factoring Polynomials

I CAN… factor a polynomial.

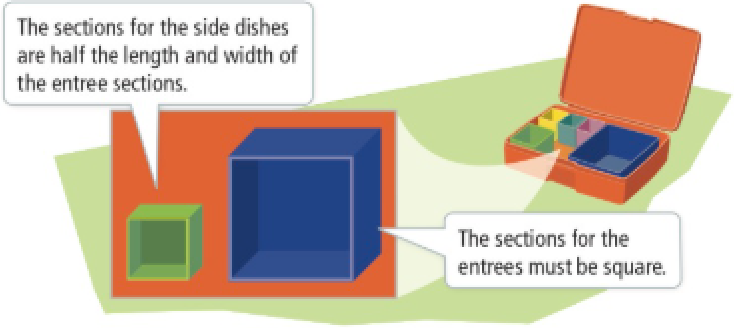
MA.912.AR.1.4–Divide a polynomial expression by a monomial expression with rational number coefficients.

Also AR.1.7

MA.K12.MTR.1.1, MTR.5.1, MTR.7.1

MODEL & DISCUSS

A food catering company has been asked to design bento boxes for entrees and side dishes.



The sections for the side dishes are half the length and width of the entree sections.

The sections for the entrees must be square.

A. Design a bento box that meets each of these requirements:

  a. Equal numbers of sections for entrees and side dishes

  b. More sections for entrees than for side dishes

  c. More sections for side dishes than for entrees

B. **Represent and Connect** For each bento box from Part A, write an algebraic expression to model the area of the bento boxes.

ESSENTIAL QUESTION

How is factoring a polynomial similar to factoring integers?

EXAMPLE 1 Divide a Monomial by a Monomial

Divide $12x^5$ by $4x^3$.

$$\frac{12x^5}{4x^3} = \left(\frac{12}{4}\right)\left(\frac{x^5}{x^3}\right)$$

$$= 3\left(\frac{x^5}{x^3}\right) \quad \text{Simplify the coefficients.}$$

$$= 3(x^{5-3}) \quad \text{Quotient of Powers Property}$$

$$= 3x^2$$

The quotient $\frac{12x^5}{4x^3}$ is equal to $3x^2$.

GENERALIZE
Dividing $12x^5$ by $4x^3$ results in a monomial. Will dividing a monomial by a monomial always result in a monomial?

Try It!

1. Simplify each quotient.

a. $\frac{45y^3}{9y^2}$ **b.** $\frac{10a^6}{25a^4}$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Relate Polynomial Division to Factoring

A. Divide $24x^3 - 12x^2 + 30x$ by $3x$.

$$\frac{24x^3 - 12x^2 + 30x}{3x} = \frac{24x^3}{3x} - \frac{12x^2}{3x} + \frac{30x}{3x}$$ Write the fraction as the sum or difference of three fractions.

$$= \frac{8x^3}{x} - \frac{4x^2}{x} + \frac{10x}{x}$$ Simplify the coefficients.

$$= 8x^{(3-1)} - 4x^{(2-1)} + 10$$ Quotient of Powers Property

$$= 8x^2 - 4x + 10$$

The quotient of $24x^3 - 12x^2 + 30x$ and $3x$ is $8x^2 - 4x + 10$.

B. Write $24x^3 - 12x^2 + 30x$ as a product of polynomials.

Use the quotient $\frac{24x^3 - 12x^2 + 30x}{3x} = 8x^2 - 4x + 10$ to rewrite $24x^3 - 12x^2 + 30x$.

$$\frac{24x^3 - 12x^2 + 30x}{3x} = 8x^2 - 4x + 10$$

$$3x\left(\frac{24x^3 - 12x^2 + 30x}{3x}\right) = 3x(8x^2 - 4x + 10)$$

Multiply each side by the divisor, 3x.

$$24x^3 - 12x^2 + 30x = 3x(8x^2 - 4x + 10)$$

So $24x^3 - 12x^2 + 30x = 3x(8x^2 - 4x + 10)$. We say $3x$ and $8x^2 - 4x + 10$ are *factors* of $24x^3 - 12x^2 + 30x$.

ANALYZE AND PERSEVERE
Can you write $24x^3 - 12x^2 + 30x$ as a product of a monomial, and a trinomial with coefficients that are less than 8, –4, and 10?

Try It! **2.** Find each quotient and write the dividend as a product.

a. $24x^3 - 12x^2 + 30x$ divided by $6x$

b. $7.2b^5 + 9.6b^3 - 12b^2$ divided by $2.4b$

EXAMPLE 3 Find the Greatest Common Factor

What is the greatest common factor (GCF) of the terms of $12x^5 + 8x^4 - 6x^3$?

Step 1 Write the prime factorization of the coefficient for each term to determine if there is a greatest common factor other than 1.

$$\begin{array}{ccc} 12 & 8 & 6 \\ \swarrow & \downarrow & \searrow \\ 2 \cdot 2 \cdot 3 & 2 \cdot 2 \cdot 2 & 2 \cdot 3 \end{array}$$

One instance of 2 is the only common factor of the numbers, so the GCF of the coefficients of this trinomial is 2.

STUDY TIP
Recall that finding the prime factorization of a number is expressing the number as a product of only prime numbers.

Step 2 Determine the greatest common factor for the variables of each term.

$$\begin{array}{ccc} x^5 & x^4 & x^3 \\ \swarrow & \downarrow & \searrow \\ x \cdot x \cdot x \cdot x \cdot x & x \cdot x \cdot x \cdot x & x \cdot x \cdot x \end{array}$$

Three instances of x are the only common factors of the terms, so the GCF of the variables is x^3.

The greatest common factor of the terms $12x^5 + 8x^4 - 6x^3$ is $2x^3$.

Try It! **3.** Find the GCF of the terms of each polynomial.

a. $15x^2 + 18$

b. $-18y^4 + 6y^3 + 24y^2$

EXAMPLE 4 Factor Out the Greatest Common Factor

Why is it helpful to factor out the GCF from a polynomial?

Consider the polynomial $-12x^3 + 18x^2 - 27x$.

Step 1 Find the GCF of the terms of the polynomial, if there is one.

Because the first term is negative, it is helpful to factor out –1.

$$-12x^3 \searrow -1 \cdot 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot x \qquad 18x^2 \searrow -1 \cdot (-2) \cdot 3 \cdot 3 \cdot x \cdot x \qquad -27x \searrow -1 \cdot 3 \cdot 3 \cdot 3 \cdot x$$

The greatest common factor is $-3x$.

COMMON ERROR
Remember to include the negative sign when factoring out the GCF of negative terms. Also, factoring out a –1 from a positive term generates two negative factors.

Step 2 Factor the GCF out of each term of the polynomial.

$$-3x(4x^2 - 6x + 9)$$

Factoring out the greatest common factor results in a polynomial with smaller coefficients and/or smaller exponents of the variable(s). This makes it easier to analyze the polynomial or factor it further.

Try It! **4.** Factor out the GCF from each polynomial.

a. $x^3 + 5x^2 - 22x$

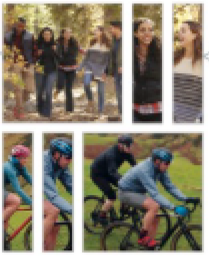
b. $-16y^6 + 28y^4 - 20y^3 + 36y^2$

APPLICATION

EXAMPLE 5 Factor a Polynomial Model

Alani is in charge of marketing for a travel company. She is designing a brochure that will have 6 photos. The photos can be arranged on the page in a number of ways.

There are 2 main square photos which have a length of x in. on each side.



There are 4 narrower photos that are each 1 in. by x in.

A. What is the total area of the photos?

First, find the area of each type of photo.

$$\begin{aligned} \text{Area} &= \text{area of square photos} + \text{area of narrower photos} \\ &= 2(x^2) + 4(1x) \\ &= 2x^2 + 4x \end{aligned}$$

There are 2 square photos, each with an area of x^2 in.^{2}. There are 4 narrower photos, each with an area of $1x$ in.^{2}.

The total area of the photos is $2x^2 + 4x$ in.^{2}.

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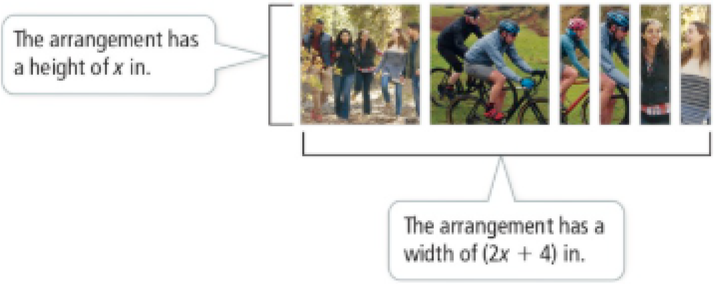
EXAMPLE 3 CONTINUED

B. Find a rectangular arrangement for the photos. What factored expression represents the area of the arrangement?

Try placing the photos in one row.

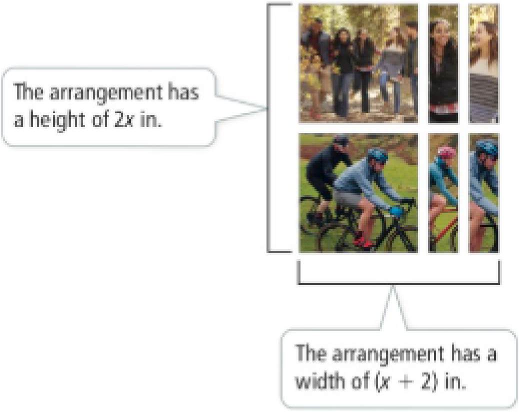


The factored form that represents the area of the arrangement is $x(2x + 4)$.

APPLY MATH MODELS
Think about how to represent this situation mathematically. How is the GCF useful in solving this problem?

C. Factor out the GCF from the polynomial. What does the GCF represent in this situation?

The GCF of $2x^2$ and $4x$ is $2x$. So you can rewrite the expression as $2x(x + 2)$.



The GCF represents the height of one possible arrangement of the photos.

D. Which of these two arrangements is a more practical use of the space on a page of the brochure?

The arrangement based on the GCF is more practical because the arrangement with the photos in one line will likely be too wide for a page.

Try It! 5. Suppose the dimensions of the narrower photos were increased to 2 in. by x in. What expression would represent the new arrangement based on the GCF?

CONCEPT SUMMARY Factoring Polynomials

WORDS Determine if a polynomial can be factored. If the polynomial can be factored, find the greatest common factor of the terms and factor it out.

ALGEBRA

$$18x^3y^2 + 12x^2y + 15x$$

Find the GCF of the terms.

$$2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot y \cdot y \qquad 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y \qquad 3 \cdot 5 \cdot x$$

The greatest common factor of $18x^3y^2 + 12x^2y + 15x$ is $3x$.

$3x(6x^2y^2 + 4xy + 5)$

Identify the remaining factors of the polynomial after factoring out the GCF, then write it in factored form.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is factoring a polynomial similar to factoring integers?

2. **Communicate and Justify** Why does the GCF of the variables of a polynomial have the *least* exponent of any variable term in the polynomial?

3. **Use Patterns and Structure** What is the greatest common factor of two monomials that do not appear to have any common factors?

4. **Error Analysis** Andrew factored $3x^2y - 6xy^2 + 3xy$ as $3xy(x - 2y)$. Describe and correct his error.

5. **Error Analysis** Wendell says that the greatest common factor of x^6 and x^8 is x^2, since the greatest common factor of 6 and 8 is 2. Is Wendell correct? Explain.

Do You KNOW HOW?

Find each quotient.

6. $\frac{42x^6}{7x^3}$

7. $\frac{42a^5 + 18a^4}{6a^4}$

8. $\frac{25x^7 - 10x^2 + 30x}{5x}$

9. $\frac{24p^6 + 12p^5 - 3.6p^4}{12p^4}$

Find the GCF of each pair of monomials.

10. x^3y^2 and x^5y

11. $8a^2$ and $28a^5$

12. $4x^3$ and $9y^5$

13. $12a^5b$ and $16a^4b^2$

Factor out the GCF from each polynomial.

14. $10a^2b + 12ab^2$

15. $x^{10} + x^9 - x^8$

16. $-33x^5 - 3x^4 + 12x^3 - 21x^2$

17. $100a^7b^5 - 150a^8b^3$

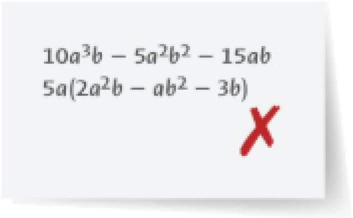
PRACTICE & PROBLEM SOLVING

UNDERSTAND

18. **Use Patterns and Structure** What term and $12x^2y$ have a GCF of $4xy$? Write an expression that shows the monomial factored out of the polynomial.

19. **Choose Efficient Methods** Write a trinomial that has a GCF of $4x^2$.

20. **Error Analysis** Describe and correct the error a student made when factoring $10a^3b - 5a^2b^2 - 15ab$ completely.



$10a^3b - 5a^2b^2 - 15ab$

$5a(2a^2b - ab^2 - 3b)$

✗

21. **Generalize** Find the quotient $12x^2 - 15x$ divided by $3x^2$. Does dividing a polynomial by a polynomial always result in a polynomial? Explain.

22. **Higher Order Thinking** In the expression $ax^2 + b$, the coefficients of a and b are multiples of 2. The coefficients c and d in the expression $cx^2 + d$ are multiples of 3. Will the GCF of $ax^2 + b$ and $cx^2 + d$ *always, sometimes,* or *never* be a multiple of 6? Explain.

23. **Analyze and Persevere** What is the GCF in the expression $x(x + 5) - 3x(x + 5) + 4(x + 5)$?

24. **Generalize** Find the greatest common factor of the terms $x^{n+1}y^n$ and x^ny^{n-2}, where n is a whole number greater than 2. How can you factor the expression $x^{n+1}y^n + x^ny^{n-2}$?

25. **Mathematical Connections** consider the following set of monomials.

$A = \{2x, 3x, 4x, 5xy, 7x, 9y, 12xy, 13x, 15x\}$

The GCF of the elements in subset $B = \{2x, 3x\}$ is x. Create 6 different subsets of A, such the GCFs of the elements are 1, $2x$, 3, $4x$, $5x$, and y.

PRACTICE

Find each quotient. SEE EXAMPLES 1 AND 2

26. $\frac{105p^6}{21p^5}$

27. $\frac{3.2m^9 - 4.8m^4}{0.4m^4}$

28. $\frac{33y^4 - 22y^3 - 77y}{11y}$

29. $\frac{96a^{10}b^3 + 51a^9b^5 + 15ab^6}{3ab}$

Find the GCF of each group of monomials.
SEE EXAMPLE 3

30. $8y^3$ and $28y$

31. $9a^2b^3$, $15ab^2$, and $21a^4b^3$

32. $18m^2$ and 25

33. x^2y^3 and x^3y^5

Factor out the GCF from each polynomial.
SEE EXAMPLE 4

34. $12x^2 - 15x$

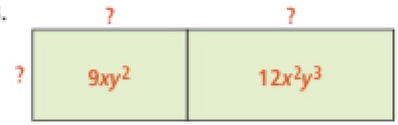
35. $-4y^4 + 6y^2 - 14y$

36. $32m^7 - 96m^6 + 144m^5 - 104m^3$

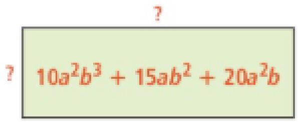
37. $24x^3y^2 - 30x^2y^3 + 12x^2y^4$

The areas of the rectangles are given. Factor out the GCF to find expressions for the missing dimensions. SEE EXAMPLE 5

38.



39.

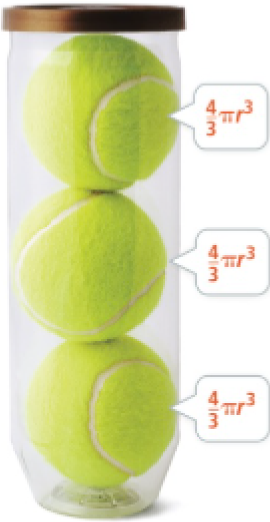


40. A farmer wants to plant three rectangular fields so that the widths are the same. The areas of the fields, in square yards, are given by the expressions $12x^2y$, $9xy^2$, and $21xy$. What is the width of the fields if $x = 3$ and $y = 4$?
SEE EXAMPLE 5

PRACTICE & PROBLEM SOLVING

APPLY

41. **Apply Math Models** Write an expression in factored form to represent the volume in the canister not occupied by the tennis balls. Assume the canister is a cylinder with volume $V = \pi r^2 h$.



42. **Use Patterns and Structure** Determine the GCF and write the expression in factored form.
$(6x^2 + 4x) + (4x^2 - 8x)$

43. **Apply Math Models** Hugo is making empanadas. He forms a sheet of dough and cuts six identical circles from it. Write an expression in factored form to represent the approximate amount of dough that is remaining. Is there enough dough for another empanada?

ASSESSMENT PRACTICE

44. Complete each factor pair for $18x^4 + 12x^3 - 24x^2$. AR.1.7

a. $\blacksquare$ and $6x^2 + 4x - 8$

b. $2x$ and $\blacksquare x^{\blacksquare} + \blacksquare x^{\blacksquare} - \blacksquare x$

c. $x^{\blacksquare}$ and $18x^2 + 12x - 24$

d. $\blacksquare x^{\blacksquare}$ and $3x^2 + 2x - 4$

45. **SAT/ACT** The area of a rectangle is $12x^3 - 18x^2 + 6x$. The width is equal to the GCF. What could the dimensions of the rectangle be?

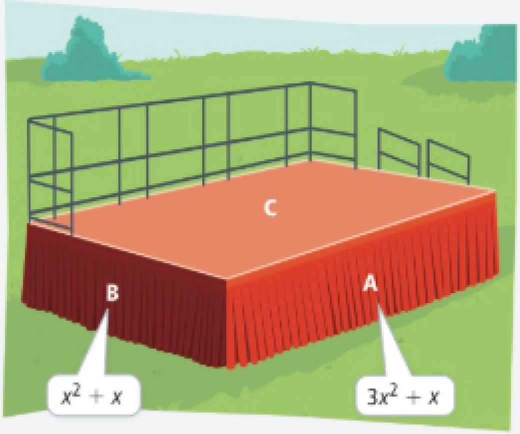
Ⓐ $6x(2x^2 - 3x)$

Ⓑ $3(4x^3 - 6x^2 + 2x)$

Ⓒ $x(12x^2 - 18x + 6)$

Ⓓ $6x(2x^2 - 3x + 1)$

46. **Performance Task** Camilla is designing a platform for an athletic awards ceremony. The areas for two of the three faces of a platform are given. The height of the platform is the greatest common factor of the polynomials representing the areas of the vertical faces.



Part A What are the dimensions of each face of the platform?

Part B What is the area of the top of the platform?

Part C What expression represents the surface area of the entire platform, excluding the bottom?

Part D What expression represents the volume of the platform?

6-5

Factoring $x^2 + bx + c$

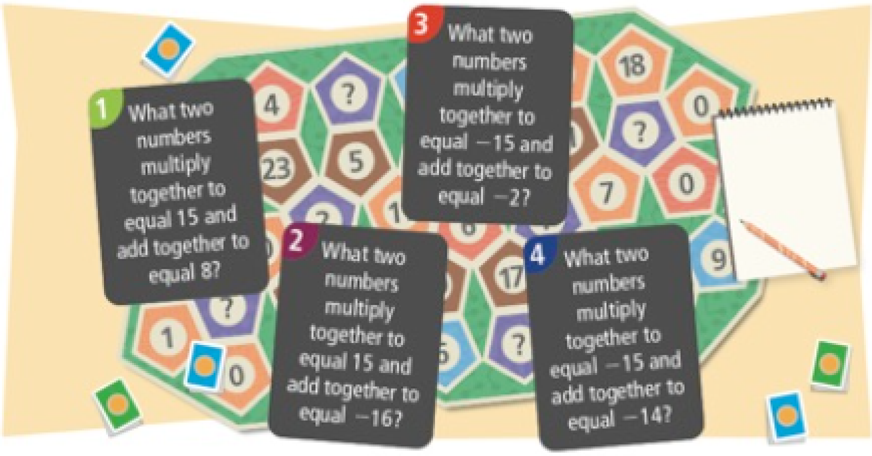
I CAN… factor a quadratic trinomial.

MA.912.AR.1.7–Rewrite a polynomial expression as a product of polynomials over the real number system.

MA.K12.MTR.1.1, MTR.3.1, MTR.5.1

EXPLORE & REASON

Consider the following puzzles.



1 What two numbers multiply together to equal 15 and add together to equal 8?

2 What two numbers multiply together to equal 15 and add together to equal −16?

3 What two numbers multiply together to equal −15 and add together to equal −2?

4 What two numbers multiply together to equal −15 and add together to equal −14?

A. Find the solutions to the four puzzles shown.

B. **Use Patterns and Structure** Write a set of four number puzzles of your own that have the same structure as these four. Describe the pattern.

ESSENTIAL QUESTION

How does recognizing patterns in the signs of the terms help you factor polynomials?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Understand Factoring a Trinomial

A. How does factoring a trinomial relate to multiplying binomials?

Consider the binomial product $(x + 2)(x + 3)$ and the trinomial $x^2 + 5x + 6$.

The product of the second terms of the binomials is equal to the last term of the trinomial.

$$(x + 2)(x + 3) = x^2 + 3x + 2x + 6$$
$$= x^2 + 5x + 6$$

The sum of the second terms of the binomials is equal to the coefficient of the second term of the trinomial.

USE PATTERNS AND STRUCTURE
How does factoring a trinomial relate to the Distributive Property?

When factoring a trinomial, you work backward to try to find the related binomial factors whose product equals the trinomial.

You can factor a trinomial of the form $x^2 + bx + c$ as $(x + p)(x + q)$ if $pq = c$ and $p + q = b$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

B. What is the factored form of $x^2 + 5x + 6$?

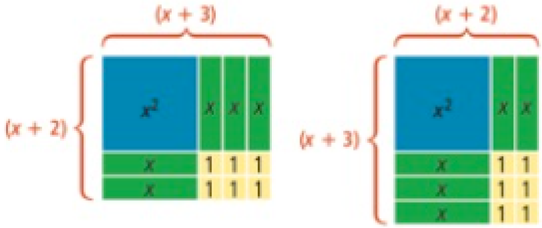
Identify a factor pair of 6 that has a sum of 5.

CHOOSE EFFICIENT METHODS
Think about different ways you could organize the factors and their sums. What organizational tools would be useful?

Factors of 6	Sum of Factors
1 and 6	7
2 and 3	5 ✓

The second term of each binomial is a factor of 6. These two factors add to 5.

If you factor using algebra tiles, the correct factor pair will form a rectangle.



The factored form of $x^2 + 5x + 6$ is $(x + 2)(x + 3)$.

Check $(x + 2)(x + 3) = x^2 + 3x + 2x + 6$

$= x^2 + 5x + 6$ ✓

The first term of each binomial is x, since $x \cdot x = x^2$.

Try It! **1.** Write the factored form of each trinomial.

a. $x^2 + 13x + 36$

b. $x^2 + 11x + 28$

EXAMPLE 2 Factor $x^2 + bx + c$, When $b < 0$ and $c > 0$

What is the factored form of $x^2 - 11x + 18$?

Identify a factor pair of 18 that has a sum of −11.

Because b is negative and c is positive, inspect only negative factors.

Factors of 18	Sum of Factors
−1 and −18	−19
−2 and −9	−11

Even though there are more factor pairs for 18, there is no need to continue once you find the correct sum.

The factored form of $x^2 - 11x + 18$ is $(x - 2)(x - 9)$.

Check $(x - 2)(x - 9) = x^2 - 9x - 2x + 18$

$= x^2 - 11x + 18$ ✓

Try It! **2.** Write the factored form of each trinomial.

a. $x^2 - 8x + 15$

b. $x^2 - 13x + 42$

EXAMPLE 3 Factor $x^2 + bx + c$, When $c < 0$

What is the factored form of $x^2 + 5x - 6$?

Identify a factor pair of -6 that has a sum of 5.

COMMON ERROR
You may think that the pairs of factors 1, −6 and −1, 6 are the same. However, the sums of the two factors are different.

Because c is negative, the factors will have opposite signs.

Factors of −6	Sum of Factors
1 and −6	−5
−1 and 6	5

The factored form of $x^2 + 5x - 6$ is $(x - 1)(x + 6)$.

Try It! **3.** Write the factored form of each trinomial.

a. $x^2 - 5x - 14$

b. $x^2 + 6x - 16$

EXAMPLE 4 Factor a Trinomial With Two Variables

A. How does multiplying binomials in two variables relate to factoring trinomials?

Consider the following binomial products.

$$(x + 2y)(x + 4y) = x^2 + 6xy + 8y^2$$
$$(x - 3y)(x + 5y) = x^2 + 2xy - 15y^2$$
$$(x - 7y)(x - 9y) = x^2 - 16xy - 63y^2$$

Each trinomial has the form $x^2 + bxy + cy^2$. Trinomials of this form are factorable when there is a factor pair of c that has a sum of b.

B. What is the factored form of $x^2 + 10xy + 24y^2$?

Identify a factor pair of 24 that has a sum of 10.

Factors of 24	Sum of Factors
3 and 8	11
4 and 6	10

STUDY TIP
When factoring a trinomial with two variables, make sure the factors contain both variables. Check your answer to determine whether you factored correctly.

The factored form of $x^2 + 10xy + 24y^2$ is $(x + 4y)(x + 6y)$.

Check $(x + 4y)(x + 6y) = x^2 + 6xy + 4xy + 24y^2$

$= x^2 + 10xy + 24y^2$ ✓

Try It! **4.** Write the factored form of each trinomial.

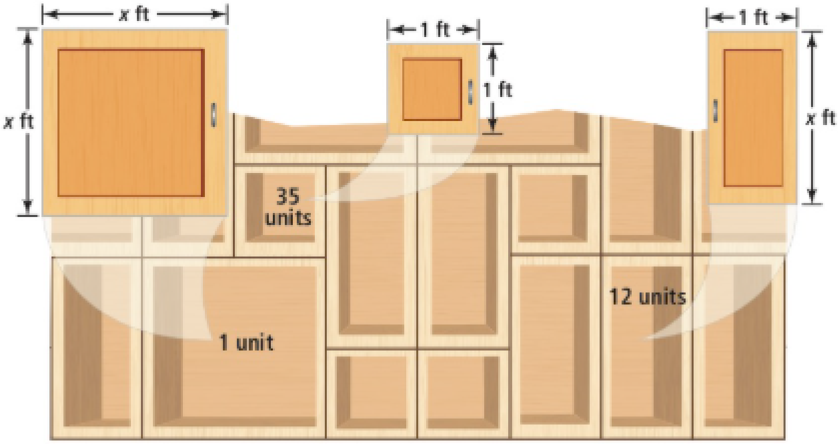
a. $x^2 + 12xy + 32y^2$

b. $x^2 - 10xy + 21y^2$

APPLICATION

EXAMPLE 5 Apply Factoring Trinomials

Benjamin is designing a new house. The bedroom closet will have one wall that contains a closet system using three different-sized storage units. The number and amount of wall space needed for each of the three types of storage units is shown. What are the dimensions of the smallest amount of wall space that will be needed?



Formulate ◀ The smallest possible closet storage system will be a rectangle that contains all of the storage units with no wasted space. Write an expression that represents the wall area of the closet in terms of the storage units.

$$x^2 + 12x + 35$$

Compute ◀ Because the area of a rectangle is the product of the length and width, factor the expression to find binomials that represent the length and width of the closet wall.

Factors of 35	Sum of Factors
1 and 35	36
5 and 7	12

$$x^2 + 12x + 35 = (x + 5)(x + 7)$$

Interpret ◀ The dimensions of the smallest amount of wall space that will be needed are $(x + 7)$ ft by $(x + 5)$ ft.

Try It! 5. What would be the dimensions of the smallest wall area you would need if you used 11 of the 1 ft-by-1 ft units while keeping the other units the same?

CONCEPT SUMMARY Factoring $x^2 + bx + c$

To factor a trinomial of the form $x^2 + bx + c$, find a factor pair of c that has a sum of b. Then use the factors you found to write the binomials that have a product equal to the trinomial.

	b and c are positive.	**b is negative and c is positive.**	**c is negative.**
WORDS	When the values of both b and c are positive, the second terms of the binomials are both positive.	When the value of b is negative and that of c is positive, the second terms of the binomials are both negative.	When the value of c is negative, the second terms of the binomials have opposite signs.
NUMBERS	b and c are positive. $x^2 + 9x + 14$ $= (x + 2)(x + 7)$	b is negative and c is positive. $x^2 - 9x + 14$ $= (x - 2)(x - 7)$	c is negative. $x^2 - 5x - 14$ $= (x + 2)(x - 7)$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does recognizing patterns in the signs of the terms help you factor polynomials?

2. **Error Analysis** A student says that since $x^2 - 5x - 6$ has two negative terms, both factors of c will be negative. Explain the error the student made.

3. **Choose Efficient Methods** What is the first step to factoring any trinomial? Explain.

4. **Communicate and Justify** To factor a trinomial $x^2 + bx + c$, why do you find the factors of c and not b? Explain.

Do You KNOW HOW?

List the factor pairs of c for each trinomial.

5. $x^2 + 17x + 16$

6. $x^2 + 4x - 21$

For each trinomial, tell whether the factor pairs of c will be both positive, both negative, or opposite signs.

7. $x^2 - 11x + 10$

8. $x^2 + 9x - 10$

9. Copy and complete the table for factoring the trinomial $x^2 - 7x + 12$.

Factors of 12	Sum of Factors
−1 and −12	?
?	−7
−2 and −6	−8

PRACTICE & PROBLEM SOLVING

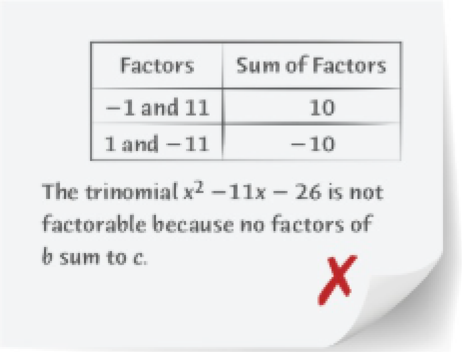
UNDERSTAND

10. **Mathematical Connections** Explain how factoring a trinomial is like factoring a number. Explain how it is different.

11. **Choose Efficient Methods** How can you use algebra tiles to factor a trinomial? How do you determine the binomial factors from an algebra tile model?

12. **Generalize** How are the binomial factors of $x^2 + 7x - 18$ and $x^2 - 7x - 18$ similar? How are they different?

13. **Error Analysis** Describe and correct the error a student made in making a table in order to factor the trinomial $x^2 - 11x - 26$.



Factors	Sum of Factors
−1 and 11	10
1 and −11	−10

The trinomial $x^2 - 11x - 26$ is not factorable because no factors of b sum to c.

✗

14. **Higher Order Thinking** Given that the trinomial $x^2 + bx + 8$ is factorable as $(x + p)(x + q)$, with p and q being integers, what are four possible values of b?

15. **Use Patterns and Structure** What is missing from the last term of the trinomial $x^2 + 5xy + 4$ so that it is factorable as the product of binomials?

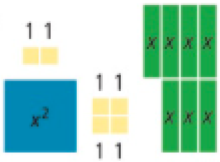
16. **Generalize** How does the sign of the last term of a trinomial help you know what type of factors you are looking for?

17. **Analyze and Persevere** A rectangle has an area of $x^2 + 7x + 12$ in.^{2}. Use factoring to find possible dimensions of the rectangle. Explain why you can use factoring to find the dimensions.

PRACTICE

Factor each trinomial represented by the algebra tiles. SEE EXAMPLE 1

18.



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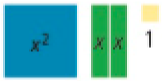


Complete the table to factor each trinomial.
SEE EXAMPLES 1 AND 3

20. $x^2 + 9x + 20$

Factors of c	Sum of Factors
?	?
?	9
?	?

21. $x^2 + 9x - 22$

Factors of c	Sum of Factors
?	?
?	?
?	9
?	?

Write the factored form of each trinomial.
SEE EXAMPLES 1, 2, 3, 4, AND 5

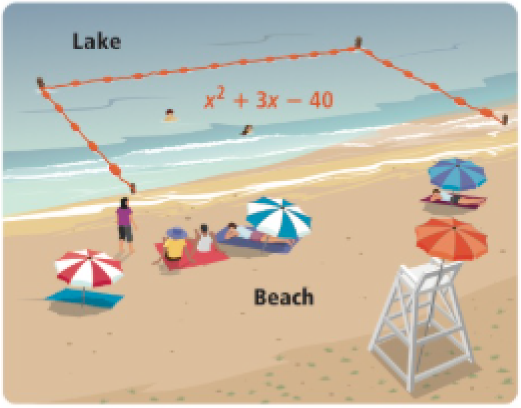
22. $x^2 + 15x + 44$
23. $x^2 - 11x + 24$
24. $x^2 + 2x - 15$
25. $x^2 - 13x + 30$
26. $x^2 + 9x + 18$
27. $x^2 - 2x - 8$
28. $x^2 + 7xy + 6y^2$
29. $x^2 - 12x + 27$
30. $x^2 + 10x + 16$
31. $x^2 - 16xy + 28y^2$
32. $x^2 - 10xy - 11y^2$
33. $x^2 + 16x + 48$
34. $x^2 - 13x - 48$
35. $x^2 + 15xy + 54y^2$

PRACTICE & PROBLEM SOLVING

APPLY

36. **Analyze and Persevere** The volume of a rectangular box is represented by $x^3 + 3x^2 + 2x$. Use factoring to find possible dimensions of the box. How are the dimensions of the box related to one another?

37. **Apply Math Models** A lake has a rectangular area roped off where people can swim under a lifeguard's supervision. The swimming section has an area of $x^2 + 3x - 40$ square feet, with the long side parallel to the lake shore.

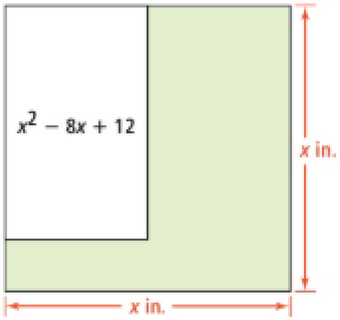


a. What are possible dimensions of the roped-off area? Use factoring.

b. How much rope is needed for the three sides that are not along the beach? Explain.

c. The rope used to mark the swimming area is 238 ft long. What is x when the total length of rope is 238 ft?

38. **Analyze and Persevere** Sarah has a large square piece of foam for an art project. The side lengths of the square are x in. To fit her project, Sarah cuts a section of foam from two of the sides so she now has a rectangle. How much foam does Sarah cut from each of the two sides?



ASSESSMENT PRACTICE

39. Maria is making a rectangular quilt. She starts with a square of fabric with side length x cm. She adds some fabric to one side of the square and a different amount of fabric to an adjacent side of the square. Her final quilt has an area of $x^2 + 13x + 30$ cm^2. How much longer is the long dimension of the quilt than the short dimension?
AR.1.7

40. **SAT/ACT** What is the factored form of $4x^3 - 24x^2 - 28x$?

Ⓐ $4x(x - 7)(x + 1)$
Ⓑ $4x(x - 1)(x + 7)$
Ⓒ $x(x - 7)(x + 4)$
Ⓓ $x(x - 4)(x + 7)$
Ⓔ $4(x - 7)(x - 1)$

41. **Performance Task** A city is designing the layout of a new park. The park will be divided into several different areas, including a field, a picnic area, and a recreation area. One design of the park is shown below.

Part A Use factoring to find the dimensions of each of the three areas of the park shown

Part B Describe two different ways to find the total area of the park.

Part C What are the dimensions of the entire park?

Part D Can you find the value of x? Explain.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.1.7–Rewrite a polynomial expression as a product of polynomials over the real number system.

MA.K12.MTR.7.1

Who's Right?

People often approach a problem in different ways. Sometimes their solutions are the same, but other times different approaches lead to very different, but still valid, solutions.

Suppose you had to solve a system of linear equations. You might solve it by graphing, while a classmate might use substitution. Is one way of solving a problem always better than another? Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

6-6

Factoring $ax^2 + bx + c$

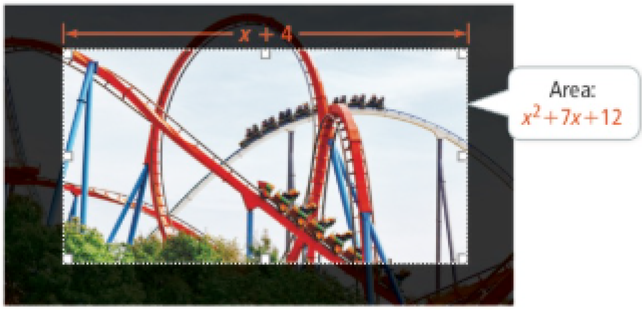
I CAN… factor a quadratic trinomial when $a \neq 1$.

MA.912.AR.1.7–Rewrite a polynomial expression as a product of polynomials over the real number system.

MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

A website design company resizes rectangular photos so they fit on the screens of various devices.



A. What expression represents the width of the photo?

B. Write three possible lengths and corresponding widths of the photo by substituting different values for x.

C. **Analyze and Persevere** Why would the company use an expression to represent the area? Explain.

ESSENTIAL QUESTION

How is factoring a quadratic trinomial when $a \neq 1$ similar to factoring a quadratic trinomial when $a = 1$?

EXAMPLE 1 Factor Out a Common Factor

What is the factored form of $3x^3 + 15x^2 - 18x$?

Before factoring the trinomial into two binomials, look for any common factors that you can factor out.

$$3x^3 \quad + \quad 15x^2 \quad - \quad 18x$$
$$3 \cdot x \cdot x \cdot x \quad + \quad 3 \cdot 5 \cdot x \cdot x \quad - \quad 2 \cdot 3 \cdot 3 \cdot x$$

There is a common factor of $3x$.

So, $3x^3 + 15x^2 - 18x = 3x(x^2 + 5x - 6)$.

Then factor the resulting trinomial, $x^2 + 5x - 6$.

Because c is negative in the trinomial $x^2 + 5x - 6$, the factors will have opposite signs.

Factors of −6	Sum of Factors
1 and −6	−5
−1 and 6	5

VOCABULARY
The answer $3x(x - 1)(x + 6)$, is $3x^3 + 15x^2 - 18x$ factored completely. There are other ways you can factor the polynomial, but only one way to factor it completely.

The factored form of $x^2 + 5x - 6$ is $(x - 1)(x + 6)$, so the factored form of $3x^3 + 15x^2 - 18x$ is $3x(x - 1)(x + 6)$.

Try It! **1.** Factor each trinomial.

a. $5x^2 - 35x + 50$ **b.** $6x^3 + 30x^2 + 24x$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Understand Factoring by Grouping

A. If $ax^2 + bx + c$ is a product of binomials, how are the values of a, b, and c related?

Consider the product $(3x + 4)(2x + 1)$.

$$\begin{aligned}(3x + 4)(2x + 1) &= (3x)(2x) + (3x)(1) + (4)(2x) + (4)(1)\\ &= 6x^2 + 3x + 8x + 4\\ &= 6x^2 + 11x + 4\end{aligned}$$

The product is $6x^2 + 11x + 4$. Notice that $ac = (6)(4)$ or $(3)(2)(4)(1)$, which is the product of all of the coefficients and constants from $(3x + 4)(2x + 1)$.

In the middle step, the coefficients of the x-terms, 3 and 8, add to form $b = 11$. They are composed of pairs of the coefficients and constants from the original product; $3 = (3)(1)$ and $8 = (4)(2)$.

If $ax^2 + bx + c$ is the product of binomials, there is a pair of factors of ac that have a sum of b.

STUDY TIP

To speed up your search, when looking for a factor pair that has a sum of b, you can rule out factor pairs with sums that are obviously far from the target.

B. How can you factor $ax^2 + bx + c$ by grouping?

Consider the trinomial $6x^2 + 11x + 4$, $a = 6$ and $c = 4$, so $ac = 24$.

Find the factor pair of 24 with a sum of 11.

Factors of 24	Sum of Factors
2 and 12	14
3 and 8	11

Rewrite $11x$ as $3x$ and $8x$.

$$\begin{aligned}6x^2 + 11x + 4 &= 6x^2 + 3x + 8x + 4\\ &= (6x^2 + 3x) + (8x + 4) && \text{Group as two binomials.}\\ &= 3x(2x + 1) + 4(2x + 1) && \text{Factor out the GCF of each binomial.}\\ &= (3x + 4)(2x + 1) && \text{Use the Distributive Property.}\end{aligned}$$

USE PATTERNS AND STRUCTURE

Common factors are not limited to monomials. Here the common factors are monomials and binomials.

The factored form of $6x^2 + 11x + 4$ is $(3x + 4)(2x + 1)$.

Check $(3x + 4)(2x + 1) = 6x^2 + 3x + 8x + 4$

$= 6x^2 + 11x + 4$ ✓

Try It! **2.** Factor each trinomial.

a. $10x^2 + 17x + 3$

b. $2x^2 + x - 21$

EXAMPLE 3 Factor a Trinomial Using Substitution

How can you use substitution to help you factor $ax^2 + bx + c$ as the product of two binomials?

Consider the trinomial $3x^2 - 2x - 8$.

Step 1 Multiply $ax^2 + bx + c$ by a to transform x^2 into $(ax)^2$.

$$3[3x^2 - 2x - 8]$$
$$= 3(3)x^2 - 2(3)x - 8(3)$$
$$= (3x)^2 - 2(3x) - 24$$

Multiply the entire trinomial by 3. The new trinomial is not equivalent to the original. Remember to divide by 3 as a last step.

$ax = 3x$

Step 2 Replace ax with a single variable. Let $p = ax$.

$$= p^2 - 2p - 24$$

Substitute p for $3x$.

Step 3 Factor the trinomial.

$$= (p - 6)(p + 4)$$

Step 4 Substitute ax back into the product. Remember $p = 3x$. Factor out common factors if there are any.

$$= (3x - 6)(3x + 4)$$
$$= 3(x - 2)(3x + 4)$$

Substitute $3x$ for p.

STUDY TIP
Because you multiplied the original expression by a new factor, the answer will not be equivalent unless you divide out the same factor at the end of your computations.

Step 5 Since you started by multiplying the trinomial by a, you must divide by a to get a product that is equivalent to original trinomial.

$$(x - 2)(3x + 4)$$

This product is equivalent to the original trinomial.

The factored form of $3x^2 - 2x - 8$ is $(x - 2)(3x + 4)$. In general, you can use substitution to help transform $ax^2 + bx + c$ with $a \neq 1$ to a simpler case in which $a = 1$, factor it, and then transform it back to an equivalent factored form.

Try It! **3.** Factor each trinomial using substitution.

a. $2x^2 - x - 6$ **b.** $10x^2 + 3x - 1$

CONCEPT SUMMARY Factoring $ax^2 + bx + c$

	Factor by Grouping	Factor Using Substitution
ALGEBRA	To factor a trinomial of the form $ax^2 + bx + c$, find a factor pair of ac that has a sum of b. Rewrite bx as a sum of those factors. Then factor out the GCFs from the expression twice to factor the original trinomial as the product of two binomials.	To factor a trinomial of the form $ax^2 + bx + c$, multiply the trinomial by a. Rewrite the first two terms using ax. Substitute a single variable for ax. Factor the trinomial. Substitute ax back in for the variable. Divide by a.
NUMBERS	$3x^2 + 22x + 7$ $= 3x^2 + 21x + 1x + 7$ $= 3x(x + 7) + 1(x + 7)$ $= (3x + 1)(x + 7)$	$3x^2 - 20x - 7$ $3[3x^2 - 20x - 7]$ $= (3x)^2 - 20(3x) - 21$ $= p^2 - 20p - 21$ $= (p - 21)(p + 1)$ $= (3x - 21)(3x + 1)$ $= 3(x - 7)(3x + 1)$ $(x - 7)(3x + 1)$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is factoring a quadratic trinomial when $a \neq 1$ similar to factoring a quadratic trinomial when $a = 1$?

2. **Error Analysis** A student says that for $ax^2 + bx + c$ to be factorable, b must equal $a + c$. Explain the error in the student's thinking.

3. **Generalize** Suppose you can factor $ax^2 + bx + c$ as $(px + q)(sx + t)$, where p, q, s, and t are integers. If $c = 1$, what do you know about the two binomial factors?

4. **Generalize** When factoring $ax^2 + bx + c$ by substitution, why is it acceptable to multiply the polynomial by a to start?

5. **Communicate and Justify** Felipe is factoring the expression $2x^2 - x - 28$ by grouping. He knows $-x$ should be rewritten as $7x$ plus $-8x$, but he is not sure which order to place the terms in the expression before grouping. Explain to Felipe why it does not matter what order the terms are in.

Do You KNOW HOW?

List the factor pairs of *ac* for each trinomial.

6. $2x^2 + 7x + 4$
7. $12x^2 - 5x - 2$

Tell whether the terms of each trinomial share a common factor other than 1. If there is a common factor, identify it.

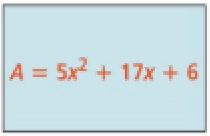
8. $15x^2 - 10x - 5$
9. $3x^3 - 2x^2 - 1$

Rewrite the *x*-term in each trinomial to factor by grouping.

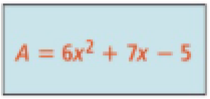
10. $35x^2 + 17x + 2$
11. $12x^2 + 20x + 3$

Factor each trinomial to find possible dimensions of each rectangle.

12.



13.



PRACTICE & PROBLEM SOLVING

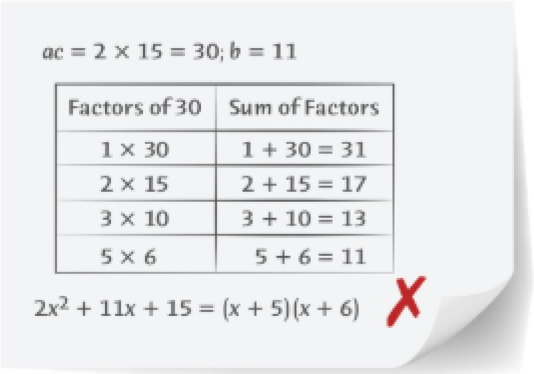
UNDERSTAND

14. **Mathematical Connections** How is factoring a common factor out of a trinomial like factoring common factors out of the numerator and denominator of a fraction? How is it different?

15. **Analyze and Persevere** What are all possible values of b for which $7x^2 + bx + 3$ is factorable, if the factors have integer coefficients and constants?

16. **Communicate and Justify** Can you factor the trinomial $3x^2 + 5x + 3$ into linear factors with integer coefficients? Explain.

17. **Error Analysis** Describe and correct the error a student made in factoring $2x^2 + 11x + 15$.



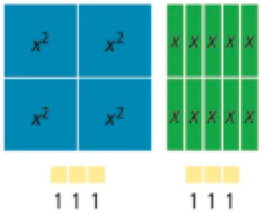
$ac = 2 \times 15 = 30; b = 11$

Factors of 30	Sum of Factors
1×30	$1 + 30 = 31$
2×15	$2 + 15 = 17$
3×10	$3 + 10 = 13$
5×6	$5 + 6 = 11$

$2x^2 + 11x + 15 = (x + 5)(x + 6)$ ✗

18. **Higher Order Thinking** Can you factor the trinomial $6x^2 + 7x - 6$ as $(px + q)(sx + t)$, where p, q, s, and t are integers? Explain why or why not.

19. **Represent and Connect** Use factoring to arrange the following algebra tiles first into one rectangle and then into two rectangles of equal size.

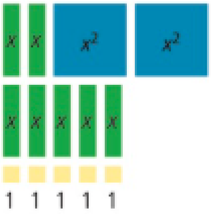


20. **Use Patterns and Structure** What is the factored form of $pqx^2 + (mp + qn)x + mn$?

PRACTICE

Factor the trinomial represented by the algebra tiles.

21.



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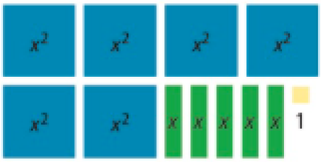


Factor each trinomial. SEE EXAMPLE 1

23. $4x^2 + 16x + 12$

24. $2x^2 - 16x + 30$

25. $3x^2 + 12x - 63$

26. $6x^2 + 12x - 48$

Identify the factor pairs of ac you could use to rewrite b to factor each trinomial by grouping.
SEE EXAMPLE 2

27. $7x^2 + 9x + 2$

28. $6x^2 + 11x - 2$

29. $8x^2 - 2x - 1$

30. $10x^2 + 19x + 6$

31. $15x^2 - 16x - 7$

32. $12x^2 + 11x + 2$

Factor each trinomial completely.
SEE EXAMPLES 1, 2, AND 3

33. $4x^2 + 13x + 3$

34. $6x^2 - 25x - 14$

35. $2x^2 + 7x - 4$

36. $12x^2 + 13x + 3$

37. $6x^3 + 9x^2 + 3x$

38. $8x^2 - 10x - 3$

39. $12x^2 + 16x + 5$

40. $16x^3 + 32x^2 + 12x$

41. $21x^2 - 35x - 14$

42. $16x^2 + 22x - 3$

43. $9x^2 + 46x + 5$

44. $24x^3 - 10x^2 - 4x$

Factor each trinomial completely.

45. $3x^2 + xy - 2y^2$

46. $2x^2 + 9xy + 10y^2$

47. $5x^2 - 4xy - y^2$

48. $2x^2 + 10xy + 12y^2$

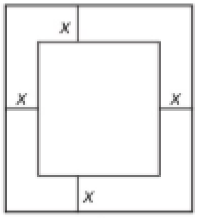
PRACTICE & PROBLEM SOLVING

APPLY

49. **Apply Math Models** A rectangular patio has an area of $2x^2 + 13x - 24$ ft^2. Use factoring to find possible dimensions of the patio. The patio is to be enlarged so that each dimension is 2 ft greater than it was originally. What are the new dimensions of the patio? What is the new area of the patio?

50. **Analyze and Persevere** Use factoring to find possible dimensions of the container shown. The container is a rectangular prism. What are the dimensions of the container if $x = 3$? What is the volume of the container if $x = 4$?

51. **Apply Math Models** A photographer is placing photos in a mat for a gallery show. Each mat she uses is x in. wide on each side. The total area of each photo and mat is shown.



Area = $4x^2 + 36x + 80$

a. Factor the total area to find possible dimensions of a photo and mat.

b. What are the dimensions of the photos in terms of x?

c. Explain why the photographer might use x to represent the width of the mat.

ASSESSMENT PRACTICE

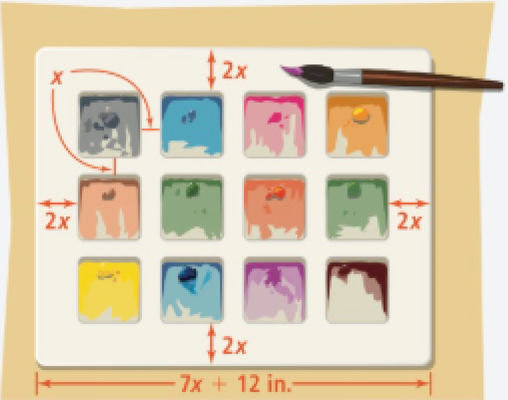
52. The area of a rectangular pool is represented by $2x^2 + 9x - 5$ ft^2. The width of the pool is represented by $x + 5$ ft. Which of the following factors represents the length of the pool, in feet? AR.1.7

Ⓐ $2x + 1$
Ⓑ $2x - 1$
Ⓒ $x - 1$
Ⓓ $2x + 4$

53. **SAT/ACT** What is the factored form of $3x^2 - 5x - 12$?

Ⓐ $(x - 4)(3x + 1)$
Ⓑ $(x - 3)(3x + 4)$
Ⓒ $(x + 4)(3x - 9)$
Ⓓ $3(x + 2)(x - 3)$
Ⓔ $3(x - 4)(x + 1)$

54. **Performance Task** A paint tray has an area of $42x^2 + 135x + 108$ in.^{2}. The square paint compartments that are all the same size and spaced evenly, though the space along the edge of the tray is twice as wide as the space between squares.



Part A What is the width of the paint tray?

Part B What is the area of each of the paint compartments in the tray?

Part C How wide are the edges of the tray if the width of the paint tray is 45 in.?

6-7

Factoring Special Cases

I CAN… factor special trinomials.

VOCABULARY

- perfect-square trinomial

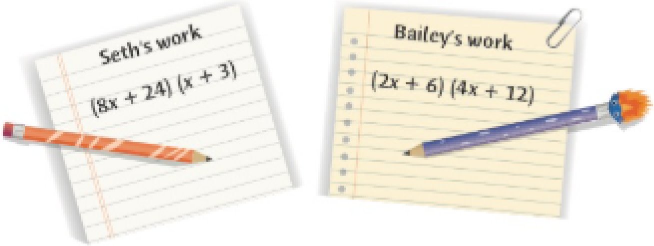
MA.912.AR.1.7–Rewrite a polynomial expression as a product of polynomials over the real number system.

Also AR.1.1

MA.K12.MTR.4.1, MTR.5.1, MTR.6.1

CRITIQUE & EXPLAIN

Seth and Bailey are given the polynomial $8x^2 + 48x + 72$ to factor.



A. Analyze each factored expression to see if both are equivalent to the given polynomial.

B. How can the product of different pairs of expressions be equivalent?

C. **Represent and Connect** Find two other pairs of binomials that are different, but whose products are equal.

ESSENTIAL QUESTION What special patterns are helpful when factoring a perfect-square trinomial and the difference of two squares?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Understand Factoring a Perfect Square

What is the factored form of a perfect-square trinomial?

A **perfect-square trinomial** results when a binomial is squared.

$$(a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)(a - b) = (a - b)^2 = a^2 - 2ab + b^2$$

The first and last terms are perfect squares. The middle term is twice the product of the first and last terms of the binomial.

A. What is the factored form of $x^2 + 14x + 49$?

Write the last term as a perfect square.

$$\begin{aligned} x^2 + 14x + 49 &= x^2 + 14x + 7^2 \\ &= x^2 + 2(7)x + 7^2 \\ &= (x + 7)(x + 7) = (x + 7)^2 \end{aligned}$$

$2ab = 2(7)x = 14x$, so the trinomial fits the pattern.

COMMON ERROR
Be careful to identify the correct values for a and b when factoring special cases. The value of a can be different from x.

B. What is the factored form of $9x^2 - 30x + 25$?

Write the first and last terms as a perfect square.

$$\begin{aligned} 9x^2 - 30x + 25 &= (3x)^2 - 30x + 5^2 \\ &= (3x)^2 - 2(3x)(5) + 5^2 \\ &= (3x - 5)(3x - 5) = (3x - 5)^2 \end{aligned}$$

$2ab = 2(3x)(5) = 30x$, so the trinomial fits the pattern.

The factored form of a perfect-square trinomial is $(a + b)^2$ when the trinomial fits the pattern $a^2 + 2ab + b^2$, and $(a - b)^2$ when the trinomial fits the pattern $a^2 - 2ab + b^2$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Try It! **1.** Factor each trinomial.

a. $4x^2 + 12x + 9$

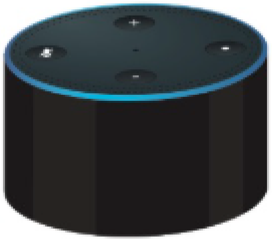
b. $x^2 - 8x + 16$

APPLICATION

EXAMPLE 2 Factor to Find a Dimension

Sasha has a tech store and needs cylindrical containers to package her voice-activated speakers. A packaging company makes two different cylindrical containers. Both are 3 in. high. The volume information is given for each type of container. Determine the radius of each cylinder. How much greater is the radius of one container than the other?

Volume:
$3\pi x^2$ in.^{3}



Volume:
$\pi(3x^2 + 30x + 75)$ in.^{3}

Formulate The formula for the volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height of the cylinder. The height of both containers is 3 in., so both expressions will have 3π in common.

Factor 3 out of the trinomial

$3\pi x^2 = 3\pi(x^2)$

$\pi(3x^2 + 30x + 75) = 3\pi(x^2 + 10x + 25)$

Factor the expressions to identify the radius of each cylinder.

Compute The expression $x^2 = x \cdot x$, so the radius of the first cylinder is x in.

Factor the expression $x^2 + 10x + 25$ to find the radius of the second cylinder.

$$\begin{aligned} x^2 + 10x + 25 &= x^2 + 2(5)x + 5^2 \\ &= (x + 5)^2 \end{aligned}$$

Rewrite the first and last terms as squares.

The radius of the second cylinder is $(x + 5)$ in.

Find the difference between the radii.

$(x + 5) - x = 5$

Interpret The larger cylinder has a radius that is 5 in. greater than the smaller one.

Try It! **2.** What is the radius of a cylinder that has a height of 3 in. and a volume of $\pi(27x^2 + 18x + 3)$ in.^{3}?

EXAMPLE 3 Factor a Difference of Two Squares

How can you factor the difference of squares using a pattern?

Recall that a binomial in the form $a^2 - b^2$ is called the difference of two squares.

$$(a - b)(a + b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

A. What is the factored form of $x^2 - 9$?

Write the last term as a perfect square.

$$x^2 - 9 = x^2 - 3^2$$
$$= (x + 3)(x - 3)$$

$a = x$ and $b = 3$, so the binomial fits the pattern.

CHECK FOR REASONABLENESS
Determine whether the factoring rule for a difference of two squares makes sense by working backward.

B. What is the factored form of $4x^2 - 81$?

Write the first and last terms as perfect squares.

$$4x^2 - 81 = (2x)^2 - 9^2$$
$$= (2x + 9)(2x - 9)$$

$a = 2x$ and $b = 9$, so the binomial fits the pattern.

The difference of two squares is a factoring pattern when one perfect square is subtracted from another. If a binomial follows that pattern, you can factor it as a sum and difference.

Try It! **3.** Factor each expression.

a. $x^2 - 64$

b. $9x^2 - 100$

EXAMPLE 4 Factor Out a Common Factor

What is the factored form of $3x^3y - 12xy^3$?

Factor out a greatest common factor of the terms if there is one. Then factor as the difference of squares.

HAVE A GROWTH MINDSET
How can you take on challenges with positivity?

$3x^3y - 12xy^3 = 3xy(x^2 - 4y^2)$ Factor out the GCF, $3xy$.

$= 3xy[x^2 - (2y)^2]$ Write each term in the brackets as a perfect square.

$= 3xy(x + 2y)(x - 2y)$ Use the difference of squares pattern.

The factored form of $3x^3y - 12xy^3$ is $3xy(x + 2y)(x - 2y)$.

Try It! **4.** Factor each expression completely.

a. $4x^3 + 24x^2 + 36x$

b. $50x^2 - 32y^2$

CONCEPT SUMMARY Factoring Special Cases of Polynomials

	Factoring a Perfect-Square Trinomial	Factoring a Difference of Two Squares
ALGEBRA	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$	$a^2 - b^2 = (a + b)(a - b)$
WORDS	Use this pattern when the first and last terms are perfect squares and the middle term is twice the product of the expressions being squared.	Use this pattern when a binomial can be written as a difference of two squares. Both terms must be perfect squares.
NUMBERS	$x^2 + 16x + 64 = x^2 + 2(8)x + 8^2$ $= (x + 8)^2$ $x^2 - 16x + 64 = x^2 - 2(8)x + 8^2$ $= (x - 8)^2$	$x^2 - 36 = x^2 - 6^2$ $= (x + 6)(x - 6)$ $2x^2 - 72 = 2(x^2 - 36)$ $= 2(x^2 - 6^2)$ $= 2(x + 6)(x - 6)$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What special patterns are helpful when factoring a perfect-square trinomial and the difference of two squares?

2. **Error Analysis** A student says that to factor $x^2 - 4x + 2$, you should use the pattern of a difference of two squares. Explain the error in the student's thinking.

3. **Vocabulary** How is a perfect square trinomial similar to a perfect square number? Is it possible to have a perfect square binomial? Explain.

4. **Use Patterns and Structure** How is the pattern for factoring a perfect-square trinomial like the pattern for factoring the difference of two squares? How is it different?

5. **Communicate and Justify** Why is it important to look for a common factor before factoring a trinomial?

Do You KNOW HOW?

Identify the pattern you can use to factor each expression.

6. $4x^2 - 9$

7. $x^2 + 6x + 9$

8. $9x^2 - 12x + 4$

9. $5x^2 - 30x + 45$

10. $100 - 16y^2$

11. $3x^2 + 30x + 75$

Write the factored form of each expression.

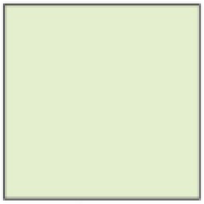
12. $49x^2 - 25$

13. $36x^2 + 48x + 16$

14. $3x^3 - 12x^2 + 12x$

15. $72x^2 - 32$

16. What is the side length of the square shown below?



Area = $x^2 + 22x + 121$

PRACTICE & PROBLEM SOLVING

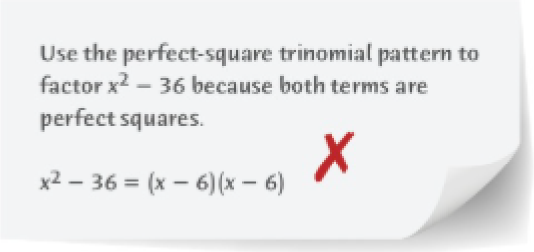
UNDERSTAND

17. **Mathematical Connections** How could you use special factoring patterns to quickly rewrite the difference $50^2 - 45^2$ as a product? Explain.

18. **Use Patterns and Structure** Is the expression $x^2 - 50$ factorable using integers? Explain why or why not.

19. **Choose Efficient Methods** What is the completely factored form of the expression $16x^4 - y^4$? Describe the method(s) of factoring you used.

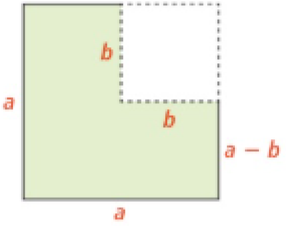
20. **Error Analysis** Describe and correct the error a student made in factoring $x^2 - 36$.



Use the perfect-square trinomial pattern to factor $x^2 - 36$ because both terms are perfect squares.

$x^2 - 36 = (x - 6)(x - 6)$ ✗

21. **Higher Order Thinking** Use the visual shown as a starting point. Describe how you can use diagrams to show that $a^2 - b^2 = (a + b)(a - b)$.



22. **Analyze and Persevere** Describe the steps you would use to factor the expression $x^4 - 8x^2 + 16$.

23. **Use Patterns and Structure** A rectangle has a width that is twice the length. If the area of the rectangle is represented by the expression $18x^2 + 48x + 32$, what expression represents the length of the rectangle? Explain.

24. **Generalize** How can you determine if a binomial of the form $x^2 - \frac{a}{b}$ is factorable using rational constants?

PRACTICE

Identify the value of c that would make the trinomial factorable using the perfect-square pattern. SEE EXAMPLE 1

25. $x^2 + 24x + c$

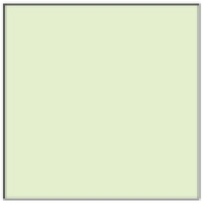
26. $x^2 - 10x + c$

27. $6x^2 - 36x + c$

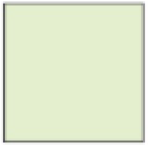
28. $3x^2 + 24x + c$

Given the area of each square, factor to find the side length. SEE EXAMPLES 1 AND 2

29. Area = $36x^2 + 120x + 100$



30. Area = $144x^2 - 24x + 1$



Factor each expression completely.
SEE EXAMPLES 1, 3, and 4

31. $x^2 + 16x + 64$

32. $x^2 - 25$

33. $x^2 - 18x + 81$

34. $x^2 - 14x + 49$

35. $100x^2 - 36$

36. $16x^2 + 40x + 25$

37. $8x^2 - 32x + 32$

38. $16x^2 - 81y^2$

39. $2x^3 + 32x^2 + 128x$

40. $7x^3y - 63xy^3$

41. $49x^3 - 16xy^2$

42. $121x^2 + 110x + 25$

43. $-3x^3 + 18x^2 - 27x$

44. $64x^2y^2 - 144z^2$

Factor each expression as the product of binomials.

45. $x^2 - \frac{1}{4}$

46. $x^2 - \frac{1}{9}$

47. $p^2 - \frac{49}{100}$

48. $x^2 + x + \frac{1}{4}$

PRACTICE & PROBLEM SOLVING

APPLY

49. **Communicate and Justify** In front of a school are several gardens in rectangular raised beds. For each of the areas of a rectangular garden given, use factoring to find possible dimensions. Could the garden be square? If so, explain why.

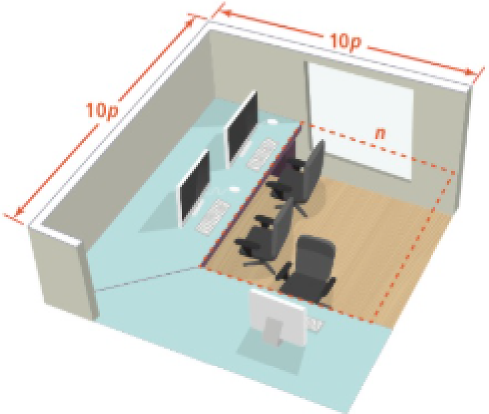
a. $x^2 + 32x + 256$

b. $x^2 - 4y^2$

c. $x^2 - 20x + 100$

50. **Analyze and Persevere** The area of a rectangular rug is $49x^2 - 25y^2$ in.^{2}. Use factoring to find possible dimensions of the rug. How are the side lengths related? What value would you need to subtract from the longer side and add to the shorter side for the rug to be a square?

51. **Apply Math Models** A furniture company created an L-shaped table by removing part of a square table.



a. Write an expression that represents the area of the L-shaped table.

b. What are all the side lengths of the L-shaped table?

c. The furniture company decides to create another table with the same area, but needs this table to be rectangular. What are the possible dimensions of the rectangular table? Explain.

ASSESSMENT PRACTICE

52. Select all polynomials that factor into a product of two binomials.
AR.1.7

☐ **A.** $25m^2 - 9n^2$

☐ **B.** $25m^2 - 30mn + 9n^2$

☐ **C.** $25m^2 - 30mn - 9n^2$

☐ **D.** $25m^2 + 30mn + 9n^2$

☐ **E.** $25m^2 + 9n^2$

53. **SAT/ACT** What is the factored form of $6x^2 - 60x + 150$?

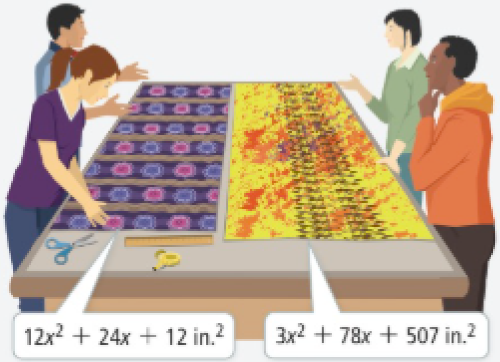
Ⓐ $6(x - 25)^2$

Ⓑ $6(x - 5)(x - 10)$

Ⓒ $6(x - 5)^2$

Ⓓ $6(x - 5)(x + 5)$

54. **Performance Task** Two pieces of fabric are being used for clothing designs for a fashion show at school. Expressions for the areas of the rectangular pieces are shown.



Part A Factor the expressions for the areas completely.

Part B Using the factorings from Part A, write all of the possible dimensions of the pieces of fabric as binomials with integer coefficients.

Part C Assume that the table is about 6 ft long. Using integer values for x, which set of binomials yields to most reasonable dimensions based on the picture?

Part D Using your result from Part C what are the dimensions in inches of the two pieces of fabric?

TOPIC
6

Topic Review

TOPIC ESSENTIAL QUESTION

1. How do you work with polynomials to rewrite expressions and solve problems?

Vocabulary Review

Choose the correct term to complete each sentence.

2. The __________ states that polynomials are closed under addition or subtraction because the result of these operations is another polynomial.
3. A(n) __________ results when a binomial is squared.
4. A(n) __________ is a real number, a variable, or the product of a real number and one or more variables with whole number exponents.
5. The product of two binomials in the form $(a + b)(a - b)$ is $a^2 - b^2$, which is called the __________.
6. The __________ is a an expression in which the terms are written in descending order according to their degree.

- Closure Property
- degree of a monomial
- degree of a polynomial
- difference of two squares
- monomial
- perfect-square trinomial
- polynomial
- standard form of a polynomial

Concepts & Skills Review

LESSON 6-1 Adding and Subtracting Polynomials

Quick Review

A **polynomial** is a monomial or the sum or difference of two or more monomials, called terms. Polynomials are named according to their degree. The **degree of a polynomial** is the greatest degree of any term of the polynomial. The **standard form of a polynomial** is a polynomial in which terms are written in descending order according to their degree.

Example

What is the difference $(5x^2 + 3x - 5) - (2x^2 + 8)$?

$(5x^2 + 3x - 5) - (2x^2 + 8)$

$= 5x^2 + 3x - 5 - 2x^2 - 8$ Apply subtraction to each term in the second expression.

$= (5x^2 - 2x^2) + (3x) + (-5 - 8)$ Use the Commutative and Associative Properties to group like terms.

$= 3x^2 + 3x - 13$ Simplify.

The difference is $3x^2 + 3x - 13$.

Practice & Problem Solving

Name each monomial based on its degree.

7. $2xy$
8. -6
9. $3x^2y$

Add or subtract to simplify each expression. Write your final answer in standard form.

10. $(5x - 1) + (2x - 3)$
11. $(2.7x^2 - 4x - 1) - (3x^2 + 8x - 4)$
12. $(5b^4 - 2 + 3b^2) + (5b^2 - 4 + 3b^4)$

13. **Use Patterns and Structure** What is the missing term in the equation? $(\_\_\_ + 5) + (3x - 2) = 8x + 3$. Explain.

14. **Analyze and Persevere** A garden center has $(3x^2 + 12x + 18)$ sq. ft of sod. One week, they receive $(4x^2 + 16x + 60)$ sq. ft of sod, and sell $(2x^2 + 9x + 27)$ sq. ft of sod. What expression represents the area of the remaining sod?

LESSON 6-2 Multiplying Polynomials

Quick Review

Use the Distributive Property to multiply polynomials as you would when multiplying integers numbers. Distribute the first polynomial to each term in the second polynomial.

Example

How can you use the Distributive Property to rewrite $(3x - 5)(4x - 9)$ as a polynomial?

Distribute the first binomial to each term in the second binomial.

$(3x - 5)(4x - 9)$

$= 3x(4x - 9) - 5(4x - 9)$ Distribute 3x and –5 to the second binomial.

$= 3x(4x) + 3x(-9) - 5(4x) - 5(-9)$ Distribute 3x and –5 to each term in the second binomial.

$= 12x^2 - 27x - 20x + 45$ Multiply.

$= 12x^2 - 47x + 45$ Combine like terms.

The product is $12x^2 - 47x + 45$.

Practice & Problem Solving

Use the Distributive Property to find each product.

15. $(x + 7)(x - 5)$

16. $(2x - 5)(3x + 1)$

Use a table to find each product.

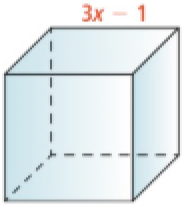
17. $(4x - 3y)(5x + y)$

18. $(x + 4)(x^2 - 3x - 1)$

19. **Analyze and Persevere** Identify the missing terms in the quotient and divisor.

$(\_\_\_ + 3)(x + \_\_\_) = x^2 + 11x + 24$

20. **Apply Math Models** The volume of a cube is calculated by multiplying the length, width and height. What is the volume of this cube in standard form?



LESSON 6-3 Multiplying Special Cases

Quick Review

The square of a binomial always follows the same pattern, $a^2 + 2ab + b^2$. The product of two binomials in the form $(a + b)(a - b)$ is $a^2 - b^2$. This is called the **difference of two squares**.

Example

What is the product $(4x - 9)(4x + 9)$?

Use the pattern.

$(4x - 9)(4x + 9)$

$= (4x)^2 - (9)^2$ Substitute 4x and 9 and for a and b in $a^2 - b^2$.

$= 16x^2 - 81$ Simplify.

The product is $16x^2 - 81$.

Practice & Problem Solving

Write each product in standard form.

21. $(b + 12)(b + 12)$

22. $(4x + 1)(4x + 1)$

23. $(6x - 9)(6x + 9)$

24. $(3x - 4y)(3x + 4y)$

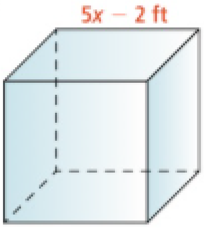
25. $(1.5x + 2)(1.5x - 2)$

26. $(3a - 5b)^2$

27. **Use Patterns and Structure** Find a value for m to make a true statement.

$mx^2 - 64 = (5x + 8)(5x - 8)$

28. **Apply Math Models** Write polynomials in standard form to represent the surface area and volume of the cube.



LESSON 6-4 Factoring Polynomials

Quick Review

To factor a common monomial factor out of a polynomial, first write the prime factorization of the coefficient for each term to determine if there is a greatest common factor other than 1. Then determine the greatest common factor for the variables of each term.

Example

What is the GCF of the terms of $16x^6 - 8x^4 + 4x^3$?

First, write the prime factorization of the coefficients for each term.

$16 = 2 \cdot 2 \cdot 2 \cdot 2$
$8 = 2 \cdot 2 \cdot 2$
$4 = 2 \cdot 2$

Each number has a common coefficient of 4, so the GCF of the coefficients is 4.

Next, determine the GCF of the variables for each term.

$x^6 = x \cdot x \cdot x \cdot x \cdot x \cdot x$
$x^4 = x \cdot x \cdot x \cdot x$
$x^3 = x \cdot x \cdot x$

Each term has the common factor of x^3, so the GCF of the variables is x^3.

The GCF of $16x^6 - 8x^4 + 4x^3$ is $4x^3$.

Practice & Problem Solving

Find the GCF of each group of monomials.

29. $6x^2, 21x$

30. bc^2, b^3c

31. $14x^2y^2, 84x^3y^5, 21xy^3$

32. $24a^2, 18$

Factor out the GCF from each polynomial.

33. $15x^3 - 42x$

34. $6y^5 - 42y^3 + 18y$

35. $6a^4 + 12a^3 + 18a^2 - 36a$

36. $49a^5b^3 - 14a^2b^2 + 35ab$

37. **Analyze and Persevere** Write a trinomial that has a GCF of $3x$.

38. **Use Patterns and Structure** Determine the GCF and write the expression in factored form.

$(8x^2 - 12x) + (6x^2 - 4x)$

LESSON 6-5 Factoring $x^2 + bx + c$

Quick Review

To factor $x^2 + bx + c$, find the factor pair of c that has a sum of b. Then use those factors to write the binomial factors of the trinomial.

Example

What is the factored form of $x^2 - 9x + 14$?

Identify a factor pair of 14 that has a sum of –9.

Factors of 14	Sum of Factors
–1 and –14	–15
–2 and –7	–9

The factored form of $x^2 - 9x + 14$ is $(x - 2)(x - 7)$.

Practice & Problem Solving

Complete the table to factor the trinomial

39. $x^2 + 7x - 18$

Factors of *c*	Sum of Factors
■	■
■	■
■	7

Write the factored form of each trinomial.

40. $x^2 + 12x + 32$

41. $x^2 + 3x - 28$

42. $x^2 - 13x - 48$

43. $x^2 + 18xy + 45y^2$

44. **Analyze and Persevere** How are the binomial factors of $x^2 + 4x - 21$ and $x^2 - 4x - 21$ similar? How are they different?

LESSON 6-6 Factoring $ax^2 + bx + c$

Quick Review

To factor a trinomial of the form $ax^2 + bx + c$, find the factor pair of ac that has a sum of b. Then use the factors you found to write the binomials that have a product equal to the trinomial.

Example

What is the factored form of $2x^2 + 9x - 5$?

For $2x^2 + 9x - 5$, $a = 2$ and $c = -5$, so $ac = -10$. Find the factor pair of -10 that has a sum of 9.

Factors of -10	Sum of Factors
-1 and 10	9

Since -1 and 10 are the correct factor pair, rewrite $9x$ as $-1x$ and $10x$.

$2x^2 + 9x - 5$

$= 2x^2 + 10x - 1x - 5$ ········ Rewrite.

$= (2x^2 + 10x) + (-1x - 5)$ ···· Group as two binomials.

$= 2x(x + 5) - 1(x + 5)$ ········ Factor out the GCFs.

$= (2x - 1)(x + 5)$ ················ Distributive Property

The factored form of $2x^2 + 9x - 5$ is $(2x - 1)(x + 5)$.

Practice & Problem Solving

Identify all of the factor pairs of *ac* you could use to rewrite *b* in order to factor each trinomial by grouping.

45. $5x^2 + 9x + 4$

46. $2x^2 + x - 15$

Write the factored form of each trinomial.

47. $3x^2 + 10x + 8$

48. $4x^2 - 3x - 10$

49. $5x^2 + 7x - 6$

50. $6x^2 + 13x + 6$

51. $10x^2 + 3x - 4$

52. $12x^2 + 22x + 6$

53. **Analyze and Persevere** What are all the possible values of b for which $3x^2 + bx - 8$ is factorable using only integer coefficients and constants?

54. **Apply Math Models** A parking lot has an area of $2x^2 + 9x - 5$ square meters. Use factoring to find possible dimensions of the parking lot. The parking lot is to be enlarged so that each dimension is 5 meters greater than it was originally. What are the new dimensions of the parking lot? What is the new area of the parking lot?

LESSON 6-7 Factoring Special Cases

Quick Review

Use the following patterns to identify and factor perfect square trinomials:

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

Use the following pattern to identify and factor the difference of two squares:

$a^2 - b^2 = (a + b)(a - b)$

Example

What is the factored form of $9x^2 - 121$?

Write the first and last term as a perfect square.

$9x^2 - 121 = (3x)^2 - 11^2$

$= (3x - 11)(3x + 11)$

Practice & Problem Solving

Identify the value of *c* that would make each trinomial factorable using the perfect-square pattern.

55. $x^2 + 16x + c$

56. $2x^2 - 28x + c$

Write the factored form of each expression.

57. $x^2 + 10x + 25$

58. $x^2 - 121$

59. $x^2 - 18x + 81$

60. $9x^2 - 49y^2$

61. $3x^2 + 18x + 27$

62. $4x^2 - 56x + 196$

63. **Use Patterns and Structure** Is the expression $3x^2 - 49$ factorable using only integer coefficients and constants? Explain why or why not.

TOPIC

7

Quadratic Functions

TOPIC ESSENTIAL QUESTION

How can you use sketches and equations of quadratic functions to model situations and make predictions?

Topic Overview

enVision® STEM Project:
Make Business Decisions

7-1 Key Features of a Quadratic Function
AR.3.4, AR.3.7, F.1.2, F.1.3, F.2.1, MTR.1.1, MTR.2.1, MTR.5.1

7-2 Quadratic Functions in Vertex Form
AR.3.4, AR.3.7, AR.3.8, F.2.1, MTR.1.1, MTR.4.1, MTR.5.1

7-3 Quadratic Functions in Standard Form
AR.1.2, AR.3.4, AR.3.7, MTR.3.1, MTR.4.1, MTR.6.1

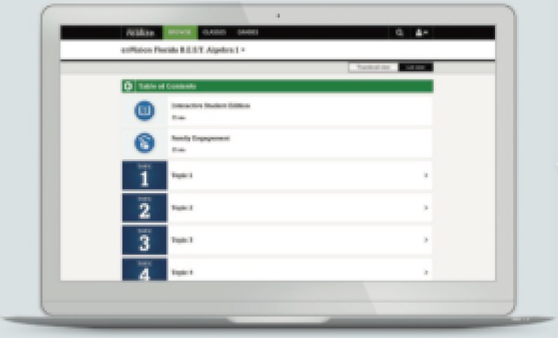
7-4 Modeling With Quadratic Functions
AR.1.1, AR.3.4, AR.3.8, F.1.2, MTR.2.1, MTR.3.1, MTR.7.1

Mathematical Modeling in 3 Acts:
The Long Shot
AR.3.4, AR.3.8, MTR.7.1

7-5 Linear, Exponential, and Quadratic Models
F.1.1, F.1.6, F.1.8, MTR.4.1, MTR.5.1, MTR.7.1

Topic Vocabulary

- parabola
- quadratic parent function
- standard form of a quadratic function
- vertex form of a quadratic function
- vertical motion model



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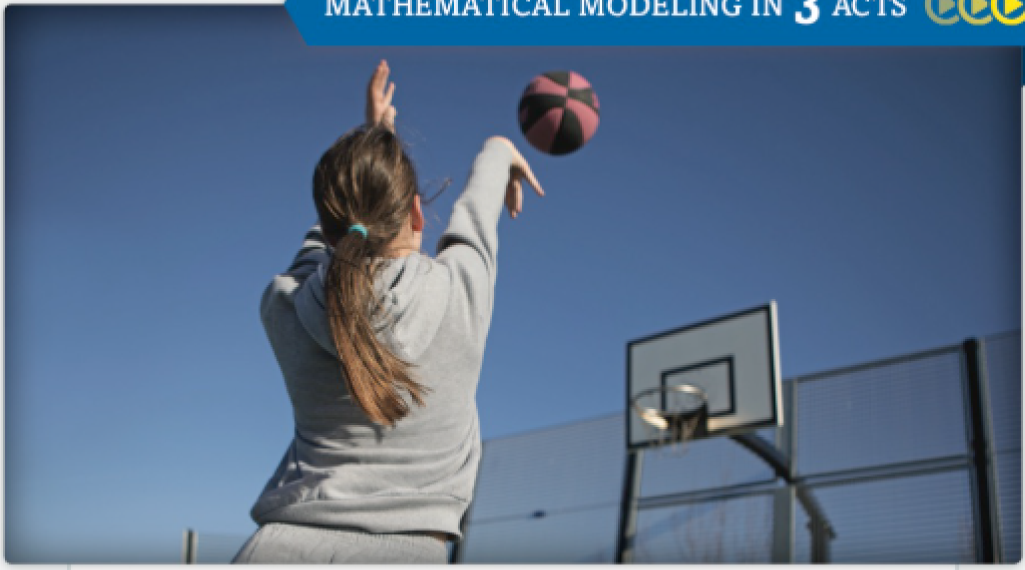
PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



The Long Shot

Have you ever been to a basketball game where they hold contests at halftime? A popular contest is one where the contestant needs to make a basket from half court to win a prize. Contestants often shoot the ball in different ways. They might take a regular basketball shot, a hook shot, or an underhand toss.

What's the best way to shoot the basketball and make a basket? In the Mathematical Modeling in 3 Acts lesson, you decide!

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ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

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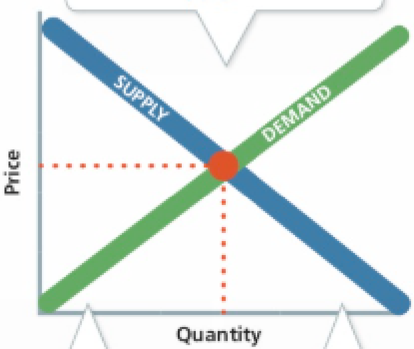
Did You Know?

The **goal of a business** owner is to **maximize profits**. Businesses have to consider many things to set the best price for their products.

A typical **small business has a net profit margin of around 10%.** This means 90% of its revenue is spent on costs, such as rent, labor, and raw materials.

The day after Thanksgiving is known as Black Friday because that is the day many retailers begin to **turn a profit for the year**. Being "in the black" is an accounting term for "making a profit."

Market equilibrium is when **supply = demand**.



Demand is how much of a product people want to buy. The **higher the demand, the higher producers can price the product.**

Supply is how much of a product is available. The **higher the supply, the lower the price** producers can charge.

Your Task: Make Business Decisions

You and your classmates will pick an industry, then suggest and defend your choice of the number of an item to make and the price at which to sell the item.

7-1

Key Features of a Quadratic Function

I CAN... identify key features of the graph of the quadratic parent function.

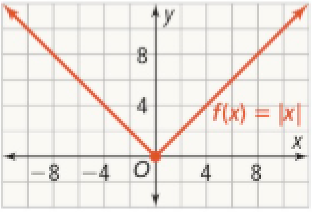
VOCABULARY

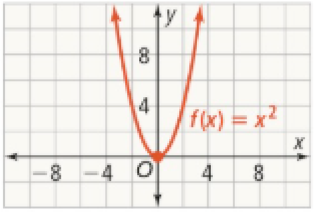
- parabola
- quadratic parent function

MA.912.AR.3.7–Given a table, equation or written description of a quadratic function, graph that function, and determine and interpret its key features. **Also AR.3.4, F.1.2, F.1.3, F.2.1**

MA.K12.MTR.1.1, MTR.2.1, MTR.5.1

EXPLORE & REASON





A. **Use Patterns and Structure** How is the graph of $f(x) = |x|$ similar to the graph of $f(x) = x^2$? How is it different?

B. What do you notice about the axis of symmetry in each graph?

ESSENTIAL QUESTION What is the quadratic parent function and how can you recognize the key features of its graph?

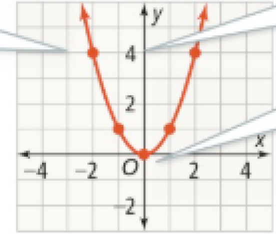
EXAMPLE 1 Identify a Quadratic Parent Function

What is a quadratic parent function and what are its characteristics?

The **quadratic parent function** is $f(x) = x^2$. It is the simplest function in the quadratic function family. The graph of the function is a curve called a **parabola**.

x	$f(x) = x^2$	(x, y)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)

The vertex is the lowest (or highest) point on the graph of a quadratic function.



The parabola opens up.

The axis of symmetry is $x = 0$.

The vertex is (0, 0). It is the turning point of the graph.

The axis of symmetry intersects the vertex, and divides the parabola in half.

GENERALIZE
Think about how the features of a quadratic function compare and contrast to those of a linear function.

Try It! **1.** When are the values of $f(x)$ from Example 1 positive and when are they negative?

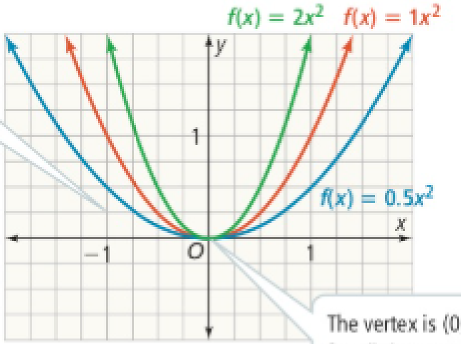
CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Understand the Graph of $f(x) = ax^2$

A. How does the value of the leading coefficient, *a*, affect the graph of $f(x) = ax^2$?

Graph some functions of the form $f(x) = ax^2$ with different positive *a*-values on the same coordinate grid and compare them.

COMMON ERROR
You may think that an *a* value with an absolute value less than 1 would decrease the width of the parabola. However, it increases the width of the parabola.



The graph of $f(x) = 0.5x^2$ is wider than the graph of $f(x) = 2x^2$.

The vertex is (0, 0) for all three graphs.

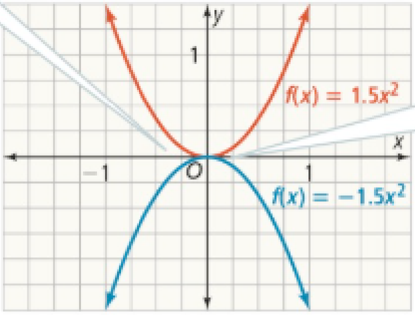
For $0 < |a| < 1$, the shape of the parabola is wider than the parent function. For $|a| > 1$, the shape of the parabola is narrower than the parent function.

B. How does the sign of *a* affect the graph of $f(x) = ax^2$?

Graph two functions of the form $f(x) = ax^2$ with opposite *a*-values on the same coordinate grid, and compare them.

REPRESENT AND CONNECT
Consider whether the value of *a* has an effect on the location of the vertex of the graph of $f(x) = ax^2$.

When $a > 0$, the *y*-coordinate of the vertex is the minimum value of the function. When $a < 0$, it is the maximum.



$f(x) = -ax^2$ is a reflection of $f(x) = ax^2$ over the *x*-axis.

When $a > 0$, the parabola opens upward.
When $a < 0$, the parabola opens downward.

Try It! 2. How does the sign of *a* affect the domain and range of $f(x) = ax^2$?

EXAMPLE 3 Interpret Quadratic Functions from Tables

Over what interval is $f(x) = 4x^2$ increasing? Over what interval is it decreasing?

Use the function to make a table of values.

x	$f(x) = 4x^2$	(x, y)
−2	16	(−2, 16)
−1	4	(−1, 4)
0	0	(0, 0)
1	4	(1, 4)
2	16	(2, 16)

The function values are **decreasing**.

The vertex (0, 0) is the turning point of the function, where it changes from decreasing to increasing.

The function values are **increasing**.

STUDY TIP
Remember that since the function has a minimum value, the parabola opens upward. If the function has a maximum value, the parabola opens downward.

The function is decreasing over the interval $x < 0$ and increasing over the interval $x > 0$.

Try It! **3.** A function of the form $g(x) = ax^2$ increases over the interval $x < 0$ and decreases over the interval $x > 0$. What is a possible value for a? Explain.

APPLICATION

EXAMPLE 4 Apply Quadratic Functions

The owner of a new dance studio is installing wooden floors in all of the dance rooms. How much should the owner expect to spend on flooring for a square room with 15-ft side lengths?

Write a function that can be used to determine the cost of the flooring.

$$c(x) = \text{price per ft}^2 \text{ of flooring} \cdot \text{area of dance floor in ft}^2$$
$$c(x) = 8.75 \cdot x^2$$

HAVE A GROWTH MINDSET
How can you use mistakes as opportunities to learn and grow?

Find the value of the function when $x = 15$.

$c(x) = 8.75x^2$

$c(15) = 8.75(15)^2$ .... Substitute 15 for x.

$c(15) = 1{,}968.75$ .... Simplify.

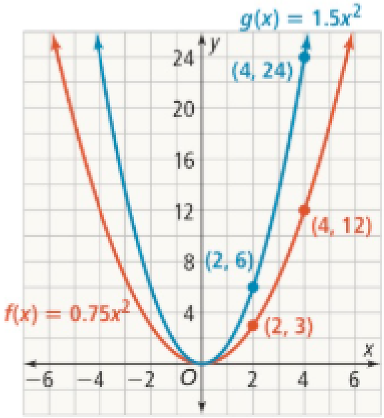
The cost for a new floor for a square dance floor with sides of 15 ft is $1,968.75.

Try It! **4.** By how much will the cost increase if the side length of the dance floor is increased by 2 ft?

EXAMPLE 5 Compare the Rate of Change

A. How do the average rates of change for $f(x) = 0.75x^2$ and $g(x) = 1.5x^2$ over the interval $2 \le x \le 4$ compare?

Step 1 Graph the two functions.



Step 2 Find the value of each function at the endpoints of the interval.

$f(2) = 0.75(2)^2 = 3$

$f(4) = 0.75(4)^2 = 12$

$g(2) = 1.5(2)^2 = 6$

$g(4) = 1.5(4)^2 = 24$

STUDY TIP
Use what you know about finding rates of change for linear functions. Think about the differences for quadratic functions.

Step 3 Find the slope of the line that passes through each pair of points.

$f(x)$: $\frac{12-3}{4-2} = \frac{9}{2} = 4.5$

$g(x)$: $\frac{24-6}{4-2} = \frac{18}{2} = 9$

The rate of change for function *g* is twice the rate of change for function *f*.

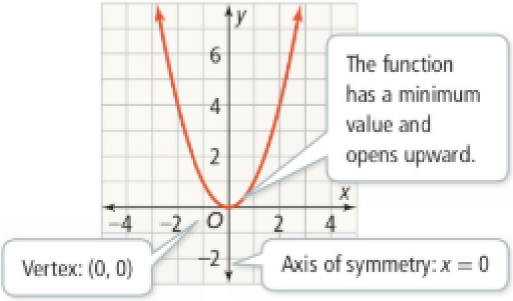
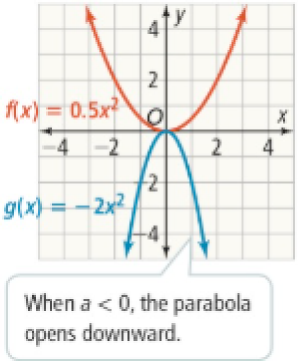
On average, the values of function *f* increase by 4.5 units and the values of function *g* increase by 9 units for each unit increase in *x* over the interval $2 \le x \le 4$.

B. How do the rates of change relate to the values of *a* in the functions?

For positive intervals, the greater the value of *a*, the greater the average rate of change. In this case the ratio of the *a*-values in the two functions is the same as the ratio of the average rates of change.

Try It! **5.** How do the average rates of change for $f(x) = -0.5x^2$ and $g(x) = -1.5x^2$ over the interval $-5 \le x \le -2$ compare?

CONCEPT SUMMARY Features of the Quadratic Function $f(x) = ax^2$

	$f(x) = x^2$	$f(x) = ax^2$
GRAPHS	 The function has a minimum value and opens upward. Vertex: (0, 0) Axis of symmetry: $x = 0$	 $f(x) = 0.5x^2$ $g(x) = -2x^2$ When $a < 0$, the parabola opens downward.
WORDS	The function $f(x) = x^2$ is the same as $f(x) = 1x^2$. It is the quadratic parent function. The function decreases over the interval $x < 0$ and increases over the interval $x > 0$.	When $0 < \lvert a \rvert < 1$, the graph of $f(x) = ax^2$ is wider than the graph of $f(x) = x^2$. When $\lvert a \rvert > 1$, graph of $f(x) = ax^2$ is narrower than the graph of $f(x) = x^2$.

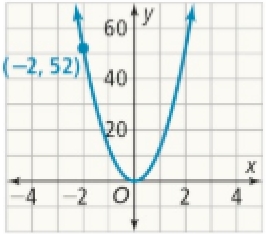
Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What is the quadratic parent function and how can you recognize the key features of its graph?

2. **Represent and Connect** How is the graph of $f(x) = ax^2$ similar to the graph of $f(x) = x^2$? How is it different?

3. **Vocabulary** Make a conjecture about why the term *quadratic parent function* includes the word "parent."

4. **Error Analysis** Abby graphed the function $f(x) = -13x^2$ by plotting the point (–2, 52). Explain the error Abby made in her graph.



Do You KNOW HOW?

How does the value of *a* in each function affect its graph when compared to the graph of the quadratic parent function?

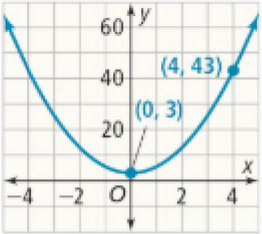
5. $g(x) = 4x^2$

6. $h(x) = 0.8x^2$

7. $j(x) = -5x^2$

8. $k(x) = -0.4x^2$

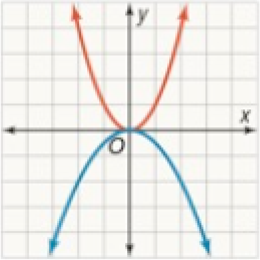
9. Given the function $f(x) = 2.5x^2 + 3$, find the average rate of change over the interval $0 \leq x \leq 4$. What does the average rate of change tell you about the function?



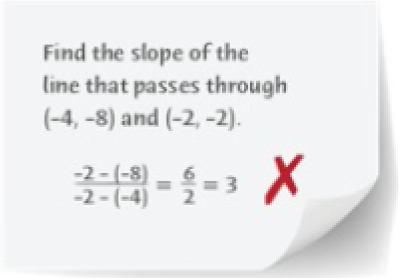
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Generalize** The graph of the parent quadratic function $f(x) = x^2$ and that of a second function of the form $g(x) = ax^2$ are shown. What conclusion can you make about the value of a in the equation of the second function?



11. **Error Analysis** Describe and correct the error a student made in finding the average rate of change for $f(x) = 0.5x^2$ over the interval $-4 \leq x \leq -2$.



Find the slope of the line that passes through (-4, -8) and (-2, -2).

$\frac{-2-(-8)}{-2-(-4)} = \frac{6}{2} = 3$ ✗

12. **Use Patterns and Structure** Use the table shown below to describe the intervals over which $f(x) = 15x^2$ is increasing and decreasing.

x	$f(x) = 15x^2$	(x, y)
−2	60	(−2, 60)
−1	15	(−1, 15)
0	0	(0, 0)
1	15	(1, 15)
2	60	(2, 60)

13. **Higher Order Thinking** Tell whether each statement about a function of the form $f(x) = ax^2$ is *always true, sometimes true,* or *never true.*

a. The graph is a parabola that opens upward.

b. The vertex of the graph is (0, 0).

c. The axis of symmetry of the graph is $x = 0$.

PRACTICE

How does the value of a in each function affect its graph when compared to the graph of the quadratic parent function? SEE EXAMPLES 1 AND 2

14. $g(x) = 6x^2$

15. $f(x) = 0.6x^2$

16. $f(x) = -7x^2$

17. $h(x) = -0.15x^2$

18. $C(x) = 0.04x^2$

19. $g(x) = 4.5x^2$

Over what interval is each function increasing and over what interval is each function decreasing? SEE EXAMPLE 3

20.

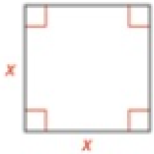
x	$f(x) = -0.3x^2$	(x, y)
−2	−1.2	(−2, −1.2)
−1	−0.3	(−1, −0.3)
0	0	(0, 0)
1	−0.3	(1, −0.3)
2	−1.2	(2, −1.2)

21.

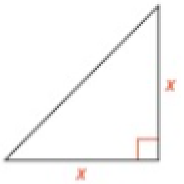
x	$f(x) = 13x^2$	(x, y)
−2	52	(−2, 52)
−1	13	(−1, 13)
0	0	(0, 0)
1	13	(1, 13)
2	52	(2, 52)

Write a quadratic function for the area of each figure. Then find the area for the given value of x. SEE EXAMPLE 4

22. $x = 13$



23. $x = 2.5$



How do the average rates of change for each pair of functions compare over the given interval? SEE EXAMPLE 5

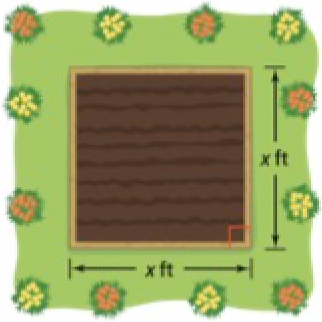
24. $f(x) = 0.1x^2$
$g(x) = 0.3x^2$
$1 \leq x \leq 4$

25. $f(x) = -2x^2$
$g(x) = -4x^2$
$-4 \leq x \leq -2$

PRACTICE & PROBLEM SOLVING

APPLY

26. **Analyze and Persevere** Some students can plant 9 carrots per square foot in the community garden shown. Write a function f that can be used to determine the number of carrots the students can plant. Give a reasonable domain for the function. How many carrots can the students plant in a garden that is square with 4-ft side lengths?



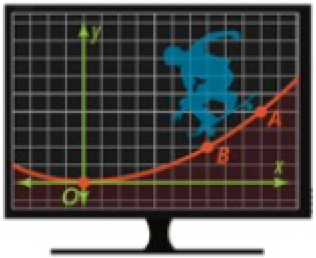
27. **Analyze and Persevere** A burrito company uses the function $C(x) = 1.74x^2$ to calculate the number of calories in a tortilla with a diameter of x inches.

    a. Find the average rates of change for the function over the intervals $6 < x < 8$ and $9 < x < 11$.

    b. Interpret the average rates of change.

    c. What does the difference in the average rates of change mean in terms of the situation?

28. **Represent and Connect** An architect uses a computer program to design a skateboard ramp. The function $f(x) = ax^2$ represents the shape of the ramp's cross section. A portion of the design is shown. The scale of each axis is 1 unit per grid line. On the ramp, a person can skateboard from point A through point B and over to a point C. If point C is the same distance above the x-axis as point B, what are its coordinates? Explain.



ASSESSMENT PRACTICE

29. The total cost, in dollars, of a square carpet can be determined by using $f(x) = 15x^2$, where x is the side length in yards. Select all the true statements. AR.3.7

    ☐ **A.** The cost of a carpet increases and then decreases as the side length increases.

    ☐ **B.** The cost of the carpet is \$15 per square yard.

    ☐ **C.** The cost of a carpet with a side length of 3 yd is \$135.

    ☐ **D.** The cost of a carpet with 6-yd sides is twice the cost of a carpet with 3-yd sides.

    ☐ **E.** The cost of a carpet increases at a constant rate as the side length increases.

30. **SAT/ACT** The graph of $f(x) = ax^2$ opens downward and is narrower than the graph of the quadratic parent function. Which of the following could be the value of a?

    Ⓐ −2 Ⓑ −0.5 Ⓒ 0.5 Ⓓ 1 Ⓔ 2

31. **Performance Task** A manufacturer has two options for making cube-shaped boxes. The cost is calculated by multiplying the surface area of the box by the cost per square inch of the cardboard.

Part A Write a quadratic function of the form $f(x) = ax^2$ for each design that can be used to determine the total cardboard cost for cubes with any side length. Interpret the value of a in each function.

Part B How do the average rates of change for the designs compare for cubes with side lengths greater than 6 in., but less than 8 in.?

Part C Make a conjecture about the packaging costs for each design when the side length of the cube is greater than 36 in. Explain your conjecture.

7-2

Quadratic Functions in Vertex Form

I CAN… graph quadratic functions using the vertex form.

VOCABULARY

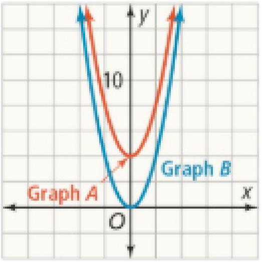
- vertex form of a quadratic function

MA.912.F.2.1–Identify the effect on the graph or table of a given function after replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$ and $f(x + k)$ for specific values of k. **Also AR.3.4, AR.3.7, AR.3.8**

MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

CRITIQUE & EXPLAIN

Allie states that the two parabolas shown may look different, but they are actually the same figure. Esteban disagrees, stating that they are different figures because they look different.



A. Give one mathematical argument to support Esteban's thinking.

B. Give one mathematical argument to support Allie's thinking.

C. **Communicate and Justify** Who do you agree with? What argument can you give to justify your reasoning?

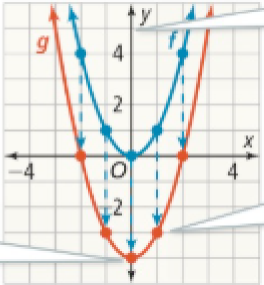
ESSENTIAL QUESTION

How can the vertex form of a quadratic function help you sketch the graph of the function?

EXAMPLE 1 Understand the Graph of $g(x) = x^2 + k$

How does the graph of $g(x) = x^2 - 4$ compare to that of $f(x) = x^2$?

Graph the function g and the parent function f.



The two graphs have the same axis of symmetry: $x = 0$.

Each point $(x, f(x))$ is translated down 4 units to the corresponding point $(x, g(x))$.

The vertex of the graph of g is $(0, -4)$.

The value of k in $g(x) = x^2 + k$ translates the graph of the parent function f, vertically k units. The vertex of the graph of g is at $(0, k)$, in this case $(0, -4)$. The value of k does not affect the axis of symmetry.

GENERALIZE
Which key features are unaffected by a vertical translation of the parent function?

Try It! 1. How does the graph of each function compare to the graph of $f(x) = x^2$?

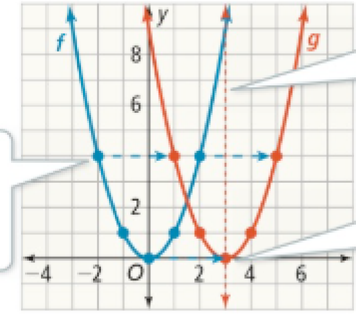
a. $h(x) = x^2 + 3$

b. $j(x) = x^2 - 2$

EXAMPLE 2 Understand the Graph of $g(x) = (x - h)^2$

How does the graph of $g(x) = (x - 3)^2$ compare to that of $f(x) = x^2$?

Graph the function g and the parent function f.



Each point $(x, f(x))$ is translated right 3 units to the corresponding point $(x, g(x))$.

The axis of symmetry of the graph of g is $x = 3$.

The vertex of the graph of g is (3, 0).

COMMON ERROR
You may think that the graph of $g(x) = (x - 3)^2$ would be a horizontal translation of the graph of $f(x) = x^2$ to the left in the negative direction along the x-axis. However, the translation is to the right in the positive direction.

The value of h in $g(x) = (x - h)^2$ translates the graph of the parent function horizontally h units. The vertex of the graph of g is at $(h, 0)$, in this case (3, 0). The value of h also translated the axis of symmetry horizontally.

Try It! 2. How does the graph of each function compare to the graph of $f(x) = x^2$?

a. $h(x) = (x + 1)^2$

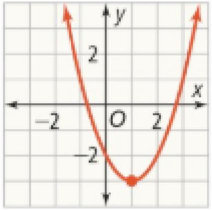
b. $j(x) = (x - 5)^2$

CONCEPTUAL UNDERSTANDING

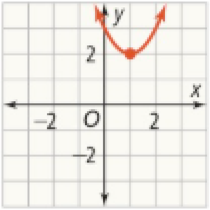
EXAMPLE 3 Understand the Graph of $f(x) = a(x - h)^2 + k$

A. What information do the values of h and k provide about the graph of $f(x) = (x - h)^2 + k$?

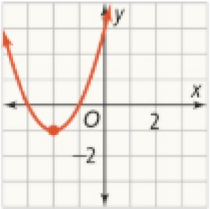
Graph several functions of the form $f(x) = (x - h)^2 + k$. Look at the location of the vertex of each graph.



$f(x) = (x - 1)^2 - 3$
vertex: (1, −3)



$f(x) = (x - 1)^2 + 2$
vertex: (1, 2)



$f(x) = (x + 2)^2 - 1$
vertex: (−2, −1)

USE PATTERNS AND STRUCTURE
Consider $f(x) = (x - 1)^2 - 3$ to be in the form $f(x) = a(x - h)^2 + k$. What is the a-value of this function?

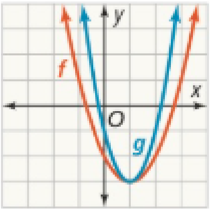
The values of h and k determine the location of the vertex and the axis of symmetry of the parabola. The vertex of the graph of $f(x) = (x - h)^2 + k$ is at (h, k). The axis of symmetry is $x = h$.

CONTINUED ON THE NEXT PAGE

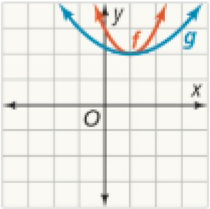
EXAMPLE 3 CONTINUED

B. How does the value of a affect the graph of $f(x) = a(x - h)^2 + k$?

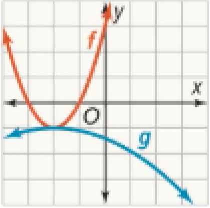
Graph each of the functions shown in part A. Then graph a new function with a different value of a to see how it affects the graph.



$f(x) = (x - 1)^2 - 3$
$g(x) = 2(x - 1)^2 - 3$



$f(x) = (x - 1)^2 + 2$
$g(x) = 0.25(x - 1)^2 + 2$



$f(x) = (x + 2)^2 - 1$
$g(x) = -0.1(x + 2)^2 - 1$

STUDY TIP
Notice that when $0 < |a| < 1$, the shape of the parabola is wider than the parent function. When $|a| > 1$, the shape of the parabola is narrower than the parent function.

The value of a does not affect the location of the vertex. The sign of a affects the direction of the parabola. The absolute value of a affects the width of the parabola.

The function $f(x) = a(x - h)^2 + k$, where $a \neq 0$ is called the **vertex form of a quadratic function**. The vertex of the graph is (h, k). The graph of $f(x) = a(x - h)^2 + k$ is a translation of the function $f(x) = ax^2$ that is translated h units horizontally and k units vertically.

Try It! 3. How does the graph of $f(x) = -3(x - 5)^2 + 7$ compare to the graph of the parent function?

EXAMPLE 4 Graph Using Vertex Form

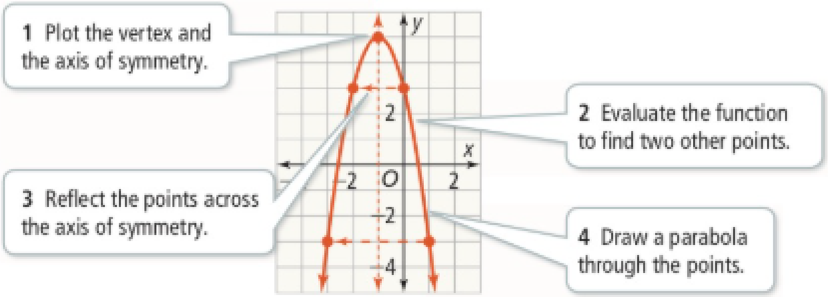
How can you use the vertex form of a quadratic function to sketch the graph of the function?

Graph $f(x) = -2(x + 1)^2 + 5$.

This is the same as $f(x) = -2(x - (-1))^2 + 5$

$h = -1$ and $k = 5$, so the vertex is $(-1, 5)$, and the axis of symmetry is $x = -1$.

COMMON ERROR
Recall that vertex form $f(x) = a(x - h)^2 + k$ includes a subtraction sign in the expression "$(x - h)$". If a quadratic function such as $f(x) = 3(x + 7)^2 - 6$ has an addition sign within that expression, then the value of h is negative.



Try It! 4. Find the vertex and axis of symmetry, and sketch the graph of the function.

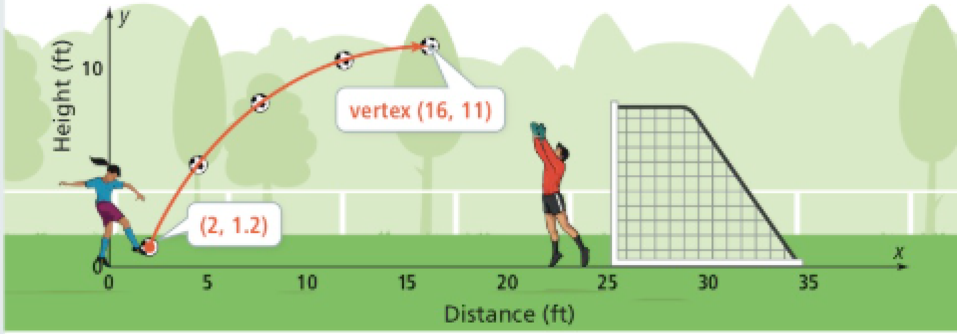
a. $g(x) = -3(x - 2)^2 + 1$

b. $h(x) = (x + 1)^2 - 4$

APPLICATION

EXAMPLE 5 Use Vertex Form to Solve Problems

Renata and Deshawn are playing soccer. Renata takes a shot on goal. Deshawn is 3 ft in front of the goal and can reach the top of the 8-ft goal when standing directly beneath it. Can he block the shot from his current position without moving or jumping?



Formulate

You can describe the parabolic path of the soccer ball using a quadratic function. The vertex of parabola is given, so write the function in vertex form.

$f(x) = a(x - h)^2 + k$

$f(x) = a(x - 16)^2 + 11$ ······ Substitute $h = 16$ and $k = 11$.

Compute

Use another point on the path of the ball to find the value of a.

The point (2, 1.2) represents the point where Renata's foot makes contact with the ball.

$1.2 = a(2 - 16)^2 + 11$ ······ Substitute $x = 2$ and $f(x) = 1.2$.

$1.2 = 196a + 11$ ······ Simplify.

$-9.8 = 196a$ ······ Simplify.

$\frac{-9.8}{196} = \frac{196a}{196}$ ······ Divide each side by 196.

$a = -0.05$

$f(x) = -0.05(x - 16)^2 + 11$ ······ Substitute $a = -0.05$ into the function.

Use the function to find the altitude of the ball at Deshawn's position.

Deshawn is 3 ft in front of the goal, so his position is 25 ft – 3 ft = 22 ft.

$f(22) = -0.05(22 - 16)^2 + 11$ ······ Substitute $x = 22$ into the function.

≈ 9.2

Interpret

When the ball reaches Deshawn it will be about 9.2 ft above the ground, which is above his 8-ft reach.

Deshawn cannot block Renata's shot from his current position without jumping or moving.

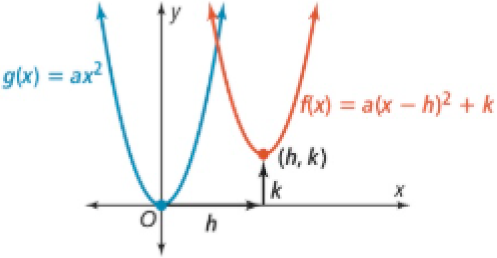
Try It! **5.** If Deshawn does not block Renata's shot, will it be a goal? Explain.

CONCEPT SUMMARY Vertex Form of a Quadratic Function

ALGEBRA

$f(x) = a(x - h)^2 + k$

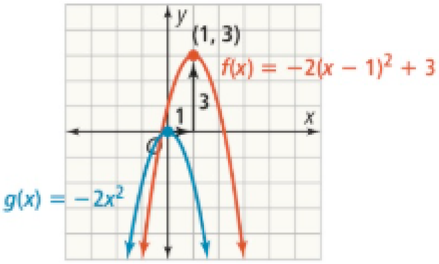
- The graph of f is the graph of $g(x) = ax^2$ translated horizontally h units and vertically k units.
- The vertex is located at (h, k).
- The axis of symmetry is $x = h$.



NUMBERS

$f(x) = -2(x - 1)^2 + 3$

- The graph of f is the graph of $g(x) = -2x^2$ translated right 1 unit and up 3 units.
- The vertex is located at (1, 3).
- The axis of symmetry is $x = 1$.



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can the vertex form of a quadratic function help you sketch the graph of the function?

2. **Communicate and Justify** A table of values for the quadratic function g is shown. Do the graphs of the functions g and $f(x) = 3(x - 1)^2 + 2$ have the same axis of symmetry? Explain.

x	$g(x)$
−4	8
−2	3
0	0
6	3

3. **Use Patterns and Structure** How are the form and the graph of $f(x) = (x - h)^2 + k$ similar to the form and graph of $f(x) = |x - h| + k$? How are they different?

4. **Error Analysis** Sarah said the vertex of the function $f(x) = (x + 2)^2 + 6$ is (2, 6). Is she correct? Explain your answer.

Do You KNOW HOW?

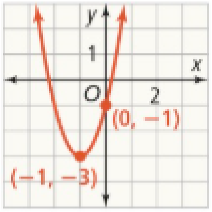
Graph each function.

5. $g(x) = x^2 + 5$

6. $f(x) = (x - 2)^2$

7. $h(x) = -2(x + 4)^2 + 1$

8. Write a function in vertex form for the parabola shown below.

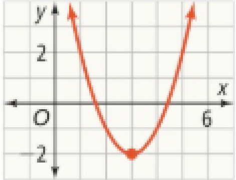


9. The height of a ball thrown into the air is a quadratic function of time. The ball is thrown from a height of 6 ft above the ground. After 1 second, the ball reaches its maximum height of 22 ft above the ground. Write the equation of the function in vertex form.

PRACTICE & PROBLEM SOLVING

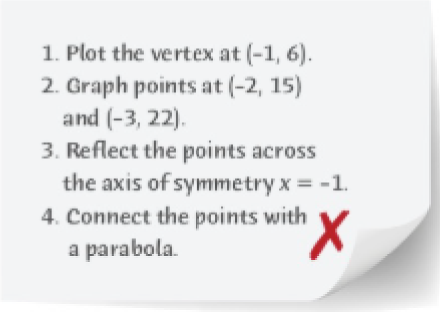
UNDERSTAND

10. **Analyze and Persevere** How can you determine the values of *h* and *k* from the graph shown? Write the function for the parabola.



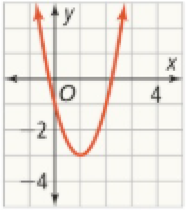
11. **Communicate and Justify** To graph the function $f(x) = (x - 5)^2 - 8$, a student translates the graph of the quadratic parent function 5 units right and 8 units down. Can a student produce the graph of $f(x) = 2(x + 3)^2 - 5$ by simply translating the quadratic parent function? Explain.

12. **Error Analysis** A student used the steps shown to graph $f(x) = (x - 1)^2 + 6$. Describe and correct the student's error.



1. Plot the vertex at (−1, 6).
2. Graph points at (−2, 15) and (−3, 22).
3. Reflect the points across the axis of symmetry x = −1.
4. Connect the points with a parabola.

13. **Mathematical Connections** The graph shown is a translation of the graph of $f(x) = 2x^2$. Write the function for the graph in vertex form.



14. **Higher Order Thinking** The graph of *h* is the graph of $g(x) = (x - 2)^2 + 6$ translated 5 units left and 3 units down.

a. Describe the graph of *h* as a translation of the graph of $f(x) = x^2$.

b. Write the function *h* in vertex form.

PRACTICE

Identify the vertex and the axis of symmetry for each function. SEE EXAMPLES 1 AND 2

15. $f(x) = x^2 + 2$

16. $f(x) = x^2 - 5$

17. $g(x) = x^2 - 1$

18. $h(x) = x^2 + 0.5$

19. $f(x) = x^2 - 2.25$

20. $f(x) = x^2 + 50$

21. $h(x) = x^2 + 7$

22. $g(x) = (x - 1)^2$

23. $g(x) = (x + 2)^2$

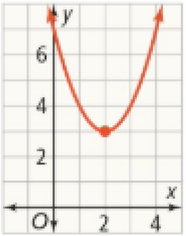
24. $f(x) = (x - 6)^2$

25. $f(x) = (x - 0.5)^2$

26. $g(x) = (x - 4)^2$

Each graph shown is a translation of the graph of $f(x) = x^2$. Write each function in vertex form. SEE EXAMPLE 3

27.



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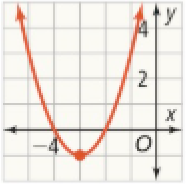


Identify the vertex, axis of symmetry, and direction of the graph of each function. Compare the width of the graph to the width of the graph of $f(x) = x^2$. SEE EXAMPLE 3

29. $f(x) = 2(x + 1)^2 + 4$

30. $g(x) = (x - 3)^2 - 3$

31. $g(x) = -0.75(x - 5)^2 + 6$

32. $h(x) = -3(x + 2)^2 - 5$

Sketch the graph of each function. SEE EXAMPLE 4

33. $f(x) = 2(x - 1)^2 + 4$

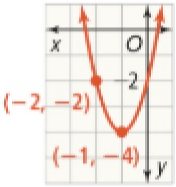
34. $g(x) = -2(x - 0.5)^2 + 1$

35. $f(x) = 0.5(x + 2)^2 + 2$

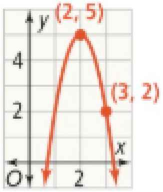
36. $h(x) = -2(x - 2)^2 - 2$

Each graph represents a quadratic function. Write each function in vertex form. SEE EXAMPLE 5

37.



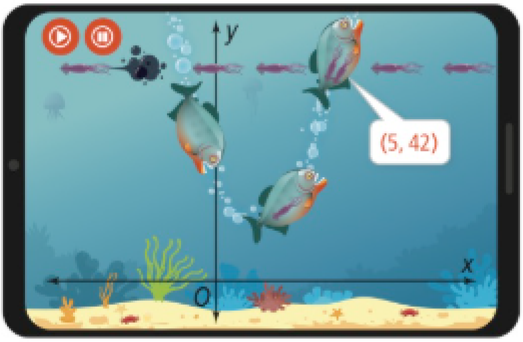
38.



PRACTICE & PROBLEM SOLVING

APPLY

39. **Analyze and Persevere** A computer game designer uses the function $f(x) = 4(x - 2)^2 + 6$ to model the path of the fish. The horizontal path of the squid intersects the path of the fish. At what other point does the squid's path intersect the path of the fish?

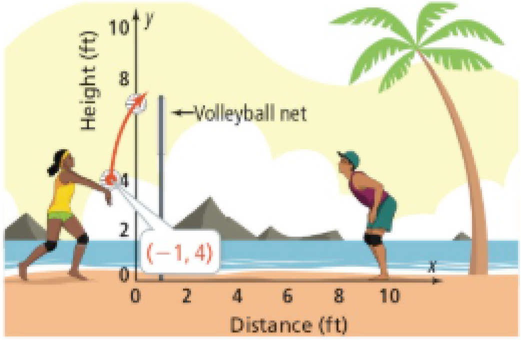


40. **Apply Math Models** Suppose a goalie kicks a soccer ball. The ball travels in a parabolic path from point (0, 0) to (57, 0).

   a. Consider a quadratic function in vertex form for the path of the ball. Which values can you determine? What values are you unable to determine? Explain.

   b. **Technology** Use a graphing calculator to explore the undetermined values. Find a set of values that generates a realistic graph. Explain how the key features of the graph correspond to the situation.

41. **Communicate and Justify** The function $f(x) = -(x - 1)^2 + 8$ models the path of a volleyball. The height of the net is 7 ft 4 in.



Will the ball go over if the player is 2 ft from the net? 4 ft, from the net? Explain.

ASSESSMENT PRACTICE

42. The function $f(x) = 2(x - 3)^2 + 9$ is graphed in the coordinate plane. Select all the true statements. AR.3.7

☐ A. The graph is a parabola that opens downward.

☐ B. The vertex of the graph is (−3, 9).

☐ C. The axis of symmetry of the graph is $x = 3$.

☐ D. The y-intercept of the graph is 9.

☐ E. The minimum of the function is 9.

43. **SAT/ACT** The graph of $g(x) = x^2$ is translated right 2 units and down 10 units. Which of the following is the function of the new graph?

Ⓐ $f(x) = (x + 2)^2 - 10$

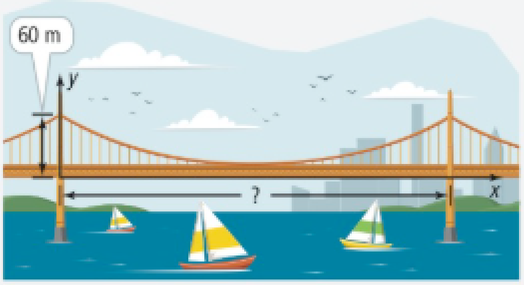
Ⓑ $f(x) = (x - 2)^2 - 10$

Ⓒ $f(x) = 2x^2 - 10$

Ⓓ $f(x) = -2x^2 - 10$

Ⓔ $f(x) = -2(x - 10)^2$

44. **Performance Task** An engineer is designing a suspension bridge with a center cable. The cable is shaped like a parabola and is attached to stability towers on both ends at the same height. For simplicity she assumes a quadratic function, and uses $f(x) = 0.0006(x - 300)^2 + 6$ to model the cable between the towers.



Part A How high above the road surface is the lowest point of the cable?

Part B How far apart are the two towers? Explain.

7-3

Quadratic Functions in Standard Form

I CAN… graph quadratic functions using standard form.

VOCABULARY

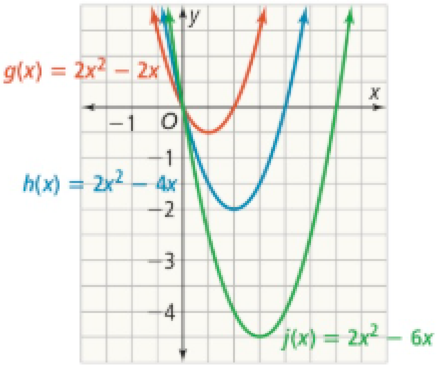
- standard form of a quadratic function

MA.912.AR.3.7–Given a table, equation or written description of a quadratic function, graph that function, and determine and interpret its key features. **Also AR.1.2, AR.3.4**

MA.K12.MTR.3.1, MTR.4.1, MTR.6.1

EXPLORE & REASON

Three functions of the form $f(x) = ax^2 + bx$ are graphed for $a = 2$ and different values of b.



A. What do the graphs have in common? In what ways do they differ?

B. What do you notice about the x-intercepts of each graph? What do you notice about the y-intercepts of each graph?

C. **Use Patterns and Structure** Look at the ratio $\frac{b}{a}$ for each function and compare it to its graph. What do you notice?

ESSENTIAL QUESTION

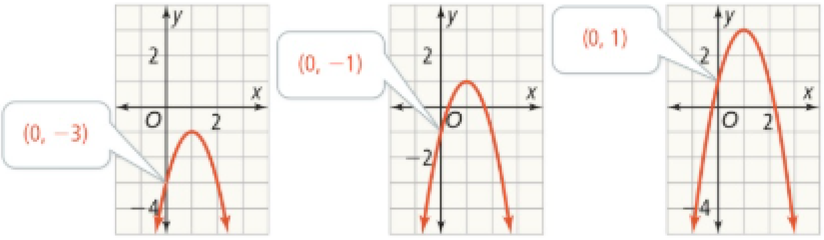
How is the standard form of a quadratic function different from the vertex form?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Relate c to the Graph of $f(x) = ax^2 + bx + c$

What information does c provide about the graph of $f(x) = ax^2 + bx + c$?

Graph several functions of the form $f(x) = ax^2 + bx + c$. Look for a connection between the graphs and the value of c for each function.



$f(x) = -2x^2 + 4x - 3$ $\quad c = -3$

$f(x) = -2x^2 + 4x - 1$ $\quad c = -1$

$f(x) = -2x^2 + 4x + 1$ $\quad c = 1$

The value of c corresponds to the y-intercept of the graph of $f(x) = ax^2 + bx + c$.

GENERALIZE
Consider the graphs of quadratic functions with the same c-values but different a- and b-values from those shown in the example.

Try It! 1. Evaluate $f(x) = ax^2 + bx + c$ for $x = 0$. How does $f(0)$ relate to the result in Example 1?

CONCEPT Standard Form of a Quadratic Equation

The **standard form of a quadratic function** is $f(x) = ax^2 + bx + c$, where $a \neq 0$. The value c is the y-intercept of the graph. The axis of symmetry of the graph is the line $x = -\frac{b}{2a}$ and the x-coordinate of the vertex is $-\frac{b}{2a}$.

EXAMPLE 2 Graph a Quadratic Function in Standard Form

Graph $f(x) = 2x^2 + 4x + 3$. What are the axis of symmetry, vertex, and y-intercept of the function?

Step 1 Find the axis of symmetry.

$$x = -\frac{b}{2a} = -\frac{4}{2(2)} = -1$$

The x-coordinate of the vertex is also −1.

The axis of symmetry is $x = -1$.

Step 2 Plot the vertex.

Use the x-coordinate of the vertex to find the y-coordinate.

$$f(x) = 2x^2 + 4x + 3$$

$$f(-1) = 2(-1)^2 + 4(-1) + 3 \quad \text{Substitute } -1 \text{ for } x.$$

$$= 1 \quad \text{Simplify.}$$

The y-coordinate of the vertex is 1. So, plot the vertex $(-1, 1)$.

CHOOSE EFFICIENT METHODS
Consider Step 3. Is there a situation where following this procedure would not yield two points on the parabola?

Step 3 Plot the y-intercept and its reflection.

$$f(x) = 2x^2 + 4x + 3$$

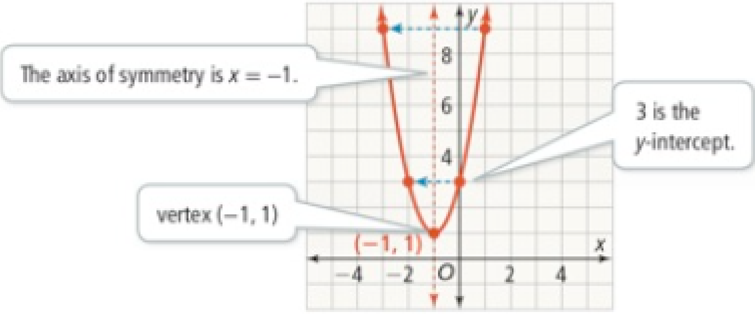
The value of c is the y-intercept.

Plot $(0, 3)$ and its reflection across the axis of symmetry $(-2, 3)$.

Step 4 Plot another point and its reflection.

Evaluate the function for another x-value. For $x = 1$, $f(1) = 9$. The reflection of $(1, 9)$ across the axis of symmetry is $(-3, 9)$. Plot $(1, 9)$ and $(-3, 9)$.

Step 5 Graph the parabola.



The axis of symmetry is $x = -1$.

3 is the y-intercept.

vertex $(-1, 1)$

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

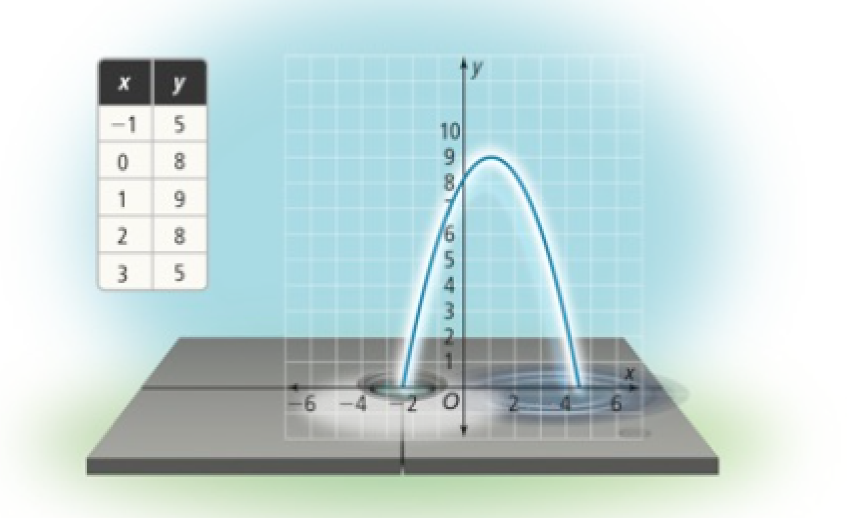
Try It! **2.** Graph each function. What are the y-intercept, the axis of symmetry, and the vertex of each function?

a. $f(x) = x^2 + 2x + 4$ **b.** $g(x) = -0.75x^2 + 3x - 4$

APPLICATION

EXAMPLE 3 Write the Quadratic Function in Standard Form

The trajectory of the water from a fountain is represented by the graph and the table of values. Find the quadratic function that represents trajectory of the water.



x	y
−1	5
0	8
1	9
2	8
3	5

Step 1 Use the vertex formula to relate a and b.

Using symmetry, you can see that the x-value of the vertex is 1.

$$-\frac{b}{2a} = 1$$

$$(-2a)\left(-\frac{b}{2a}\right) = (-2a)1$$

$$b = -2a$$

Step 2 Use the y-intercept, $b = -2a$, and the point (3, 5) to solve for a.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= ax^2 + bx + 8 \\ &= ax^2 - 2ax + 8 \\ 5 &= a(3)^2 - 2a(3) + 8 \\ 5 &= 9a - 6a + 8 \\ a &= -1 \end{aligned}$$

CHECK FOR REASONABLENESS
Does it make sense that $a < 0$ when modeling the trajectory of a water fountain with a quadratic function?

Step 3 Use a to solve for b.

$$\begin{aligned} b &= -2a \\ &= -2(-1) \\ &= 2 \end{aligned}$$

Step 4 Substitute the values for a, b, and c into the standard form.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= -1x^2 + 2x + 8 \\ &= -x^2 + 2x + 8 \end{aligned}$$

The function $f(x) = -x^2 + 2x + 8$ models the trajectory of the water from the fountain.

CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

Try It! **3.** Write a quadratic function to represent the values in each table.

a.

x	–1	0	1	2	3
y	19	9	3	1	3

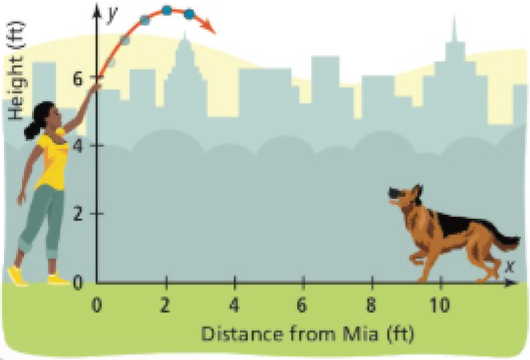
b.

x	1	2	3	4	5
y	2	1	2	5	10

APPLICATION

EXAMPLE 4 Analyze the Structure of Different Forms

Mia tosses a ball to her dog. The function $f(x) = -0.5(x - 2)^2 + 8$ represents the ball's path.



A. What does the vertex form of the function tell you about the situation?

$f(x) = a(x - h)^2 + k$

$f(x) = -0.5(x - 2)^2 + 8$

$a = -0.5$. Since $a < 0$, the parabola opens downward.

$h = 2$ and $k = 8$, so the vertex is (2, 8).

The vertex form tells you the vertex of the graph of the function, which is (2, 8). The ball reaches a maximum height of 8 ft above the ground, 2 ft away from where Mia releases it.

B. What does the standard form of a function tell you about the situation?

Rewrite the function in standard form.

$$\begin{aligned} f(x) &= -0.5(x - 2)^2 + 8 \\ &= -0.5(x^2 - 4x + 4) + 8 && \text{Expand } (x - 2)^2. \\ &= -0.5x^2 + 2x + 6 && \text{Use the Distributive Property and simplify.} \end{aligned}$$

$f(x) = ax^2 + bx + c$

$f(x) = -0.5x^2 + 2x + 6$

The y-intercept is 6.

The standard form tells you the y-intercept of the graph of the function, which is (0, 6). The ball was 6 ft above the ground when Mia threw it.

CHECK FOR REASONABLENESS
Think about the reasonableness of the domain and range when you graph the function. Do both positive and negative values make sense?

Try It! **4.** Suppose the path of the ball in Example 4 is $f(x) = -0.25(x - 1)^2 + 6.25$. Find the ball's initial and maximum heights.

CONCEPT SUMMARY Standard Form of a Quadratic Function

ALGEBRA Standard form: $f(x) = ax^2 + bx + c$, where $a \neq 0$.

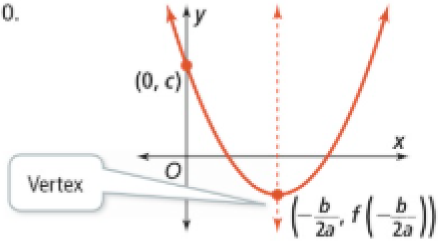
y-intercept: c

Axis of symmetry: $x = -\frac{b}{2a}$

x-coordinate of the vertex: $-\frac{b}{2a}$

y-coordinate of the vertex: $f\left(-\frac{b}{2a}\right)$

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$



NUMBERS Standard form: $f(x) = 2x^2 + 8x + 5$.

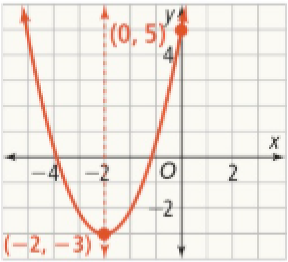
y-intercept: 5

Axis of symmetry: $x = -\frac{8}{2(2)} = -2$

x-coordinate of the vertex: $-\frac{8}{2(2)} = -2$

y-coordinate of the vertex: $f(-2) = -3$

Vertex: $(-2, -3)$



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is the standard form of a quadratic function different from the vertex form?

2. **Communicate and Justify** How are the form and graph of $f(x) = ax^2 + bx + c$ similar to the form and graph of $g(x) = ax^2 + bx$? How are they different?

3. **Vocabulary** How can you write a quadratic function in *standard form,* given its vertex form?

4. **Error Analysis** Sage began graphing $f(x) = -2x^2 + 4x + 9$ by finding the axis of symmetry $x = -1$. Explain the error Sage made.

Do You KNOW HOW?

Graph each function. For each, identify the axis of symmetry, the y-intercept, and the coordinates of the vertex.

5. $f(x) = 2x^2 + 8x - 1$

6. $f(x) = -0.5x^2 + 2x + 3$

7. $f(x) = -3x^2 - 6x - 5$

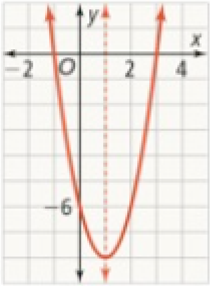
8. $f(x) = 0.25x^2 - 0.5x - 6$

9. A water balloon is tossed into the air. The function $h(x) = -0.5(x - 4)^2 + 9$ gives the height, in feet, of the balloon from the surface of a pool as a function of the balloon's horizontal distance from where it was first tossed. Will the balloon hit the ceiling 12 ft above the pool? Explain.

PRACTICE & PROBLEM SOLVING

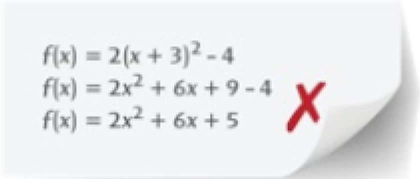
UNDERSTAND

10. **Analyze and Persevere** The graph of the function $f(x) = 2x^2 - bx - 6$ is shown. What is the value of b? Explain.



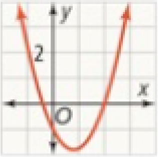
11. **Communicate and Justify** To identify the y-intercept of a quadratic function, would you choose to use vertex form or standard form? Explain.

12. **Error Analysis** Describe and correct the error a student made when writing the quadratic function $f(x) = 2(x + 3)^2 - 4$ in standard form.



$f(x) = 2(x + 3)^2 - 4$
$f(x) = 2x^2 + 6x + 9 - 4$
$f(x) = 2x^2 + 6x + 5$

13. **Choose Efficient Methods** Estimate the coordinates of the vertex of the graph of $f(x) = 1.25x^2 - 2x - 1$ below. Then explain how to find the exact coordinates.



14. **Higher Order Thinking** Points (2, −1), (−2, 7), (1, −2), (0, −1), and (4, 7) lie on the graph of a quadratic function.

a. What is the axis of symmetry of the graph?

b. What is the vertex?

c. What is the y-intercept?

d. Over what interval does the function increase?

PRACTICE

What is the y-intercept of each function? SEE EXAMPLE 1

15. $f(x) = 2x^2 - 4x - 6$

16. $f(x) = 0.3x^2 + 0.6x - 0.7$

17. $f(x) = -2x^2 - 8x - 7$

18. $f(x) = 3x^2 + 6x + 5$

19. $f(x) = -x^2 - 2x + 3$

20. $f(x) = -0.5x^2 + x + 2$

Find the y-intercept, the axis of symmetry, and the vertex of the graph of each function. SEE EXAMPLE 2

21. $f(x) = 2x^2 + 8x + 2$

22. $f(x) = -2x^2 + 4x - 3$

23. $f(x) = 0.4x^2 + 1.6x$

24. $f(x) = -x^2 - 2x - 5$

25. $f(x) = 5x^2 + 5x + 12$

26. $f(x) = 4x^2 + 12x + 5$

27. $f(x) = x^2 - 6x + 12$

28. $f(x) = -2x^2 + 16x + 40$

Write a quadratic function to represent the values in each table. SEE EXAMPLE 3

29.

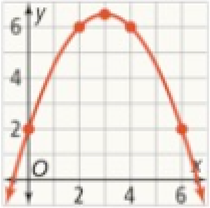
x	y
−1	13
0	3
1	−3
2	−5
3	−3

30.

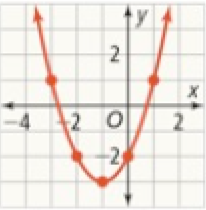
x	y
−4	10
−3	11
−2	10
−1	7
0	2

Write a quadratic function to represent each parabola. SEE EXAMPLE 3

31.



32.



Write each function in standard form. SEE EXAMPLE 4

33. $f(x) = 4(x + 1)^2 - 3$

34. $f(x) = 0.1(x - 2)^2 - 0.1$

35. $f(x) = -2(x - 9)^2 + 15$

36. $f(x) = -(x + 3)^2 + 8$

PRACTICE & PROBLEM SOLVING

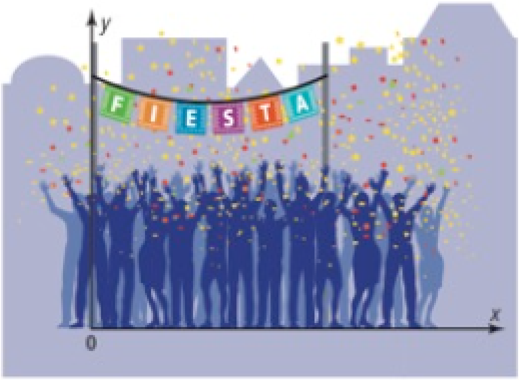
APPLY

37. Analyze and Persevere Two balls are tossed up into the air, and caught at the same heights they were tossed from. The function $f(x) = -4.9x^2 + 14.7x + 0.975$ models the path of Ball A. The path of Ball B over time is shown in the table. Compare the ranges of the two functions. Which ball reaches a greater height? Explain how you can answer without graphing either function.

Time (*s*)	Height (*m*)
x	*g*(*x*)
0	1.975
1	11.775
1.5	13
2	11.775
2.5	1.975

38. Represent and Connect The position of a ball after it is kicked can be determined by using the function $f(x) = -0.11x^2 + 2.2x + 1$, where y is the height, in feet, above the ground and x is the horizontal distance, in feet, of the ball from the point at which it was kicked. What is the height of the ball when it is kicked? What is the highest point of the ball in the air?

39. Apply Math Models A banner is hung for a party. The distance from a point on the bottom edge of the banner to the floor can be determined by using the function $f(x) = 0.25x^2 - x + 9.5$, where x is the distance, in feet, of the point from the left end of the banner. How high above the floor is the lowest point on the bottom edge of the banner? Explain.



ASSESSMENT PRACTICE

40. An object is launched at 64 ft per second from an elevated platform. The quadratic function shown in the table models its trajectory $f(x)$ over time, x. Select all the true statements. AR.3.7

x	0	1	2	4
f(*x*)	6	54	70	6

- ☐ **A.** The height of the platform is 6 ft.
- ☐ **B.** The object reaches its maximum height after 2 seconds.
- ☐ **C.** The maximum height of the object is 70 ft.
- ☐ **D.** The object will be lower than 40 feet at 3 seconds.
- ☐ **E.** The height of the object increases and then decreases.

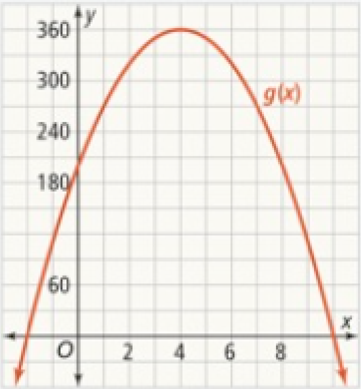
41. SAT/ACT What is the maximum value of $f(x) = -4x^2 + 16x + 12$?

Ⓐ 12 Ⓑ 16 Ⓒ 24 Ⓓ 28 Ⓔ 64

42. Performance Task Two models are used to predict monthly revenue for a new sports drink. In each model, x is the number of \$1-price increases from the original \$2 per bottle price.

Model A $f(x) = -12.5x^2 + 75x + 200$

Model B



Part A Identify the price you would set for each model to maximize monthly revenue. Explain.

Part B A third model includes the points (9, 605), (8, 600), (10, 600), (7, 585), and (11, 585). What price maximizes revenue according to this model? Explain.

7-4

Modeling With Quadratic Functions

I CAN… use quadratic functions to model real-world situations.

VOCABULARY

- vertical motion model

MA.912.AR.3.4–Write a quadratic function to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. **Also AR.1.1, AR.3.8, F.1.2**

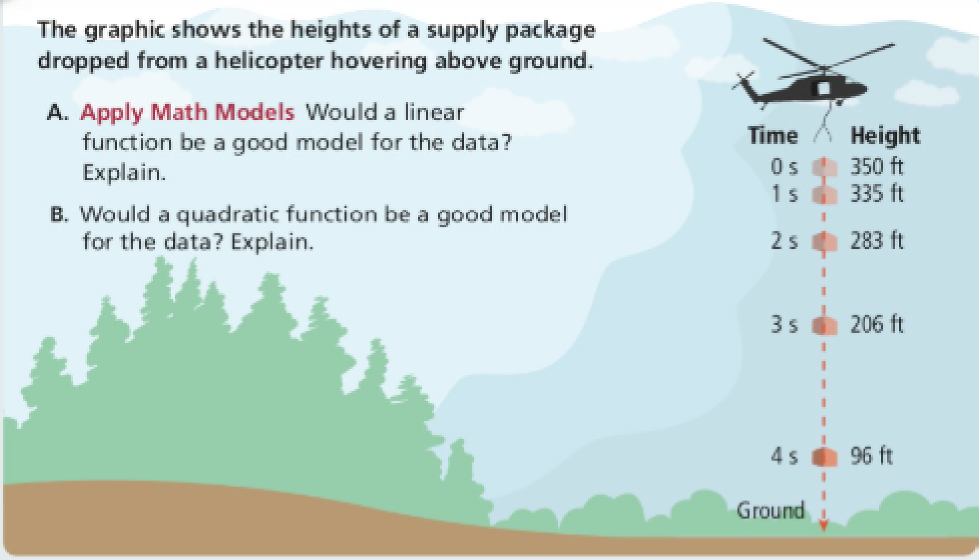
MA.K12.MTR.2.1, MTR.3.1, MTR.7.1

MODEL & DISCUSS

The graphic shows the heights of a supply package dropped from a helicopter hovering above ground.

A. **Apply Math Models** Would a linear function be a good model for the data? Explain.

B. Would a quadratic function be a good model for the data? Explain.



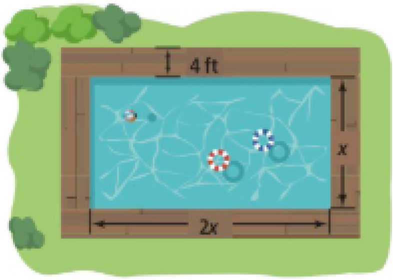
ESSENTIAL QUESTION

What kinds of real-world situations can be modeled by quadratic functions?

APPLICATION

EXAMPLE 1 Use Quadratic Functions to Model Area

A company offers rectangular pool sizes with dimensions as shown. Each pool includes a deck around it. If Carolina wants a 15-ft wide pool with a deck, how many square feet will she need to have available in her yard?



A. **Write a quadratic function to represent the area of the pool and deck.**

REPRESENT AND CONNECT
To write the function, think about how the length of the pool is related to its width. Then write expressions for the length and width of the rectangular area that contains both the pool and the deck.

Let x be the width of the pool.

$$f(x) = (2x + 8)(x + 8)$$

Area = Length × Width

$$= 2x^2 + 24x + 64$$

The quadratic function $f(x) = 2x^2 + 24x + 64$ can be used to find the area of the rectangular pool and the deck.

B. **Find the area of the pool and the deck.**

$$f(15) = 2(15)^2 + 24(15) + 64$$

Substitute 15 for x.

$$= 874$$

Carolina needs 874 ft^2 to build a 15-ft wide pool with deck.

Try It! 1. Suppose the length of the pool in Example 1 is 3 times the width. How does the function that represents the combined area of the pool and the deck change? Explain.

CONCEPT Vertical Motion Model

The equation $h(t) = -16t^2 + v_0t + h_0$ is the **vertical motion model**. The variable h represents the height of an object, in feet, t seconds after it is launched into the air. The term v_0 is the object's initial vertical velocity and h_0 is its initial height.

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Model Vertical Motion

A diver jumps off a high platform at an initial vertical velocity of 16 ft/s.

A. What quadratic function represents the height *h* of the diver after *t* seconds of the dive?

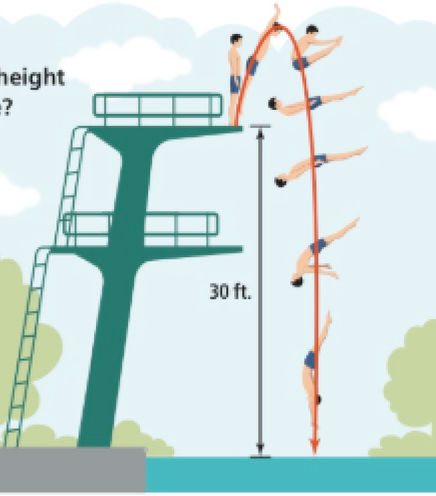
Use the vertical motion model to write the quadratic function.

$$h(t) = -16t^2 + v_0t + h_0$$

$$h(t) = -16t^2 + 16t + 30$$

Initial vertical velocity is 16 ft/s.

height of the platform is 30 ft.



B. How many feet above the platform will the diver be at the highest point of his dive?

Find the maximum value of the graph described by $h(t) = -16t^2 + 16t + 30$.

$$t = -\frac{b}{2a}$$

The maximum value is located at t-value of the axis of symmetry.

$$t = -\frac{16}{2(-16)}$$

$$= \frac{1}{2}$$

$$h\left(\frac{1}{2}\right) = -16\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right) + 30$$

Substitute the t-value into the function to find $h(t)$, the y-value of the vertex.

$$= 34$$

The vertex is $\left(\frac{1}{2}, 34\right)$.

The platform is at a height of 30 ft and the vertex is at 34 ft. So the diver will be about 34 − 30, or 4 feet above the platform.

USE PATTERNS AND STRUCTURE
What do you notice about the structure of the vertical motion model and the standard form of a quadratic function? How are they similar?

Try It! 2. Find the diver's maximum height above the water if he dives from a 20-ft platform with an initial velocity of 8 ft/s.

APPLICATION

EXAMPLE 3 Use Quadratic Functions to Model Revenue

For the past six years, ticket prices for a school play have increased by $1. The table shows the average revenue per show for the last four years based on the increases in the ticket prices. The director wants to determine how much she can raise the ticket prices and still have the school earn at least $800 in revenue.

Price Increase x	Revenue $f(x)$
3	964
4	991
5	1000
6	991

A. Model the data using a quadratic function.

The vertex is at (5, 1000). Use vertex form.

$$f(x) = a(x - h)^2 + k$$

$$= a(x - 5)^2 + 1000$$ Substitute (5, 1000) for (h, k).

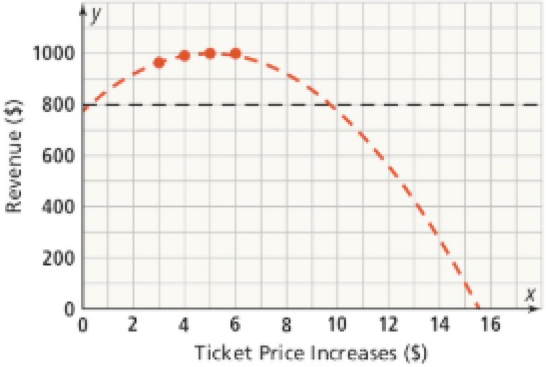
$$991 = a(6 - 5)^2 + 1000$$ Use (6, 991) to solve for a.

$$991 = a + 1000$$

$$a = -9$$

Substitute the values for a, h, and k into vertex form. The function $f(x) = -9(x - 5)^2 + 1000$ models the data.

B. Graph the function and determine the greatest integer x-value with a y-value greater than $800.



An increase of $9 is the greatest increase that would still yield a revenue above $800 per show.

APPLY MATH MODELS
There is another value that would yield the same revenue as the $9 increase. What possible advantage might there be reducing the ticket price to that level?

C. Describe any constraints on the domain.

Because of the director's requirements on revenue a reasonable domain would be $\{x \mid x \text{ is an integer and } 1 \leq x \leq 9\}$.

Try It! **3.** Jordan sells jewelry online. Monthly revenue varies as a function of the single price that Jordan sets for all pieces. Use vertex form to create a quadratic model for Jordon's monthly revenue.

Price ($)	12	13	14	15	16
Revenue ($)	864	884	896	900	896

CONCEPT SUMMARY Modeling With Quadratic Functions

AREA

When the length and width of rectangle are each variable expressions, a quadratic function can be used to model the rectangle's area.

$$A = (3x + 4)(x - 2)$$
$$= 3x^2 - 2x - 8$$

VERTICAL MOTION

The vertical motion model gives the height h, in feet, of an object t seconds after launch.

$$h(t) = -16t^2 + v_0t + h_0$$

v_0 is the initial velocity

h_0 is the initial height

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What kinds of real-world situations can be modeled by quadratic functions?

2. **Represent and Connect** How is the function $h(t) = -16t^2 + bt + c$ related to vertical motion?

3. **Vocabulary** What does it mean in a real-world situation when the *initial velocity* is 0?

4. **Error Analysis** Chen uses $h(t) = -16t^2 + 6t + 16$ to determine the height of a ball t seconds after it is thrown at an initial velocity of 16 ft/s from an initial height of 6 ft. Describe the error Chen made.

Do You KNOW HOW?

Write a vertical motion model in the form $h(t) = -16t^2 + v_0t + h_0$ for each situation presented. For each situation, determine how long, in seconds, it takes the thrown object to reach maximum height.

5. Initial velocity: 32 ft/s; initial height: 20 ft

6. Initial velocity: 120 ft/s; initial height: 50 ft

7. A rectangular patio has a length four times its width. It also has a 3-ft wide brick border around it. Write a quadratic function to determine the area of the patio and border.

8. Write a quadratic function to represent the data in the table.

x	y
1	55
2	65
3	71
4	73
5	71

PRACTICE & PROBLEM SOLVING

UNDERSTAND

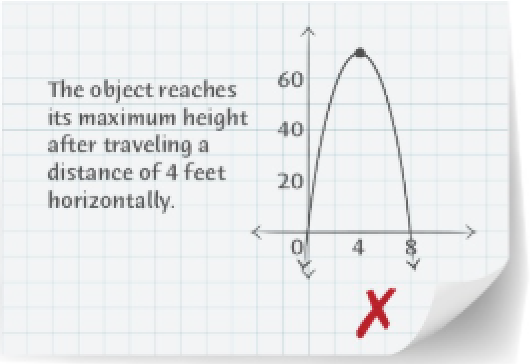
9. **Analyze and Persevere** For each vertical motion model, identify the maximum height reached by the object and the amount of time for the object to reach the maximum height.

   a. $h(t) = -16t^2 + 200t + 25$

   b. $h(t) = -16t^2 + 36t + 4$

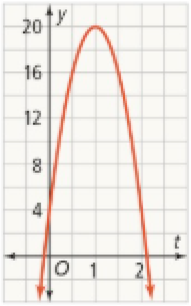
10. **Choose Efficient Methods** If a data set includes the maximum or minimum value is it more efficient to use vertex form or standard form to find a quadratic model for the data? Explain.

11. **Error Analysis** Describe and correct the error a student made when interpreting the graph of the vertical motion model $h(t) = -at^2 + bt + c$.



The object reaches its maximum height after traveling a distance of 4 feet horizontally.

12. **Represent and Connect** In the graph of a vertical motion model shown, how is the initial velocity related to the vertex of the parabola?



13. **Higher Order Thinking** The function $f(x) = x^2 + 3x - 10$ models the area of a rectangle.

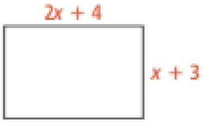
    a. Describe the length and width of the rectangle in terms of x.

    b. What is a reasonable domain and range for the situation? Explain.

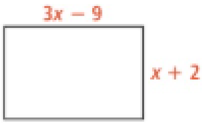
PRACTICE

Use a quadratic function to model the area of each rectangle. Graph the function. Evaluate each function for $x = 8$. SEE EXAMPLE 1

14.



15.



Write a function h to model the vertical motion for each situation, given $h(t) = -16t^2 + v_0t + h_0$. Find the maximum height. SEE EXAMPLE 2

16. initial vertical velocity: 32 ft/s; initial height: 75 ft

17. initial vertical velocity: 200 ft/s; initial height: 0 ft

18. initial vertical velocity: 50 ft/s; initial height: 5 ft

19. initial vertical velocity: 48 ft/s; initial height: 6 ft

Write a quadratic function to represent the data in each table. SEE EXAMPLE 3

20.

x	y
0	2
1	−3
2	−6
3	−7
4	−6

21.

x	y
−3	4
−2	5
−1	4
0	1
1	−4

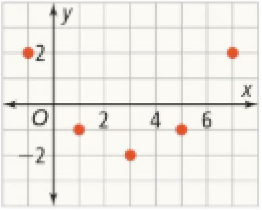
22.

x	y
3	49
4	55
5	57
6	55
7	49

23.

x	y
−3	27
−2	15
−1	11
0	15
1	27

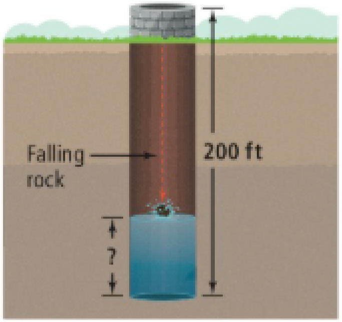
24. Write a quadratic function to represent the data in the graph. SEE EXAMPLE 3



PRACTICE & PROBLEM SOLVING

APPLY

25. Apply Math Models A student drops a rock over the edge of the well and hears it splash into water after 3 seconds. Write a function in the form $h(t) = -16t^2 + v_0t + h_0$ to determine the height of the rock above the bottom of the well t seconds after the student drops the rock. What is the distance from the surface of the water to the bottom of the well?



26. Apply Math Models Becky tosses a ball up in the air. She releases it at a height of 6 ft above the ground. Two seconds later, Becky catches it at the same height, 6 ft.

a. Write a quadratic function to model the height of the ball as a function of time.

b. What was the maximum height of the ball?

c. What was the average rate of change in the height of the ball for the first half of the toss?

d. What does average rate of change mean in terms of the situation?

27. Mathematical Connections Dakota bought 120 ft of wire fencing at \$0.50/ft to enclose a rectangular playground. The playground surface will be covered with mulch at a cost of \$1.25/ft^2. Write a quadratic function that can be used to determine the total cost of fencing and mulch for a playground with side length x. What is the cost if one side is 20 ft?

ASSESSMENT PRACTICE

28. The function $h(t) = -16t^2 + 96t + 10$ models the path of a projectile. Select all the true statements. AR.3.8

☐ **A.** The initial height of the projectile is 10 ft because $h(0) = 10$.

☐ **B.** The initial height of the projectile is 90 ft because $h(0) = 90$.

☐ **C.** The projectile reaches a maximum height at time 3 s.

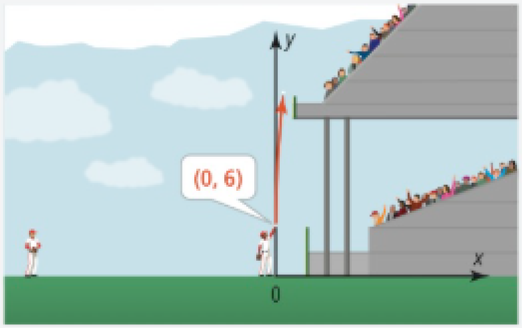
☐ **D.** The projectile reaches a maximum height at time 6 s.

☐ **E.** The maximum height of the projectile is $h(3)$.

29. SAT/ACT A basketball is thrown straight up into the air from a height of 2.1 ft with an initial velocity of 7 ft/s. Which function models the height of the ball after t seconds?

Ⓐ $h(t) = -16t^2 + 2.1t + 7$
Ⓑ $h(t) = -16t^2 - 2.1t + 7$
Ⓒ $h(t) = -16t^2 + 2.1t - 7$
Ⓓ $h(t) = -16t^2 + 7t + 2.1$
Ⓔ $h(t) = -16t^2 - 7t + 2.1$

30. Performance Task A baseball player is standing 1.5 ft away from the edge of the upper deck that is 20 ft above the baseball field. He throws a ball into the air for the fans sitting in the upper deck.



Part A Write a quadratic function that can be used to determine the height of the ball if it is thrown at an initial velocity of 35 ft/s from a height of 6 ft. Graph the function.

Part B The seats for the upper deck start 2 ft from the edge. Will the ball travel high enough to land on the upper deck?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.3.8–Solve and graph mathematical and real-world problems that are modeled with quadratic functions. Interpret key features and determine constraints in terms of the context.
Also AR.3.4
MA.K12.MTR.7.1

The Long Shot

Have you ever been to a basketball game where they hold contests at halftime? A popular contest is one where the contestant needs to make a basket from half court to win a prize. Contestants often shoot the ball in different ways. They might take a regular basketball shot, a hook shot, or an underhand toss.

What is the best way to shoot the basketball to make a basket? Think about this during this Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the Main Question you will answer.
3. Make an initial conjecture that answers this Main Question.
4. Explain how you arrived at your conjecture.

ACT 2 Develop a Model

5. Use the math that you have learned in the topic to refine your conjecture.

ACT 3 Interpret the Results

6. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

7-5

Comparing Linear, Exponential, and Quadratic Functions

I CAN… determine whether a linear, exponential, or quadratic function best models a data set.

MA.912.F.1.8–Determine whether a linear, quadratic or exponential function best models a given real-world situation. **Also F.1.1, F.1.6**
MA.K12.MTR.4.1, MTR.5.1, MTR.7.1

MODEL & DISCUSS

Jacy and Emma use different functions to model the value of a bike x years after it is purchased. Each function models the data in the table.

Jacy's function: $f(x) = -14.20x + 500$

Emma's function: $f(x) = 500(0.85)^x$

Time (yr)	Value ($)
0	500.00
1	485.20
2	472.13
3	461.00
4	452.10

A. **Analyze and Persevere** Why did Jacy and Emma not choose a quadratic function to model the data?

B. Whose function do you think is a better model? Explain.

C. Do you agree with this statement? Explain why or why not.

To ensure that you are finding the best model for a table of data, you need to find the values of the functions for the same values of x.

ESSENTIAL QUESTION

How can you determine whether a linear, exponential, or quadratic function best models data?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Determine Which Function Type Represents Data

A. How can you determine whether the data in the table can be modeled by a linear function?

Recall that linear functions have a constant rate of change. Confirm that the data has a constant amount of change per unit interval.

	x	y	Change in y-values
	−3	−2	
+3	0	7	$7 - (-2) = 9$
+3	3	16	$16 - 7 = 9$
+3	6	25	$25 - 16 = 9$
+3	9	34	$34 - 25 = 9$

COMMUNICATE AND JUSTIFY
Is a constant change in the y-values sufficient to show that data are linear?

The y-values increase by 9 units for every 3 units of increase of the x-values. The data shows a constant amount of change per interval, so a linear function models this data.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

B. How can you determine whether the data in the table can be modeled by an exponential function?

Exponential data grows or decays by a constant ratio or a constant percent per unit interval.

COMMON ERROR
Both quadratic and exponential functions may model nonlinear data, however quadratic functions do not model data with a constant percent change per unit interval.

x	y	Ratios	Percent Change
1	400		
2	456	$\frac{456}{400} = 1.14$	$\frac{456-400}{400} = 0.14$; 14%
3	520	$\frac{520}{456} \approx 1.14$	$\frac{520-456}{456} \approx 0.14$; 14%
4	593	$\frac{593}{520} \approx 1.14$	$\frac{593-520}{520} \approx 0.14$; 14%
5	676	$\frac{676}{593} \approx 1.14$	$\frac{676-593}{593} \approx 0.14$; 14%

The data shows approximately equal ratios and percent changes between consecutive *y*-values. An exponential function best models the data.

C. How can you determine if data is more appropriately modeled by a quadratic function rather than a linear or exponential function?

Determine if the data changes from increasing to decreasing or decreasing to increasing.

x	y
1	100.2
2	83.9
3	101.2
4	152.1
5	236.6

The *y*-values are decreasing

The *y*-values are increasing.

The *y*-values appear to decrease to a minimum and then increase. A quadratic function is an appropriate model.

When data values increase to a maximum then decrease, or when they decrease to a minimum and then increase, a quadratic function is more appropriate than a linear or exponential function.

LEARN TOGETHER
How do you listen actively as others share ideas?

Quadratic functions may also model other strictly increasing or strictly decreasing situations where the data do not have a constant amount change per unit interval or a constant percent change per unit interval.

Try It! **1.** Does a linear, quadratic, or exponential function best model the data? Explain.

a.

x	1	2	3	4	5
y	16	8	6	10	20

b.

x	−2	−1	0	1	2
y	4	12	36	108	324

APPLICATION

EXAMPLE 2 Choose a Function Type for Real-World Data

Determine whether a linear, quadratic or exponential function best models each situation.

A. Emaan runs a fruit smoothie stand and tracks the revenue from January to December. Emaan finds that revenue increases from the beginning of the year into the summer months and then decreases as the weather gets colder towards the end of the year.

Revenue increases and then decreases.
A quadratic function best models the situation.

B. Luis is training to cycle at a consistent pace. He uses an app to track the distance he covers on his training ride. Luis checks his distance every 5 minutes during a 20 minute ride.

The unit interval is 5 minutes, and the total distance Luis rides will increase by a constant amount because of his steady pace over each interval.
A linear function best models the situation.

APPLY MATH MODELS
If Luis checked his total distance at 8, 12, 14, and 20 minutes would the data still appear linear?

C. Raul's social media account had 842 followers. Over the course of two months there was a 12% increase in the number of followers each week.

The unit interval is 1 week, there is a constant percent change in followers per unit interval so an exponential function best models the situation.

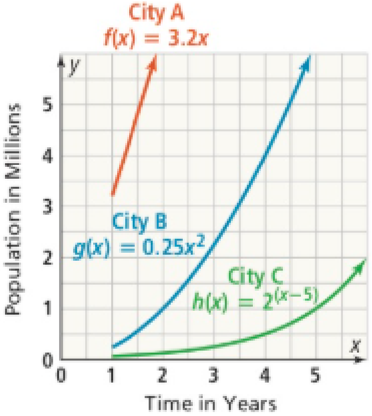
Try It! **2.** Determine whether a linear, quadratic or exponential function best models each situation.

a. Noemi earns \$16 per hour at her job. She works 40 hours per week. Noemi tracks her total income over the first 6 weeks of her employment.

b. Ines gives surfing lessons. Over the course of several years she tracks her revenue based on different rates for private lessons. Ines finds that she earns the most revenue if she charges \$65 per hour for a lesson. Ines earns less revenue if she charges either more or less than \$65 per hour.

EXAMPLE 3 Compare Linear, Exponential, and Quadratic Growth

The graph shows population models for three cities, based on data over a five-year period. If the populations continue to increase in the same ways, when will the population of City C exceed the populations of the other two cities?

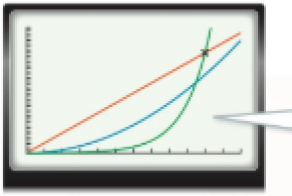


Method 1 Use the table of values.

x	$f(x)$	$g(x)$	$h(x)$
4	12.8	4.0	0.5
5	16.0	6.25	1
6	19.2	9.0	2
7	22.4	12.25	4
8	25.6	16.0	8
9	28.8	20.25	16
10	32.0	25.0	32
11	35.2	30.25	64

The population of City C is greater than those of City A and City B.

Method 2 Use a graphing calculator to determine the points of intersection.



x scale: 10 *y scale*: 32

Use your calculator to find the point where function h exceeds functions f and g.

USE PATTERNS AND STRUCTURE
Look at the structure of each of the graphs. Notice that a quantity that increases exponentially will eventually exceed a quantity that increases linearly or quadratically.

After 10 years, the population of City C will exceed the populations of City A and City B. It will continue to outgrow the other cities because it is growing exponentially.

Try It! 3. Compare the functions $f(x) = 3x + 2$, $g(x) = 2x^2 + 3$, and $h(x) = 2^x$. Show that as x increases, $h(x)$ will eventually exceed $f(x)$ and $g(x)$.

CONCEPT SUMMARY Linear, Quadratic, and Exponential Functions

	Linear	Quadratic	Exponential
WORDS	The rate of change is constant.	The *y*-values increase to a maximum then decrease, or they decrease to a minimum then increase.	The percent growth or decay is constant.

TABLES

$f(x) = 0.5x + 1$

x	y	Change Per Unit Interval
0	1	
2	2	$\frac{2-1}{2-0} = 0.5$
4	3	$\frac{3-2}{4-2} = 0.5$
6	4	$\frac{4-3}{6-4} = 0.5$

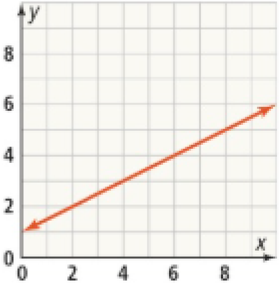
$f(x) = 0.5x^2 - 4x + 5$

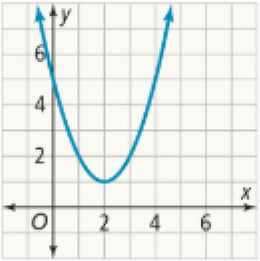
x	y	Behavior
0	5	
1	2	decreasing
2	1	minimum
3	2	increasing

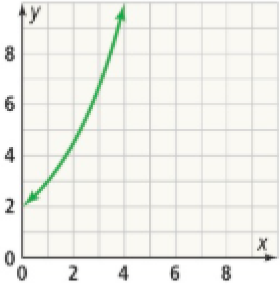
$f(x) = 2(1.5)^x$

x	y	Percent Change
0	2	
1	3	$\frac{3-2}{2} = 0.5$; 50%
2	4.5	$\frac{4.5-3}{3} = 0.5$; 50%
3	6.75	$\frac{6.75-4.5}{4.5} = 0.5$; 50%

GRAPHS







Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you determine whether a linear, exponential, or quadratic function best models data?

2. **Check for Reasonableness** The average rate of change of a function is less from $x = 1$ to $x = 4$ than from $x = 5$ to $x = 8$. What type of function could it be? Explain.

3. **Error Analysis** Kiyo states that data that decreases can only be modeled using an exponential decay function. Explain the error Kiyo made.

Do You KNOW HOW?

Determine whether the data are best modeled by a linear, quadratic, or exponential function.

4.

x	0	1	2	3	4
y	162	54	18	6	2

5.

x	−2	−1	0	1	2
y	2	7	12	17	22

6. A company's profit from a certain product is represented by $P(x) = -5x^2 + 1{,}125x - 5{,}000$, where x is the price of the product. Compare the growth in profits from $x = 120$ to $x = 140$ and from $x = 140$ to $x = 160$. What do you notice?

PRACTICE & PROBLEM SOLVING

UNDERSTAND

7. **Choose Efficient Methods** Create a flow chart to show the process to determine whether a given data set represents a function that is linear, quadratic, exponential, or none of these.

8. **Generalize** For linear functions the differences between consecutive y-values, or the *1st differences*, are constant.

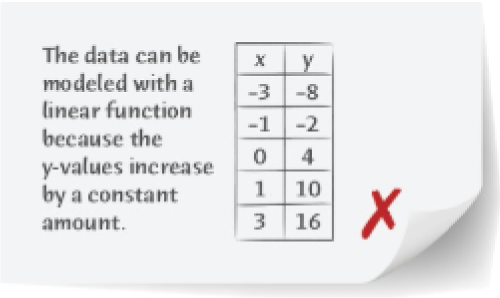
   a. Copy and complete the table for the *1st* and *2nd differences* of the quadratic function $f(x) = 3x^2 + x + 5$.

   b. Find the 1st and 2nd differences using the same x-values for $f(x) = 2x^2 + 3x + 7$.

   c. Make a conjecture about the 1st and 2nd differences of the y-values for consecutive values of quadratic functions.

$f(x) = 3x^2 + x + 5$			
x	y	1st Differences	2nd Differences
0	5		
1	9	$9 - 5 = 4$	
2	19	$19 - 9 = 10$	$10 - 4 = 6$
3	35	$35 - 19 = 16$	$16 - 10 = 6$
4	57	■	■
5	85	■	■

9. **Error Analysis** What is the error in the student's reasoning below? Describe how to correct the statement.



The data can be modeled with a linear function because the y-values increase by a constant amount.

x	y
-3	-8
-1	-2
0	4
1	10
3	16

10. **Higher Order Thinking** A savings account has a balance of \$1. Savings Plan A will add \$1,000 to an account each month, and Plan B will double the amount each month.

    a. Which plan is better in the short run? For how long? Explain.

    b. Which plan is better in the long run? Explain.

PRACTICE

Determine whether a linear, quadratic, or exponential function is the best model for the data in each table. SEE EXAMPLE 1

11.

x	y
0	1
1	3
2	9
3	27
4	81

12.

x	y
1	13
2	16
3	13
4	4
5	−11

13.

x	y
0	56
1	57
2	50
3	35
4	12

14.

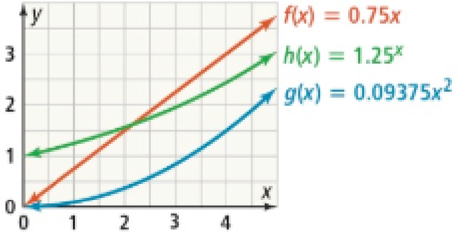
x	y
0	−6
1	−3
2	0
3	3
4	6

Determine whether a linear, quadratic or exponential function best models each situation. SEE EXAMPLE 2

15. The first year Denzel started his company he had sales of about \$26,000. Over the past 4 years sales have increased by 17% each year.

16. Murphy measures the air temperature over a 24-hour period. Starting at 4 A.M., he sees that the temperature rises to a high in the early afternoon, before it begins dropping over the rest of the day.

17. Use the functions shown. SEE EXAMPLE 3



   a. Evaluate each function for $x = 6$, $x = 8$ and $x = 12$.

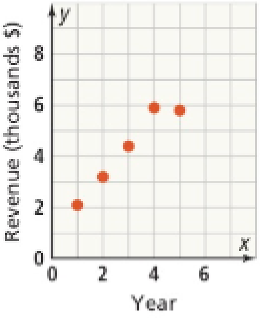
   b. When will function h exceed function f and function g?

PRACTICE & PROBLEM SOLVING

APPLY

18. **Apply Math Models** Emery, Deondra, and Kelley run an airboat tour company. They are studying their revenues from the past 5 years and want to predict their future revenues.

Year	Revenue (thousands $)
1	2.1
2	3.2
3	4.4
4	5.9
5	5.8



The team creates the following models from their data:

Emery: $f(x) = 0.9x + 1.3$

Deondra: $g(x) = 1.4(1.38)^x$

Kelley: $h(x) = -0.24x^2 + 2.3x - 0.10$

a. Whose model would you expect to be the most optimistic? Whose model would you expect to be the least optimistic? Explain.

b. Graph the revenue data and the three models on the same coordinate grid.

c. What are the predicted revenues for the upcoming year in each model?

d. Whose model do you think is the most realistic? Explain.

19. **Communicate and Justify** Carmen is considering two plans to pay off a $10,000 loan. The tables show the amount remaining on the loan after x years.

Plan A	
Year	Amount Remaining
0	10,000
1	9,000
2	8,100
3	7,290
4	6,561

Plan B	
Year	Amount Remaining
0	10,000
1	9,500
2	9,000
3	8,500
4	8,000

Which plan should Carmen use to pay off the loan as soon as possible? Justify your answer using a function model.

ASSESSMENT PRACTICE

20. Function f represents the population, in millions, of Franklin x years from now. Exponential function g represents the population, in millions, of Georgetown x years from now. If the patterns shown in the table continue, will Franklin always have a greater population than Georgetown? Explain. F.1.6

x	0	1	2	3
$f(x)$	3.4	5.6	7.8	10.0
$g(x)$	2.4	3.6	5.4	8.1

21. **SAT/ACT** At what point will $f(x) = 3^x$ exceed $g(x) = 2x + 5$ and $h(x) = x^2 + 4$?

Ⓐ (1, 7)

Ⓑ (1.8, 7.3)

Ⓒ (2, 9)

Ⓓ (2.4, 9.8)

22. **Performance Task** Ella is studying the population trends of three neighboring towns. Five years ago each town had a population of about 20,000 people. Ella recorded the percent changes in the population for each town.

Town A Population		Town B Population		Town C Population	
Year	Percent change	Year	Percent change	Year	Percent change
1	6.25	1	3.75	1	4.00
2	5.88	2	1.20	2	3.99
3	5.56	3	−1.19	3	4.02
4	5.26	4	−3.61	4	4.00
5	5.00	5	−6.25	5	3.97

Part A Based on the percent changes in population which towns have increasing populations? Which towns have decreasing populations?

Part B What type of function would best model the population data for each town?

Part C Use the percent changes to determine the approximate populations for the three towns over the 5-year period.

TOPIC
7

Topic Review

TOPIC ESSENTIAL QUESTION

1. How can you use sketches and equations of quadratic functions to model situations and make predictions?

Vocabulary Review

Choose the correct term to complete each sentence.

2. The graph of a quadratic function is a(n) ________.
3. The function $f(x) = x^2$ is called the ________.
4. To model the height of an object launched into the air t seconds after it is launched, you can use the ________.
5. The ________ is $f(x) = ax^2 + bx + c$.

- parabola
- quadratic parent function
- standard form of a quadratic function
- vertex form of a quadratic function
- vertical motion model

Concepts & Skills Review

LESSON 7-1 Key Features of Quadratic Functions

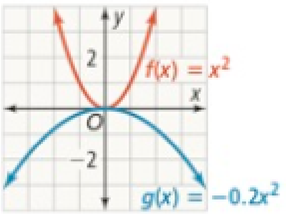
Quick Review

The graph of $f(x) = ax^2$ is a **parabola** with **vertex** (0, 0) and **axis of symmetry** $x = 0$. When $a > 0$, the parabola opens upward and the function has a minimum at the vertex. When $a < 0$, the parabola opens downward and the function has a maximum at the vertex.

Example

Compare the graph of $g(x) = -0.2x^2$ with the graph of $f(x) = x^2$.

The graph of g opens downward and is wider than the graph of f. For both graphs, the axis of symmetry is $x = 0$ and the vertex is (0, 0).



Practice and Problem Solving

Compare the graph of each function with the graph of $f(x) = x^2$.

6. $g(x) = 1.5x^2$
7. $h(x) = -9x^2$

Over what interval is each function increasing and over what interval is each function decreasing?

8. $f(x) = -2.5x^2$
9. $f(x) = 6x^2$

10. **Communicate and Justify** Explain how you can you tell whether a function of the form $f(x) = ax^2$ has a minimum or a maximum value and what that value is.

11. **Apply Math Models** Artificial turf costs \$15/sq ft to install, and sod costs \$0.15/sq ft to install. Write a quadratic function that represents the cost of installing artificial turf on a square plot with a side length of x feet, and a second quadratic function that represents the cost of installing sod on the same plot. How do the graphs of the two functions differ?

LESSON 7-2 Quadratic Functions in Vertex Form

Quick Review

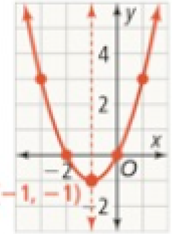
The **vertex form of a quadratic function** is $f(x) = a(x - h)^2 + k$. The vertex of the graph is at (h, k) and the axis of symmetry is $x = h$.

Example

Graph the function $f(x) = (x + 1)^2 - 1$.

The vertex is $(-1, -1)$ and the axis of symmetry is $x = -1$.

Use the points $(0, 0)$ and $(1, 3)$ to find two other points. Reflect each point across the axis of symmetry.



Practice and Problem Solving

12. **Use Patterns and Structure** Graph the functions below. How are the graphs alike? How are the graphs different from each other?

$$f(x) = -5(x - 3)^2 + 2$$
$$g(x) = -2(x - 3)^2 + 2$$

Identify the vertex and axis of symmetry of the graph of each function.

13. $g(x) = (x + 8)^2 + 1$
14. $h(x) = (x - 5)^2 - 2$

15. **Apply Math Models** An astronaut on the moon throws a moon rock into the air. The rock's height, in meters, above the moon's surface x seconds after it is thrown can be determined by the function $h(x) = -1.6(x - 2.5)^2 + 15$. What is the maximum height of the rock above the moon's surface? How many seconds after being thrown does the rock reach this height?

LESSON 7-3 Quadratic Functions in Standard Form

Quick Review

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$. The y-intercept is c and the axis of symmetry, which is also the x-coordinate of the vertex, is $x = -\frac{b}{2a}$.

Example

Graph the function $f(x) = 3x^2 - 6x + 2$.

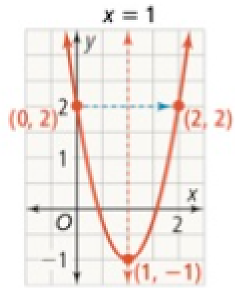
The y-intercept is 2.

Find the axis of symmetry.

$$x = -\frac{b}{2a} = -\frac{-6}{2(3)} = 1$$

Find the y-coordinate of the vertex.

$$f(1) = 3(1)^2 - 6(1) + 2 = -1$$



Plot the vertex $(1, -1)$ and identify the axis of symmetry.

Plot the y-intercept $(0, 2)$. Reflect that point across the axis of symmetry.

Practice and Problem Solving

Identify the y-intercept, axis of symmetry, and vertex of the graph of each function.

16. $g(x) = -x^2 + 4x + 5$
17. $h(x) = -3x^2 + 7x + 1$

18. Graph the function $f(x) = -3x^2 + 12x + 5$.

19. Write the quadratic function in standard form that represents the table of values.

x	−2	−1	0	1	2
y	3	1	3	9	19

20. **Use Patterns and Structure** When given a function in standard form, how can you determine if the parabola has a minimum or maximum value?

21. **Represent and Connect** A ball is tossed into the air. The function $f(x) = -16x^2 + 4x + 5$ represents the height in feet of the ball x seconds after it is thrown. At what height was the ball tossed into the air?

LESSON 7-4 Modeling With Quadratic Functions

Quick Review

Quadratic functions can model situations. For example, the vertical motion model $h(t) = -16t^2 + v_0t + h_0$ is a quadratic function.

Example

Alberto launches an emergency flare at an initial velocity of 64 ft/s from an initial height of 6 ft. The flare must reach a height of 100 ft to be seen by a rescue team. Is Alberto's launch successful?

Substitute 64 for v_0 and 6 for h_0 in the vertical motion model.

$$\begin{aligned} h(t) &= -16t^2 + v_0t + h_0 \\ &= -16t^2 + 64t + 6 \end{aligned}$$

Find the vertex $(t, h(t))$.

$$\begin{aligned} t &= -\frac{b}{2a} \\ &= -\frac{64}{2(-16)} \\ &= 2 \end{aligned}$$

$$\begin{aligned} h(2) &= -16(2)^2 + 64(2) + 6 \\ &= 70 \end{aligned}$$

The vertex is (2, 70).

The flare will reach a maximum height of 70 ft, so Alberto's launch is not successful.

Practice and Problem Solving

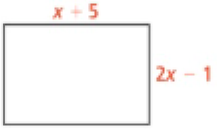
Write a function h to model the vertical motion for each situation, given $h(t) = -16t^2 + v_0t + h_0$. Find the maximum height.

22. initial velocity: 54 ft/s
initial height: 7 ft

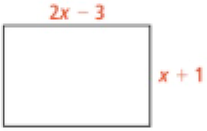
23. initial velocity: 18 ft/s
initial height: 9 ft

Write a quadratic function to represent the area of each rectangle. Graph the function. Interpret the vertex and intercepts. Identify a reasonable domain and range.

24.



25.



26. A company tracks their monthly revenue based on the price they charge for their product. It is reasonable to conclude from the data that the company makes a maximum revenue when the product sells for $24.

Price ($)	Revenue ($)
20	3220
22	3580
24	3700
26	3580
28	3550

a. Find a quadratic model for the monthly revenue based on the price of the product.

b. If the company puts their product on sale for $18 how much revenue would they expect to generate?

27. **Analyze and Persevere** Given a vertical motion model, how can you identify the amount of time an object is in the air before it reaches the ground?

LESSON 7-5 Linear, Exponential, and Quadratic Models

Quick Review

To determine whether a linear, quadratic, or exponential function best models a real-world situation consider how the data changes per unit interval.

A linear function best models the data when values change by a constant amount per unit interval.

An exponential function best models the data when values grow or decay by a constant percent per unit interval.

When data values increase to a maximum then decrease, or when they decrease to a minimum and then increase, a quadratic function is more appropriate than a linear or exponential function.

Quadratic functions may model other strictly increasing or strictly decreasing situations where the data do not have a constant amount change per unit interval or a constant percent change per unit interval.

Example

Determine whether the function below is linear, quadratic, or exponential.

x	y	Percent Change
0	1	
1	3	$\frac{3-1}{1} = \frac{2}{1} = 2$; 200%
2	9	$\frac{9-3}{3} = \frac{6}{3} = 2$; 200%
3	27	$\frac{27-9}{9} = \frac{18}{9} = 2$; 200%
4	81	$\frac{81-27}{27} = \frac{54}{27} = 2$; 200%

Since the y-values are increasing by a constant percent per unit interval the function is exponential.

Practice and Problem Solving

28. **Analyze Sense and Persevere** What is the first step in determining whether a table shows a linear, quadratic, or exponential function?

Determine whether the data in the tables represent a linear, quadratic, or exponential function.

29.

x	y
7	4
8	−1
9	−4
10	−5
11	−4

30.

x	y
−2	−20
0	−6
2	8
4	22
6	36

Determine whether a linear, quadratic or exponential function best models each situation.

31. Esteban's family buys a new car. For the next five years, the car's value decreases by about 20% each year.

32. Abby plans on renting a jet ski for 1 to 4 hours. Jet ski rentals cost $80 per hour.

33. A local coffee shop adjusts the price of a medium coffee as demand changes. In the early season, when demand is low, they charge less. During peak season, the shop increases the price of a coffee to a maximum. As demand drops off, the shop lowers their coffee price. The coffee shop owner wants to model price as a function of time.

TOPIC

8

Solving Quadratic Equations

TOPIC ESSENTIAL QUESTION

How do you use quadratic equations to model situations and solve problems?

Topic Overview

enVision® STEM Project:
Designing a T-Shirt Launcher

8-1 Solving Quadratic Equations Using Graphs and Tables
AR.3.6, AR.3.7, AR.3.8, MTR.1.1, MTR.4.1, MTR.5.1

8-2 Solving Quadratic Equations by Factoring
AR.1.2, AR.3.1, AR.3.5, AR.3.6, AR.3.7, AR.3.8, MTR.2.1, MTR.5.1, MTR.7.1

8-3 Solving Quadratic Equations Using Square Roots
AR.3.1, MTR.1.1, MTR.5.1, MTR.6.1

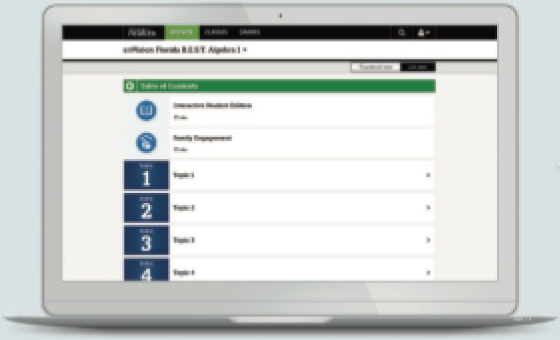
8-4 Completing the Square
AR.1.1, AR.1.2, AR.3.1, AR.3.6, MTR.3.1, MTR.4.1, MTR.5.1

8-5 The Quadratic Formula and the Discriminant
AR.3.1, AR.3.6, MTR.1.1, MTR.3.1, MTR.5.1

Mathematical Modeling in 3 Acts:
Unwrapping Change
AR.3.8, MTR.7.1

Topic Vocabulary

- completing the square
- discriminant
- quadratic equation
- quadratic formula
- root
- standard form of a quadratic equation
- Zero-Product Property
- zeros of a function



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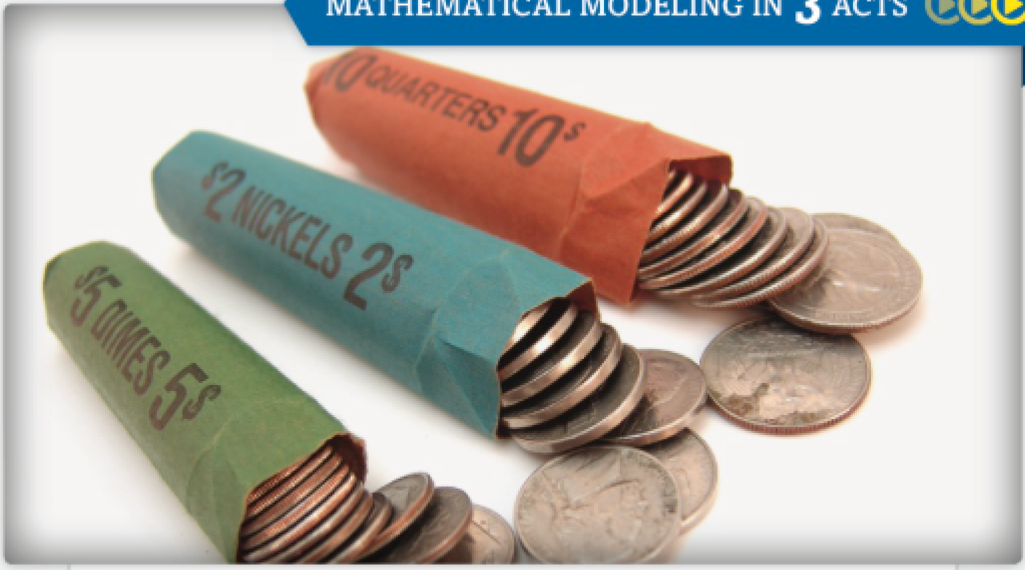
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Unwrapping Change

When you arrange a group of objects in different ways, it seems like the space they take up has changed. But, the number of objects didn't change!

We use coin wrappers to store coins in an efficient way. How much more efficient is it than the alternative? Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision® *STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

MATH TOOLS Explore math with digital tools and manipulatives.

DESMOS Use Anytime and as embedded Interactives in Lesson content.

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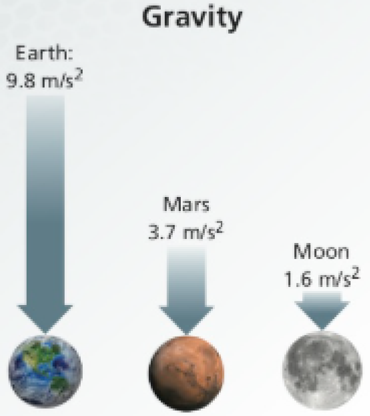
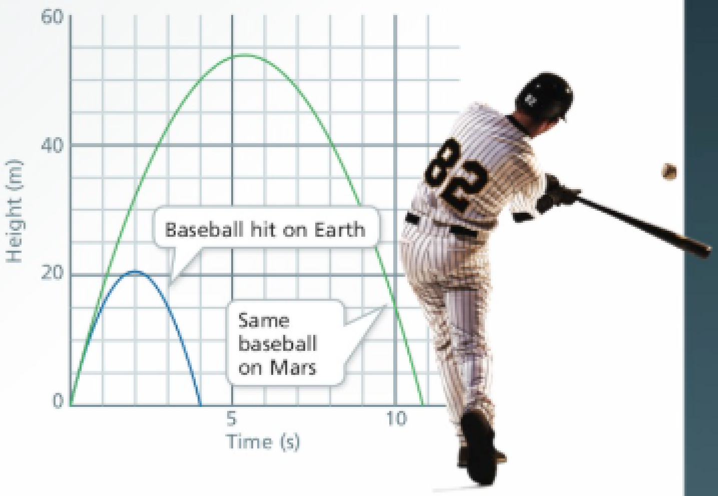
TOPIC 8 PROJECT enVision STEM

Did You Know?

Objects launched or thrown into the air follow a **parabolic path**. The force of gravity and the horizontal and vertical velocities determine a quadratic function for an object's path.



The **weaker the gravity, the higher an object will fly** and the longer it will remain airborn.



T-shirt launchers are used at sporting events to send shirts to fans high in the stands. Some t-shirts can travel as far as **400 feet**.

Your Task: Designing a T-Shirt Launcher

You and your classmates will design a t-shirt launcher and determine possible heights and distances on Earth and other planets.

8-1

Solving Quadratic Equations Using Graphs and Tables

I CAN… use graphs and tables to find solutions of quadratic equations.

VOCABULARY

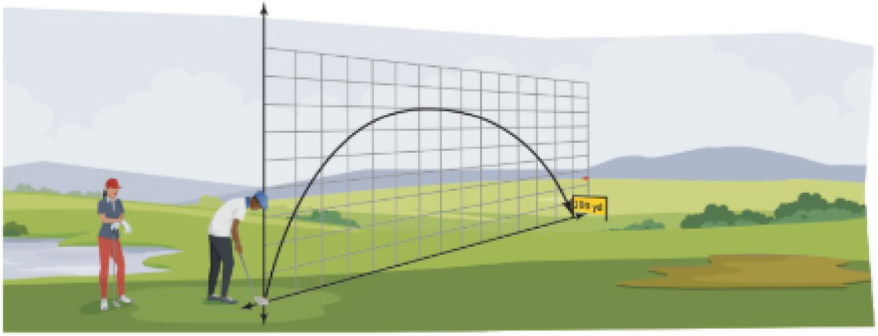
- quadratic equation
- zeros of a function

MA.912.AR.3.6–Given an expression or equation representing a quadratic function, determine the vertex and zeros and interpret them in terms of a real-world context. **Also AR.3.7, AR.3.8**

MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

The path of a golf ball hit from the ground resembles the shape of a parabola.



A. What point represents the golf ball before it is hit off the ground?

B. What point represents the golf ball when it lands on the ground?

C. **Use Patterns and Structure** Explain how the points in Part A and B are related to the ball's distance from the ground.

ESSENTIAL QUESTION

How can graphs and tables help you solve quadratic equations?

CONCEPTUAL UNDERSTANDING

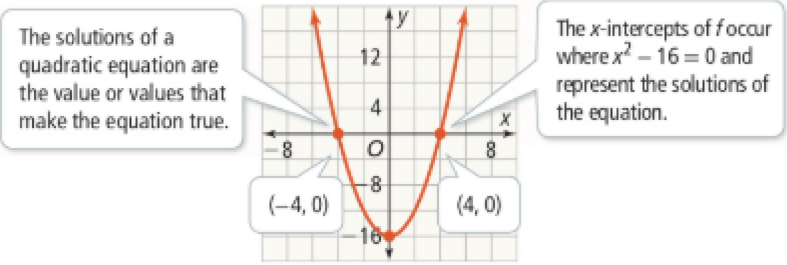
EXAMPLE 1 Recognize Solutions of Quadratic Equations

Why are the *x*-intercepts of the graph of a quadratic function important to the solution of a related equation?

A. Find the solutions of the quadratic equation $x^2 - 16 = 0$.

A **quadratic equation** is an equation of the second degree. The related function of a quadratic equation with 0 on one side is the quadratic expression given on the other side.

Graph the related function, $f(x) = x^2 - 16$.



COMMUNICATE AND JUSTIFY
Think about the connection between the graph of the function and the solutions to the quadratic equation. What evidence can you give that the *x*-intercept is the solution of the quadratic equation?

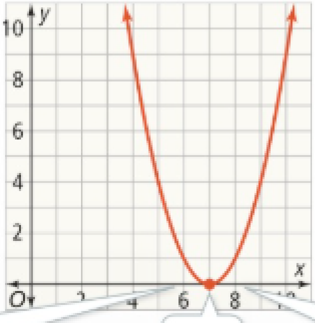
The graph of the function has two *x*-intercepts, so the equation has two real solutions. The solutions of the equation $x^2 - 16 = 0$ are $x = -4$ and $x = 4$. Solutions to an equation of the form $f(x) = 0$, are called the **zeros of a function**. The zeros of a function correspond to the *x*-intercepts of the function.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

B. Find the solutions of $x^2 - 14x + 49 = 0$.

Graph the related function $f(x) = x^2 - 14x + 49$.



The graph touches the x-axis but does not cross it.

(7, 0)

There is only one x-intercept.

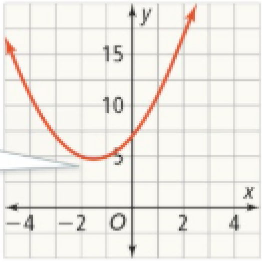
The graph of the function has only one x-intercept, so the equation has only one real solution, $x = 7$.

C. Find the solutions of $x^2 + 3x + 7 = 0$.

Graph the related function $f(x) = x^2 + 3x + 7$.

The graph of the function has no x-intercepts, so the equation has no real solutions.

STUDY TIP
A quadratic equation can have 0, 1, or 2 real solutions.



The graph does not cross the x-axis.

Try It! **1.** What are the solutions of each equation?

a. $x^2 - 36 = 0$

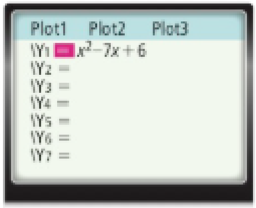
b. $x^2 + 6x + 9 = 0$

EXAMPLE 2 Solve Quadratic Equations Using Tables

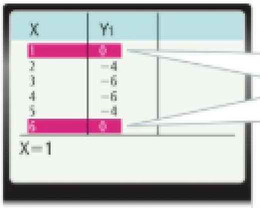
A. How can you use a table to find the solutions of $x^2 - 7x + 6 = 0$?

Enter the function $y = x^2 - 7x + 6$ into a graphing calculator.

USE PATTERNS AND STRUCTURE
How would the solutions of this quadratic equation appear in a graph?



```
Plot1  Plot2  Plot3
\Y1 = x²−7x+6
\Y2 =
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```



X	Y1
1	0
2	−4
3	−6
4	−6
5	−4
6	0

X=1

Use the table to identify the values of x when y = 0.

There are two real solutions, $x = 1$ and $x = 6$.

CONTINUED ON THE NEXT PAGE

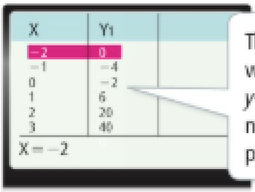
EXAMPLE 2 CONTINUED

B. How can you use a table to estimate the solutions of $3x^2 + 5x - 2 = 0$?

CHOOSE EFFICIENT METHODS
The solution may not always appear in a table when $y = 0$. Would a graph be more useful in finding the solution for this quadratic equation?

Enter the function $y = 3x^2 + 5x - 2$ into a graphing calculator.

The table shows one solution, $x = -2$.



X	Y_1
−2	0
−1	−4
0	−2
1	6
2	20
3	40

X = −2

The other solution occurs where the signs of the y-values change from negative to positive or positive to negative.

Refine the table settings to find the other solution of the equation. Change the table settings to show steps of 0.25.

The other solution is between 0.25 and 0.5.

Using a table has limitations. When the corresponding x-values for $y = 0$ are not shown in the table, you can estimate the solution.

X	Y_1
0	−2
0.25	−0.5625
0.5	1.25
0.75	3.4375
1	6

X = 0.25

Approximate the solution by the value of y when x goes from 0.25 to 0.5.

Try It! 2. Find the solutions for $4x^2 + 3x - 7 = 0$ using a table. If approximating, give the answer to the nearest tenth.

APPLICATION

EXAMPLE 3 Use Approximate Solutions

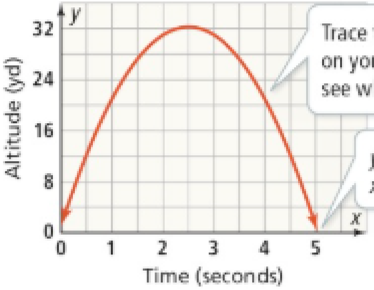
Tama hits her golf ball off the tee. The height of the golf ball is modeled by the function $f(x) = -5x^2 + 25x + 1$, where x is the number of seconds after the golf ball is hit. How long is the golf ball in the air?

$y = 0$ when the ball lands.

Graph $f(x) = -5x^2 + 25x + 1$ to find when $y = 0$.

COMMUNICATE AND JUSTIFY
What are the benefits of using a graph to approximate a solution? When is a table more useful than a graph?

The graph of the function shows the x-intercept at 5.04. This means the golf ball was in the air about 5 seconds before it hit the ground.



Trace the graph on your screen to see when $y = 0$.

$y = 0$ when $x = 5.04$

Try It! 3. At the next tee, a golf ball was hit and modeled by $-16x^2 + 11x + 6 = 0$. When will the golf ball hit the ground?

CONCEPT SUMMARY Solving Quadratic Equations Using Graphs and Tables

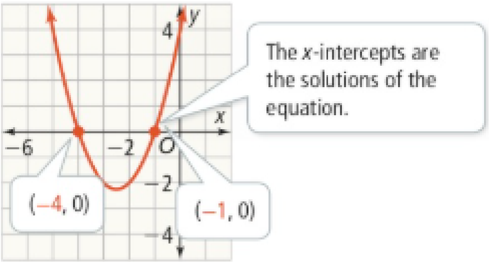
WORDS A quadratic equation can be written in standard form $ax^2 + bx + c = 0$, where $a \neq 0$.

A quadratic equation can have 0, 1, or 2 real solutions.

Zeros of the function related to a quadratic equation are the solutions of the equation.

ALGEBRA $x^2 + 5x + 4 = 0$ The solutions are $x = -4$ and $x = -1$.

GRAPH $f(x) = x^2 + 5x + 4$



The x-intercepts are the solutions of the equation.

TABLE

x	y
−5	4
−4	0
−3	−2
−2	−2
−1	0

The solutions are the x-values when the y-values are 0.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can graphs and tables help you solve quadratic equations?

2. **Choose Efficient Methods** In a table that shows no exact solutions, how do you know if there are any solutions? How can you find an approximate solution?

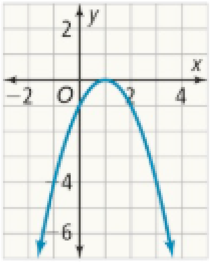
3. **Error Analysis** Eli says that the solutions to $x^2 + 100 = 0$ are −10 and 10 because 10^2 is 100. What is the error that Eli made? Explain.

4. **Check for Reasonableness** When you graph a quadratic function, the y-intercept appears to be 1, and the x-intercepts appear to be −4 and 2.5. Which values represent the solution(s) to the related quadratic equation of the function? How can you verify this? Explain.

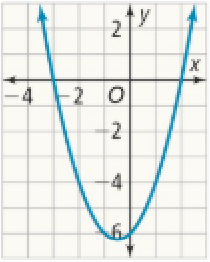
Do You KNOW HOW?

Use each graph to find the solution of the equation.

5. $-x^2 + 2x - 1 = 0$



6. $x^2 + x - 6 = 0$



Solve each quadratic equation by graphing the related function.

7. $x^2 - 2x - 3 = 0$

8. $x^2 + x + 1 = 0$

Find the solutions of each equation using a table. Round approximate solutions to the nearest tenth.

9. $x^2 + 3x - 4 = 0$

10. $3x^2 - 2x + 1 = 0$

11. What are the solutions of $-5x^2 + 10x + 2 = 0$? Round approximate solutions to the nearest tenth.

PRACTICE & PROBLEM SOLVING

UNDERSTAND

12. **Analyze and Persevere** Consider the quadratic equation $x^2 + 2x - 24 = 0$.

    a. How could you solve the equation using a graph? Explain.

    b. How could you solve the equation using a table? Explain.

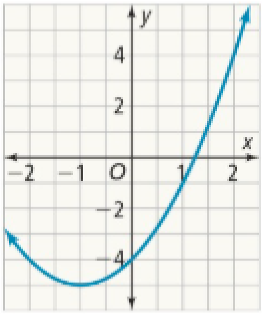
13. **Generalize** For an equation of the form $ax^2 + bx + c = 0$, where the graph crosses the y-axis once and does not intersect the x-axis. Describe the solution(s) of the equation.

14. **Error Analysis** Describe and correct the error a student made in stating the number of solutions of a quadratic equation. Explain.

> A quadratic equation has either two solutions or no solution. ✗

15. **Higher Order Thinking** Infinitely many quadratic equations of the form $ax^2 + bx + c = 0$ can have the same two solutions. Sketch the graphs of two quadratic functions on the same grid to show how this could be true.

16. **Use Patterns and Structure** How many zeros does the quadratic function shown have? Explain.

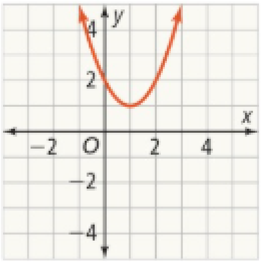


17. **Mathematical Connections** If a quadratic function has a maximum value that is greater than 0, how many zeros does the function have? Explain.

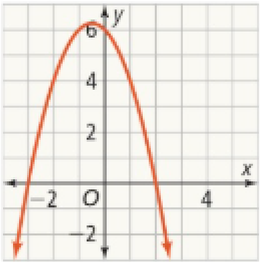
PRACTICE

Use each graph to find the solution of the related equation. SEE EXAMPLE 1

18. $x^2 - 2x + 2 = 0$



19. $-x^2 - x + 6 = 0$



Solve each quadratic equation by graphing the related function. Round approximate solutions to the nearest tenth. SEE EXAMPLES 1 AND 3

20. $x^2 - 121 = 0$

21. $x^2 - 4x + 4 = 0$

22. $x^2 + 3x + 7 = 0$

23. $x^2 - 5x = 0$

24. $-x^2 + 6x + 7 = 0$

25. $-x^2 + 8x - 7 = 0$

26. $x^2 - 2 = 0$

27. $2x^2 - 11x + 12 = 0$

28. $-3x^2 + 5x + 7 = 0$

29. $-16x^2 + 70 = 0$

Find the solutions for each equation using a table. Round approximate solutions to the nearest tenth. SEE EXAMPLE 2

30. $x^2 - 16 = 0$

31. $x^2 + 8x + 16 = 0$

32. $x^2 + 3x + 1 = 0$

33. $x^2 + 4x + 6 = 0$

PRACTICE & PROBLEM SOLVING

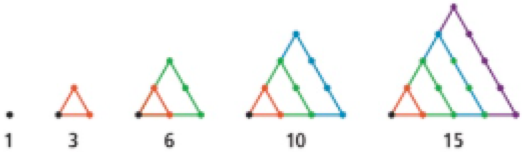
APPLY

34. **Apply Math Models** A small company shows the profits from their business with the function $P(x) = -0.01x^2 + 60x + 500$, where x is the number of units they sell and P is the profit in dollars.

a. How many units are sold by the company to earn the maximum profit?

b. How many units are sold when the company starts showing a loss?

35. **Analyze and Persevere** A pattern of triangular numbers is shown. The first is 1, the second is 3, the third is 6, and so on.



The formula $0.5n^2 + 0.5n$ can be used to find the nth triangular number. Is 50 a triangular number? Explain.

36. **Analyze and Persevere** The equation $-16x^2 + 10x + 15 = 0$ represents the height, in feet, of a flotation device above the water after x seconds. The coefficient of the linear term represents the initial velocity. The constant term represents the initial height.

a. If the initial velocity is 0, when should the flotation device land in the water?

b. If the initial height is 0, when does the flotation device land in the water?

ASSESSMENT PRACTICE

37. Select all the solutions of the quadratic equation $0 = 2x^2 + 5x$. AR.3.1

☐ **A.** $x = -10$

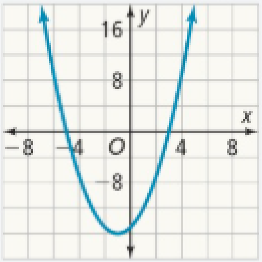
☐ **B.** $x = -2.5$

☐ **C.** $x = 0$

☐ **D.** $x = 2.5$

☐ **E.** $x = 10$

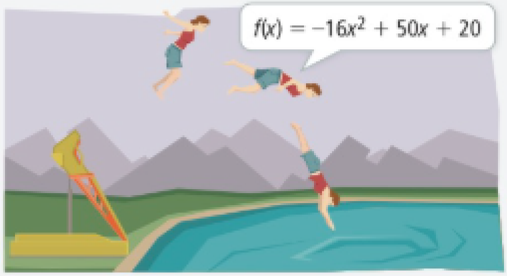
38. **SAT/ACT** What are the solutions of $x^2 + 2x - 15 = 0$ using the graph shown?



Ⓐ −3, 3 Ⓑ −5, 3

Ⓒ −8, 5 Ⓓ −16, 0

39. **Performance Task** A human catapult is used to launch a person into a lake. The height, in feet, of the person is modeled as shown, where x is the time in seconds from the launch.



Part A What equation can you use to find when the person touches the lake? Find the solution.

Part B Are both solutions to the quadratic equation valid solutions to the lake problem? Explain.

Part C What is the greatest height reached?

8-2

Solving Quadratic Equations by Factoring

I CAN… find the solution of a quadratic equation by factoring.

VOCABULARY

- standard form of a quadratic equation
- Zero-Product Property

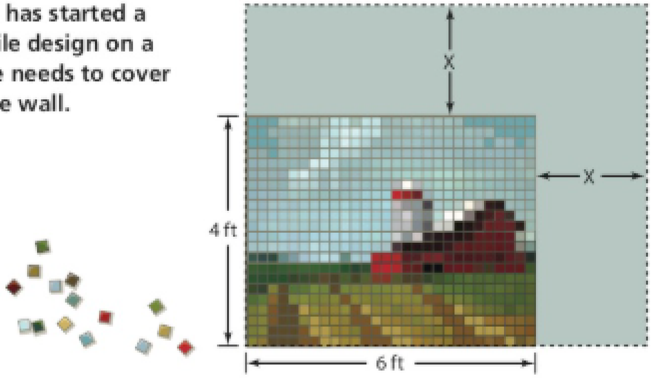
MA.912.AR.3.1–Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real number system.

Also AR.1.2, AR.3.5, AR.3.6, AR.3.7, AR.3.8

MA.K12.MTR.2.1, MTR.5.1, MTR.7.1

MODEL & DISCUSS

An artist has started a mosaic tile design on a wall. She needs to cover the entire wall.



A. Write expressions to represent the length of the wall and width of the wall.

B. **Use Patterns and Structure** What expression represents the area of the entire wall? Explain.

C. How can you determine the area of the part of the wall that the artist has not yet covered?

ESSENTIAL QUESTION

How does factoring help you solve quadratic equations?

EXAMPLE 1 Use the Zero-Product Property

How can you find the solution of the equation $(x - 9)(5x + 2) = 0$?

The **Zero-Product Property** states that for all real numbers a and b, if $ab = 0$, then either $a = 0$ or $b = 0$.

Set each factor of the equation equal to zero to find the solution.

$(x - 9) = 0$ or $(5x + 2) = 0$

$x = 9$ $x = -\frac{2}{5}$

Either $(x - 9)$ or $(5x + 2)$ is equal to 0 according to the Zero-Product Property.

USE PATTERNS AND STRUCTURE
Think about solving this equation using a table and graph. How would the solutions appear in a table? In a graph?

Check each solution.

Substitute 9 for x.

$$(9 - 9)(5(9) + 2) = 0$$
$$(0)(47) = 0$$
$$0 = 0 \checkmark$$

Substitute $-\frac{2}{5}$ for x.

$$\left(-\frac{2}{5} - 9\right)\left(5\left(-\frac{2}{5}\right) + 2\right) = 0$$
$$\left(-9\frac{2}{5}\right)(0) = 0$$
$$0 = 0 \checkmark$$

The solutions of $(x - 9)(5x + 2) = 0$ are $x = 9$ and $x = -\frac{2}{5}$.

Try It! **1.** Solve each equation.

a. $(2x - 1)(x + 3) = 0$

b. $(2x + 3)(3x - 1) = 0$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Solve by Factoring

How can you use factoring to solve $x^2 + 9x = -20$?

The **standard form of a quadratic equation** is $ax^2 + bx + c = 0$, where $a \neq 0$.

Step 1 Write the equation in standard form.

$$x^2 + 9x = -20$$
$$x^2 + 9x + 20 = 0$$

When solving a quadratic equation by factoring, begin by writing the equation in standard form.

Step 2 Make a table to find the set of factors to solve $x^2 + 9x + 20 = 0$. The set of factors that have a product of 20 and a sum of 9 can be used to solve the equation.

Factors of 20	Sum of Factors
1, 20	21
2, 10	12
4, 5	9

The factors 4 and 5 have a product of **20** and a sum of **9.**

STUDY TIP
If you can factor the standard form of the equation then you can find the solution.

Step 3 Rewrite the standard form of the equation in factored form.

$$(x + 4)(x + 5) = 0$$

Step 4 Use the Zero-Product Property to solve the equation.

$$(x + 4) = 0 \quad \text{or} \quad (x + 5) = 0$$
$$x = -4 \qquad\qquad x = -5$$

The solutions of $x^2 + 9x + 20 = 0$ are $x = -4$ and $x = -5$.

Try It! **2.** Solve each equation by factoring.

a. $x^2 + 16x + 64 = 0$ **b.** $x^2 - 12x = 64$

APPLICATION

EXAMPLE 3 Use Factoring to Solve a Real-World Problem

A museum vault has an outer steel wall with a uniform width of x. The area of the museum vault ceiling and the outer steel wall is 1,664 ft^2. What is the width of the outer steel wall?

Formulate ◀ Write an equation to represent the area of the vault.

$$(2x + 20)(2x + 40) = 1{,}664$$

length × width = area

Compute ◀ Use the Distributive Property. Write the equation in standard form.

$$(2x + 20)(2x + 40) = 1{,}664$$
$$4x^2 + 120x - 864 = 0$$
$$\frac{4x^2}{4} + \frac{120x}{4} - \frac{864}{4} = \frac{0}{4}$$
$$x^2 + 30x - 216 = 0$$
$$(x - 6)(x + 36) = 0$$

Divide each term by 4 to simplify the equation.

CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

Interpret ◀ The solutions of the equation are $x = 6$ and $x = -36$.

The length of the wall cannot be negative. Therefore -36 cannot be a solution. The width of the wall is 6 ft.

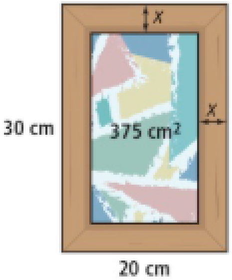
Check the solution.

Substitute 6 for x in the original equation.

$$[2(6) + 20]\ [2(6) + 40] = 1{,}664$$

$$(32)(52) = 1{,}664\ ✓$$

Try It! **3.** A picture inside a frame has an area of 375 cm^2. What is the width of the frame?



EXAMPLE 4 Use Factored Form to Graph a Quadratic Function

How can you use factoring to graph the function $f(x) = x^2 - 2x - 8$?

Step 1 Factor the related quadratic equation.

$$x^2 - 2x - 8 = 0$$

$$(x + 2)(x - 4) = 0$$

Step 2 Determine the solutions of the equation.

$$(x + 2) = 0 \quad \text{or} \quad (x - 4) = 0$$

$$x = -2 \qquad\qquad x = 4$$

STUDY TIP
A parabola is symmetrical so the vertex is halfway between the two x-intercepts.

Step 3 Find the coordinates of the vertex. Find the average of the x-intercepts 4 and -2.

$$\frac{4 + (-2)}{2} = 1$$

The x-coordinate of the vertex is 1.

Find the y-coordinate of the vertex.

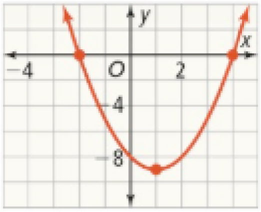
$$f(x) = (1)^2 - 2(1) - 8 = -9$$

Substitute the x-coordinate in the quadratic function.

The vertex is $(1, -9)$.

Step 4 Plot the vertex and the x-intercepts.

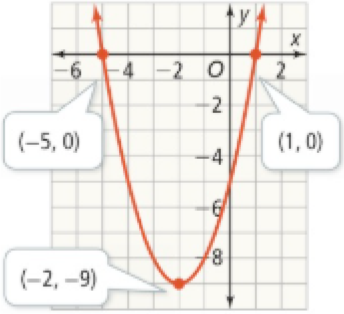
Use the vertex and x-intercepts to sketch the graph.



Try It! **4.** Use factoring to graph the function $f(x) = 2x^2 + 5x - 3$.

EXAMPLE 5 Write the Factored Form of a Quadratic Function

How can you write the factored form of the quadratic function related to a graph?



Step 1 Find the x-intercepts.

The x-intercepts are −5 and 1.

Step 2 Write a quadratic equation in factored form.

Use the x-intercepts, which are also the solutions of the quadratic equation, as the factors.

$$a(x - p)(x - q) = 0$$
$$a[x - (-5)]\,[x - (1)] = 0$$
$$a(x + 5)(x - 1) = 0$$

Substitute the x-intercepts for p and q.

COMMON ERROR
There are an infinite number of parabolas that pass through (−5, 0) and (1, 0). Be sure to determine the value of a to find the one parabola that also passes through (−2, −9).

Step 3 Write the function in factored form.

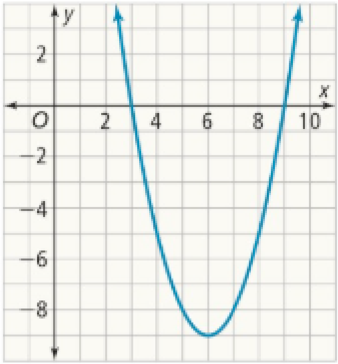
Use a third point to solve for a.

$$f(x) = a(x + 5)(x - 1)$$
$$-9 = a(-2 + 5)(-2 - 1)$$
$$a = 1$$

Use the vertex. Substitute −2 for x and −9 for $f(x)$.

The factored form of the quadratic function is $f(x) = 1(x + 5)(x - 1)$ or $f(x) = (x + 5)(x - 1)$.

Try It! **5.** What is the factored form of the function?



EXAMPLE 6 Use Factored Form of a Quadratic Function

Malcom has 80 ft of fencing to enclose a rectangular area for his garden plot.

HAVE A GROWTH MINDSET
After receiving constructive feedback, how do you use it as an opportunity to improve?

A. Write a function for the area of the garden in terms of the width.

Write an equation for the length in terms of the width.

$$2\ell + 2w = P$$
$$2\ell + 2w = 80$$
$$2\ell = 80 - 2w$$
$$\ell = 40 - w$$

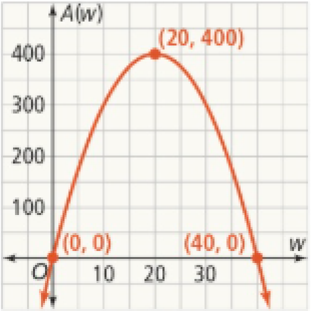
Then write the area function.

$$A = \ell w$$
$$= (40 - w)w$$
$$A(w) = w(40 - w)$$

Substitute the expression for length in terms of width.

B. Graph the function for the area and determine any constraints on the domain and range in terms of the context.

The function is written in factored form, so the x-intercepts are $w = 0$ and $w = 40$. The vertex is midway between them at $w = 20$. $A(20) = 400$. The vertex is located at (20, 400).



The garden plot needs to have a width greater than 0. However, a width of 40 would use all of the fencing for two sides, leaving no fencing for the other two sides.

A reasonable domain given the context is $0 < w < 40$.

The range for the function is $0 < A(w) < 400$.

C. What dimensions would give Malcom the greatest area for his garden? Explain.

Because the parabola is opening downward, the y-value of the vertex is the maximum value of the function.

The greatest area rectangular plot Malcom can enclose with 80 ft of fencing is 400 ft^2, which corresponds to a width of 20 feet. Using the equation $\ell = 40 - w$, the length is also 20 ft.

Try It! **6. Suppose Malcom had 144 feet of fencing.**

a. Write the function for the area in terms of the length of the plot.

b. What dimensions would give Malcom the greatest area for his garden?

CONCEPT SUMMARY Solving Quadratic Equations by Factoring

WORDS The **Zero-Product Property** states that for all real numbers a and b, if $ab = 0$, then either $a = 0$ or $b = 0$. You can apply the Zero-Product Property to a factored quadratic equation to help you find the x-intercepts of the graph of the related function.

ALGEBRA $x^2 + 2x - 3 = 5$

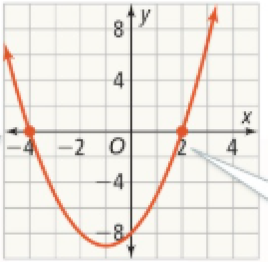
$$x^2 + 2x - 8 = 0$$
$$(x + 4)(x - 2) = 0$$

Write the equation in standard form and factor.

$$(x + 4) = 0 \quad \text{or} \quad (x - 2) = 0$$
$$x = -4 \qquad\qquad x = 2$$

The solutions of the quadratic equation are $x = -4$ and $x = 2$.

GRAPH $f(x) = x^2 + 2x - 8$



The x-intercepts are −4 and 2.

The x-intercepts of the graph correspond to the zeros of the function.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does factoring help you solve quadratic equations?

2. **Use Patterns and Structure** Compare the solutions of $2x^2 + 5x - 7 = 0$ and $4x^2 + 10x - 14 = 0$. What do you notice? Explain.

3. **Vocabulary** What is the *Zero-Product Property*? When can you use it to solve a quadratic equation? Explain.

4. **Generalize** If a perfect-square trinomial has a value of 0, how many solutions does the equation have? Explain.

Do You KNOW HOW?

Solve each equation.

5. $(x - 10)(x + 20) = 0$
6. $(3x + 4)(x - 4) = 0$

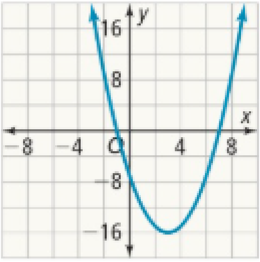
Solve each equation by factoring.

7. $x^2 + 18x + 32 = 0$
8. $x^2 - 4x - 21 = 0$

Solve each equation.

9. $x^2 + 2x = -1$
10. $x^2 - 8x = 9$
11. $2x^2 + x = 15$
12. $5x^2 - 19x = -18$

13. Write a quadratic equation, in factored form, whose solutions correspond to the x-intercepts of the quadratic function shown at the right.



14. Factor the equation $x^2 - 6x + 5 = 0$. Find the coordinates of the vertex of the related function, and graph the function $y = x^2 - 6x + 5$.

PRACTICE & PROBLEM SOLVING

UNDERSTAND

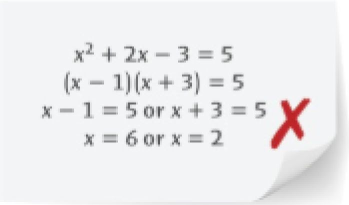
15. **Represent and Connect** One solution of a quadratic equation is 8. What do you know about the quadratic equation? What are two ways you would know if a quadratic equation could have this solution?

16. **Communicate and Justify** Write a quadratic equation for each condition below. Explain your reasoning.

    a. The equation has solutions that are opposites.

    b. The equation has one solution.

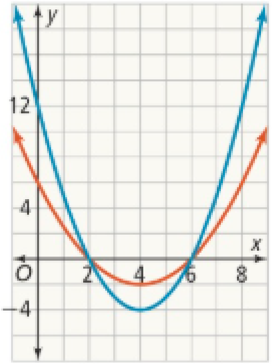
17. **Error Analysis** Describe and correct the error a student made in factoring.



$$x^2 + 2x - 3 = 5$$
$$(x - 1)(x + 3) = 5$$
$$x - 1 = 5 \text{ or } x + 3 = 5$$
$$x = 6 \text{ or } x = 2$$

18. **Analyze and Persevere** Explain how you would factor $2x^2 + 8x + 6 = 0$.

19. **Higher Order Thinking** Both parabolas are graphs of quadratic functions.



    a. Write the factored form of the equation related to one of the functions. Which curve is related to your function?

    b. Use a constant factor to find the equation related to the other function.

    c. What relationship do you see between the two functions? How are these reflected in the constant?

PRACTICE

Solve each equation. SEE EXAMPLE 1

20. $(x - 5)(x + 2) = 0$

21. $(2x - 5)(7x + 2) = 0$

22. $3(x + 2)(x - 2) = 0$

23. $(3x - 8)^2 = 0$

Solve each equation by factoring.
SEE EXAMPLES 2 AND 3

24. $x^2 + 2x + 1 = 0$

25. $x^2 - 5x - 14 = 0$

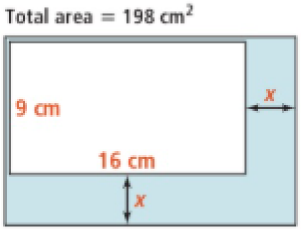
26. $x^2 + 7x = 0$

27. $2x^2 - 5x + 2 = 0$

28. $2x^2 + 3x = 5$

29. $5x^2 + 16x = -3$

30. Write an equation to represent the shaded area. Then find the value of x. SEE EXAMPLE 3



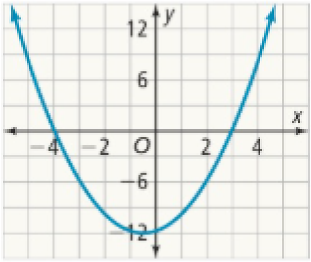
Factor, find the coordinates of the vertex of the related function, then graph. SEE EXAMPLE 4

31. $x^2 - 2x - 63 = 0$

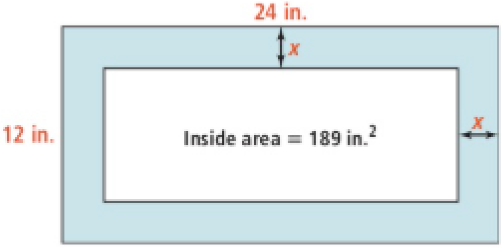
32. $x^2 + 16x + 63 = 0$

Write the factored form for the quadratic function. SEE EXAMPLE 5

33.



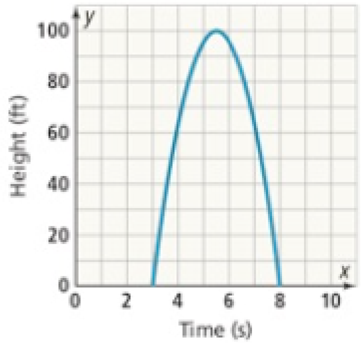
34. Write the function $A(x)$ for the unshaded area of the figure below. Describe any constraints on the domain. SEE EXAMPLE 6



PRACTICE & PROBLEM SOLVING

APPLY

35. **Mathematical Connections** A streamer is launched 3 s after a fuse is lit and lands 8 s after it is lit.

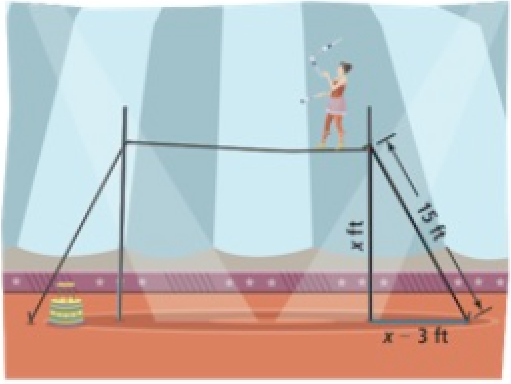


a. What is a quadratic equation in factored form that models the situation?

b. What is the vertex of the function related to your equation? How does this compare with the vertex of the graph?

c. What can you multiply your factored form by to get the function for the graph? Explain your answer.

36. **Apply Math Models** A 15 ft long cable is connected from a hook to the top of a pole that has an unknown height. The distance from the hook to the base of the pole is 3 ft shorter than the height of the pole.



a. What can you use to find the height of the pole?

b. Write and solve a quadratic equation to find the height of the pole.

c. How far is the hook from the base of the pole?

ASSESSMENT PRACTICE

37. Select all equations that you can solve easily by factoring. AR.3.1

☐ A. $x^2 + 6x = -8$

☐ B. $2x^2 + x = 5$

☐ C. $x^2 + 2x = 8$

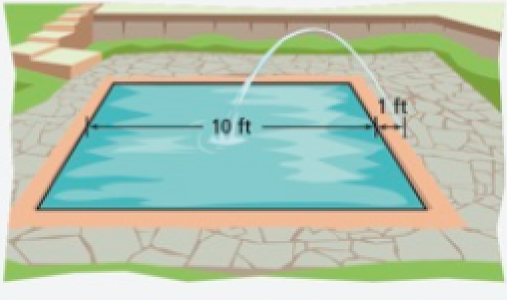
☐ D. $2x^2 + 5x = 10$

☐ E. $2x^2 - 11x = -12$

38. **SAT/ACT** A quadratic equation of the form $x^2 + bx + c = 0$ has a solution of −2. Its related function has a vertex at (2.5, −20.25). What is the other solution to the equation?

Ⓐ −11
Ⓑ −4.5
Ⓒ 0.5
Ⓓ 7
Ⓔ 9

39. **Performance Task** An engineer is designing a water fountain that starts 1 ft off of the edge of a 10 ft wide pool. The water from the fountain needs to project into the center of the pool. The path of the water from the fountain is in the shape of a parabola.



Part A Let the the point (1, 0) be the location of the starting point of the water. Write a quadratic equation to model the path of the water.

Part B What is the maximum height of the water? Use your equation from Part A.

Part C What is the equation for the path of the water if the maximum height of the water must be 4 ft?

8-3

Solving Quadratic Equations Using Square Roots

I CAN… solve quadratic equations by taking square roots.

MA.912.AR.3.1–Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real number system.

MA.K12.MTR.1.1, MTR.5.1, MTR.6.1

EXPLORE & REASON

A developer is building three square recreation areas on a parcel of land. He has not decided what to do with the enclosed triangular area in the center.

A. How can you determine the side lengths of the enclosed triangle?

B. What relationships do you notice among the areas of the squares?

C. **Use Patterns and Structure** How can the developer adjust this plan so that each recreation area covers less area but still has a similar triangular section in the middle? Explain.

ESSENTIAL QUESTION

How can square roots be used to solve quadratic equations?

EXAMPLE 1 Solve Equations of the Form $x^2 = a$

A. What are the solutions of the equation $x^2 = 49$?

Solve by inspection.

$$x^2 = 49$$

$$x = \pm 7$$

Remember that 49 is the square of 7 and −7.

The solutions of the equation are 7 and −7.

COMMON ERROR

When taking a square root of both sides of an equation, always consider the positive and negative square root solutions.

B. What are the solutions of the equation $x^2 = -121$?

Solve by inspection.

$$x^2 = -121$$

$$\sqrt{x^2} = \sqrt{-121}$$

The solutions to the equation are not real numbers. There is no real number that can be multiplied by itself to produce a negative number. A negative radicand indicates that there is no real solution to an equation of the form $x = \sqrt{a}$.

Try It! 1. Solve each equation by inspection.

a. $x^2 = 169$

b. $x^2 = -16$

EXAMPLE 2 Solve Equations of the Form $ax^2 = c$

A. What are the solutions of the equation $7x^2 = 112$?

Isolate the variable using properties of equality.

$$7x^2 = 112$$
$$\frac{7x^2}{7} = \frac{112}{7}$$
$$x^2 = 16$$
$$\sqrt{x^2} = \sqrt{16}$$
$$x = \pm 4$$

16 is a perfect square.

B. What are the solutions of the equation $-3x^2 = -24$?

Isolate the variable using properties of equality.

$$-3x^2 = -24$$
$$\frac{-3x^2}{-3} = \frac{-24}{-3}$$
$$x^2 = 8$$
$$x = \pm\sqrt{8}$$

Take the square root of both sides to solve for x.

CHOOSE EFFICIENT METHODS
You could rewrite $\pm\sqrt{8}$ as $\pm 2\sqrt{2}$. What is the advantage of using $\pm\sqrt{8}$? What would be the advantage of using $\pm 2\sqrt{2}$?

Try It! **2.** What are the solutions for each equation? If the solution is not an integer, state what two integers the solution is between.

a. $5x^2 = 125$ **b.** $-\frac{1}{2}x^2 = -36$

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Solve Equations of the Form $ax^2 + b = c$

How can you solve the quadratic equation $3x^2 - 5 = 22$?

Rewrite the equation in the form $x^2 = a$.

$$3x^2 - 5 = 22$$
$$3x^2 = 27$$
$$x^2 = 9$$

Write in the form $x^2 = a$, where a is a real number.

$$\sqrt{x^2} = \sqrt{9}$$

Take the square root of each side of the equation.

$$x = \pm 3$$

You can use the properties of equality to write the equation $3x^2 - 5 = 22$ in the form $x^2 = a$. Since a is a perfect square there are two integer answers. The solutions of this quadratic equation are -3 and 3.

USE PATTERNS AND STRUCTURE
Compare the steps to solve a quadratic equation to those of solving a linear equation. How are the steps similar? How are they different?

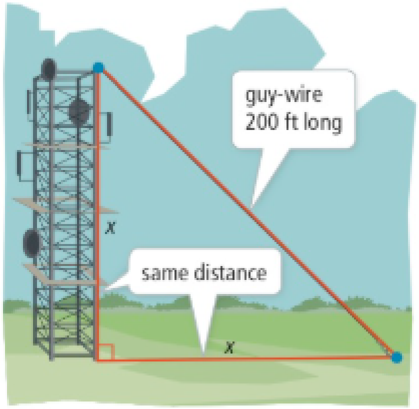
Try It! **3.** Solve the quadratic equations.

a. $-5x^2 - 19 = 144$ **b.** $3x^2 + 17 = 209$

APPLICATION

EXAMPLE 4 Determine a Reasonable Solution

A cell phone tower has a guy-wire for support as shown. The height of the tower and the distance from the tower to where the guy-wire is secured on the ground are the same distance. What is the height of the tower?



Formulate ◀ Write an equation that relates the lengths of the sides of the triangle formed by the guy-wire, the tower, and the distance on the ground from the tower to where the guy-wire is secured.

Let x represent the height of the tower and the distance on the ground.

$$x^2 + x^2 = 200^2$$

Use x in the Pythagorean Theorem for the side lengths since the two lengths are the same.

Compute ◀ Solve the equation for x.

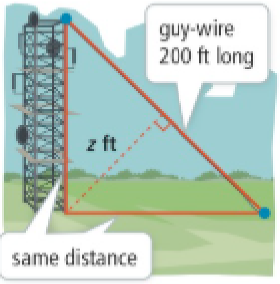
$$x^2 + x^2 = 200^2$$
$$2x^2 = 40{,}000$$
$$x^2 = 20{,}000$$
$$\sqrt{x^2} = \sqrt{20{,}000}$$
$$x = \sqrt{2 \cdot 100 \cdot 100}$$
$$x = \pm 100\sqrt{2}$$

Rewrite 20,000 using 100 as a factor.

Interpret ◀ The height of the tower must be positive, so the solution is $100\sqrt{2} \approx 141$.

The height of the cell phone tower is approximately 141 ft.

Try It! 4. Find the distance from the base of the tower to the midpoint of the guy-wire.



CONCEPT SUMMARY Solving Quadratic Equations Using Square Roots

WORDS To solve a quadratic equation using square roots, isolate the variable, and take the square root of both sides of the equation.

NUMBERS

$x^2 = 25$

$x = \pm 5$

Solve by inspection. Remember that 25 is the square of 5 and −5.

$5x^2 - 8 = 12$

$5x^2 = 20$

$x^2 = 4$

$\sqrt{x^2} = \sqrt{4}$

$x = \pm 2$

Use ± to indicate there are two solutions, one positive and one negative.

$2x^2 + 9 = 1$

$2x^2 = -8$

$x^2 = -4$

$\sqrt{x^2} = \sqrt{-4}$

$x = \sqrt{-4}$

No solution

A negative radicand indicates no solution.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can square roots be used to solve quadratic equations?

2. **Generalize** How many solutions does $ax^2 = c$ have if a and c have different signs? Explain.

3. **Use Patterns and Structure** How do you decide when to use the ± symbol when solving a quadratic equation?

4. **Error Analysis** Trey solved $2x^2 = 98$ and said that the solution is 7. Is he correct? Why or why not?

5. **Communicate and Justify** How is solving an equation in the form $ax^2 = c$ similar to solving an equation in the form $ax^2 + b = c$? How are they different?

Do You KNOW HOW?

Solve each equation by inspection.

6. $x^2 = 400$

7. $x^2 = -25$

Solve each equation.

8. $3x^2 = 400$

9. $-15x^2 = -90$

10. $2x^2 + 7 = 31$

11. $2x^2 - 7 = 38$

12. $-4x^2 - 1 = 48$

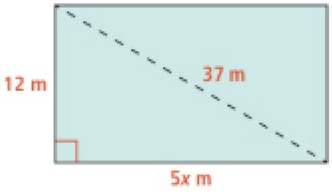
13. $-4x^2 + 50 = 1$

14. $3x^2 + 2x^2 = 150$

15. $3x^2 + 18 = 5x^2$

Solve for *x*.

16.



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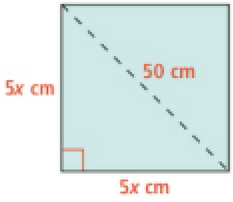
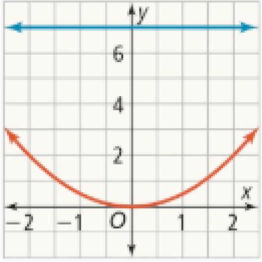


PRACTICE & PROBLEM SOLVING

UNDERSTAND

18. Analyze and Persevere Where will the parabola intersect the line? What equation did you solve to find the intersection?



19. Check for Reasonableness When solving an equation of the form $ax^2 + b = c$, what does the error message indicate? What situation may cause this error?

```
ERR: NONREAL ANS
1: Quit
2: Goto
```

20. Generalize When does solving a quadratic equation of the form $ax^2 = c$ yield the given result?

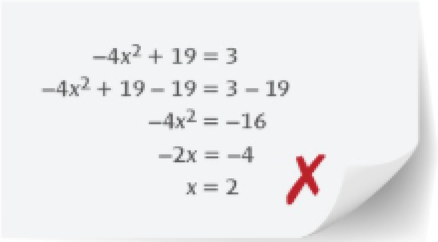
a. a rational solution

b. an irrational solution

c. one real solution

d. non-real solutions

21. Error Analysis Describe and correct the errors a student made in solving $-4x^2 + 19 = 3$.



$$-4x^2 + 19 = 3$$
$$-4x^2 + 19 - 19 = 3 - 19$$
$$-4x^2 = -16$$
$$-2x = -4$$
$$x = 2$$

✗

22. Higher Order Thinking

a. Solve $(x - 5)^2 - 100 = 0$. Show the steps for your solution.

b. Explain how you could solve an equation of the form $(x - d)^2 - c = 0$ for x.

PRACTICE

Solve each equation by inspection. SEE EXAMPLE 1

23. $x^2 = 256$

24. $x^2 = 144$

25. $x^2 = -20$

26. $x^2 = -27$

27. $x^2 = 91$

28. $x^2 = 0.25$

Solve each equation. SEE EXAMPLE 2

29. $12x^2 = 300$

30. $-x^2 = 0$

31. $0.1x^2 = 100$

32. $227x^2 = 1{,}816$

33. $-36x^2 = -36$

34. $-16x^2 = 200$

Solve each equation. SEE EXAMPLE 3

35. $x^2 + 65 = 90$

36. $x^2 - 65 = 90$

37. $3x^2 + 8 = 56$

38. $3x^2 - 8 = 56$

39. $\frac{4x^2 + 10}{2} = 5$

40. $\frac{8x^2 - 40}{4} = 470$

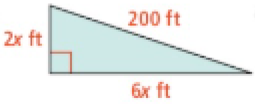
Solve each equation. Approximate irrational solutions to the nearest hundredth. SEE EXAMPLE 4

41. $6x^2 + 2x^2 = 80$

42. $6x^2 + (2x)^2 = 80$

Solve for x. Then find the side lengths of each triangle to the nearest tenth. SEE EXAMPLE 4

43.



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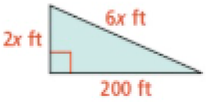


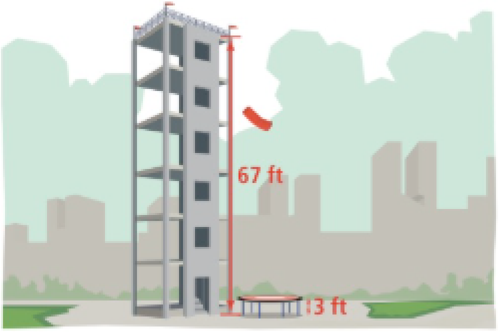
45. Use two methods to solve $x^2 - 900 = 0$. Explain.

46. At a certain time of day, the sun shines on a large flagpole causing a shadow that is twice as long as the flagpole is tall. What is the height of the flagpole to the nearest tenth of a foot?

PRACTICE & PROBLEM SOLVING

APPLY

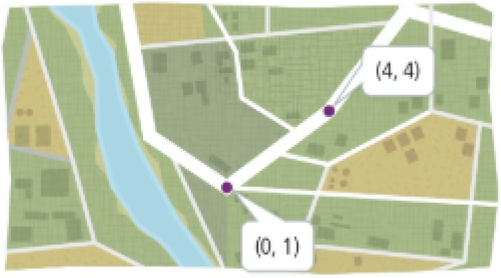
47. **Apply Math Models** A test weight is dropped from the top of a fire department training tower onto a net three feet off of the ground. Use $-16t^2$ for the change in height per second.



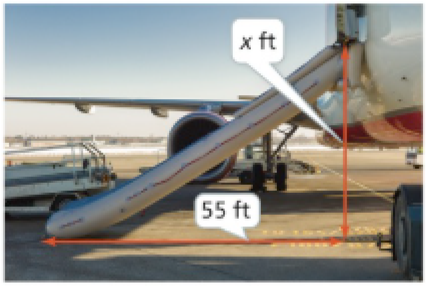
   a. Write an equation to determine the time it takes for the test weight to drop on to the net.

   b. How long does it take before the test weight is caught by the net? Explain.

48. **Analyze and Persevere** Calculate the distance in miles between the two points shown on the map.



49. **Analyze and Persevere** The evacuation slide from an aircraft is shown. If the slide is 73 feet long, what is its height at the top in feet?



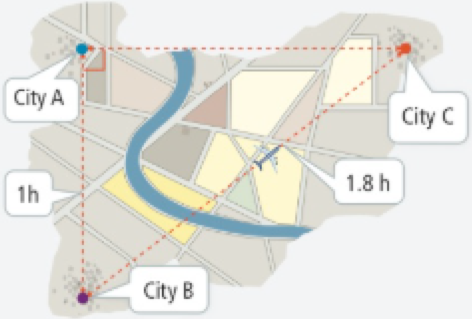
ASSESSMENT PRACTICE

50. Find the solutions of the equation $2{,}900 - 5x^2 = 840$. AR.3.1

51. **SAT/ACT** A park has an area of 280 m^2. A rectangular region with a length three times its width will be added to give the park a total area of 435 m^2. Which equation can be solved to find the width of the region?

Ⓐ $x + 3x + 280 = 435$
Ⓑ $(x \cdot 3x) + 280 = 435$
Ⓒ $(x^2 + 3x) + 280 = 435$
Ⓓ $x^2 + (3x)^2 + 280 = 435$

52. **Performance Task** A CEO flies to three different company locations. The flight times for two of her legs are shown.



Part A The plane travels at an average speed of 120 mph. Find the distance between City A and City B and the distance between City B and City C.

Part B Write and solve a quadratic equation that can be used to find the distance between City A and City C.

Part C How long will the flight between City C and City A last?

8-4

Completing the Square

I CAN… use completing the square to solve quadratic equations.

VOCABULARY

- completing the square

MA.912.AR.3.1–Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real number system. Also AR.1.1, AR.1.2, AR.3.6

MA.K12.MTR.3.1, MTR.4.1, MTR.5.1

CRITIQUE & EXPLAIN

Enrique and Nadeem used different methods to solve the equation $x^2 - 6x + 9 = 16$.

Enrique

$x^2 - 6x + 9 = 16$

$x^2 - 6x - 7 = 0$

$(x - 7)(x + 1) = 0$

$x - 7 = 0$ OR $x + 1 = 0$

$x = 7$ OR $x = -1$

The solutions are 7 and -1.

Nadeem

$x^2 - 6x + 9 = 16$

$(x - 3)^2 = 16$

$x - 3 = \pm 4$

$x - 3 = 4$ OR $x - 3 = -4$

$x = 7$ OR $x = -1$

The solutions are 7 and -1.

A. Critique Enrique's work. If his method is valid, explain the reasoning he used. If his method is not valid, explain why not.

B. Critique Nadeem's work. If his method is valid, explain the reasoning he used. If his method is not valid, explain why not.

C. **Use Patterns and Structure** Can you use either Enrique's or Nadeem's method to solve the equation $x^2 + 10x + 25 = 3$? Explain.

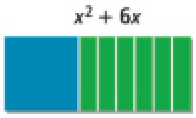
ESSENTIAL QUESTION

How is the technique of completing the square helpful for analyzing quadratic functions?

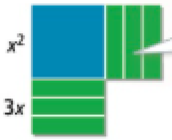
EXAMPLE 1 Complete the Square

What value of c makes $x^2 + 6x + c$ a perfect-square trinomial?

Step 1 Use algebra tiles to model the trinomial.

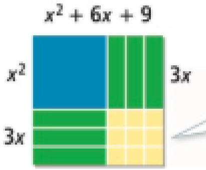


Rearrange the tiles.



Since $\frac{6}{2} = 3$, there are three x tiles on each side of the x^2 tile.

Step 2 Add 1-tiles to complete the square of algebra tiles.



Since $\frac{6}{2} = 3$ and $3^2 = 9$, there are nine 1-tiles.

Notice that the number of 1-tiles added to the square, 9, is equal to the square of half of the coefficient of x, or $\left(\frac{b}{2}\right)^2$.

USE PATTERNS AND STRUCTURE
How would the algebra tiles be rearranged if the coefficient of x were 12?

CHECK FOR REASONABLENESS
In the expression $x^2 + 6x + c$, consider the possible values for c. Can c have a negative value in a perfect-square trinomial?

CONTINUED ON THE NEXT PAGE

COMMON ERROR
Make sure you include the sign of the x-term when finding the value that completes the square.

EXAMPLE 1 CONTINUED

Step 3 Write $x^2 + 6x + 9$ as a binomial squared.

$$x^2 + 6x + 9 = (x + 3)^2$$

The constant term, 3, of the binomial is half the coefficient, 6, of the x-term.

The value of c that makes $x^2 + 6x + c$ a perfect-square trinomial is 9.

The process of adding $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to form a perfect-square trinomial is called **completing the square**. Completing the square is useful when a quadratic expression is not factorable and can be used with any quadratic expression.

Try It! **1.** What value of c completes the square?

a. $x^2 + 12x + c$ **b.** $x^2 + 8x + c$

EXAMPLE 2 Solve $x^2 + bx + c = 0$

How can you find the solutions of $x^2 - 14x + 16 = 0$?

Step 1 Write the equation in the form $ax^2 + bx = d$.

$$x^2 - 14x + 16 = 0$$
$$x^2 - 14x = -16$$

Step 2 Complete the square.

$$x^2 - 14x + 49 = -16 + 49$$
$$x^2 - 14x + 49 = 33$$

Use $\left(\frac{14}{2}\right)^2 = 49$ and add 49 to each side.

Step 3 Write the trinomial as a binomial squared.

$$x^2 - 14x + 49 = (x - 7)^2 = 33$$

LEARN TOGETHER
How can you respectfully disagree and manage your emotions?

Step 4 Solve for x.

$$x - 7 = \pm\sqrt{33}$$
$$x = 7 \pm \sqrt{33}$$

The solutions are $x = 7 + \sqrt{33}$ and $x = 7 - \sqrt{33}$.

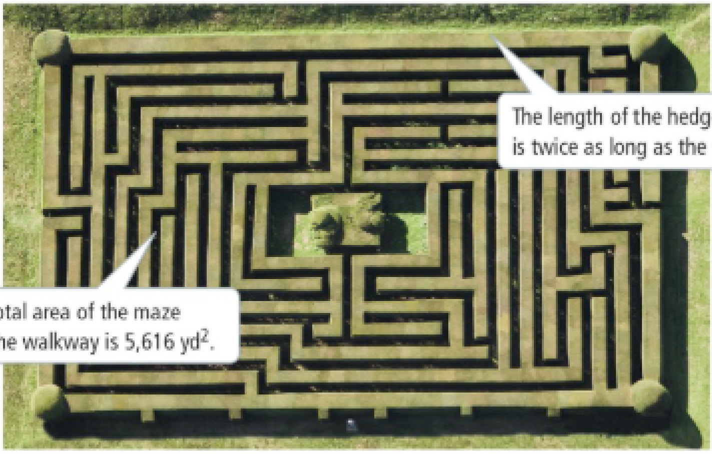
Try It! **2.** What are the solutions of each quadratic equation? Solve by completing the square.

a. $x^2 + 10x - 9 = 0$ **b.** $x^2 - 8x - 6 = 0$

APPLICATION

EXAMPLE 3 Complete the Square When $a \neq 1$ Initially

The hedge maze has a 2-yard wide walkway around it. What are the dimensions of the maze?



The length of the hedge maze is twice as long as the width.

The total area of the maze and the walkway is 5,616 yd^2.

Formulate Let x be the width of the maze. So, $(x + 4)$ is the width of the maze and walkway, and $(2x + 4)$ is the length of the maze and walkway.

Compute

$$\text{length} \times \text{width} = \text{area}$$

$$(2x + 4) \times (x + 4) = 5{,}616$$

$$(2x + 4)(x + 4) = 5{,}616$$

$$2x^2 + 12x + 16 = 5{,}616$$

Use completing the square since the quadratic is not factorable.

$$2x^2 + 12x = 5{,}600$$

$$x^2 + 6x = 2{,}800$$

Put the equation in the form $x^2 + bx = d$ by dividing each side by 2.

$$x^2 + 6x + 9 = 2{,}800 + 9$$

$$(x + 3)^2 = 2{,}809$$

Add: $\left(\frac{6}{2}\right)^2 = 9$ to each side.

$$x + 3 = \pm\sqrt{2{,}809}$$

$$x = -3 + \sqrt{2{,}809} \quad \text{or} \quad x = -3 - \sqrt{2{,}809}$$

$$= -3 + 53 \qquad\qquad = -3 - 53$$

$$= 50 \qquad\qquad = -56$$

Interpret The negative solution does not make sense in this situation, so $x = 50$.

The width of the maze is 50 yd. and the length of the maze is 100 yd. With the walkway, the total width is 54 yd, and the total length is 104 yd.

Try It! 3. A maze and walkway with the same total area of 5,616 square yards has a walkway that is one yard wide. What are the dimensions of this maze?

CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Use Completing the Square to Write a Quadratic Function In Vertex Form

How can you use completing the square to rewrite the quadratic function $y = x^2 - 8x + 11$ in vertex form?

Recall that the vertex form of a quadratic function is $y = a(x - h)^2 + k$, where (h, k) is the vertex. Completing the square is useful in order to identify $(x - h)^2$.

$$\begin{aligned} y &= x^2 - 8x + 11 \\ y - 11 &= x^2 - 8x \quad \text{.......... Isolate } x^2 + bx. \\ y - 11 + 16 &= x^2 - 8x + 16 \quad \text{.......... Complete the square.} \\ y + 5 &= (x - 4)^2 \\ y &= (x - 4)^2 - 5 \end{aligned}$$

The vertex form of the quadratic function is $y = (x - 4)^2 - 5$.

Try It! **4.** What is the vertex form of each function?

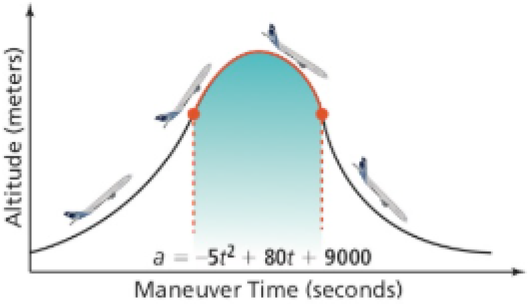
a. $y = x^2 - 2x + 3$

b. $y = x^2 + 6x + 25$

APPLICATION

EXAMPLE 5 Write Vertex Form When $a \neq 1$

Astronauts train for flying in zero gravity using a special plane that flies in parabolic arcs. The graph shown approximates the altitude *a*, in meters, of a plane in relation to the time *t*, in seconds during a training session.



What is the maximum altitude reached by the plane? At what time does the plane reach its maximum altitude?

$$\begin{aligned} a &= -5t^2 + 80t + 9{,}000 \\ a - 9{,}000 &= -5t^2 + 80t \\ a - 9{,}000 &= -5(t^2 - 16t) \\ a - 9{,}000 - 5(64) &= -5(t^2 - 16t + 64) \\ a - 9{,}320 &= -5(t - 8)^2 \\ a &= -5(t - 8)^2 + 9{,}320 \end{aligned}$$

Factor out -5 so the expression in parentheses has the form $t^2 + bt$.

Add $\left(-\frac{16}{2}\right)^2 = 64$ to complete the square.

COMMON ERROR
When you complete the square inside the parentheses on the right side, add the product of $-5(64)$ to the other side of the equation.

The vertex is (8, 9,320). The maximum altitude of the plane is 9,320 m. That altitude is reached after 8 s.

Try It! **5.** Find the minimum value of the function $y = 7x^2 + 168x + 105$.

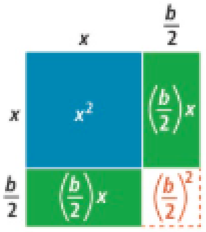
CONCEPT SUMMARY Completing the Square

WORDS To complete the square, add the square of half of the coefficient of x to both sides of a quadratic equation.

Completing the square is useful for changing $ax^2 + bx + c$ to the form $a(x - h)^2 + k$.

ALGEBRA

$$x^2 + bx \rightarrow x^2 + bx + \left(\frac{b}{2}\right)^2$$
$$= \left(x + \frac{b}{2}\right)^2$$



NUMBERS Solve $x^2 + 14x + 19 = 0$.

$$x^2 + 14x = -19$$
$$x^2 + 14x + 19 - 19 = 0 - 19$$
$$x^2 + 14x + 49 = -19 + 49$$
$$(x + 7)^2 = 30$$
$$x + 7 = \pm\sqrt{30}$$
$$x = -7 \pm \sqrt{30}$$
$$x = -7 + \sqrt{30} \text{ and } x = -7 - \sqrt{30}$$

Write $y = 3x^2 - 24x + 2$ in vertex form.

$$y = 3x^2 - 24x + 2$$
$$y - 2 = 3x^2 - 24x$$
$$y - 2 = 3(x^2 - 8x)$$
$$y - 2 + 3(16) = 3(x^2 - 8x + 16)$$
$$y + 46 = 3(x - 4)^2$$
$$y = 3(x - 4)^2 - 46$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is the technique of completing the square helpful for analyzing quadratic functions?

2. **Vocabulary** Why does it make sense to describe adding 25 to $x^2 + 10x$ as *completing the square*?

3. **Error Analysis** A student began solving $x^2 + 8x = 5$ by writing $x^2 + 8x + 16 = 5$. Explain the error the student made.

4. **Communicate and Justify** How is changing a quadratic function from standard form to vertex form like solving a quadratic equation by completing the square? How is it different?

5. **Choose Efficient Methods** Why is it helpful to factor out the coefficient of x^2 before completing the square?

Do You KNOW HOW?

Find the value of c that makes each expression a perfect-square trinomial.

6. $x^2 + 26x + c$

7. $x^2 + 2x + c$

8. $x^2 + 18x + c$

Solve each equation.

9. $x^2 + 8x = -1$

10. $2x^2 - 24x - 4 = 0$

11. $x^2 - 4x = 7$

Write each function in vertex form.

12. $y = x^2 + 4x - 5$

13. $y = 5x^2 - 10x + 7$

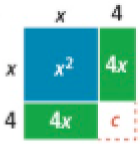
14. $y = x^2 + 8x - 15$

PRACTICE & PROBLEM SOLVING

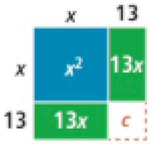
UNDERSTAND

15. **Use Patterns and Structure** What value of c completes the square for each area model below? Represent the area model as a perfect-square trinomial and as a binomial squared.

a.

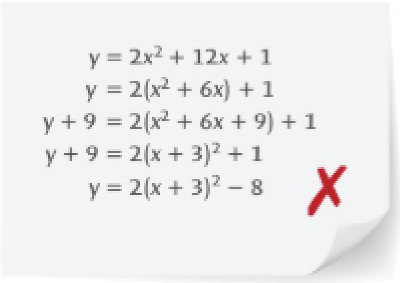


b.



16. **Choose Efficient Methods** To solve the equation $x^2 - 7x - 9 = 0$, would you use graphing, factoring, or completing the square if you want exact solutions? Explain.

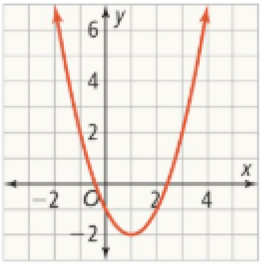
17. **Error Analysis** Describe and correct the error a student made in writing the quadratic function $y = 2x^2 + 12x + 1$ in vertex form.



$$y = 2x^2 + 12x + 1$$
$$y = 2(x^2 + 6x) + 1$$
$$y + 9 = 2(x^2 + 6x + 9) + 1$$
$$y + 9 = 2(x + 3)^2 + 1$$
$$y = 2(x + 3)^2 - 8$$

18. **Communicate and Justify** Find the solution to the equation $x^2 + 4x = -12$. Explain your reasoning.

19. **Mathematical Connections** Use the graph of $f(x) = x^2 - 2x - 1$ to estimate the solutions of $f(x) = 5$. Then find the exact solutions.



PRACTICE

Find the value of c that makes each expression a perfect-square trinomial. Write each expression as a binomial squared. SEE EXAMPLE 1

20. $x^2 + 16x + c$

21. $x^2 + 22x + c$

22. $p^2 - 30p + c$

23. $k^2 - 5k + c$

24. $g^2 + 17g + c$

25. $q^2 - 48q + c$

Solve each equation by completing the square. SEE EXAMPLES 2 AND 3

26. $x^2 + 6x = 144$

27. $x^2 - 4x = 30$

28. $m^2 + 16m = -59$

29. $x^2 - 2x - 35 = 0$

30. $5n^2 - 3n - 15 = 0$

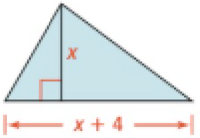
31. $4w^2 + 12w - 44 = 0$

32. $3r^2 + 18r = 21$

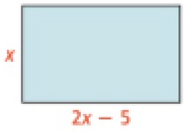
33. $2v^2 - 10v - 20 = 8$

Find the value of x. If necessary, round to the nearest hundredth. SEE EXAMPLE 3

34. Area of triangle = 8



35. Area of rectangle = 50



Write each function in vertex form, and identify the vertex. SEE EXAMPLES 4 AND 5

36. $y = x^2 + 4x - 3$

37. $y = x^2 + 12x + 27$

38. $y = x^2 - 6x + 12$

39. $y = x^2 - 14x - 1$

40. $y = 3x^2 - 6x - 2$

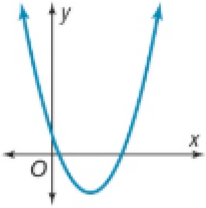
41. $y = 2x^2 - 20x + 35$

42. $y = -x^2 - 8x - 7$

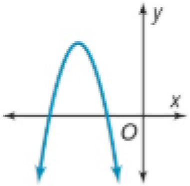
43. $y = -4x^2 + 16x + 5$

Write each function in vertex form. Tell whether each graph could represent the function.

44. $y = x^2 + 6x + 3$



45. $f(x) = -x^2 - 10x - 21$

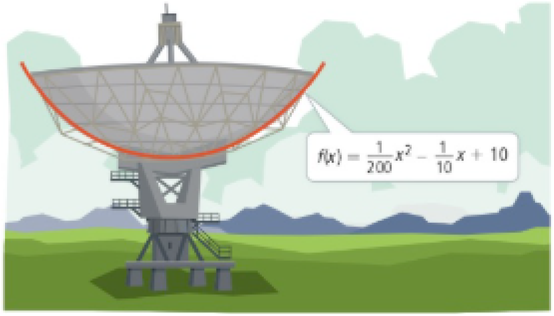


PRACTICE & PROBLEM SOLVING

APPLY

46. **Represent and Connect** You are designing a square banner for a school assembly. You want the banner to be gold with vertical purple bars as shown. You have enough material to make the area of the rectangular gold section 36 ft^2. What are the dimensions of the banner?

47. **Use Patterns and Structure** The profile of a satellite dish is shaped like a parabola. The bottom of the dish can be modeled by the function shown, where x and $f(x)$ are measured in meters. Use the vertex form of the quadratic function to determine the vertex or the lowest point of the dish. How wide is the dish at 10 m off of the ground? Explain.



$f(x) = \frac{1}{200}x^2 - \frac{1}{10}x + 10$

48. **Higher Order Thinking** The kicker on a football team uses the function, $h = -16t^2 + v_0t + h_0$, to model the height of a football being kicked into the air.

a. Show that for any values of v_0 and h_0, the maximum height of the object is $\frac{(v_0)^2}{64} + h_0$.

b. The kicker performs an experiment. He thinks if he can double the initial upward velocity of the football kicked from the ground, the maximum height will also double. Is the kicker correct? If not, how does the maximum height change? Explain.

ASSESSMENT PRACTICE

49. A rectangle is 8 cm longer than it is wide. Its area is 250 cm^2. Find the width and perimeter of the rectangle. AR.3.1

50. **SAT/ACT** The expressions $f(x) = x^2 + 12x + c$ and $g(x) = x^2 - 20x + d$ are perfect-square trinomials. What is the value of $f(0) - g(0)$?

Ⓐ −256
Ⓑ −64
Ⓒ 0
Ⓓ 32

51. **Performance Task** An electronics manufacturer designs a smartphone with an aspect ratio (the ratio of the screen's height h to its width w) of 16 : 9.

Part A Write the width in terms of h. What is the area of the phone, including the border, in terms of h?

Part B The total area of the screen and border is about 21.48 in.^{2}. What is the value of h?

Part C What are the height and width of the screen? What is the total height and width of the phone including the border?

8-5

The Quadratic Formula and the Discriminant

I CAN… use the quadratic formula to solve quadratic equations.

VOCABULARY

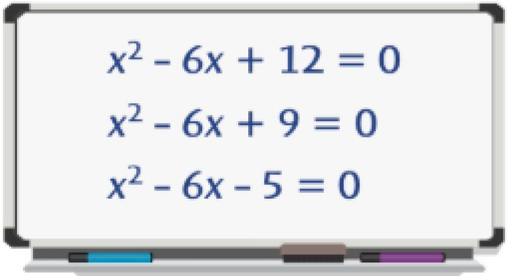
- discriminant
- quadratic formula
- root

MA.912.AR.3.1–Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real number system. **Also AR.3.6**

MA.K12.MTR.1.1, MTR.3.1, MTR.5.1

EXPLORE & REASON

Three quadratic equations are shown on the whiteboard.



$$x^2 - 6x + 12 = 0$$
$$x^2 - 6x + 9 = 0$$
$$x^2 - 6x - 5 = 0$$

A. How many real solutions are there for each of the quadratic equations shown? Explain your answer.

B. **Analyze and Persevere** Use your graphing calculator to graph the related function for each equation. What are the function equations for each graph's reflection over the *x*-axis? Explain how you found the function equations.

C. What do you notice about the graphs that have zero *x*-intercepts? One *x*-intercept? Two *x*-intercepts?

ESSENTIAL QUESTION

When should you use the quadratic formula to solve equations?

EXAMPLE 1 Derive the Quadratic Formula

How can you use completing the square to create a general formula that solves every quadratic equation?

Step 1 Complete the square for the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

Isolate $ax^2 + bx$.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Divide by a. Note that a cannot be 0.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{4a}{4a}\right)\left(-\frac{c}{a}\right) + \frac{b^2}{4a^2}$$

Multiply $-\frac{c}{a}$ by $\frac{4a}{4a}$ to get like denominators.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

STUDY TIP

Recall that in order to complete the square of a quadratic equation you need to take half of the coefficient of the linear term and square it.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Step 2 Solve for x.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Take the square root of each side of the equation.

$$x=\pm\sqrt{\frac{b^2-4ac}{4a^2}}-\frac{b}{2a}$$

Subtract $\frac{b}{2a}$ from each side of the equation.

$$x=\frac{\pm\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The **quadratic formula** gives solutions of quadratic equations in the form $ax^2+bx+c=0$ for real values of a, b, and c. The quadratic formula is a useful method to find the solutions of any quadratic equations.

Try It! **1.** What is the maximum number of solutions the quadratic formula can give? Explain.

EXAMPLE 2 Use the Quadratic Formula

How can you use the quadratic formula to find the solutions of $x^2-7=4x$?

Write the equation in standard form and identify a, b, and c.

$$x^2-4x-7=0$$

$a=1$, $b=-4$, $c=-7$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-7)}}{2(1)}$$

Substitute the values for a, b, and c into the quadratic formula.

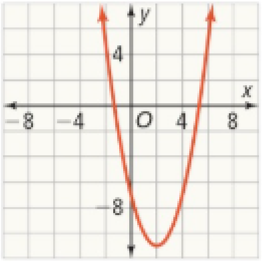
$$x=\frac{4\pm\sqrt{44}}{2}$$

$$x=\frac{4+\sqrt{44}}{2}\approx 5.32 \text{ and } x=\frac{4-\sqrt{44}}{2}\approx -1.32$$

COMMON ERROR
You might get the sign of $-b$ wrong if you forget to use parentheses when substituting negative values for b.

The solutions of $x^2-7=4x$ are $x\approx 5.32$ and $x\approx -1.32$.

Graphing the equation helps verify that the solutions found using the quadratic formula are correct.



Try It! **2.** Find the solutions of each equation using the quadratic formula.

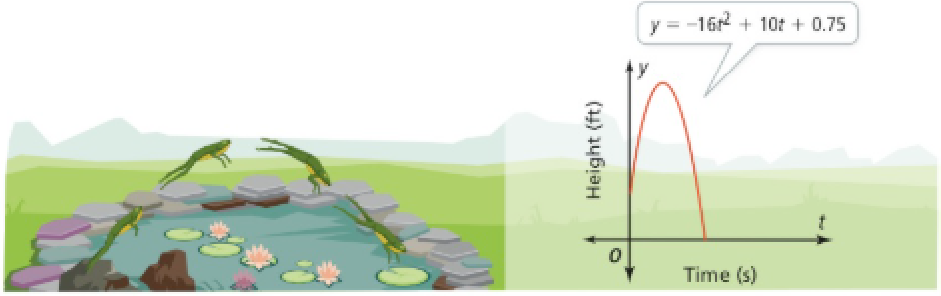
a. $21-4x=x^2$

b. $x^2-2x=24$

APPLICATION

EXAMPLE 3 Find Approximate Solutions

The function shown represents the height of a frog x seconds after it jumps off a rock. How many seconds is the frog in the air before it lands on the ground?



Formulate Write a related quadratic equation with $y = 0$ to find when the frog lands on the ground. Although the path of the frog's jump is a parabolic curve, you are being asked to find the time of the jump and not the distance of the jump. The function representing the time of the jump is also parabolic, but the parabola is not the same.

$-16t^2 + 10t + 0.75 = 0$

Find the values of a, b, and c in the equation.
$a = -16$, $b = 10$, $c = 0.75$

Compute Substitute the values of a, b, and c into the quadratic formula.

$$t = \frac{-(10) \pm \sqrt{(10)^2 - 4(-16)(0.75)}}{2(-16)}$$

$$= \frac{-10 \pm \sqrt{148}}{-32}$$

$$= \frac{-10 \pm 12.17}{-32}$$

$$t = \frac{-10 + 12.17}{-32} = -\frac{2.17}{32} \approx -0.068 \text{ and}$$

$$t = \frac{-10 - 12.17}{-32} = \frac{-22.17}{-32} \approx 0.693$$

Interpret The negative value for t is not a realistic answer in this situation because time is positive.

The frog is in the air about 0.7 s before it lands on the ground.

Try It! **3.** The height of another frog over time is modeled by the function $y = -16t^2 + 10t + 0.3$. How many seconds is this frog in the air before landing on the ground? Round your answer to the nearest hundredth.

CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Understand and Use the Discriminant

How can you determine the number of solutions of a quadratic equation without solving it?

In the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the **discriminant** is the expression $b^2 - 4ac$. The discriminant indicates the number of real solutions of the equation. The solutions of a quadratic equation are also called its **roots**. Roots are the input values for which the related function is zero.

If $b^2 - 4ac > 0$, there are two real solutions.

If $b^2 - 4ac = 0$, there is one real solution.

If $b^2 - 4ac < 0$, there are no real solutions.

GENERALIZE
What does the graph of the related function look like when a quadratic equation has two real solutions? One real solution? No real solutions?

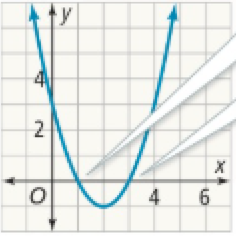
A. Find the number of solutions for $x^2 - 4x + 3 = 0$.

$x^2 - 4x + 3 = 0$

$(-4)^2 - 4(1)(3) = 4$

The discriminant is > 0, so there are two real roots.

$y = x^2 - 4x + 3$



The graph of the related function intersects the x-axis at two points.

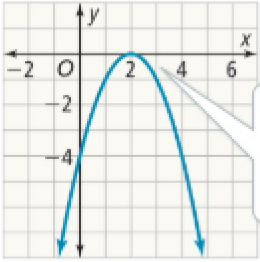
B. Find the solutions for $-x^2 + 4x - 4 = 0$.

$-x^2 + 4x - 4 = 0$

$(4)^2 - 4(-1)(-4) = 0$

The discriminant is 0, so there is one real root.

$y = -x^2 + 4x - 4$



The graph of the related function intersects the x-axis at one point.

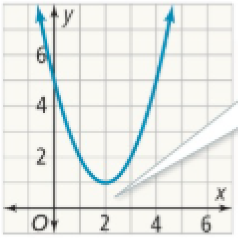
C. Find the solutions for $x^2 - 4x + 5 = 0$.

$x^2 - 4x + 5 = 0$

$(-4)^2 - 4(1)(5) = -4$

The discriminant is < 0, so there are no real roots.

$y = x^2 - 4x + 5$



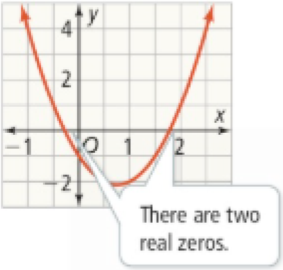
The graph of the related function does not intersect the x-axis.

Try It! **4.** Use the discriminant to find the number of roots of each equation.

a. $x^2 - 10x + 25 = 0$

b. $-x^2 - 6x - 10 = 0$

CONCEPT SUMMARY Using the Quadratic Formula

	Equation	Quadratic Formula	Discriminant
ALGEBRA	$ax^2 + bx + c = 0$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$b^2 - 4ac$
NUMBERS	$2x^2 - 3x - 1 = 0$	$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$ $x = \frac{3 + \sqrt{17}}{4} \approx 1.78$ and $x = \frac{3 - \sqrt{17}}{4} \approx -0.28$	$(-3)^2 - 4(2)(-1) = 17$ 17 > 0, two real solutions
	Related Function		
GRAPH	$y = 2x^2 - 3x - 1$  There are two real zeros.		The discriminant of the related equation is > 0.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** When should you use the quadratic formula to solve equations?

2. **Communicate and Justify** What value of b^2 is needed for there to be exactly one real solution of a quadratic equation? Explain.

3. **Vocabulary** How are the *roots* of a quadratic equation related to its *discriminant*?

4. **Error Analysis** A student says that the quadratic formula cannot be used to solve $-23x^2 + 5 = 0$. Explain the error the student made.

5. **Choose Efficient Methods** When is completing the square better than using the quadratic formula?

Do You KNOW HOW?

Identify *a*, *b*, and *c* in each of the quadratic equations.

6. $4x^2 + 2x - 1 = 0$
7. $-x^2 + 31x + 7 = 0$
8. $2x^2 - 10x - 3 = 0$
9. $x^2 + x - 1 = 0$

Given the discriminant of a quadratic equation, determine the number of real solutions.

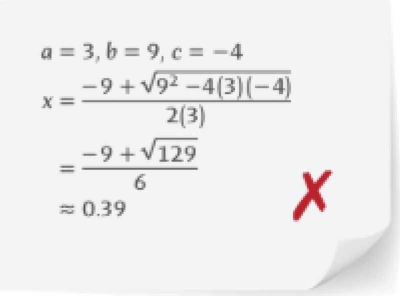
10. 8
11. −3
12. 0
13. 1

PRACTICE & PROBLEM SOLVING

UNDERSTAND

14. **Mathematical Connections** Why does a quadratic equation have to be in standard form before applying the quadratic formula to find solutions?

15. **Error Analysis** Describe and correct the error a student made in solving $3x^2 + 9x - 4 = 0$.



$a = 3, b = 9, c = -4$

$$x = \frac{-9 + \sqrt{9^2 - 4(3)(-4)}}{2(3)}$$

$$= \frac{-9 + \sqrt{129}}{6}$$

$$\approx 0.39$$

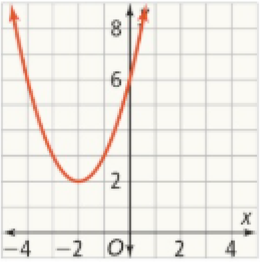
16. **Choose Efficient Methods** Which method would you use to solve each equation? Explain.

a. $x^2 + 9x = 0$

b. $11x^2 - 4 = 0$

c. $7x^2 + 11x - 6 = 0$

17. **Use Patterns and Structure** The graph of a quadratic function is shown below. Describe how you could change the graph so that the discriminant of the new related quadratic equation is positive.



18. **Higher Order Thinking** Use the quadratic formula to prove the axis of symmetry can be found using $-\frac{b}{2a}$. What does the discriminant of a quadratic equation tell you about the vertex of the graph of the related function?

PRACTICE

Solve each equation using the quadratic formula. Round to the nearest hundredth. SEE EXAMPLES 1, 2, AND 3

19. $-2x^2 + 12x - 5 = 0$

20. $x^2 + 19x - 7 = 0$

21. $3x^2 + 18x - 27 = 0$

22. $-7x^2 + 2x + 1 = 0$

23. $2x^2 + 9x + 7 = 0$

24. $-x^2 + 9x + 5 = -3$

25. $4x^2 + 17x - 5 = 4$

26. $5x^2 + 10x + 7 = 2$

27. $-6x^2 + 5x - 2 = -11$

28. $-2x^2 + 4x + 9 = -3$

Use the discriminant to determine the real roots for each equation. SEE EXAMPLE 4

29. $3x^2 - 9x - 16 = 0$

30. $-4x^2 + 7x - 11 = 0$

31. $2x^2 - 6x + 3 = 0$

32. $5x^2 - 20x + 20 = 0$

33. $7x^2 - 14x + 12 = 5$

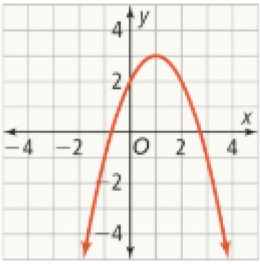
34. $9x^2 + 5x - 2 = -4$

35. $-8x^2 - 3x - 1 = 5$

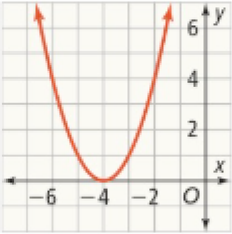
36. $2x^2 - 21x - 7 = 4$

For each graph, determine the number of roots the related quadratic equation has. Then determine whether its discriminant is greater than, equal to, or less than zero. SEE EXAMPLE 4

37.



38.

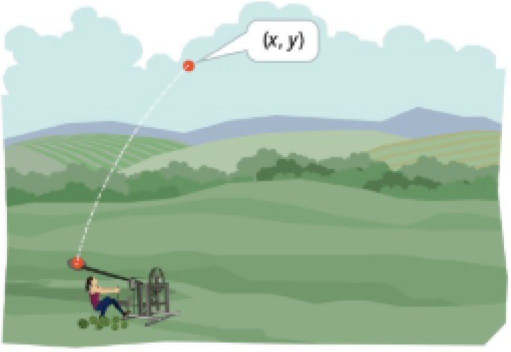


PRACTICE & PROBLEM SOLVING

APPLY

39. **Analyze and Persevere** A quadratic function can be used to model the height y of an object that is thrown over time x. What are the values of the discriminant of the related equation of the function $f(x) = -16x^2 + 35x + 5$, which models a ball being thrown into the air?

40. **Apply Math Models** The function $f(x) = -16x^2 + 64x + 5$ models the height y, in feet, of a watermelon from a watermelon launcher after x seconds.



a. Write a quadratic equation that can be used to determine when the watermelon reaches 20 ft.

b. Use the discriminant to predict the number of solutions to the equation from part (a).

c. What are the approximate solutions of the equation you wrote in part a?

41. **Analyze and Persevere** The student council is raising money for school dances by selling spirit T-shirts. The function $R = -5n^2 + 85n + 1{,}000$ models the revenue R in dollars they expect per increase of n dollars over the original price of each T-shirt. The goal is $1,250.

a. Write a quadratic equation to find the dollar increase n in price needed to meet this goal.

b. Solve the equation using the quadratic formula. What price(s) will result in the student council meeting their goal?

ASSESSMENT PRACTICE

42. Solve the equation $2x^2 + x - 21 = 0$ using the quadratic formula. AR.3.1

43. **SAT/ACT** What is the discriminant of $x^2 - x - 3 = 0$?

Ⓐ −11

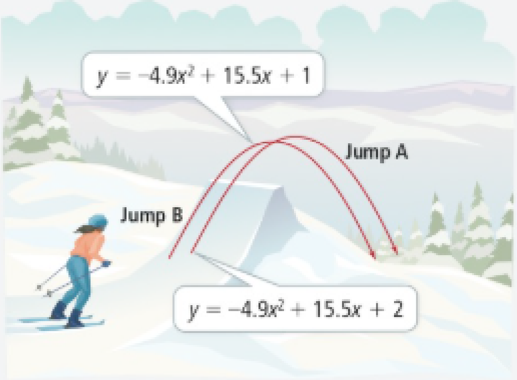
Ⓑ 0

Ⓒ 11

Ⓓ 13

Ⓔ −13

44. **Performance Task** A skier made 2 jumps that were recorded by her coach. A function that models the height y, in meters, at x seconds for each jump is shown.



Part A Predict which jump kept the skier in the air for the greatest number of seconds.

Part B Use the quadratic formula to find how long the skier was in the air during Jump A.

Part C Use the quadratic formula to find how long the skier was in the air during Jump B.

Part D Do your results support your prediction? Explain.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.3.8–Solve and graph mathematical and real-world problems that are modeled with quadratic functions. Interpret key features and determine constraints in terms of the context.

MA.K12.MTR.7.1

Unwrapping Change

When you arrange a group of objects in different ways, it seems like the space they take up has changed. But, the number of objects didn't change!

We use coin wrappers to store coins in an efficient way. How much more efficient is it than the alternative? Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC

8

Topic Review

TOPIC ESSENTIAL QUESTION

1. How do you use quadratic equations to model situations and solve problems?

Vocabulary Review

Choose the correct term to complete each sentence.

2. The ____________ is $ax^2 + bx + c = 0$, where $a \neq 0$.
3. The process of adding $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to form a perfect-square trinomial is called ____________.
4. The x-intercepts of the graph of the function are also called the ____________.
5. The ____________ is the expression $b^2 - 4ac$, which indicates the number of real solutions of a quadratic equation.
6. The ____________ states that for all real numbers a and b, if $ab = 0$, then either $a = 0$ or $b = 0$.

- completing the square
- discriminant
- quadratic equation
- quadratic formula
- standard form of a quadratic equation
- Zero-Product Property
- zeros of a function

Concepts & Skills Review

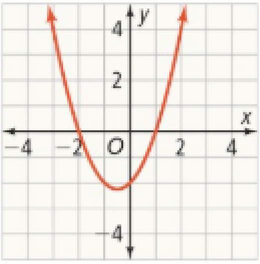
LESSON 8-1 Solving Quadratic Equations Using Graphs and Tables

Quick Review

A **quadratic equation** is an equation of the second degree. A quadratic equation can have 0, 1 or 2 solutions, which are known as the **zeros of the related function**.

Example

Find the solutions of $0 = x^2 + x - 2$.



The x-intercepts of the related function are −2, and 1, so the equation has two real solutions.

From the graph, the solutions of the equation $x^2 + x - 2 = 0$ appear to be $x = -2$ and $x = 1$.

It is important to verify those solutions by substituting into the equation.

$(-2)^2 + (-2) - 2 = 0$ $\qquad$ $1^2 + 1 - 2 = 0$

$0 = 0$ $\qquad$ $0 = 0$

Practice & Problem Solving

Solve each quadratic equation by graphing.

7. $x^2 - 16 = 0$
8. $x^2 - 6x + 9 = 0$
9. $x^2 + 2x + 8 = 0$
10. $2x^2 - 11x + 5 = 0$

Find the solutions for each equation using a table. Round to the nearest tenth.

11. $x^2 - 64 = 0$
12. $x^2 - 6x - 16 = 0$
13. **Apply Math Models** A video game company uses the profit model $P(x) = -x^2 + 14x - 39$, where x is the number of video games sold, in thousands, and $P(x)$ is the profit earned in millions of dollars. How many video games would the company have to sell to earn a maximum profit? How many video games would the company have to sell to not show a profit?

LESSON 8-2 Solving Quadratic Equations by Factoring

Quick Review

The **standard form of a quadratic equation** is $ax^2 + bx + c = 0$, where $a \neq 0$. The **Zero-Product Property** states that for all real numbers a and b, if $ab = 0$, then either $a = 0$ or $b = 0$. The solutions of a quadratic equation can often be determined by factoring.

Example

How can you use factoring to solve $x^2 + 4x = 12$?

First write the equation in standard form.

$x^2 + 4x - 12 = 0$

Then, rewrite the standard form of the equation in factored form.

$(x - 2)(x + 6) = 0$

Use the Zero-Product Property. Set each factor equal to zero and solve.

$x - 2 = 0$ or $x + 6 = 0$

$x = 2$ $x = -6$

The solutions of $x^2 + 4x - 12 = 0$ are $x = 2$ and $x = -6$.

Practice & Problem Solving

Solve each equation by factoring.

14. $x^2 + 6x + 9 = 0$

15. $x^2 - 3x - 10 = 0$

16. $x^2 - 12x = 0$

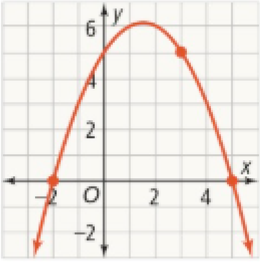
17. $2x^2 - 7x - 15 = 0$

Factor, find the coordinates of the vertex of the related function, and then graph it.

18. $x^2 - 12x + 20 = 0$

19. $x^2 - 8x + 15 = 0$

20. Write the factored form for the quadratic function.



21. **Error Analysis** Describe and correct the error a student made in factoring.

$$2x^2 - 8x + 8 = 0$$
$$2(x^2 - 4x + 4) = 0$$
$$2(x - 2)(x - 2) = 0$$
$$x = -2$$

LESSON 8-3 Solving Quadratic Equations Using Square Roots

Quick Review

To solve a quadratic equation using square roots, isolate the variable and find the square root of both sides of the equation.

Example

Use the properties of equality to solve the quadratic equation $4x^2 - 7 = 57$.

Rewrite the equation in the form $x^2 = a$.

$4x^2 - 7 = 57$

$4x^2 = 64$ Rewrite using the form $x^2 = a$, where a is a real number.

$x^2 = 16$

$\sqrt{x^2} = \sqrt{16}$ Take the square root of each side of the equation.

$x = \pm 4$

Since 16 is perfect square, there are two integer answers. The solutions of the quadratic equation $4x^2 - 7 = 57$ are $x = -4$ and $x = 4$.

Practice & Problem Solving

Solve each equation by inspection.

22. $x^2 = 289$

23. $x^2 = -36$

24. $x^2 = 155$

25. $x^2 = 0.64$

Solve each equation.

26. $5x^2 = 320$

27. $x^2 - 42 = 358$

28. $4x^2 - 18 = 82$

29. **Higher Order Thinking** Solve $(x - 4)^2 - 81 = 0$. Explain the steps in your solution.

30. **Analyze and Persevere** Use the equation $d = \sqrt{(12 - 5)^2 + (8 - 3)^2}$ to calculate the distance between the points (3, 5) and (8, 12). What is the distance?

LESSON 8-4 Completing the Square

Quick Review

The process of adding $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to form a perfect-square trinomial is called **completing the square**. This is useful for changing $ax^2 + bx + c$ to the form $a(x - h)^2 + k$.

Example

Find the solutions of $x^2 - 16x + 12 = 0$.

First, write the equation in the form $ax^2 + bx = d$.

$x^2 - 16x = -12$

Complete the square.

$b = -16$, so $\left(\frac{-16}{2}\right)^2 = 64$

$x^2 - 16x + 64 = -12 + 64$

$x^2 - 16x + 64 = 52$

Write the trinomial as a binomial squared.

$(x - 8)^2 = 52$

Solve for x.

$x - 8 = \sqrt{52}$

$x = 8 \pm 2\sqrt{13}$

$x = 8 + 2\sqrt{13}$ and $x = 8 - 2\sqrt{13}$.

Practice & Problem Solving

Find the value of *c* that makes each expression a perfect-square trinomial. Then write the expression as a binomial squared.

31. $x^2 + 18x + c$

32. $x^2 - 6x + c$

33. $x^2 - 15x + c$

34. $x^2 + 24x + c$

Solve each equation by completing the square.

35. $x^2 + 18x = 24$

36. $x^2 - 10x = 46$

37. $x^2 + 22x = -39$

38. $3x^2 + 42x + 45 = 0$

39. **Choose Efficient Methods** To solve the equation $x^2 - 9x - 15 = 0$, would you use graphing, factoring, or completing the square if you want exact solutions? Explain.

LESSON 8-5 The Quadratic Formula and the Discriminant

Quick Review

The **quadratic formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, gives solutions of quadratic equations in the form $ax^2 + bx + c = 0$ for real values of a, b, and c where $a \neq 0$. The quadratic formula is a useful method to find the solutions of quadratic equations that are not factorable.

The **discriminant** is the expression $b^2 - 4ac$, which indicates the number of solutions of the equation. The solutions of a quadratic equation are also called its **roots**, which are the input values when the related function's output value is zero.

If $b^2 - 4ac > 0$, there are 2 real solutions.

If $b^2 - 4ac = 0$, there is 1 real solution.

If $b^2 - 4ac < 0$, there are no real solutions.

Example

Use the quadratic formula to find the solutions of $x^2 - 9 = 5x$.

Write the equation in standard form $ax^2 + bx + c = 0$ and identify a, b and c.

$x^2 - 5x - 9 = 0$

$a = 1$, $b = -5$, $c = -9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-9)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{61}}{2}$$

$$x = \frac{5 + \sqrt{61}}{2} \approx 6.41 \text{ and}$$

$$= \frac{5 - \sqrt{61}}{2} \approx -1.41$$

The approximate solutions of $x^2 - 9 = 5x$ are $x \approx 6.41$ and $x \approx -1.41$.

Practice & Problem Solving

Solve each equation using the quadratic formula.

40. $2x^2 + 3x - 5 = 0$

41. $-5x^2 + 4x + 12 = 0$

42. $3x^2 + 6x - 1 = 4$

43. $4x^2 + 12x + 6 = 0$

Use the discriminant to determine the number of real solutions for each equation.

44. $3x^2 - 8x + 2 = 0$

45. $-4x^2 - 6x - 1 = 0$

46. $7x^2 + 14x + 7 = 0$

47. $2x^2 + 5x + 3 = -5$

48. **Error Analysis** Describe and correct the errors a student made in solving $3x^2 - 5x - 8 = 0$.

$a = 3$, $b = -5$, $c = 8$

$$x = \frac{-5 \pm \sqrt{(-5)^2 - 4(3)(8)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{-71}}{6}$$

There are no real solutions.

49. **Communicate and Justify** The function $f(x) = -5x^2 + 20x + 55$ models the height of a ball x seconds after it is thrown into the air. What are the possible solutions to the related equation? Explain.

TOPIC

9

Working With Functions

TOPIC ESSENTIAL QUESTION

What are some operations on functions that you can use to create models and solve problems?

Topic Overview

enVision® STEM Project:
Program a Square Root Algorithm

9-1 Square Root Functions
F.1.1, F.1.2, F.1.3, MTR.2.1, MTR.5.1, MTR.6.1

9-2 Cubic and Cube Root Functions
F.1.1, F.1.2, MTR.4.1, MTR.5.1, MTR.7.1

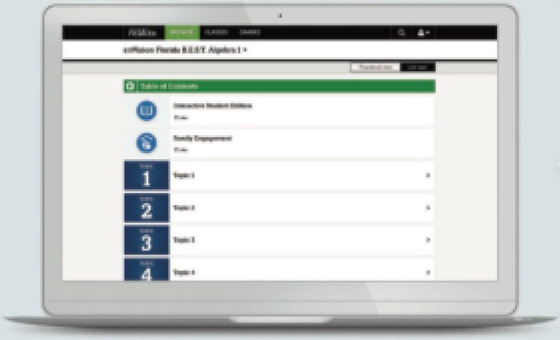
9-3 Analyzing Functions
AR.3.7, AR.4.3, AR.5.6, F.1.1, F.1.5, F.1.6, F.2.3, MTR.1.1, MTR.4.1, MTR.5.1

Mathematical Modeling in 3 Acts:
Edgy Tiles
F.1.1, F.1.2, MTR.7.1

9-4 Operations With Functions
F.3.1, MTR.1.1, MTR.5.1, MTR.7.1

Topic Vocabulary

- cube root function
- cubic function
- square root function



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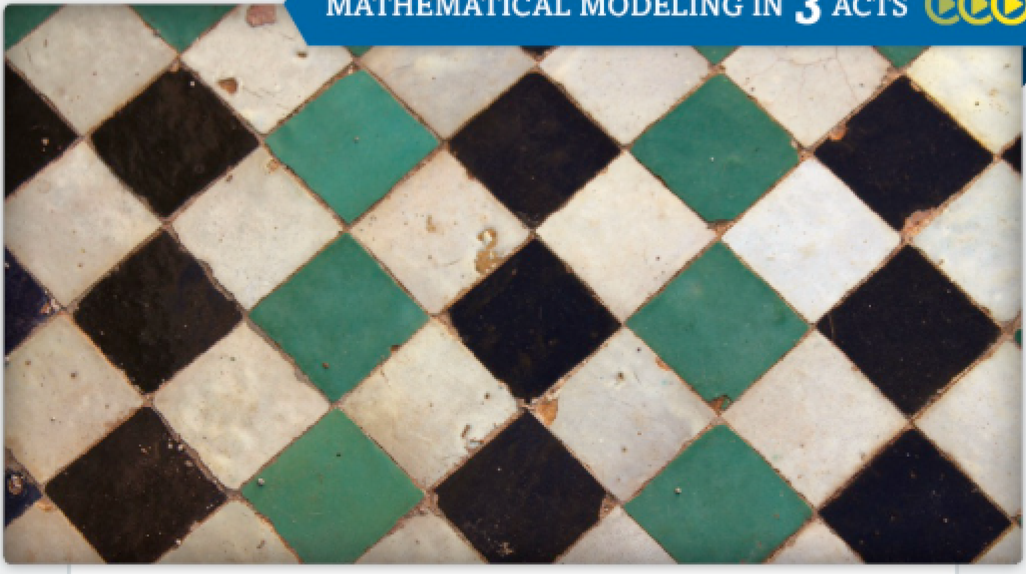
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Edgy Tiles

For more than 3,000 years, people have glazed ceramics and other materials to make decorative tile patterns. Tiles used to be used only in important buildings or by the very rich, but now you can find tiles in almost any house.

Before you start tiling a wall, floor, or other surface, it's important to plan out how your design will look. Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision® *STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

MATH TOOLS Explore math with digital tools and manipulatives.

DESMOS Use Anytime and as embedded Interactives in Lesson content.

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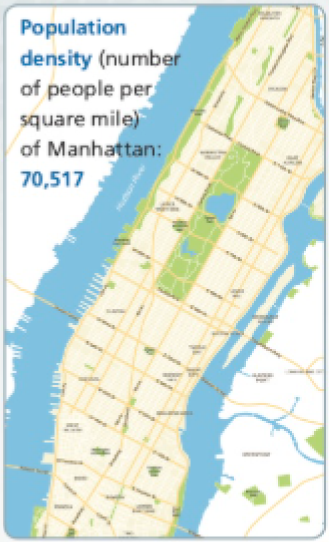
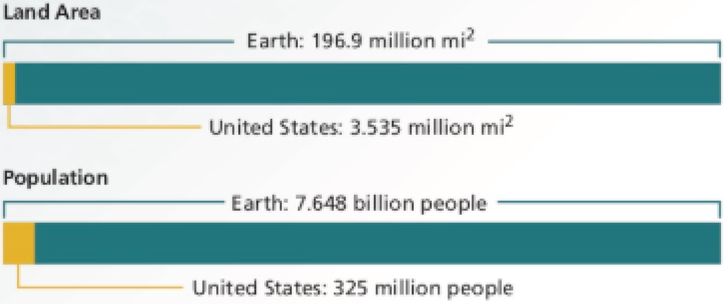
Did You Know?

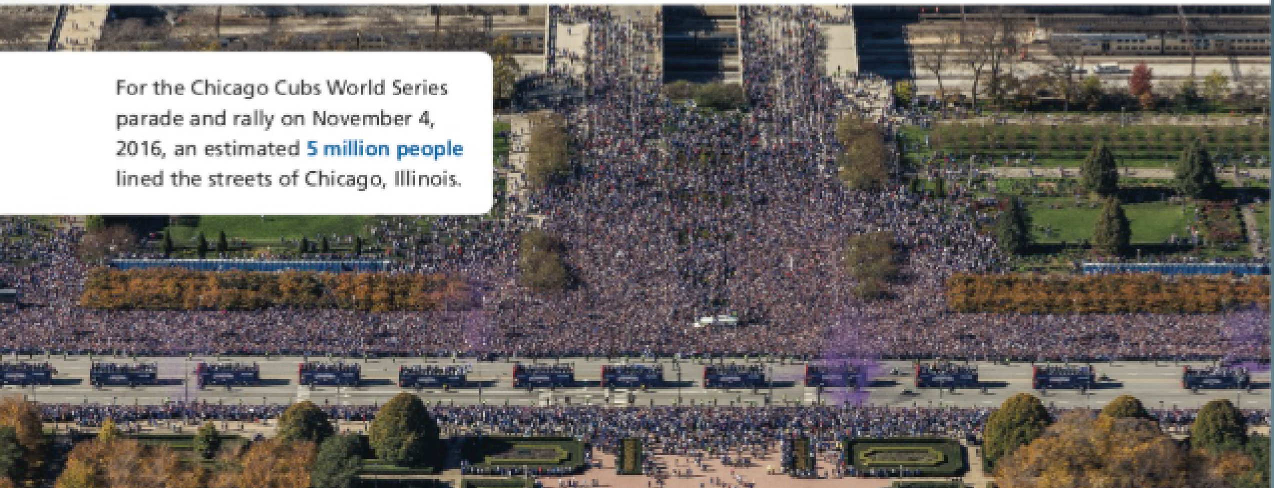
Standing up, the average **person** takes up **2 square feet** of space.



Population density (number of people per square mile) of Manhattan: **70,517**



The cities of San Francisco, Chicago, and New York all have **population density greater than 10,000.**



For the Chicago Cubs World Series parade and rally on November 4, 2016, an estimated **5 million people** lined the streets of Chicago, Illinois.

Your Task: Program a Square Root Algorithm

Sports arenas and open spaces are designed to hold great numbers of people. You and your classmates will design a square building to hold a given number of people.

9-1

Square Root Functions

I CAN… describe the key features of the square root function.

VOCABULARY

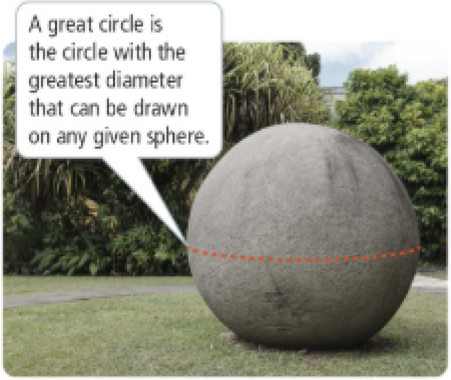
- square root function

MA.912.F.1.1–Given an equation or graph that defines a function, determine the function type. Given an input-output table, determine a function type that could represent it. **Also F.1.2, F.1.3**

MA.K12.MTR.2.1, MTR.5.1, MTR.6.1

EXPLORE & REASON

One of the strangest mysteries in archaeology was discovered in the Diquís Delta of Costa Rica. Hundreds of sphere-shaped stones were found.



A great circle is the circle with the greatest diameter that can be drawn on any given sphere.

A. The formula for the surface area of a sphere is $SA = 4\pi r^2$. What is the surface area of the stone in terms of the circumference of the great circle?

B. The circumferences of the great circles of the spheres range in size from about 6 cm to 6 m. Make a graph that represents circumference as a function of surface area.

C. **Use Patterns and Structure** What similarities and differences do you notice between the graph from Part B and the graph of a quadratic function?

ESSENTIAL QUESTION

What key features are shared among the square root function and translations of the square root function?

CONCEPTUAL UNDERSTANDING

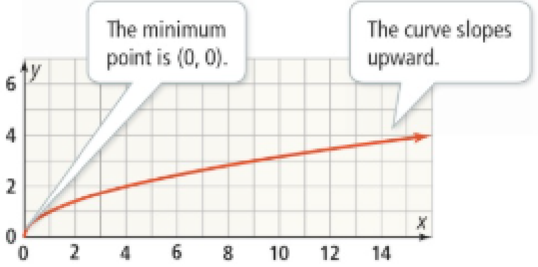
EXAMPLE 1 Key Features of the Square Root Function

What are the key features of $f(x) = \sqrt{x}$?

The function $f(x) = \sqrt{x}$ is the **square root function**.

Make a table and graph the function.

x	$f(x) = \sqrt{x}$
0	0
1	1
4	2
9	3
16	4



COMMON ERROR
Recall that $\sqrt{x}$ is equal to the *positive* square root of x.

The domain is restricted to $x \geq 0$, because only nonnegative numbers have a real square root. Since $\sqrt{x}$ indicates the nonnegative or principal square root, the range is $f(x) \geq 0$.

For $f(x) = \sqrt{x}$, the x- and y-intercepts of the graph of the function are both 0. The graph is increasing for all values in the domain of f.

Try It! 1. Graph each function. What are the intercepts, domain, and range of the function?

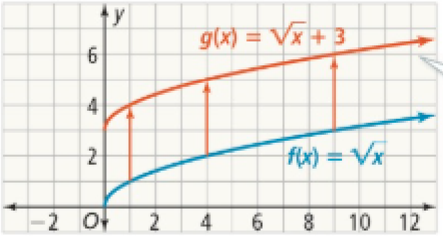
a. $p(x) = -\sqrt{x}$

b. $q(x) = \sqrt{\frac{x}{10}}$

EXAMPLE 2 Translations of the Square Root Function

A. How does the graph of $g(x) = \sqrt{x} + 3$ compare to the graph of $f(x) = \sqrt{x}$?

Graph each function.



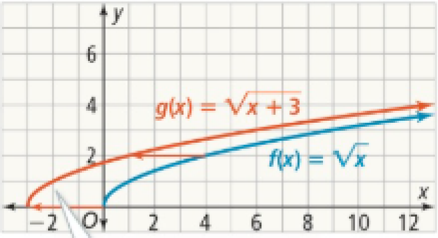
For each x-value, the corresponding y-value is 3 units greater for g than it is for f.

The graph of $g(x) = \sqrt{x} + 3$ is a vertical translation of $f(x) = \sqrt{x}$.

The translation is a result of adding a constant to the output of a function. The domain for both functions is $x \geq 0$. The range for function f is $y \geq 0$, so the range for function g is $y \geq 3$.

B. How does the graph of $g(x) = \sqrt{x + 3}$ compare to the graph of $f(x) = \sqrt{x}$?

Graph each function.



For each y-value, the corresponding x-value is 3 units less for g than it is for f.

The graph of $g(x) = \sqrt{x + 3}$ is a horizontal translation of $f(x) = \sqrt{x}$.

The translation is the result of adding a constant to the input of a function. The domain of f is $x \geq 0$, and the domain of g is $x \geq -3$. The range for both functions is $y \geq 0$.

USE PATTERNS AND STRUCTURE

Notice that the graph of $g(x) = \sqrt{x + 3}$ is a horizontal shift of $f(x) = \sqrt{x}$ left 3 units. Do the quadratic functions $g(x) = (x + 3)^2$ and $f(x) = x^2$ follow the same pattern?

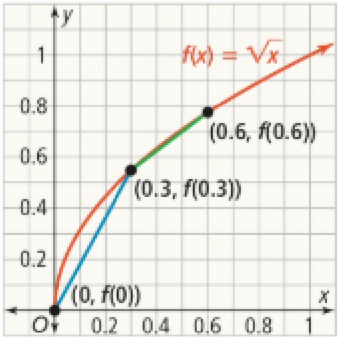
Try It! **2.** How does each graph compare to the graph of $f(x) = \sqrt{x}$?

a. $g(x) = \sqrt{x} - 4$

b. $p(x) = \sqrt{x - 10}$

EXAMPLE 3 Rate of Change of the Square Root Function

For the function $f(x) = \sqrt{x}$, how does the average rate of change from $x = 0$ to $x = 0.3$ compare to the average rate of change from $x = 0.3$ to $x = 0.6$?



Step 1 Evaluate the function for the x–values that correspond to the endpoints of each interval.

$$\begin{aligned} f(0) &= \sqrt{0} \\ &= 0 \\ f(0.3) &= \sqrt{0.3} \\ &\approx 0.548 \\ f(0.6) &= \sqrt{0.6} \\ &\approx 0.775 \end{aligned}$$

Step 2 Find the average rate of change over each interval.

From $x = 0$ to $x = 0.3$:

$$\frac{f(0.3) - f(0)}{0.3 - 0} \approx \frac{0.548 - 0}{0.3 - 0} = \frac{0.548}{0.3} \approx 1.83$$

CHECK FOR REASONABLENESS
You can see the difference in the average rates of change in the graph. The line through (0, f(0)) and (0.3, f(0.3)) is steeper than the line through (0.3, f(0.3)) and (0.6, f(0.6)).

From $x = 0.3$ to $x = 0.6$:

$$\frac{f(0.6) - f(0.3)}{0.6 - 0.3} \approx \frac{0.775 - 0.548}{0.6 - 0.3} = \frac{0.227}{0.3} \approx 0.757$$

The average rate of change over the interval $0 \le x \le 0.3$ is greater than the average rate of change over the interval $0.3 \le x \le 0.6$.

Try It! **3.** For the function $h(x) = \sqrt{2x}$, find $h(8)$, $h(10)$, and $h(12)$. Then find the average rate of change of the function over each interval.

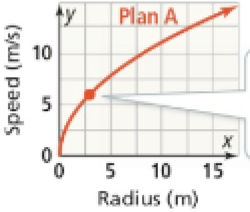
a. $8 \le x \le 10$

b. $10 \le x \le 12$

APPLICATION

EXAMPLE 4 Compare Functions

Two plans are being considered to determine the speed of a theme park ride with a circular wall that spins. Plan A is represented by the function with the graph shown. The ride shown in the photo is an example of Plan B. If the ride has a radius of 3 m, which plan would result in a greater speed for the ride?



When the radius is 3 m, the speed is about 6 m/s.

Plan B is represented by the function $f(r) = 5\sqrt{r}$, where r is the radius of the ride, and $f(r)$ is the speed.

CHOOSE EFFICIENT METHODS
How could you determine the radius in meters for both plans given a corresponding speed of 7.5 m/s?

Compare the plans.

Plan A

The graph of Plan A shows that the corresponding speed at a radius of 3 meters is about 6 m/s.

OR

Plan B

Evaluate $f(r) = 5\sqrt{r}$ for $r = 3$.

$$f(3) = 5\sqrt{(3)}$$
$$\approx 8.7$$

The ride using Plan B has a speed of about 8.7 m/s when the radius is 3 m.

With a radius of 3 m, the speed of the ride using Plan A is 6 m/s, and the speed of the ride using Plan B is about 8.7 m/s.

So, the ride using Plan B has a greater speed for a radius of 3 m.

Try It! 4. To the nearest thousandth, evaluate each function for the given value of the variable.

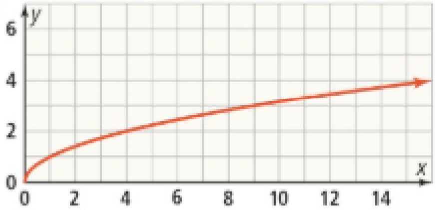
a. $v(x) = \frac{\sqrt{x}}{10}; x = 17$

b. $w(x) = \sqrt{\frac{x}{10}}; x = 17$

CONCEPT SUMMARY Key Features of the Square Root Function

ALGEBRA $f(x) = \sqrt{x}$

GRAPH



KEY FEATURES

- domain: $x \geq 0$
- range: $y \geq 0$
- x-intercept: 0
- y-intercept: 0
- increasing: $0 \leq x < \infty$
- decreasing: never
- positive: $0 < x < \infty$
- negative: never

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What key features are shared among the square root function and translations of the square root function?

2. **Use Patterns and Structure** Explain why each function is, or is not, a translation of the square root function $f(x) = \sqrt{x}$.

   a. $h(x) = 2\sqrt{x + 1}$

   b. $g(x) = \sqrt{x + 2} - 3$

3. **Error Analysis** A student identified (6, 12) and (9, 27) as points on the graph of the function $f(x) = \sqrt{3x}$. What error did the student make?

4. **Communicate and Justify** What is the domain of $f(x) = \sqrt{x + 3}$?

Do You KNOW HOW?

How does the graph of each function compare to the graph of $f(x) = \sqrt{x}$?

5. $g(x) = \sqrt{x} - 2$

6. $h(x) = \sqrt{x - 5}$

7. $p(x) = 5 + \sqrt{x}$

8. $q(x) = \sqrt{7 + x}$

For the given function, find the average rate of change to the nearest hundredth over the given interval.

9. $f(x) = \sqrt{x + 7}$; $2 \leq x \leq 10$

10. $g(x) = \sqrt{x + 7}$; $-3 \leq x \leq 5$

11. $h(x) = \sqrt{2x}$; $0 \leq x \leq 10$

PRACTICE & PROBLEM SOLVING

UNDERSTAND

12. **Represent and Connect** Use a graphing calculator to graph $f(x) = -\sqrt{x+7}$. Describe the domain and range of the function.

13. **Error Analysis** Describe and correct the error a student made when comparing the graph of $g(x) = \sqrt{x+3}$ to the graph of $f(x) = \sqrt{x}$.

> 1. The expression under the radical in $g(x)$ is $x + 3$.
> 2. $x + 3$ is to the right of x, so the graph of g is a translation of the graph of f by 3 units to the right.
>
> ✗

14. **Use Patterns and Structure** Write a function involving a square root expression that meets each requirement or set of requirements.
    - a. The domain is $x \geq 0$ and the range is $y \geq 2$.
    - b. The function is increasing over the interval $-9 \leq x < \infty$.
    - c. The domain is $x \geq 11$ and the range is $y \geq 0$.
    - d. The function is positive over the interval $4 < x < \infty$ and negative over the interval $0 \leq x < 4$.
    - e. The function is decreasing over the interval $0 \leq x < \infty$.

15. **Use Patterns and Structure** For a function of the form $f(x) = \sqrt{x-h} + k$, why are some real numbers excluded from the domain and the range?

16. **Communicate and Justify** Explain the steps of each calculation.
    - a. Find $f(10)$ if $f(x) = \frac{\sqrt{2x}}{7}$.
    - b. Find $f(10)$ if $f(x) = \sqrt{\frac{2x}{7}}$.

PRACTICE

Find the *x*- and *y*-intercepts of each function. If there is no intercept, write *Does not exist.* SEE EXAMPLE 1

17. $f(x) = \sqrt{x} - 2$

18. $g(x) = \sqrt{x-9}$

19. $h(x) = \sqrt{x+9}$

20. $k(x) = \sqrt{x} + 2$

How does each graph compare to the graph of $f(x) = \sqrt{x}$? SEE EXAMPLE 2

21. $q(x) = \sqrt{x} + 11$

22. $r(x) = \sqrt{x+11}$

23. $s(x) = \sqrt{x-2} + 5$

24. $t(x) = \sqrt{x+3} - 6$

Write an expression for each function. SEE EXAMPLE 2

25. a translation by 6 units up of $f(x) = \sqrt{x}$.

26. a translation by $\frac{1}{2}$ unit to the right of $f(x) = \sqrt{x}$.

Find the value of the given function at each end of the range of values of the variable. Then calculate the average rate of change of the function between the two values of the variable. SEE EXAMPLES 3 AND 4

27. $p(x) = \sqrt{15x}$; $0.01 \leq x \leq 1.01$

28. $q(x) = \sqrt{x+11}$; $-3 \leq x \leq 0$

29. $r(x) = \sqrt{2x-7}$; $5 \leq x \leq 10$

30. $t(x) = \sqrt{\frac{x-4}{2}}$; $4 \leq x \leq 8$

Describe the domain and range for each function. SEE EXAMPLE 1

31. Function *p* from Exercise 27

32. Function *q* from Exercise 28

33. Function *r* from Exercise 29

34. function *t* from Exercise 30

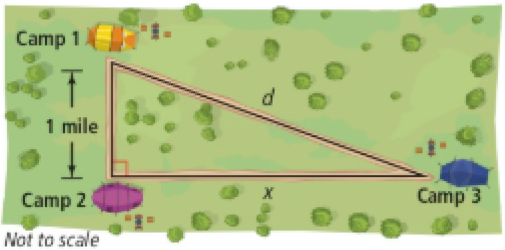
PRACTICE & PROBLEM SOLVING

APPLY

35. **Apply Math Models** A teacher adjusts the grades of an exam using a curve. If a student's raw score on a test is x, the score based on the curve is given by the function $c(x) = 10\sqrt{x}$.

Five students received raw scores of 49, 42, 55, and 72. What are their scores according to the curve?

36. **Analyze and Persevere** A group of campers leave Camp 2 and hike x miles along the path to Camp 3. The distance d between the group of campers and Camp 1 is given by $d(x) = \sqrt{x^2 + 1}$.



Not to scale

a. Use the function to find the distance d of the campers when $x = 1$, 10, 15, 18.5, 25, and 50.

b. When the campers have hiked 5 miles from Camp 2, their distance from Camp 1 is $\sqrt{5^2 + 1} = \sqrt{26} \approx 5.1$ miles. How much farther do they need to hike until they double their distance from Camp 1? Show your work.

37. **Represent and Connect** The distance to the horizon is a function of height above sea level. If the height h above sea level is measured in feet and the distance d to the horizon is measured in miles, then $d(h) \approx 1.22\sqrt{h}$.

On a hot-air balloon ride, a passenger looks out from 54 ft above sea level. What is the distance from the passenger to the horizon?

ASSESSMENT PRACTICE

38. Which of the following functions are vertical translations of $f(x) = \sqrt{x}$? Select all that apply.
F.1.1

Ⓐ $g(x) = \sqrt{x} - 4$

Ⓑ $h(x) = 3 + \sqrt{x}$

Ⓒ $k(x) = \sqrt{-5 + x}$

Ⓓ $m(x) = -\sqrt{x}$

Ⓔ $n(x) = -7 + \sqrt{x}$

39. **SAT/ACT** For the square root function $p(x) = \sqrt{x}$, the average rate of change between $x = 13$ and $x = a$ is 0.155. What is the value of a?

Ⓐ −4

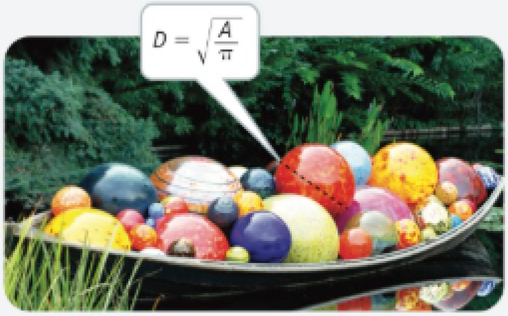
Ⓑ 0

Ⓒ 5

Ⓓ 8

Ⓔ 11

40. **Performance Task** The relationship between the surface area A and the diameter D of each glass sphere can be described using the equation shown.



$D = \sqrt{\frac{A}{\pi}}$

PART A Find the average rate of change in the diameter for surface areas between 20 in.^{2} and 10 in.^{2}.

PART B Find the average rate of change in the diameter when the surface area decreases from 16 in.^{2} to 14 in.^{2}.

PART C Find the average rate of change in D when A increases from 14.9 to 15.1 in.^{2}, and find the average rate of change in D when A increases from 14.99 to 15.01 in.^{2}.

PART D Describe a pattern in parts A, B, and C.

9-2

The Cubic and Cube Root Functions

I CAN… identify the key features of the cubic and cube root functions.

VOCABULARY

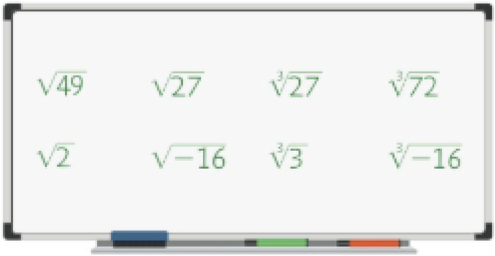
- cube root function
- cubic function

MA.912.F.1.1–Given an equation or graph that defines a function, determine the function type. Given an input-output table, determine a function type that could represent it. Also F.1.2

MA.K12.MTR.4.1, MTR.5.1, MTR.7.1

CRITIQUE & EXPLAIN

Emilia wrote several radical expressions on the whiteboard.



$\sqrt{49}$ $\sqrt{27}$ $\sqrt[3]{27}$ $\sqrt[3]{72}$

$\sqrt{2}$ $\sqrt{-16}$ $\sqrt[3]{3}$ $\sqrt[3]{-16}$

A. Evaluate each expression, and explain how to plot each value on a real number line.

B. Explain how evaluating a cube root function is different from evaluating a square root function.

C. **Communicate and Justify** Emilia states that it is not possible to plot either $\sqrt{-16}$ or $\sqrt[3]{-16}$ on the real number line. Do you agree? Explain.

ESSENTIAL QUESTION

What are the key features of the cubic and cube root functions?

EXAMPLE 1 Key Features of the Cubic Function

What are the key features of $f(x) = x^3$?

The function $f(x) = x^3$ is the **cubic function**.

Make a table of values.

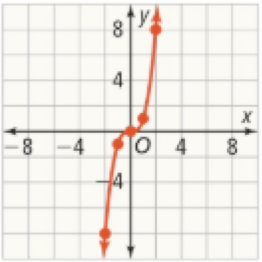
x	-2	-1	0	1	2
y	-8	-1	0	1	8

Since $f(0) = 0$, both the x- and y-intercepts are 0.

USE PATTERNS AND STRUCTURE

Does the graph of $f(x) = x^3$ have an axis of symmetry?

Graph the fuznction.



As the value of x increases so does its cube. The cubic function is always increasing.

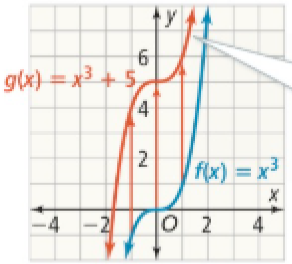
There are no restrictions on the x- or y-values, so the domain is $\{x \mid x \text{ is a real number}\}$ And the range is $\{y \mid y \text{ is a real number}\}$.

Try It! 1. Over what interval is $f(x) = x^3$ positive? Over what interval is it negative?

EXAMPLE 2 Translations of the Cubic Function

A. How does the graph of $g(x) = x^3 + 5$ compare to the graph of $f(x) = x^3$?

Graph each function.

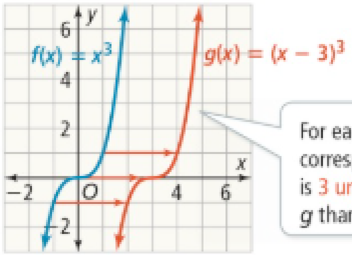


For each x-value, the corresponding y-value is 5 units greater for g than it is for f.

The graph of $g(x) = x^3 + 5$ is a vertical translation of $f(x) = x^3$.

The translation is a result of adding a constant to the output of a function. Because the translation is vertical, the interval over which the function g is increasing or decreasing is unchanged. It is increasing over its entire domain.

B. How does the graph of $g(x) = (x - 3)^3$ compare to the graph of $f(x) = x^3$?



For each y-value, the corresponding x-value is 3 units greater for g than it is for f.

The graph of $g(x) = (x - 3)^3$ is a horizontal translation of $f(x) = x^3$. The domain and range for both functions are all real numbers.

COMMON ERROR

You may think that because some key features of cubic functions are unchanged by translations that all key features are unchanged. Be sure to investigate before arriving at that conclusion.

Try It! 2. Over what interval is each function positive? Over what interval is it negative?

a. $g(x) = (x + 2)^3$

b. $h(x) = x^3 - 1$

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Key Features of the Cube Root Function

What are the key features of $f(x) = \sqrt[3]{x}$?

The function $f(x) = \sqrt[3]{x}$ is the **cube root function**.

Make a table of values.

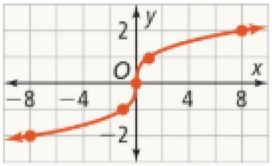
x	−8	−1	0	1	8
$f(x)$	−2	−1	0	1	2

$(-2)^3 = -8$, so $\sqrt[3]{-8} = -2$.

Since $x^3 = 0$ only when $x = 0$, the origin is the only point where the graph of $f(x) = \sqrt[3]{x}$ intercepts both the x- and y-axes.

USE PATTERNS AND STRUCTURE
How does the cube root affect the domain and range of the function?

Graph the function.



As the value of x increases, so does its cube root. So, the cube root function is always increasing.

There are no restrictions on the x- or y-values, so the domain and range are all real numbers.

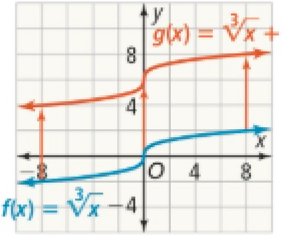
Try It! 3. Over what interval is $f(x) = \sqrt[3]{x}$ positive? Over what interval is it negative?

EXAMPLE 4 Translations of the Cube Root Function

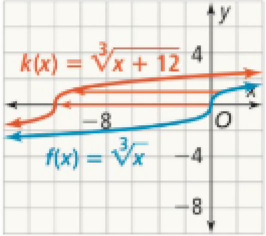
Powered By desmos

How do the graphs of $g(x) = \sqrt[3]{x} + 6$ and $k(x) = \sqrt[3]{x + 12}$ compare to the graph of $f(x) = \sqrt[3]{x}$?

Graph each function.



The graph of g is a translation of the graph of f 6 units up.



The graph of k is a translation of the graph of f 12 units left.

STUDY TIP
Recall that subtracting the negative value, −12, makes the operation appear as addition in the function $k(x) = \sqrt[3]{x + 12}$.

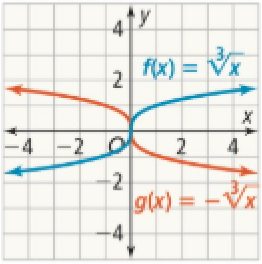
As with other functions, when you add a constant to the output of $f(x) = \sqrt[3]{x}$ the result is a vertical translation of the graph of f. When you subtract a constant from the input of the function f, the result is a horizontal translation of the graph of f.

Try It! 4. How do the graphs of $g(x) = \sqrt[3]{x} - 2$ and $p(x) = \sqrt[3]{x + 1}$ compare to the graph of $f(x) = \sqrt[3]{x}$?

EXAMPLE 5 Reflections of Functions

How do the key features of $g(x) = -\sqrt[3]{x}$ compare with the key features of $f(x) = \sqrt[3]{x}$?

Graph the functions.



The graph of g is the graph of f, reflected over the x-axis. Because the graph of f passes through the origin, so does the graph of g, so the x- and y-intercepts of g are unchanged. They are both 0.

Reflecting the graph over the x-axis does not affect the domain or range of this function. The domain and range for both functions are all real numbers.

LEARN TOGETHER
How can you share your ideas and communicate your thinking with others?

Try It! 5. Graph each pair of functions. Compare their intercepts and their domains and ranges.

a. $f(x) = x^3$ and $g(x) = -x^3$

b. $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$

APPLICATION

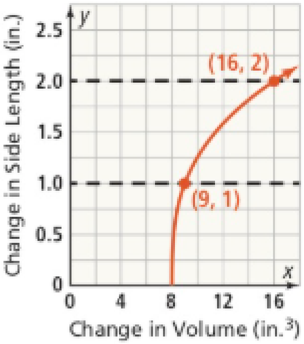
EXAMPLE 6 Model a Problem Using the Cube Root Function

Creative Clays is increasing the package size for its art clay. Designers are considering different sizes. Assume that the new package will be a cube with volume x in.^{3}. For what increases in volume would the side length increase between 1 in. and 2 in.?

Each orginal clay cube contains 8 in.^{3} of clay.

Since the volume of the new package is x in.^{3} and the volume of old package is 8 in.^{3}, the increase in volume is $x - 8$ in.^{3}. The change in side length of the cube is $f(x) = \sqrt[3]{x - 8}$.

Graph $f(x) = \sqrt[3]{x - 8}$.

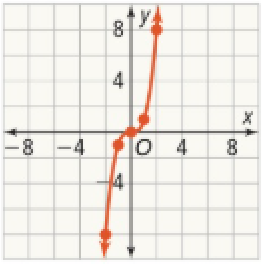
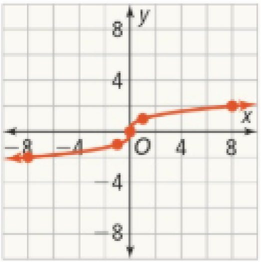


The graph shows that $f(9) = 1$ and $f(16) = 2$. So for increases in volume between 9 and 16 in.^{3} the side length would increase by 1 to 2 in.

APPLY MATH MODELS
Mathematics has many industrial and commercial applications, such as the development of product packaging. Modeling before production begins can prevent expensive mistakes.

Try It! 6. A cube has a volume of 10 cm^3. A larger cube has a volume of x cm^3. Consider the function $f(x) = \sqrt[3]{x - 10}$. What do the values $f(14)$ and $f(19)$ represent?

CONCEPT SUMMARY Cubic and Cube Root Functions

	The Cubic Function	The Cube Root Function
ALGEBRA	$f(x) = x^3$	$f(x) = \sqrt[3]{x}$
GRAPHS		

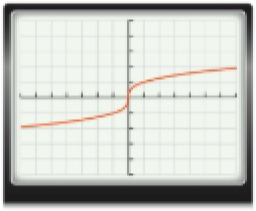
KEY FEATURES

- **x-intercept:** 0
- **y-intercept:** 0
- **increasing:** $-\infty < x < \infty$
- **positive:** $0 < x < \infty$
- **domain:** $\{x \mid x \text{ is a real number}\}$
- **range:** $\{y \mid y \text{ is a real number}\}$
- **decreasing:** never
- **negative:** $-\infty < x < 0$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are the key features of the cubic and cube root functions?

2. **Error Analysis** Timothy uses his calculator to investigate the domain and range of $f(x) = \sqrt[3]{x}$. He estimates the range as $-2 \leq y \leq 2$. What is the error that Timothy made?



3. **Use Patterns and Structure** Is the graph of $g(x) = -\sqrt[3]{x}$ increasing and decreasing over the same intervals as the graph of $f(x) = \sqrt[3]{x}$? Explain.

Do You KNOW HOW?

4. Identify the domain and range of $s(x) = \sqrt[3]{3x}$.

5. Describe how the graph of $g(x) = \sqrt[3]{x} - 3$ is related to the graph of $f(x) = \sqrt[3]{x}$.

6. Identify the domain and range of $g(x) = x^3 - 4$.

7. Identify the intercepts of $g(x) = (x - 4)^3$.

8. Calculate the average rate of change of $g(x) = \sqrt[3]{x} + 3$ for $4 \leq x \leq 7$.

9. Describe how the graph of $g(x) = \sqrt[3]{x - 4}$ is related to the graph of $f(x) = \sqrt[3]{x}$.

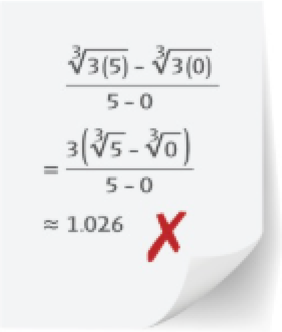
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Communicate and Justify** Explain why the x- and y-intercepts of $f(x) = \sqrt[3]{x}$ are the same.

11. **Use Patterns and Structure** Compare the average rates of change for $f(x) = \sqrt[3]{x}$ and $f(x) = \sqrt[3]{x} + 5$ for $0 \leq x \leq 4$.

12. **Error Analysis** Hugo calculated that the average rate of change of $f(x) = \sqrt[3]{3x}$ for $0 \leq x \leq 5$ is 1.026. Explain the error that Hugo made.



$$\frac{\sqrt[3]{3(5)} - \sqrt[3]{3(0)}}{5 - 0}$$

$$= \frac{3\left(\sqrt[3]{5} - \sqrt[3]{0}\right)}{5 - 0}$$

$$\approx 1.026 \quad ✗$$

13. **Higher Order Thinking** Copy and complete the tables shown below.

x	$f(x) = x^3$
−2	■
−1	■
0	■
1	■
2	■

x	$g(x) = \sqrt[3]{x}$
−8	■
−1	■
0	■
1	■
8	■

a. Describe what you notice about the x- and y-values for two functions.

b. Graph both functions on the same coordinate plane.

c. Could you use the graph of one function to create the graph of the other function? Explain.

14. **Use Patterns and Structure** For each condition, describe a translation of $f(x) = \sqrt[3]{x}$ that results in the graph of function g.

a. The y-intercept of the graph of g is −2.

b. The x-intercept of the graph of g is −1.

PRACTICE

For each function, identify the domain, range, and intercepts. SEE EXAMPLES 1 AND 2

15. $g(x) = x^3 + 1$

16. $g(x) = (x - 5)^3$

17. $g(x) = (x + 3)^3$

18. $g(x) = x^3 - 8$

For each function, describe the interval over which it is positive and the interval over which it is negative. SEE EXAMPLES 1 AND 2

19. $g(x) = x^3 + 8$

20. $g(x) = (x - 4)^3$

21. $g(x) = (x + 1)^3$

22. $g(x) = x^3 - 8$

For each function, identify domain, range, and intercepts. SEE EXAMPLES 3 AND 4

23. $f(x) = \sqrt[3]{x - 3}$

24. $f(x) = \sqrt[3]{2x}$

25. $f(x) = \sqrt[3]{x} - 1$

26. $f(x) = \sqrt[3]{x + 2}$

Describe translations that transform the graph of $f(x) = \sqrt[3]{x}$ into the graph of the given function. SEE EXAMPLE 4

27. $g(x) = \sqrt[3]{x - 3}$

28. $p(x) = \sqrt[3]{x} + 2$

29. $p(x) = \sqrt[3]{x} - 10$

30. $q(x) = \sqrt[3]{x + 7}$

For each pair of functions, describe the intervals on which the function is positive or negative. SEE EXAMPLE 5

31. $f(x) = x^3$ and $g(x) = -x^3$

32. $f(x) = \sqrt[3]{x}$ and $g(x) = -\sqrt[3]{x}$

33. $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$

34. $f(x) = x^3 + 1$ and $g(x) = -(x^3 + 1)$

Graph each function. Use the graph to estimate the values of x that satisfy each condition. SEE EXAMPLE 6

35. $f(x) = \sqrt[3]{x};\ 1 \leq f(x) \leq 2$

36. $g(x) = \sqrt[3]{x - 2};\ 1 \leq g(x) \leq 2$

37. $p(x) = \sqrt[3]{x - 1} + 3;\ 2 \leq p(x) \leq 5$

PRACTICE & PROBLEM SOLVING

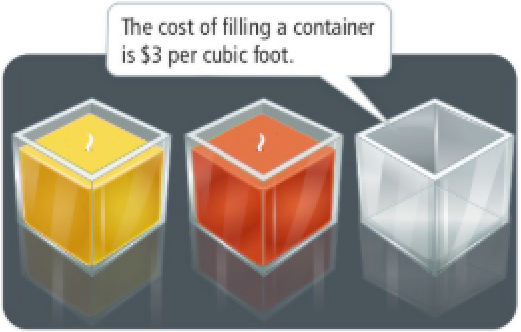
APPLY

38. **Represent and Connect** Weekly sales at Tamika's Auto Sales are shown in the table.

Week	1	2	3	4	5	6
Cars Sold	9	13	14	15	17	18

Plot the sales on a graph and write a cube root function that approximately models the sales. Explain what the features of the cube root function mean for the dealer's sales in the long run.

39. **Apply Math Models** Max Wax Company packages colored wax to make homemade candles in cube-shaped containers. The production line needs to plan sizes of the containers based on the associated costs. Write a cube root function that tells the side lengths of the container, x, in inches for a given cost, C.



40. **Mathematical Connections** A cube has the same volume as a box that is 4 ft 5 in. long, 3 ft 2 in. wide, and 4 ft 3 in. deep.

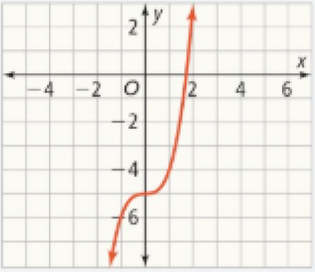
a. Write an expression that models the length of one side of the cube.

b. Find the side length of the cube.

c. Does the cube or the box have a greater surface area? How much greater?

ASSESSMENT PRACTICE

41. Which of the following best describes the graph? F.1.1



Ⓐ A horizontal translation of $f(x) = \sqrt[3]{x}$

Ⓑ A vertical translation of $f(x) = \sqrt[3]{x}$

Ⓒ A horizontal translation of $f(x) = x^3$

Ⓓ A vertical translation of $f(x) = \sqrt{x}$

Ⓔ A vertical translation of $f(x) = x^3$

42. **SAT/ACT** Which shows the average rate of change of $f(x) = \sqrt[3]{x} - 2$ over $1 \leq x \leq 4$?

Ⓐ 0.47

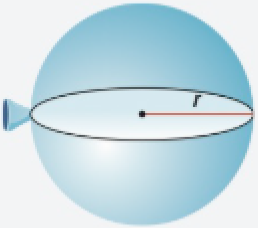
Ⓑ 0.20

Ⓒ −0.20

Ⓓ −0.47

Ⓔ −1.53

43. **Performance Task** Paul is filling spherical water balloons for an experiment. It is important that each balloon holds exactly the same volume of water, but Paul does not have a good instrument for measuring capacity.



Part A Write a cube root function that allows Paul to predict the radius associated with a given volume using $V = \frac{4}{3}\pi r^3$.

Part B Describe a reasonable domain and range.

Part C If each balloon should have a volume of 72 in.^{3}, what radius should the balloon have?

9-3

Analyzing Functions

I CAN… identify the function type when given an equation, graph, or table.

MA.912.F.1.1–Given an equation or graph that defines a function, determine the function type. Given an input-output table, determine a function type that could represent it. **Also AR.3.7, AR.4.3, AR.5.6, F.1.5, F.1.6, F.2.3**

MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

MODEL & DISCUSS

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Each table represents part of a function.

x	$f(x)$
−2	1
−1	4
0	5
1	4
2	1

x	$g(x)$
−2	20
−1	10
0	5
1	2.5
2	1.25

x	$h(x)$
−2	11
−1	8
0	5
1	2
2	−1

x	$j(x)$
−2	2
−1	1
0	0
1	1
2	2

x	$k(x)$
−2	21
−1	11
0	5
1	3
2	5

A. Plot the points of each function on a graph. Describe what you know about each function.

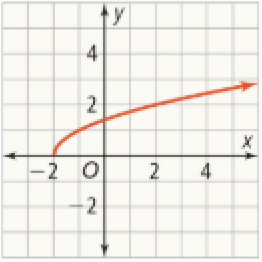
B. **Use Patterns and Structure** Which functions are related? Explain your reasoning.

ESSENTIAL QUESTION

What can you learn about a function by analyzing its graph?

EXAMPLE 1 Analyze Domain and Range

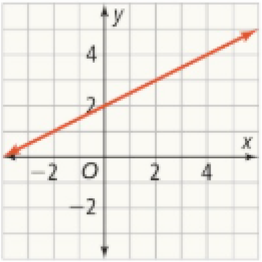
The graphs of three functions are shown. What are their domains and ranges?



$f(x) = \sqrt{x + 2}$

Domain: $x \geq -2$

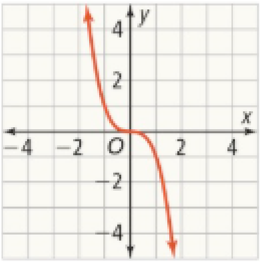
Range: $y \geq 0$



$g(x) = \frac{1}{2}x + 2$

Domain: $-\infty < x < \infty$

Range: $-\infty < y < \infty$



$h(x) = -x^3$

Domain: $-\infty < x < \infty$

Range: $-\infty < y < \infty$

For the square root function f, any values in the radicand must be nonnegative, so $x + 2 \nless 0$, meaning $x \geq -2$. The domain is restricted and the output of the function, the range, is always nonnegative.

The linear function g and the cubic function h have no restrictions on their domains or their ranges.

COMMON ERROR

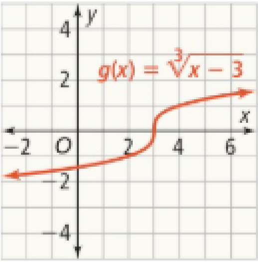
Remember to extend the function, beyond the edges of the sketch when the domain is all real numbers. The graph of g, for example, continues up and to the right out of view as x increases.

Try It! 1. Find the domain and range of each function. How are they the same? How are they different?

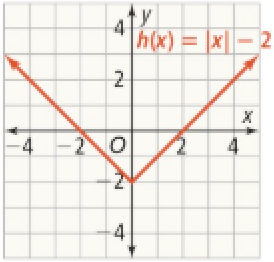
$f(x) = 8 - 3x$ $\qquad$ $g(x) = (x - 5)^2 - 7$

EXAMPLE 2 Analyze Key Features

The graphs of three functions are shown. Determine the intervals over which each function is increasing and decreasing.

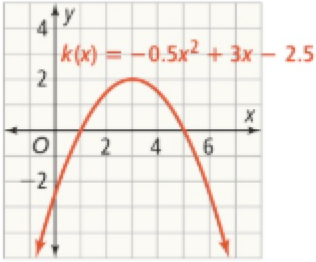


The cube root parent function $f(x) = \sqrt[3]{x}$ is increasing over its entire domain, all real numbers. The horizontal translation of the cube root function, $g(x) = \sqrt[3]{x - 3}$, is also always increasing.



GENERALIZE
How does a vertex on a graph relate to increasing and decreasing intervals?

The function $h(x) = | x | - 2$ is a vertical translation of the absolute value parent function. The graph of h has a vertex at (0, −2). It is increasing over the interval $0 < x < \infty$ and decreasing over the interval $-\infty < x < 0$.



The graph of function $k(x) = -0.5x^2 + 3x - 2.5$ is a parabola that opens downward. It has a vertex at (3, 2). Function k is increasing over the interval $-\infty < x < 3$ and decreasing over the interval $3 < x < \infty$.

Try It! **2. a.** For each function in Example 2, determine the intervals over which the function is positive and the intervals over which it is negative.

b. For each linear function, determine the intervals over which the function is positive and over which it is negative.

x	$g(x)$
−1	7
0	5
1	3
2	1
3	−1

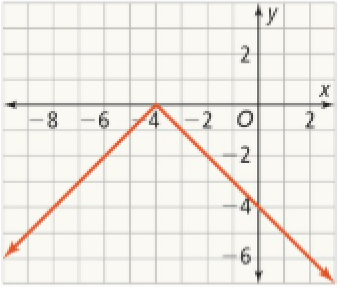
x	$h(x)$
−1	−5
0	−4
1	−3
2	−2
3	−1

EXAMPLE 3 Analyze End Behaviors of Graphs

What is the end behavior of each function?

End behavior describes what happens to the ends of the graph of a function as x approaches infinity or negative infinity (written as $x \to \infty$ and $x \to -\infty$).

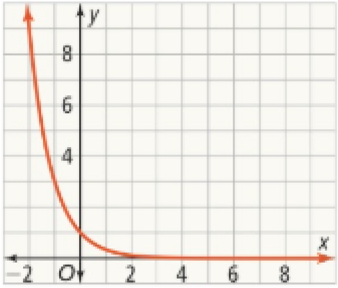
As $x \to \infty$, the values of $f(x)$ decrease without bound, or $f(x) \to -\infty$. The same is true as $x \to -\infty$.



$f(x) = -|x + 4|$

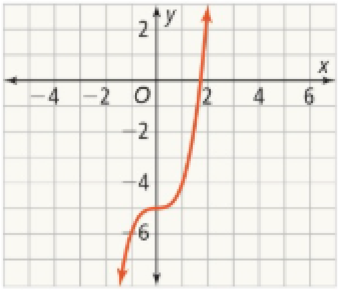
COMMUNICATE AND JUSTIFY
If the y-value of a function does not approach infinity as x approaches either negative or positive infinity what key feature might the graph include?

For this exponential function, there is a horizontal asymptote at $y = 0$. So as $x \to \infty$, the values of $g(x)$ approach 0. But as $x \to -\infty$, the values of $g(x)$ increase without bound, or $g(x) \to \infty$.



$g(x) = \left(\frac{1}{3}\right)^x$

As $x \to \infty$, the values of $h(x)$ grow more and more steeply, but they do not approach any asymptote, so $h(x) \to \infty$. As $x \to -\infty$, values of $h(x)$ decrease, and $h(x) \to -\infty$.



$h(x) = x^3 - 5$

Try It! **3. a.** Compare the end behaviors of the two linear functions.

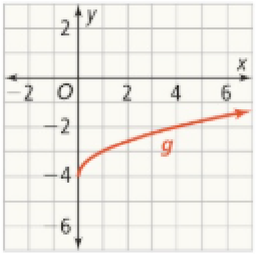
$f(x) = \frac{2}{17}x - 8956$ $\qquad$ $g(x) = 23x + 348$

b. Compare the end behaviors of the functions.

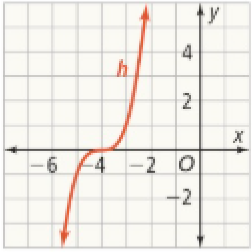
$f(x) = \frac{1}{3}x^2 + 4$ $\qquad$ $g(x) = \left(\frac{1}{3}\right)^x + 4$ $\qquad$ $h(x) = \frac{1}{3}x + 4$

EXAMPLE 4 Recognize Function Types

Given each graph, determine the function's type.

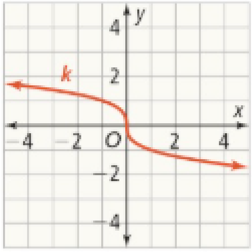


Function *g* has an endpoint at (0, −4). It is increasing throughout its domain, as $x \to \infty$. Function *g* is square root function.



Function *h* is always increasing and does not have an asymptote. It rises increasingly steeply as $x \to \infty$, and decreases steeply as $x \to -\infty$, Function *h* is a cubic function.

ANALYZE AND PERSEVERE
If a graph is a translation of a parent function why might looking at end behavior be more helpful than positive and negative intervals in determining the function type?



Function *k* is decreasing over its entire domain. It is positive over the interval $-\infty < x < 0$ and negative over the interval $0 < x < \infty$. This is the opposite of the key features of the parent cube root function. Function *k* is a cube root function.

Try It! 4. Determine a function type that could represent the values in each table.

a.

x	$f(x)$
0	13
1	17
2	21
3	25
4	29

b.

x	$g(x)$
0	7
1	14
2	28
3	56
4	112

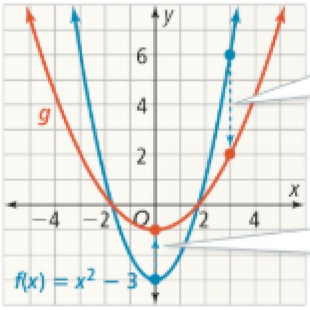
c.

x	$h(x)$
0	5
1	0
2	−3
3	−4
4	−3

EXAMPLE 5 Analyze Transformations

The graph of g is a transformation of function f. Determine the type of transformation and describe the amount of translation, stretch, or compression.

A.

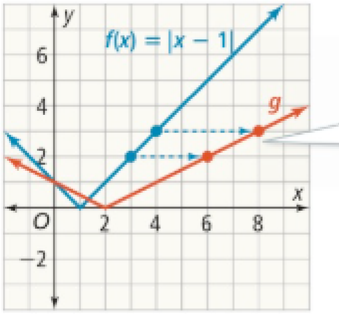


The point (3, 6) is on graph of f. The corresponding point on g is (3, 2). The y-value of g is $\frac{1}{3}$ the distance to the x-axis.

The point (0, −3) is on graph of f. The corresponding point on g is (0, −1). The y-value of g is $\frac{1}{3}$ the distance to the x-axis.

The graph g of is vertical compression of $f(x) = x^2 - 3$ by a factor of $\frac{1}{3}$. You can write g as $\frac{1}{3}f(x)$.

B.

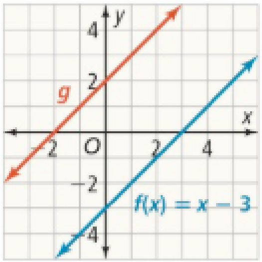


Each point on the graph of g is twice the distance from the y-axis as the corresponding points on the graph of f.

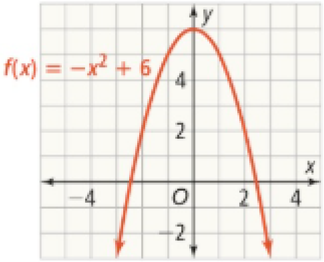
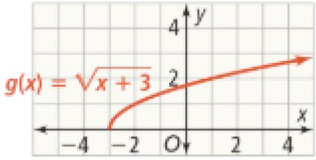
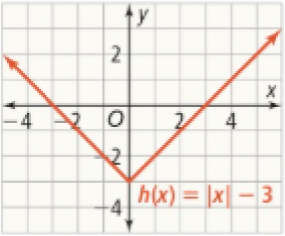
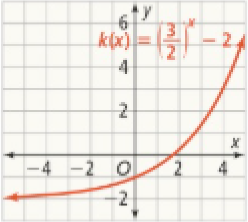
The graph g of is horizontal stretch of $f(x) = |x - 1|$ by a factor of 2. You can write g as $f\left(\frac{1}{2}x\right)$.

COMMON ERROR
You might think that multiplying the input of the function stretches the graph horizontally, but it compresses the graph. To represent a horizontal stretch by a factor of n you need to multiply the input by $\frac{1}{n}$.

Try It! 5. The graph of g is a transformation of function f. Determine the type of transformation and describe the amount of translation, stretch, or compression.



CONCEPT SUMMARY Common Features of Functions

	WORDS	GRAPHS
Domain and Range	The domain of f is the set of all real numbers. The range of f is the set of all real numbers less than or equal to 6.	
Increasing and Decreasing Intervals	g has no maximum value. The function is increasing over the interval $-3 \leq x < \infty$. It is never decreasing.	
Positive and Negative Intervals	The x-intercepts are −3 and 3. The function is positive for $\{x \mid x < -3 \text{ or } x > 3\}$ and negative for $\{x \mid -3 < x < 3\}$.	
End Behavior	End behavior describes what happens to the ends of the graph. As $x \to \infty$, $k(x) \to \infty$. As $x \to -\infty$, $k(x) \to -2$.	

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What can you learn about a function by analyzing its graph?

2. **Error Analysis** Kona states that the maximum value of $f(x) = -2^x$ is 0. Explain Kona's error.

3. **Use Patterns and Structure** How are behaviors of quadratic functions like those of the absolute value function?

Do You KNOW HOW?

For each function identify the domain and range, state the intervals over which is it increasing or decreasing, the intervals over which it is positive or negative, asymptotes if they exist, and describe the end behavior.

4. $f(x) = 10(0.5)^x + 2$

5. $g(x) = x^2 + 2x + 1$

6. $h(x) = \frac{2}{5}x - 6$

PRACTICE & PROBLEM SOLVING

UNDERSTAND

7. **Choose Efficient Methods** Without sketching the graph, how can you identify the domain and range of $f(x) = \sqrt{2x - 5}$?

8. **Communicate and Justify** Function f has a vertex. Can the function be increasing over its entire domain? Can it be decreasing over its entire domain? Explain.

9. **Error Analysis** Describe and correct the error a student made in describing the end behavior of the function $y = 1{,}000{,}000 - x^2$.

> Every number that I enter for x gives a great big value for y, so as $x \to \infty$, $y \to \infty$.
>
> ✗

10. **Communicate and Justify** The linear functions $f(x) = mx + b$ and $g(x) = px + q$ have the same end behavior. What relationships must exist, if any, between the values of m, b, p, and q?

11. **Error Analysis** If a function is increasing throughout its domain, the y-values are greater and greater as x approaches infinity. Libby claims that any function that has all real numbers as its domain and is increasing everywhere must have all real numbers as its range as well. Is Libby correct? Explain why or why not.

12. **Higher Order Thinking** Consider the function $f(x) = x^2$. For each of the following transformations describe which of these key features would change: domain, range, intercepts, increasing and decreasing intervals, and end behavior.

    a. The graph of g is a horizontal translation of f.

    b. The graph of h is a vertical translation of f.

    c. The graph of k is a reflection of f over the y-axis.

    d. The graph of p is a reflection of f over the x-axis.

PRACTICE

Sketch the graph of each function. Determine the domain and range, intercepts if they exist, the intervals over which the function is increasing or decreasing, and end behavior. SEE EXAMPLES 1, 2, AND 3

13. $g(x) = -x^2 + 4$

14. $g(x) = -x + 4$

15. $g(x) = -4^x$

16. $g(x) = x^2 + 4$

Determine each function's type. SEE EXAMPLE 4

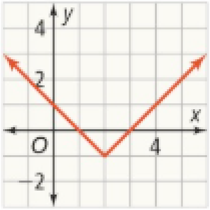
17. $g(x) = -3 + \sqrt{x}$

18. $g(x) = -\sqrt[3]{x}$

19. $g(x) = 2^x + 10$

20. $g(x) = |x - 7|$

Given each graph, determine the function's type.
SEE EXAMPLE 4

21.


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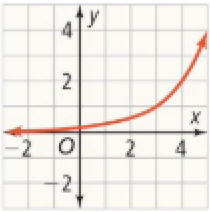

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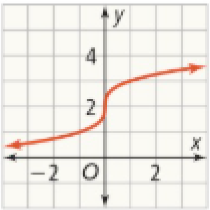

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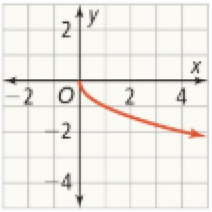
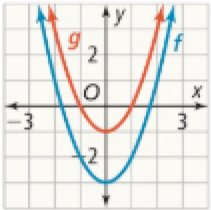


The graph of g is a transformation of the graph of f. Describe each transformation.
SEE EXAMPLE 5

25.


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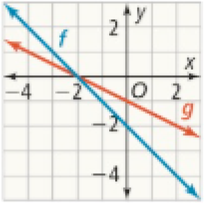

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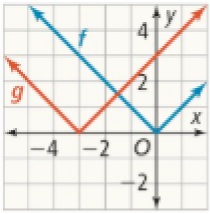

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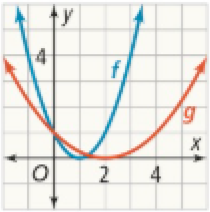
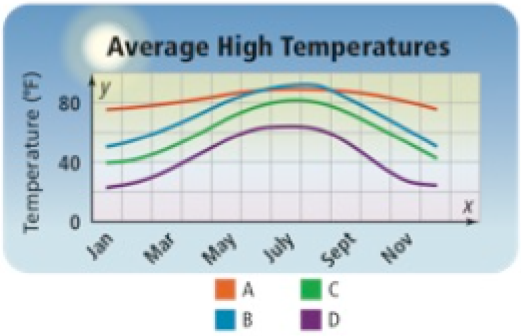


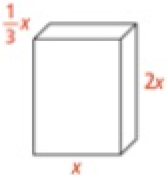
PRACTICE & PROBLEM SOLVING

APPLY

29. **Check for Reasonableness** The average high temperatures for four different cities, Anchorage, AK, Kansas City, MO, Miami, FL, and New York, NY, have been used to create the graph. Use information about maximum and minimum values to match A, B, C and D with the appropriate city. Explain your reasoning.



30. **Analyze and Persevere** A marketing company is designing a new package for a box of cereal. They have determined that the function $C(x) = 4.5x^2$ models the cost of a box with side lengths as shown (measured in inches). Identify a reasonable domain and range for the function.



31. **Apply Math Models** Yumiko is an animator. She uses computer-generated imagery (CGI) to create scenes for a movie. The shapes and features she uses are defined by functions. What types of functions might she use to create a scene such as the one below?

ASSESSMENT PRACTICE

32. Consider the functions $g(x) = x + 1$, $h(x) = x^2 + 1$, and $k(x) = 2^x$. Select all of the true statements. F.1.6

Ⓐ The functions have the same y-intercept.

Ⓑ The range of the functions is same.

Ⓒ The domain of the function is the same.

Ⓓ The functions are always increasing.

Ⓔ Two of the functions have no negative values.

33. **SAT/ACT** Which function has an axis of symmetry at $x = 1$ and a maximum value of 3?

Ⓐ $y = 1 - |x - 3|$

Ⓑ $y = |x - 1| + 3$

Ⓒ $y = |x + 1| - 3$

Ⓓ $y = 3 - |x - 1|$

Ⓔ $y = |x - 3| + 1$

34. **Performance Task** Jack started a small business recently, and he has been tracking his monthly profits, summarized in the table below.

Jan	$3
Feb	$10
Mar	$25
Apr	$40

Mar	$100
June	$180
July	$415
Aug	$795

Part A Create a graph to show Jack's profits over time. Determine the type of function that will best model Jack's profits based on data collected so far.

Part B Evaluate features of the function that will be relevant to Jack's business. Explain what those features mean in this context.

Part C Write an equation that models the growth of Jack's business. Use your function to predict Jack's profits for August of the following year. Is your prediction reasonable? Explain why or why not.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.F.1.1–Given an equation or graph that defines a function, determine the function type. Given an input-output table, determine a function type that could represent it. **Also F.1.2**

MA.K12.MTR.7.1

Edgy Tiles

For more than 3,000 years, people have glazed ceramics and other materials to make decorative tile patterns. Tiles used to be used only in important buildings or by the very rich, but now you can find tiles in almost any house.

Before you start tiling a wall, floor, or other surface, it's important to plan out how your design will look. Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

9-4

Operations With Functions

I CAN… perform operations on functions to answer real-world questions.

MA.912.F.3.1–Given a mathematical or real-world context, combine two functions, limited to linear and quadratic, using arithmetic operations. When appropriate, include domain restrictions for the new function.

MA.K12.MTR.1.1, MTR.5.1, MTR.7.1

MODEL & DISCUSS

In business, the term *profit* is used to describe the difference between the money the business earns (revenue) and the money the business spends (cost).

A. Grooming USA charges \$25 for every pet that is groomed. Let x represent the number of pets groomed in a month. Define a revenue function for the business.

B. Materials and labor for each pet groomed cost \$15. The business also has fixed costs of \$1,000 each month. Define a cost function for this business.

C. Last month, Grooming USA groomed 95 pets. Did they earn a profit? What would the profit be if the business groomed 110 pets in a month?

D. **Generalize** Explain your procedure for calculating the profit for Grooming USA. Suppose you wanted to calculate the profit for several different scenarios. How could you simplify your process?

ESSENTIAL QUESTION

How do you combine, multiply, and divide functions, and how do you find the domain of the resulting function?

EXAMPLE 1 Add and Subtract Functions

How do you define the sum, $f + g$, and the difference, $f - g$, of the functions $f(x) = 3x + 4$ and $g(x) = x^2 - 5x + 2$?

A. What is the sum of $f(x) = 3x + 4$ and $g(x) = x^2 - 5x + 2$?

To define the sum of two functions with known rules, add their rules.

$$(f + g)(x) = f(x) + g(x)$$

$$= (3x + 4) + (x^2 - 5x + 2)$$ Substitute the rule of each function.

$$= x^2 + (3x - 5x) + (4 + 2)$$ Group like terms.

$$= x^2 - 2x + 6$$ Combine like terms.

ANALYZE AND PERSEVERE
Defining a function includes describing its domain. The domain of $f \pm g$ is the intersection of the domains of f and g.

The domain of f is $\{x \mid x \text{ is a real number}\}$.

The domain of g is $\{x \mid x \text{ is a real number}\}$.

So the domain of $f + g$ is $\{x \mid x \text{ is a real number}\}$.

The sum of the two functions is $(f + g)(x) = x^2 - 2x + 6$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

B. What is the difference $f - g$ of $f(x) = 3x + 4$ and $g(x) = x^2 - 5x + 2$?

To define the difference of two functions with known rules, subtract their rules.

$$(f - g)(x) = f(x) - g(x)$$

$= (3x + 4) - (x^2 - 5x + 2)$ Substitute the rule of each function.

$= 3x + 4 - x^2 + 5x - 2$ Use the Distributive Property.

$= -x^2 + 8x + 2$ Combine like terms.

HAVE A GROWTH MINDSET
In what ways do you give your best effort and persist?

The domain of f is $\{x \mid x \text{ is a real number}\}$. The domain of g is $\{x \mid x \text{ is a real number}\}$. So the domain of $f - g$ is $\{x \mid x \text{ is a real number}\}$.

The difference of the two functions is $(f - g)(x) = -x^2 + 8x + 2$.

Try It!

1. Let $f(x) = 2x^2 + 7x - 1$ and $g(x) = 3 - 2x$. Identify rules for the following functions.

a. $f + g$ **b.** $f - g$

APPLICATION

EXAMPLE 2 Multiply Functions

The demand d, in units sold, for a company's new brand of cell phone at price x, in dollars, is $d(x) = 5{,}000 - 10x$. What is the company's expected revenue from cell phone sales in terms of the price, x?

The company's revenue will equal the price of its cell phones multiplied by the demand for its cell phones.

Revenue = price × demand

The demand is the function $d(x)$. The price is the function $p(x)$.

The product of two functions is the product of their rules: $(p \cdot d) = p(x) \cdot d(x)$.

USE PATTERNS AND STRUCTURE
You can use the Associative and Commutative Properties to add and multiply functions, since these operations are based on addition and multiplication of real numbers.

		Domains
Price:	$p(x) = x$	$p(x)$: $0 \le x$
Demand:	$d(x) = 5{,}000 - 10x$	$d(x)$: $x \le 500$
Revenue:	$R(x) = p(x) \cdot d(x)$	$R(x)$: $0 \le x \le 500$
	$= x(5{,}000 - 10x)$	
	$= 5{,}000x - 10x^2$	

Price cannot be negative.

Demand cannot be negative: $0 \le 5{,}000 - 10x$

Domain is the intersection of the domains of p and d.

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

The revenue the company will earn in terms of the cell phone price x is represented by $R(x) = 5{,}000x - 10x^2$.

Try It! **2.** Suppose demand, d, for a company's product at cost, x, is predicted by the function $d(x) = -0.25x^2 + 1{,}000$, and the price, p, that the company can charge for the product is given by $p(x) = x + 16$. Find the company's revenue function.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Divide Functions

A. How do you define the quotient of functions $f(x) = x^2 - 5x + 17$ and $g(x) = 4x - 3$?

To define the quotient of two functions, take the quotient of their rules: $h(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$h(x) = \frac{f(x)}{g(x)}$$

$$= \frac{x^2 - 5x + 17}{4x - 3}$$ Substitute the rule for each function.

If possible, factor the numerator and denominator. In this case both are prime polynomials and cannot be factored.

The quotient of $\frac{f}{g}$ is $h(x) = \frac{x^2 - 5x + 17}{4x - 3}$.

The domain of h is the set of all values for which f, g, and $\frac{f}{g}$ are defined, so g cannot be 0. The domain of h is $-\infty < x < \frac{3}{4}$ and $\frac{3}{4} < x < \infty$.

B. Define the quotient $\frac{f}{g}$ of $f(x) = x - 7$ and $g(x) = 2x^2 - 13x - 7$.

Take the quotient of their rules.

$$\frac{f(x)}{g(x)} = \frac{x - 7}{2x^2 - 13x - 7}$$ Substitute the rule of each function.

$$= \frac{x - 7}{(2x + 1)(x - 7)}$$ Factor the denominator.

$$= \frac{1}{2x + 1}$$ Simplify.

The quotient $\frac{f(x)}{g(x)} = \frac{1}{2x + 1}$. The domain is $\left\{x \mid x \neq 7 \text{ and } x \neq -\frac{1}{2}\right\}$.

COMMON ERROR

You may think that the domain of $\frac{f}{g}$ is the set of real numbers. However, both $x \neq 7$ and $x \neq -\frac{1}{2}$. Remember to identify the domain ***before*** simplifying the rational function.

Try It! **3.** Identify the rule and domain for $\frac{f}{g}$ for each pair of functions.

a. $f(x) = x^2 - 3x - 18$, $g(x) = x + 3$

b. $f(x) = x - 3$, $g(x) = x^2 - x - 6$

CONCEPT SUMMARY Function Operations

	Add or Subtract Functions	Multiply Functions	Divide Functions
ALGEBRA	$(f + g)(x) = f(x) + g(x)$ $(f - g)(x) = f(x) - g(x)$	$(f \cdot g)(x) = f(x) \cdot g(x)$	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
WORDS	The domain of the sum or difference of f and g is the intersection of the domain of f and the domain of g.	The domain is the set of all real numbers for which f and g and $f \cdot g$ are defined.	The domain is the set of all real numbers for which f and g and $\frac{f}{g}$ are defined.
NUMBERS	For $f(x) = 3x + 5$ and $g(x) = x - 3$, $f + g = (3x + 5) + (x - 3)$ $= 4x + 2$ $f - g = (3x + 5) - (x - 3)$ $= 2x + 8$ The domains of $f + g$ and of $f - g$ are both $\{x \mid x \text{ is a real number}\}$.	For $f(x) = 3x + 5$ and $g(x) = x - 3$, $f \cdot g = (3x + 5)(x - 3)$ $= 3x^2 - 4x - 15$ The domain of $f \cdot g$ is $\{x \mid x \text{ is a real number}\}$.	For $f(x) = 3x + 5$ and $g(x) = x - 3$, $\frac{f}{g} = \frac{3x+5}{x-3}$. The domain is $-\infty < x < 3$ and $3 < x < \infty$.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do you combine, multiply, and divide functions, and how do you find the domain of the resulting function?

2. **Use Patterns and Structure** What property is useful when subtracting a function that has multiple terms?

3. **Error Analysis** Reagan said the domain of $\frac{f}{g}$ when $f(x) = 5x^2$ and $g(x) = x + 3$ is the set of real numbers. Explain why Reagan is incorrect.

4. **Use Patterns and Structure** Explain why the order of the functions affects the result when subtracting or dividing two functions.

Do You KNOW HOW?

Let $f(x) = 3x^2 + 5x + 1$ and $g(x) = 2x - 1$.

5. Identify the rule for $f + g$.

6. Identify the rule for $f - g$.

7. Identify the rule for $g - f$.

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x - 4$.

8. Identify the rule for $f \cdot g$.

9. Identify the rule for $\frac{f}{g}$, and state the domain.

10. Identify the rule for $\frac{g}{f}$, and state the domain.

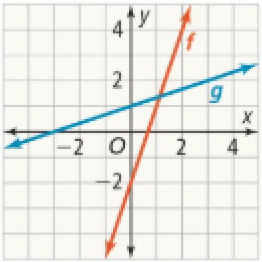
PRACTICE & PROBLEM SOLVING

UNDERSTAND

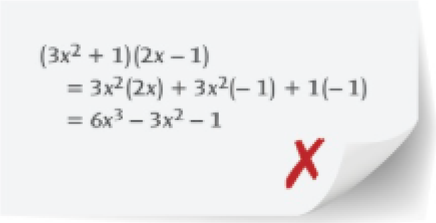
11. **Mathematical Connections** How is adding functions like adding polynomials? How is it different?

12. **Communicate and Justify** Explain why the domain for the quotient of functions might not be the set of all real numbers.

13. **Analyze and Persevere** Given the graphs of f and g, sketch the graphs of $f + g$ and $f \cdot g$.

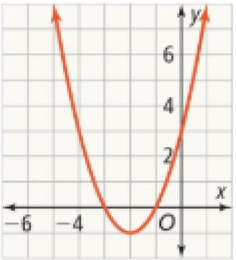


14. **Error Analysis** Describe and correct the error a student made in multiplying the two functions, $f(x) = 3x^2 + 1$ and $g(x) = 2x - 1$.



$$\begin{aligned}&(3x^2 + 1)(2x - 1)\\ &\quad = 3x^2(2x) + 3x^2(-1) + 1(-1)\\ &\quad = 6x^3 - 3x^2 - 1\end{aligned}$$

15. **Higher Order Thinking** What two functions could you multiply to create the function shown in the graph? How do the domain and range of each of the functions compare to the domain and range of the graphed function?



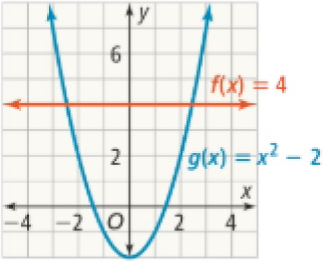
PRACTICE

Let $f(x) = 2x^2 + 5x - 1$ and $g(x) = 3x + 2$. Identify the rules for the following functions. SEE EXAMPLE 1

16. $f + g$

17. $f - g$

18. Given the graphs of f and g, graph $f + g$. Compare the domain and range of $f + g$ to the domains and ranges of f and g. SEE EXAMPLE 1



19. A florist charges \$10 for delivery plus an additional \$2 per mile from the flower shop. The florist pays the delivery driver \$0.50 per mile and \$5 for handling each delivery. If x is the number of miles a delivery location is from the flower shop, what expression models the amount of money the florist earns for each delivery? SEE EXAMPLE 2

20. Suppose the demand d, in units sold, for a company's jeans at price x, in dollars, is $d(x) = 600 - 4x$.

a. If revenue = price × demand, write the rule for the function $r(x)$, which represents the company's expected revenue in jean sales. Then state the domain of this function.

b. If the price is \$40, how much revenue will the company earn? SEE EXAMPLE 2

Identify the rule and domain for $\frac{f}{g}$. SEE EXAMPLE 3

21. $f(x) = x^2 + 3x - 28$ and $g(x) = x + 7$.

22. $f(x) = x - 3$ and $g(x) = x^2 + 2x - 15$.

23. $f(x) = 3x^2 - x + 11$ and $g(x) = 2x - 5$.

24. $f(x) = 2x^2 + 5x - 18$ and $g(x) = x - 2$.

PRACTICE & PROBLEM SOLVING

APPLY

25. **Analyze and Persevere** A laser tag center charges \$50 to set up a party, and \$75 per hour. The center pays its employees that work the party a total of \$36 per hour.

    a. Write a function f that represents the amount of revenue from a party that runs for x hours.

    b. Write a function g that represents the expenses for a party that runs for x hours.

    c. Write a combined function that represents the amount of profit the laser tag center makes on a party that runs x hours.

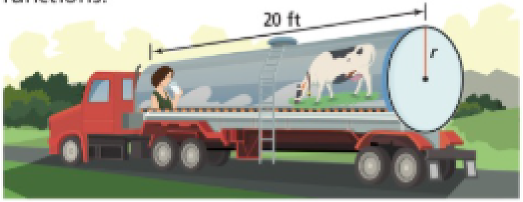
26. **Represent and Connect** A store is selling bumper stickers in support of a local sports team. The function $h(x) = -20x^2 + 80x + 240$ models the revenue, in dollars, the store expects to make by increasing the price of a bumper sticker x dollars over the original price of \$2. The store paid a total of \$200 for the bumper stickers.

    a. Write a function that represents the amount of money the store paid for the bumper stickers. What kind of function is it?

    b. What function models the store's profit from the bumper stickers?

    c. What is the price per bumper sticker when the store makes a profit of \$20?

27. **Apply Math Models** The surface of a cylindrical tank is being painted. The total surface area of a cylindrical tank is the sum of two area functions.



    a. Write a function that gives the total area of the two circular ends as a function of radius.

    b. Write a function that gives the lateral surface area of the cylinder as a function of radius.

    c. Combine the functions from parts (a) and (b) to get the total surface area of the cylinder as a function of radius.

ASSESSMENT PRACTICE

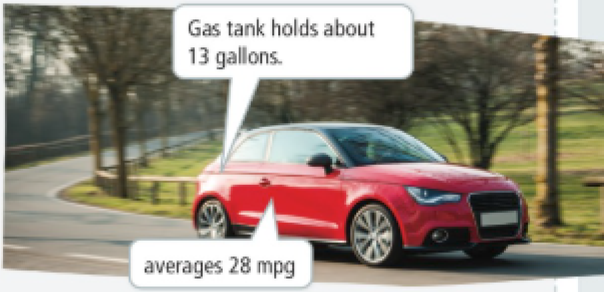
28. Given the functions $f(x) = x + 8$ and $g(x) = x^2 - 9$, which of the following are true statements about $f - g$? Select all that apply. F.3.1

    Ⓐ It is a linear function.
    Ⓑ It is a quadratic function.
    Ⓒ The domain is $-\infty < x < \infty$.
    Ⓓ The range is $-\infty < y < \infty$.
    Ⓔ The range is $y \geq 17$.

29. **SAT/ACT** The function h is the sum of the functions $f(x) = 3x + 5$ and $g(x) = 2x^2 - 6x - 2$. Which represents h?

    Ⓐ $h(x) = 5x^2 - x - 2$
    Ⓑ $h(x) = 2x^2 - 3x + 3$
    Ⓒ $h(x) = 2x^2 + 9x + 7$
    Ⓓ $h(x) = -3x + 3$

30. **Performance Task** A fuel-efficient car can travel 6 miles further per gallon than average while driving on the highway, and about 4 miles less than average while in the city.



Gas tank holds about 13 gallons.

averages 28 mpg

Part A Write two functions to determine the distance the driver could travel in the city or on the highway, using x gallons of gasoline.

Part B Assuming that the car has full tank of gas, what is the domain and range of each function?

Part C Suppose the driver does a combination of city and highway driving. Using the functions you found in Part A, write one function that could represent the distance traveled on x gallons of gasoline.

Part D Assume that the car has full tank of gas, what is the domain and range of the function you found in Part C?

TOPIC

9

Topic Review

TOPIC ESSENTIAL QUESTION

1. What are some operations on functions that you can use to create models and solve problems?

Vocabulary Review

Choose the correct term to complete each sentence.

2. The __________ is the function $f(x) = \sqrt[3]{x}$.

3. The __________ is the function $f(x) = x^3$.

4. The __________ is the function $f(x) = \sqrt{x}$.

- square root function
- cubic function
- cube root function

Concepts & Skills Review

LESSON 9-1 Square Root Functions

Quick Review

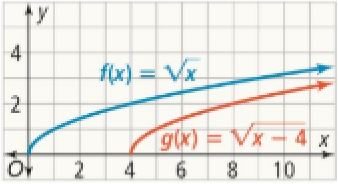
The **square root function** is $f(x) = \sqrt{x}$.

Example

How does the graph of $g(x) = \sqrt{x - 4}$ compare to the graph of $f(x) = \sqrt{x}$?

Graph each function.

For each *y*-value, the corresponding *x*-value is 4 units more for function *g* than it is for function *f*.



The graph of $g(x) = \sqrt{x - 4}$ is a horizontal translation of $f(x) = \sqrt{x}$ 4 units to the right. The domain for function *f* is $\{x \mid x \geq 0\}$, and the domain for function *g* is $\{x \mid x \geq 4\}$. The range for both functions is the same, $\{y \mid y \geq 0\}$.

Practice & Problem Solving

How does each graph compare to the graph of $f(x) = \sqrt{x}$?

5. $g(x) = \sqrt{x} + 4$

6. $g(x) = \sqrt{x - 8}$

7. $g(x) = \sqrt{x} - 5$

8. $g(x) = \sqrt{x + 2}$

Write an expression that represents each function.

9. $g(x)$, which is a translation 5 units down of $f(x) = \sqrt{x}$

10. $h(x)$, which is a translation 2 units left of $f(x) = \sqrt{x}$

11. **Choose Efficient Methods** Use a graphing calculator to graph $f(x) = -\sqrt{x - 3}$. Describe the domain and range of the function.

12. **Apply Math Models** The maximum speed of a sailboat is measured in knots and is estimated using the equation $s(\ell) = 1.34\sqrt{\ell}$, where ℓ is the length of the sailboat in feet. What is the approximate speed of a sailboat that has a length of 45 ft?

LESSON 9-2 Cubic and Cube Root Functions

Quick Review

The cubic function is $f(x) = x^3$.

The x- and y-intercepts are both 0. The function is increasing over its entire domain, which is all real numbers.

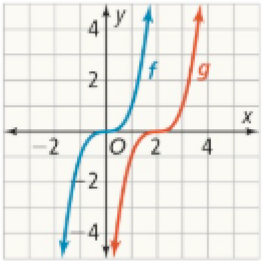
The **cube root function** is $f(x) = \sqrt[3]{x}$.

The x- and y-intercepts are both 0 and, like the cubic function the function is increasing over its entire domain.

Example

Compare the key features of $g(x) = (x - 2)^3$ with $f(x) = x^3$.

Graph both functions.



The graph of g is a horizontal translation of the graph of f, 2 units to the right.

The domain of f and g is $\{x \mid x \text{ is a real number}\}$. The range of each function is $\{y \mid y \text{ is a real number}\}$.

The x-intercept of g is 2, and its y-intercept is -8.

Both functions are increasing for all real numbers.

Function f is positive over the interval

$0 < x < \infty$ and negative over $-\infty < x < 0$. Function g is positive over the interval $2 < x < \infty$ and negative over the interval $-\infty < x < 2$.

Practice and Problem Solving

Describe translations that transform the graph of $f(x) = x^3$ into the graph of the given function.

13. $g(x) = x^3 - 5$

14. $p(x) = (x - 10)^3$

15. $h(x) = (x + 8)^3$

16. $m(x) = 10 + x^3$

Describe translations that transform the graph of $f(x) = \sqrt[3]{x}$ into the graph of the given function.

17 $g(x) = \sqrt[3]{x + 5}$

18. $h(x) = \sqrt[3]{x} + 4$

19. $j(x) = \sqrt[3]{x - 1}$

20. $p(x) = \sqrt[3]{x} - 2.5$

For each function, identify the intercepts and describe the intervals over which the function is positive and negative.

21. $g(x) = x^3 - 1$

22. $q(x) = (x - 5)^3$

23. $r(x) = \sqrt[3]{x - 1}$

24. $s(x) = \sqrt[3]{x} + 2$

25. **Represent and Connect** A fish store needs more cube-shaped fish tanks for its display shelves. Each fish requires 1 ft^3 of water, and each tank costs \$5 per ft^3. Write a cube root function that gives the side lengths of the container x, in inches for a given cost C.

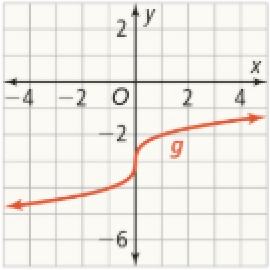
LESSON 9-3 Analyzing Functions

Quick Review

Different function types have characteristic key features such as domain, range, intercepts, intervals where the function is increasing, decreasing, positive or negative; end behavior and asymptotes. These key features can be used to help identify the function type given the graph of a function.

Example

Given the graph, determine the function type.



The domain of function g is $\{x \mid x \text{ is a real number}\}$, and its range is $\{y \mid y \text{ is a real number}\}$.

The function is increasing over its entire domain and does not have an asymptote.

Function g has no maximum, $g(x)$ increases less steeply as $x \to \infty$, but still approaches infinity. It decreases less steeply as $x \to -\infty$, but still approaches negative infinity.

Function h is a cube root function.

Practice & Problem Solving

Sketch the graph of each function and identify its domain and range.

26. $f(x) = x^2 + 6$

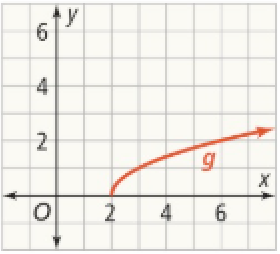
27. $g(x) = -4(2)^x$

28. $j(x) = -2(x - 5)^2 + 12$

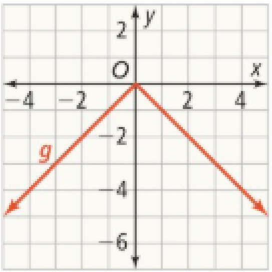
29. $k(x) = -2x + 12$

For each graph, identify the function type and determine the intervals over which the function is increasing or decreasing.

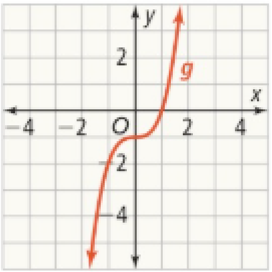
30.



31.



32.



33. **Use Patterns and Structure** Without sketching the graph, how can you identify the end behavior of $f(x) = x^2 - 5x + 8$?

34. **Apply Math Models** The height of a ball thrown from the top of a building is modeled by $h(t) = -16t^2 + 48t + 80$, where $h(t)$ is the height of the ball in feet after t seconds. The height of another ball hit by a bat on a small hill is modeled by $g(t) = -16t^2 + 96t + 20$. Give the maximum values and the axes of symmetry for both functions.

LESSON 9-4 Operations With Functions

Quick Review

You can add, subtract, multiply, or divide functions. When adding, subtracting, and multiplying functions, the domain is the intersection of the domains of the two functions. When dividing functions, the domain is the set of all real numbers for which both original functions and the new function are defined.

Example

What is the product of the two functions $f(x) = 2x + 1$ and $g(x) = 2x - 5$? Determine the domain and range of $f \bullet g$.

Use the Distributive Property when multiplying polynomials.

$$\begin{aligned} f(x) \bullet g(x) &= (2x + 1)(2x - 5) \\ &= 2x(2x) + 2x(-5) + 1(2x) + 1(-5) \\ &= 4x^2 - 10x + 2x - 5 \\ (f \bullet g)(x) &= 4x^2 - 8x - 5 \end{aligned}$$

The domain and range of the original functions are all real numbers. The domain of $f \bullet g$ is $\{x \mid x \text{ is a real number}\}$. You can find the average of the zeros $-\frac{1}{2}$ and $\frac{5}{2}$, to locate the vertex of the parabola $(1, -9)$, and determine the range of the quadratic function. The range of $f \bullet g$ is $\{y \mid y \geq -9\}$.

Practice & Problem Solving

Find $f + g$.

35. $f(x) = 3x^2 + 5x$
$g(x) = 2x - 8$

36. $f(x) = 3x^2 - 5x + 1$
$g(x) = x^2 - 8x - 3$

Find $f \bullet g$.

37. $f(x) = 5x^2 + 2x$
$g(x) = 3x - 1$

38. $f(x) = x^2 + 2x - 5$
$g(x) = x - 4$

Identify the rule and the domain for $\frac{f}{g}$.

39. $f(x) = x^2$
$g(x) = x^2 + 1$

40. $f(x) = x + 1$
$g(x) = 2x - 5$

41. $f(x) = x^2 + 5x - 14$
$g(x) = x + 7$

42. $f(x) = x - 3$
$g(x) = x^2 + 3x - 18$

43. **Analyze and Persevere** Write two functions that, when combined by multiplying, have a different range than at least one of the functions.

44. **Apply Math Models** A clothing company has determined that the revenue function for selling x thousands of hats is $R(x) = -5x^2 + 23x$. The cost function for producing those hats is $C(x) = 2x + 9$. Write a combined function that represents the profit for selling x thousands of hats, and determine the clothing company's profit from selling 3,000 hats.

TOPIC

10

Analyzing Data

TOPIC ESSENTIAL QUESTION

How do you use statistics to model situations and solve problems?

Topic Overview

enVision® STEM Project:
Take an Energy Survey

10-1 Representing Numerical Data
DP.1.1, DP.1.2, DP.1.4, MTR.2.1, MTR.3.1, MTR.5.1

10-2 Representing Categorical Data
DP.1.1, DP.1.2, MTR.2.1, MTR.4.1, MTR.7.1

10-3 Representing Bivariate Data
DP.1.1, DP.1.2, DP.3.1, MTR.2.1, MTR.3.1. MTR.4.1

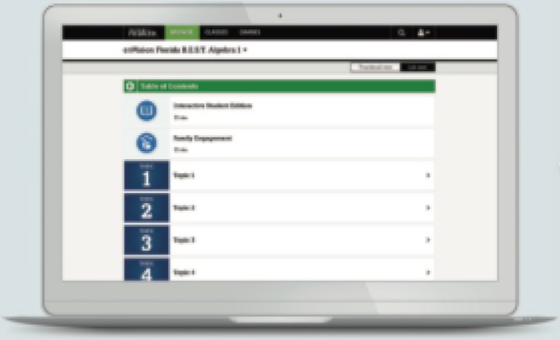
10-4 Analyzing Lines of Fit
DP.1.3, DP.2.4, DP.2.5, DP.2.6, MTR.1.1, MTR.5.1, MTR.6.1

10-5 Analyzing Two-Way Frequency Tables
DP.1.3, DP.3.1, DP.3.2, DP.3.3, MTR.2.1, MTR.4.1, MTR.5.1

Mathematical Modeling in 3 Acts:
Text Message
DP.1.1, DP.1.2, MTR.7.1

Topic Vocabulary

- bivariate categorical data
- causation
- conditional relative frequency
- extrapolation
- interpolation
- joint frequency
- joint relative frequency
- line of best fit
- linear regression
- marginal frequency
- marginal relative frequency
- negative correlation
- positive correlation
- residual



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Digital Experience

INTERACTIVE STUDENT EDITION
Access online or offline.

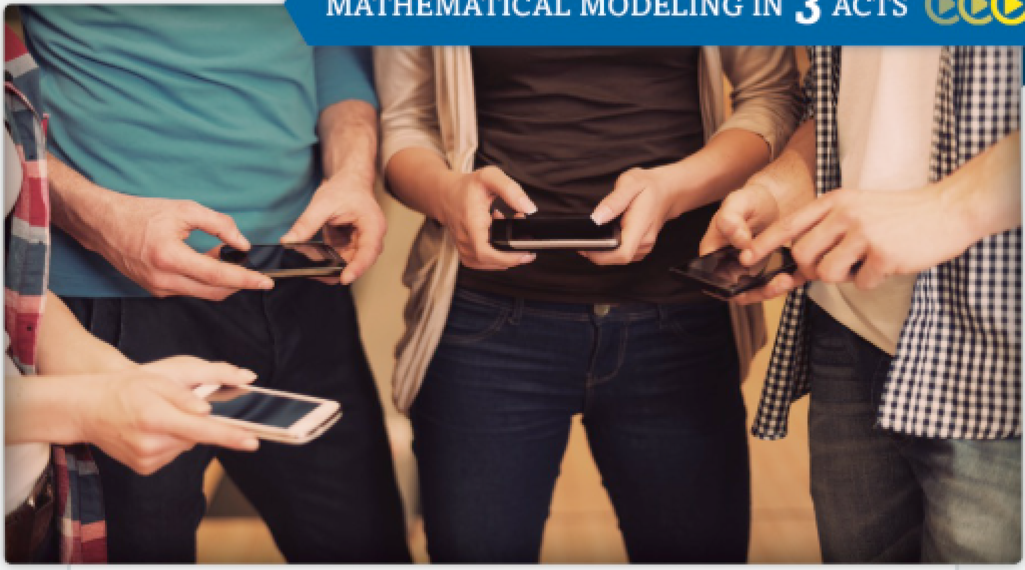
FAMILY ENGAGEMENT
Involve family in your learning.

ACTIVITIES Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its***.

ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Text Message

Text messages used to be just that: text only. Now you can send multimedia messages (or MMS) with emojis, images, audio, and videos. Did you know Finland was the first country to offer text messaging to phone customers?

Some people send and receive so many texts that they use textspeak to make typing faster. RU 1 of them? You will see one person keep track of his text messages in this Modeling Mathematics in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

MATH TOOLS Explore math with digital tools and manipulatives.

DESMOS Use Anytime and as embedded Interactives in Lesson content.

QR CODES Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

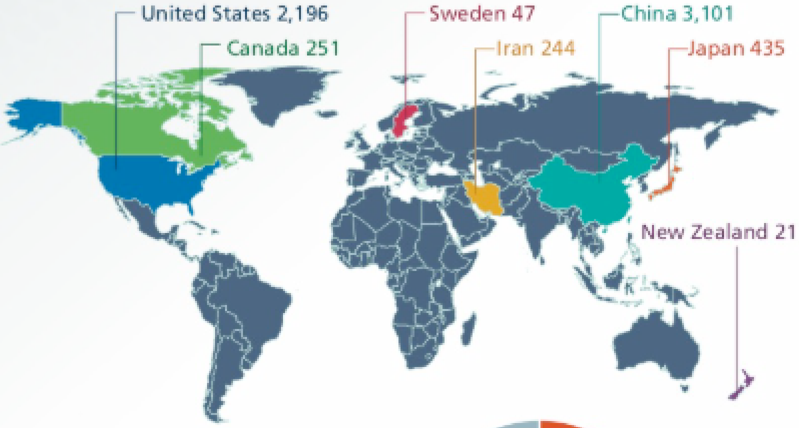
TOPIC 10 PROJECT enVision® STEM

Did You Know?

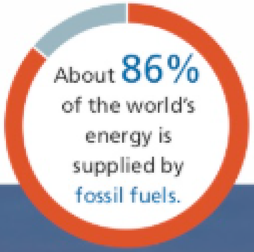
The average energy consumption in U.S. households is many times greater than countries in the rest of the world.

About 80% of American homes have a clothes dryer, which uses around **12% of the home's electricity** to dry about 300 loads of laundry each year.

Total Annual Energy Consumption for Select Countries (in million tons of oil equivalent)



In the United States, petroleum, natural gas, and coal have provided most of the energy for more than 100 years.



About 86% of the world's energy is supplied by fossil fuels.

In 2013, winds generated almost 3% of the world's electricity. World-wide, wind-generated power grows at a rate of about 17% per year.

Your Task: Take an Energy Survey

You and your classmates will develop a survey, and then gather and analyze data looking for ways to reduce energy consumption.

10-1

Representing Numerical Data

I CAN… select appropriate representations for numerical data.

VOCABULARY

- margin of error

MA.912.DP.1.1–Given a set of data, select an appropriate method to represent the data, depending on whether it is numerical or categorical data and on whether it is univariate or bivariate.
Also DP.1.2, DP.1.4

MA.K12.MTR.2.1, MTR.3.1, MTR.5.1

APPLICATION

MODEL & DISCUSS

A new shoe store is opening, and aims to target their products at teenagers. Survey your class to help determine the appropriate sizes of shoes to stock.

A. Explain why you chose to organize the data the way that you did.

B. How do you think the store could use these data?

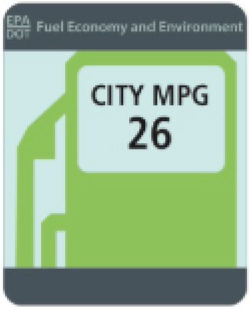
C. **Choose Efficient Methods** How would you display these data in a presentation?

ESSENTIAL QUESTION

What information about data sets can you get from different data displays?

EXAMPLE 1 Represent and Interpret Data in a Line Plot

Manuel plans to buy a new car with the gas mileage shown. To determine if the gas mileage of this car is good, he gathers data on the estimated city driving fuel efficiency, in miles per gallon, of several other cars. How does the fuel efficiency of the car he wants to buy compare to the fuel efficiency of the other cars he researched?

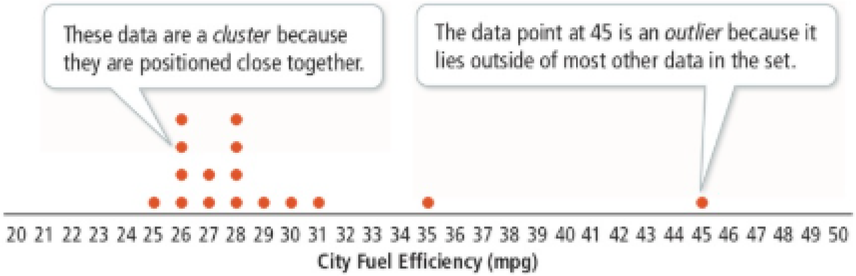


25	45	26	35	31	26	30	28
29	26	28	26	27	27	28	28

REPRESENT AND CONNECT
Why is a line plot a good way to show individual values? What kind of data do other types of displays show?

Manuel wants to compare individual values, so a line plot is a good way to show that information.

Create a line plot of the data by first drawing a number line that represents the range of the data. Plot each value from the table as a dot or mark above the number line.


These data are a *cluster* because they are positioned close together.

The data point at 45 is an *outlier* because it lies outside of most other data in the set.

Use the line plot to interpret the data.

The line plot shows that most of the values are clustered between 25 and 28.

The car Manuel plans to buy has about the same city fuel efficiency as comparable cars.

Try It! 1. A stem and leaf plot also displays individual data values. Why might a line plot better for displaying Manuel's data than a stem and leaf plot?

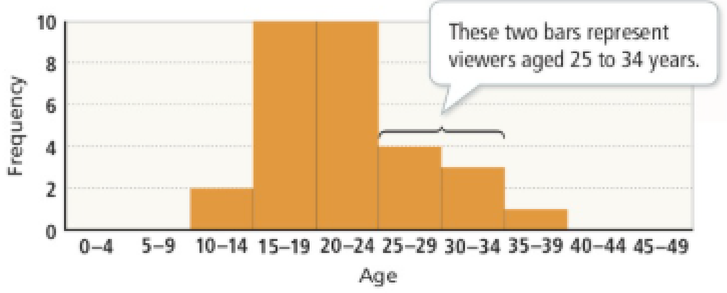
EXAMPLE 2 Represent and Interpret Data in a Histogram

A marketing team is about to launch a campaign for a new product that is targeted at adults aged 25–34 years. The team is researching the age range of viewers of a certain TV show to decide whether to advertise during the show. The data show the ages of a random sample of 30 viewers of the show. Based on the findings, should the marketing team launch their campaign during this particular show?

14	21	22	17	24	20
26	15	20	22	14	24
26	15	17	21	32	30
16	31	25	25	19	16
21	37	17	20	15	16

Histograms are often used to represent data over ranges of numbers.

To create a histogram, first decide on an appropriate interval for the data. Intervals of width 5 that include 25-29 and 30-34 will capture the targeted adults.



These two bars represent viewers aged 25 to 34 years.

COMMON ERROR

Be careful not to misinterpret the intervals. Each interval represents 5 possible values. Since the intervals start at 0, they are 0–4, 5–9, 10–14, and so on.

Use the histogram to interpret the data.

The histogram reveals among other things, that fewer than 25% of the viewers are between the ages of 25 and 34.

Based on this information, the marketing team should not launch their campaign during this particular show.

Try It! **2. a.** What age group would be a good match for products advertised on this TV show? Explain.

b. What additional information can be revealed by the histogram?

c. What is one possible drawback to presenting data using a histogram?

EXAMPLE 3 Represent and Interpret Data in a Box Plot

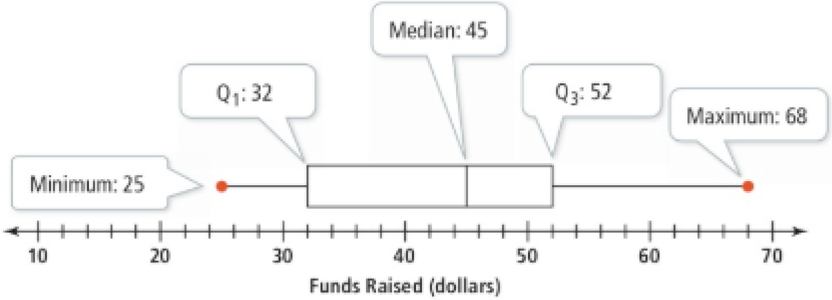
Students at a local high school organized a fundraiser for charity. Kaitlyn, the student council president, announces that more than half of the students raised over $50 each. The amounts of money raised by a random sample of 24 students is shown. Do the data support Kaitlyn's claim?

$59	$42	$25	$38	$45	$54	$68	$32
$26	$54	$50	$45	$42	$48	$50	$25
$45	$36	$55	$27	$31	$32	$49	$54

Kaitlyn's claim is about the distribution of values, so a box plot will reveal the information needed.

List the data from least to greatest to identify the mimimum, maximum, median, and the first and third quartiles. Create a box plot of the data.

VOCABULARY
Recall that the first quartile is the middle number in the distribution between the minimum value and the median. The third quartile is the middle number in the distribution between the median and the maximum value.



Use the box plot to interpret the data. The box plot shows that, based on the sample, the median amount of money collected was $45. Since half of this population collected $45 or less, the data do not support Kaitlyn's claim.

Try It! **3. a.** Suppose Kaitlyn wants to make the statement that 25% of the students raised over a certain amount. What is that amount? Explain.

b. Suppose Kaitlyn made a histogram of the fundraising data. Is it easier or harder to use this display to support her claim? Explain.

CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Choose a Data Display

Helena's dance team scores 68 points at a competition. The scores for all of the teams that competed are shown.

66	89	81	75
90	79	82	68
80	82	65	80
81	66	81	83

A. Helena wants to know in what place her team finished. Should Helena use a line plot, histogram, or box plot to display the data?

Compare the features of the three types of displays that could be used for this numerical data.

- A line plot has dots or marks for each value in the data. It shows clusters of data and outliers.
- A histogram groups values in a data set into ranges or intervals. Individual values are not displayed but trends are observable.
- A box plot shows center and spread of a data set. A box plot does not show individual data but summarizes the data using 5 key pieces of information.

Helena is interested in displaying individual scores. A line plot will display the data in a way that is most helpful to Helena.

CHOOSE EFFICIENT METHODS
What would be a good display choice if Helena wanted to compare her team's score with the median?

B. What place did Helena's team finish in the competition?

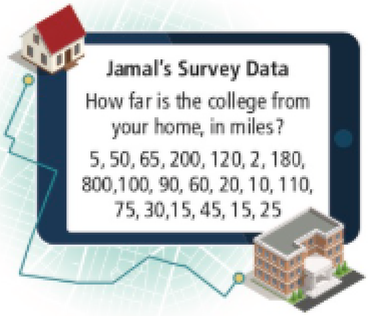
Create and analyze a line plot of the data.

A line plot reveals that Helena's team score was low compared to the other teams. Only three teams scored lower, so Helena's team placed 13th out of 16.

Try It! 4. Which data display should Helena use if she wants to know what percent of the teams scored higher than her team? Explain.

EXAMPLE 5 Estimate Population Data

Jamal is considering offering rides home for hire to college students on weekends and holidays. He would like to keep roundtrips from the college to fewer than 200 miles, so he surveys a random sample of students at nearby local colleges.



Jamal's Survey Data

How far is the college from your home, in miles?

5, 50, 65, 200, 120, 2, 180, 800,100, 90, 60, 20, 10, 110, 75, 30,15, 45, 15, 25

A. What is the estimated mean number of miles from home for local college students?

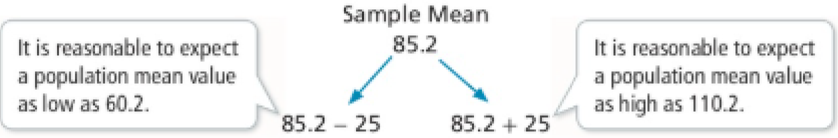
You can use the mean of the sample data to estimate the mean for the entire population.

$$\text{sample mean} = \frac{2017}{20} = 100.85 \text{ miles}$$

VOCABULARY
Recall that *sample mean* is mean of a sampled data set.

B. Jamal surveys 100 additional local college students and combines the data with the original survey results. The sample mean of the combined data is 85.2 miles with a margin of error of ±25. What does this tell Jamal about how far, on average, students are from home?

The **margin of error** is the maximum expected difference between the sample mean and the population mean. You can use the margin of error to estimate the actual population mean.



It is reasonable to expect a population mean value as low as 60.2.

Sample Mean
85.2

$85.2 - 25$ $\qquad$ $85.2 + 25$

It is reasonable to expect a population mean value as high as 110.2.

This tells Jamal that, on average, the local college student population lives between 60.2 and 110.2 miles from college.

C. According to school data, the combined sample represents 5% of the students enrolled at the local colleges. What is the estimated number of local college students?

Let x = the estimated number of local college students.

$$0.05x = 120$$
$$x = 2400$$

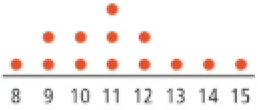
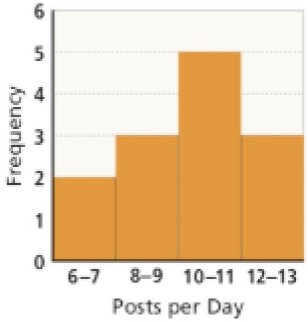
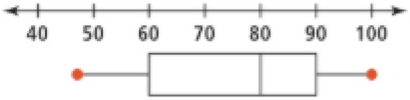
120 is 5% of the number of local college students.

D. Based on the survey results, would you recommend that Jamal offers a ride service to local college students? Explain.

Yes, there are 2400 students who, on average, live about 100 miles or less from college. This meets Jamal's criteria and offers a large number of potential customers.

Try It! **5.** Based on a survey of 150 of the 1,659 households in Wachula, Florida, a marketing firm determined that the average number of cell phones in a household is 2.27 with a margin of error of ±0.3. What is a reasonable estimate of the number of cell phones owned by residents in Wachula, Florida?

CONCEPT SUMMARY Univariate Numerical Data Displays

	Line Plots	Histograms	Box Plots
WORDS	Line plots display each data value from a set of data. They show clusters, gaps, and outliers in a data set.	Histograms do not show individual values, but show clearly the shape of the data. The data are organized into intervals. The bars show the frequency, or number of times, that the data within that interval occur.	Box plots show the center (median) and spread of a distribution. Box plots provide the following information about a data set: minimum, maximum, and median values, and the first and third quartile.
GRAPHS			

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What information about data sets can you get from different data displays?

2. **Communicate and Justify** How is a line plot different from a box plot? How are they similar?

3. **Choose Efficient Methods** If you want to see data values grouped in intervals, which data display should you choose? Explain.

4. **Error Analysis** Taylor says you can determine the mean of a data set from its box plot. Is Taylor correct? Explain your reasoning.

5. **Use Patterns and Structure** Can you determine the minimum and maximum values of a data set simply by looking at its line plot? Histogram? Box plot? Explain.

Do You KNOW HOW?

Use the data set shown for Exercises 6–11.

7	5	8	15	4
9	10	1	12	8
13	7	11	8	10

6. Make a line plot for the data. What information does the display reveal about the data set?

7. Make a histogram for the data. What information does the display reveal about the data set?

8. Make a box plot for the data. What information does the display reveal about the data set?

Identify the most appropriate data display to answer each question about the data set. Justify your response.

9. What is the median of the data set?

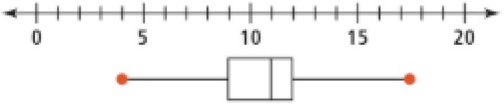
10. How many data values are greater than 7?

11. How many values fall in the interval 10 to 12?

PRACTICE & PROBLEM SOLVING

UNDERSTAND

12. **Use Patterns and Structure** A data set is represented by the box plot shown. Between which two values would the middle 50% of the data be found? Explain.

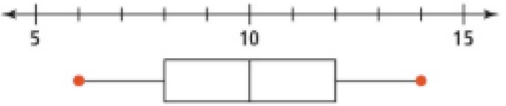


13. **Generalize** Write a scenario for which a line plot would be the best display for a data set. Explain your thinking.

14. **Error Analysis** Describe and correct the errors a student made in analyzing the histogram shown.

The median of the data set is 6.

15. **Higher Order Thinking** The box plot represents a data set with 12 values. The minimum, first quartile, median, third quartile, and maximum values are 6, 8, 10, 12, and 14, respectively.



a. Is it possible to create a line plot for the data set using just the box plot and the values given? Explain.

b. Is it possible to create a histogram using just the box plot and the values given? Explain.

PRACTICE

For each data set, create the data display that best reveals the answer to the question. Explain your reasoning. SEE EXAMPLES 1–4

16. What is the median value of the data set?

40	47	43	35	42	32	40	47
49	46	50	42	48	43	34	45

17. What is the frequency of the data value 83?

85	81	83	84	83	80
84	86	76	83	82	83
82	82	84	89	85	83

18. How many data values are between 7 and 9?

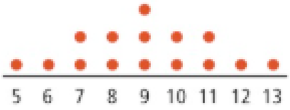
9.6	5.5	8.4	9.1	6.7
7.2	11.5	9.2	5.2	7.6
11.1	6.1	7.2	14.8	12.5
8.4	10.5	10.2	8.4	13.5

Choose whether a line plot, histogram, or box plot is the most appropriate data display to answer each question about a data set. Explain. SEE EXAMPLE 4

19. How many data values are greater than any given value in the data set?

20. What are the frequencies for each interval of 5 points?

Consider the data set represented by the line plot. Create a different data display that better reveals the answer to each question. SEE EXAMPLES 1–4



21. How many data values are in the interval between 8 and 10 inclusive?

22. What is the first quartile of the data set?

23. Based the on a survey of 120 of the 1,352 households in a local town, a marketing firm determined that the average number of computers in a household is 2.42 with a margin of error of ±0.4. What is a reasonable estimate of the number of computers owned by residents in the town? SEE EXAMPLE 5

PRACTICE & PROBLEM SOLVING

APPLY

24. **Apply Math Models** Isabel knits scarves and sells them online. The table shows the prices of the scarves she sold last month. At what prices were the middle 50% of the scarves sold? Create a data display that will reveal the answer.

Prices of Scarves ($)

35	32	60	80	36
90	45	76	96	92
100	120	60	38	75
36	36	100	100	100
95	58	100	85	40

25. **Analyze and Persevere** Lucy usually pays between $0.40 and $0.60 per ounce for her favorite shampoo. She gathers prices of the same shampoo at different stores near her home. Prices are shown in dollars in the table. Create a data display that allows Lucy to easily compare the price she is paying to the other prices. How does the price she is currently paying compare?

Shampoo Pricing Comparison

0.55	0.95	0.29	0.65	0.39
0.99	0.42	1.10	0.99	0.75
0.65	0.99	0.34	0.85	0.99
0.95	0.75	0.95	0.50	0.75

26. **Represent and Connect** Aaron scores 82 points at his karate tournament. He wants to compare his score to the others in the competition to see how many competitors scored higher than he did. The table shows all scores for the competition. What type of data display is appropriate to answer his question? Create the data display and analyze Aaron's performance.

Karate Scores

78	66	82	86	72	70
74	86	30	80	89	80
82	68	100	84	84	42
86	82	80	94	78	82

ASSESSMENT PRACTICE

27. Kwame recorded all of his math test scores and made a box plot of his data. Select all the features of the data set that his box plot shows.

D.P.1.1

☐ **A.** Median of the data set

☐ **B.** Individual values in the data set

☐ **C.** Outliers

☐ **D.** Minimum of the data set

☐ **E.** Maximum of the data set

28. **SAT/ACT** From which display(s) can the median of a data set be determined?

Ⓐ Line plot only

Ⓑ Box plot only

Ⓒ Line plot and box plot

Ⓓ Histogram and box plot

Ⓔ Line plot, histogram, and box plot

29. **Performance Task** A group of students use a stopwatch to record times for a 100-yard dash. Tell whether each student should choose a line plot, a histogram, or a box plot to display the data. Explain your reasoning. Then create the display.

12.5	13.5	14.1	12.8	13.4
14.0	11.5	14.2	13.9	14.4
13.3	14.5	13.2	13.6	12.0
14.5	13.5	14.4	14.1	13.9

Part A Neil wants a data display that clearly shows the shape of the data distribution.

Part B Yuki wants a display that shows the spread of data above and below the median.

Part C Thato wants a display that groups the data by intervals.

Part D Edwin wants a display that he could use to find the mean of the data set.

10-2

Representing Categorical Data

I CAN… identify and represent categorical data.

VOCABULARY

- relative frequency table

MA.912.DP.1.1–Given a set of data, select an appropriate method to represent the data, depending on whether it is numerical or categorical data and on whether it is univariate or bivariate.
Also DP.1.2
MA.K12.MTR.2.1, MTR.4.1, MTR.7.1

MODEL & DISCUSS

MARKET RESEARCHERS WANTED!

A clothing company is designing a new line of shirts. Look around your classroom and collect data about the color of top worn by each student. If a student's top has multiple colors, choose the most prevalent one.

A. Explain why you chose to organize the data the way that you did.

B. How do you think the company could use these data?

C. **Represent and Connect** How would you display these data?

ESSENTIAL QUESTION

How do you determine the best method to represent categorical data?

EXAMPLE 1 Represent Categorical Data

Venetta's class voted to determine who should be on the leadership committee. Each student voted for one candidate and the top 3 students would make up the committee. The votes are shown below.

Would a frequency table or a line plot be more helpful in representing who gets on the committee? Explain.

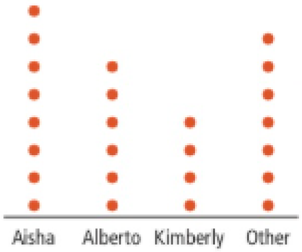
Aisha	Nadeem	Alberto	Aisha	Alberto
Kimberly	Alberto	Aisha	Alberto	Tavon
Alberto	Aisha	Kimberly	Venetta	Aisha
Tavon	Teo	Alberto	Aisha	Kimberly
Aisha	Yuson	Nadeem	Kimberly	Aisha

VOCABULARY
Recall that *categorical data* refers to data which can be divided into groups.

Because the data can be placed into categories (student) and there is one variable (number of votes), it is *univariate categorical data*. Vennetta could use either display to represent the data.

Name	Votes
Aisha	𝍸 III
Alberto	𝍸 I
Kimberly	IIII
Nadeem	II
Tavon	II
Teo	I
Venetta	I
Yuson	I

The frequency table is useful in organizing the data and helping to build other displays. However, it may show more details than are needed for the situation.



The needed information is which three got the most votes. The "Other" column shows that each vote was counted but simplifies the display.

The line plot is an appropriate way to show that Aisha, Alberto, and Kimberly received the votes to become members of the leadership committee.

CONTINUED ON THE NEXT PAGE

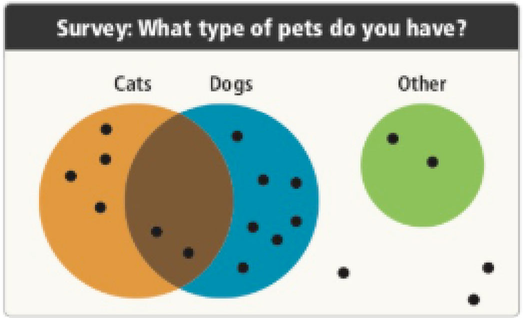
EXAMPLE 1 CONTINUED

Try It! 1. How is the line plot of the categorical data in Example 1 similar to a line plot of numerical data? How is it different?

EXAMPLE 2 Choose a Data Display

Ms. Lyons class completes the Venn diagram shown below to record the type of pets they have at home.

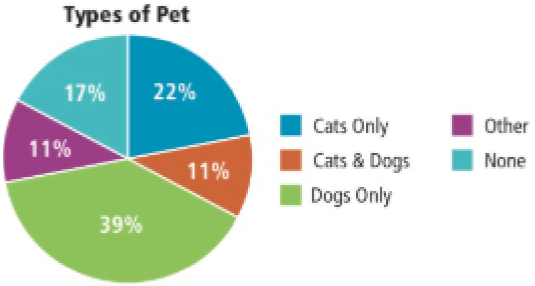
Which display would be better suited to highlight the portion of students who have a cat for a pet, a bar graph or a circle graph? Explain.



The categories are the *type of pets* and there is one measure (*number of students*), so the data is univariate categorical data.

Ms. Lyons could use either type of graph to represent univariate categorical data. However, the circle graph displays the data set as a whole so it is easier to see the portion or percentage of each category.

COMMON ERROR
Make sure that you do not count overlapping categories more than once; otherwise the total will be more 100%.



You can interpret the circle graph to see that 33% of the students have a cat for a pet.

Try It! 2. Ms. Lyons wants to create a display that will highlight how many more students have only dogs as compared with students who have only cats. Which display would be more appropriate, a bar graph or a circle graph? Explain.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Extrapolate Using Relative Frequencies

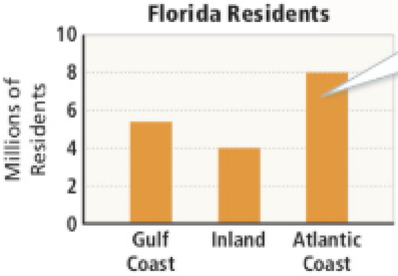
Mariana has the Florida population data by area from 2004, shown at the right. In 2019 the total population of Florida was 21.48 million. She wants to make a display that will help her extrapolate these regional values for 2019.

Florida Residents

Geographic Area	2004 Population (millions)
Gulf Coast	5.39
Inland	4.00
Atlantic Coast	7.99

A. Would a bar graph or a relative frequency table be more helpful to Mariana in making predictions? Explain.

The population data by area is univariate categorical data, so either display could represent the data. but they have different strengths.



A bar graph makes it easy to identify the maximum and minimum values. However, it shows the counts only.

To construct a relative frequency table, find the total of the data, then divide the data in each category by the total to find percent.

REPRESENT AND CONNECT
Why does it make sense that the total of the relative frequencies is 1?

Geographic Area	2004 Population (millions)	Relative Frequency
Gulf Coast	5.39	$\frac{5.39}{17.38} \approx 0.31$
Inland	4.00	$\frac{4}{17.38} \approx 0.23$
Atlantic Coast	7.99	$\frac{7.99}{17.38} \approx 0.46$
Total	17.38	1.00

The **relative frequency table** shows the relationship between the count of each category and the total.

The relative frequency table provides a profile of the distribution of data across the categories that can be extrapolated. It is a more appropriate display for Mariana's task.

B. What is a good estimate for how many people reside in inland Florida in 2019?

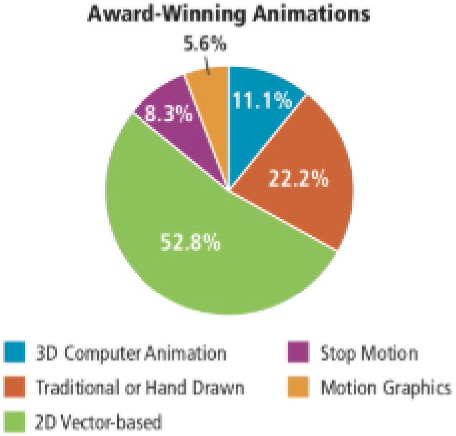
Assume the distribution of population is fairly stable over the years. Use the relative frequency from the modified bar chart to find an estimate for the number of inland residents in 2019.

There were likely about 4.94 million inland residents in Florida in 2019.

Try It! **3.** How would a bar graph constructed using relative frequencies be similar to the bar graph in Example 3? How would it differ?

EXAMPLE 4 Interpret Data Displays

A group of animators had 36 award-winning films over a five-year period. The graph shows the distribution of the main types of animation in the award-winning films.



A. What type of data is represented?

The categories are the types of animation. The counts are numbers of awards won in each format. The display represents univariate categorical data.

REPRESENT AND CONNECT
How can a circle graph help you see proportional relationships among the data?

B. What type of animated film won about half of all of the awards?

This circle graph does not show the counts directly. However because of the structure of the graph you can interpret the data display and see that the 2D Vector-based section accounts for more than half of the circle, meaning it accounts for about half of all the awards.

C. Which type of animated film won roughly as many awards as three other types combined?

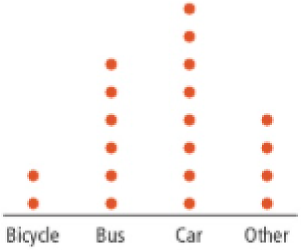
The Traditional or Hand Drawn animation films account for roughly one quarter of the circle, which is about the same portion of awards as the remaining three categories.

Try It! **4.** How could you determine the actual counts in the data set?

CONCEPT SUMMARY Categorical Data

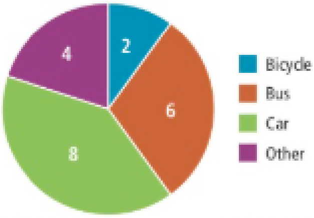
Categorical data is data organized by categories.

LINE PLOTS



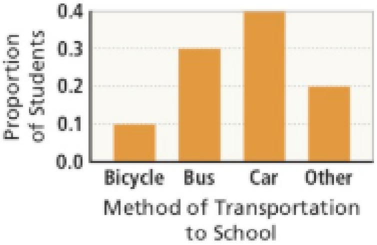
Line plots provide a simple visual display that makes it easy to identify the category with greatest number of data points. Unlike line plots the categories can be placed in any order and categories can be combined.

CIRCLE GRAPHS



Circle graphs can make is easier to compare the frequency of data within each category with total data set.

BAR GRAPHS



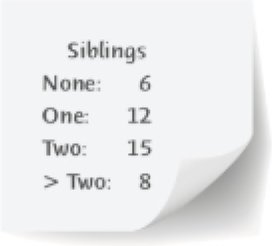
You can use bar graphs to display frequencies or relative frequencies. Relative frequencies can be used to make predictions about similar data sets.

Do You Understand?

1. **ESSENTIAL QUESTION** How can you display categorical data?

2. **Vocabulary** How is categorical different from numerical data?

3. **Error Analysis** Terry records the number of siblings his classmates have. He says the data are numerical because he recorded numbers. Explain his error.



Siblings

None: 6

One: 12

Two: 15

> Two: 8

Do You KNOW HOW?

Which type(s) of display (frequency table, line plot, circle graph, or bar graph) would be appropriate to highlight the following? Explain.

4. the minimum value in a data set

5. the percent of the total data set that a certain category represents

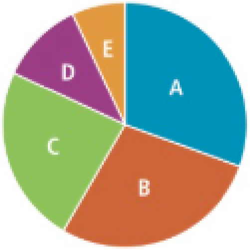
6. the most popular category in a data set

7. relative frequencies used to make predictions about similar data sets

PRACTICE & PROBLEM SOLVING

UNDERSTAND

8. **Represent and Connect** A data set is represented by the circle graph shown.

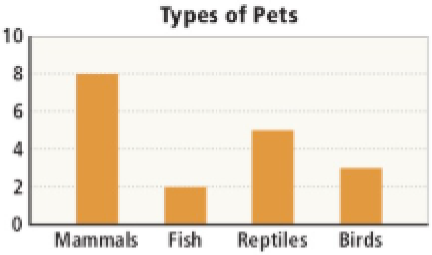


   a. What does the size of each section tell you about that portion of the data?

   b. What would be a better data display for comparing Categories A and B?

9. **Generalize** Write a scenario for which a line plot could be used to represent a data set, but a histogram could not. Explain your thinking.

10. **Error Analysis** Using the bar graph, a student says that dogs and birds are their favorite pets. Describe and correct the errors she made in analyzing the data shown.



11. **Higher Order Thinking** Think about the similarities and differences between these similar-looking data displays. Is it possible to create a categorical line plot from a numerical line plot? Explain.

12. **Communicate and Justify** Given categorical univariarate data displayed in a circle graph is it possible to determine the following? Explain.

   a. Mean

   b. Mode

   c. Total

   d. Range

PRACTICE

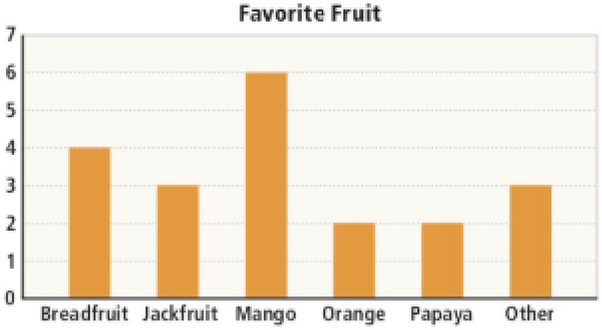
For situation, determine which representation is more appropriate to represent the data. Explain your answer. Tell whether the data is categorical or numerical. SEE EXAMPLES 1 AND 2

13. An advertising agency surveys a group of 50 people to determine which of 8 new car colors they prefer: *histogram* or *bar graph*

14. Alyssa surveys her freshman Math class to determine their favorite TV show. She wants to predict how many students in all 8 Math classes like the same show: *line plot* or *frequency table*

15. Tiana records the weather in her town each day for a month. She describes it as sunny, cloudy, rainy, or mixed. Tiana wants to show that there are about twice as many sunny days as rainy days: *circle graph* or *line plot*

Students in one freshman science class made a graph showing the results of a survey where they were asked to state their favorite fruit. Use the display to answer the following questions.
SEE EXAMPLES 1, 2, 3, AND 4



16. What type of data is represented in the display? Explain.

17. How many students preferred either papayas or breadfruit?

18. Does the display show all of the students' favorite fruits? Explain.

19. There are a total of 72 students in all of the Freshman science classes. If their preferences are consistent with those in the class shown above roughly how many students will say that mango is their favorite?

PRACTICE & PROBLEM SOLVING

APPLY

20. Apply Math Models Francisco is working with a local community group to form a steelpan band. He makes a list of the musicians they would like to have in the band.

Musicians Wanted

Steelpan Players
Double Second: 3
Double Tenor: 3
Tenor: 6
Guitar: 2
Triple Cello: 2
Tenor Bass: 2
Six Bass: 2

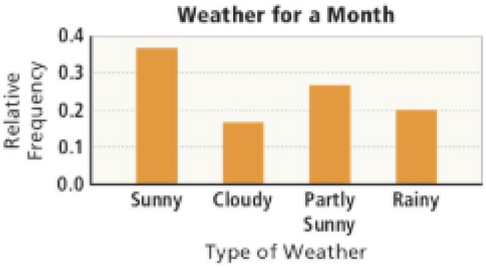
Others
Drum Set: 1
Engine Room (percussion): 2

a. Which display could Francisco use to highlight the greatest need in the band, a line plot or a circle graph? Explain.

b. Francisco wants to represent the instrument players he needs in a display. He decides to focus on pan players, so what categories might he aggregate or combine?

c. The frontline section is closest to the audience. They are comprised of the tenor, double tenor and double second pans. Make a display that shows the distribution of the frontline players based on the instruments they play.

21. Apply Math Models Samantha tracks the weather in her town for one month and makes the following relative frequency bar graph. If the weather patterns stay the same, roughly how many days would Samantha expect to see at least some sun over the next two months? Explain.



ASSESSMENT PRACTICE

22. Univariate categorical data represents _______ variable(s). DP.1.2

23. SAT/ACT Which display would not be appropriate to use to represent categorical univariate data?

Ⓐ box plot

Ⓑ bar chart

Ⓒ frequency table

Ⓓ line plot

24. Performance Task Xavier traveled to the coast with some friends and went birdwatching for an hour. He recorded the number and types of birds he saw.

Bird	Count
Great Blue Heron	13
Ibis	9
Snowy Egret	15
Oystercatcher	3
Pelican	24
Wilet	16

Part A What kind of data did Xavier collect? Explain.

Part B What type of display would best highlight the percentage of birds that Xavier spotted that were snowy egrets? Explain your answer and construct the display.

Part C Xavier's friend spotted 120 birds nearby. If the distribution of species is similar, how many of the birds might be great blue herons?

10-3

Representing Bivariate Data

I CAN… identify, represent, and interpret bivariate data.

VOCABULARY

- bivariate categorical data

MA.912.DP.1.2–Interpret data distributions represented in various ways. State whether the data is numerical or categorical, whether it is univariate or bivariate and interpret the different components and quantities in the display.
Also DP.1.1, DP.3.1
MA.K12.MTR.2.1, MTR.3.1, MTR.4.1

CONCEPTUAL UNDERSTANDING

MODEL & DISCUSS

In Miami-Dade County, the percent of Hispanic households has changed over time, sourced from the U.S Census.

Decade	Percent
1960	5.3
1970	23.6
1980	35.7
1990	49.2
2000	57.3
2010	64
2020	71

A. Can the data be displayed on a number line? Why or why not?

B. Does the table show numerical or categorical data?

C. **Represent and Connect** How would you represent the data?

ESSENTIAL QUESTION What representations may be used to display bivariate data?

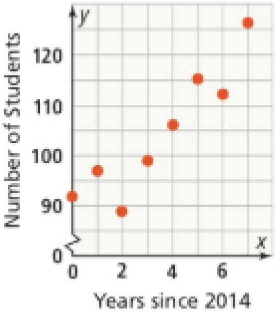
EXAMPLE 1 Identify Bivariate Data

Enrollment at the Horizons Vocational School has changed over the last eight years. How can you describe and display the data?

The coordinates of each data point are numbers, but the points themselves cannot be placed along a single number line. The data is formed of two sets of numerical data, so it is called bivariate numerical data.

Bivariate numerical data can be displayed on a coordinate plane, which is formed by two perpendicular number lines. Each data point has two coordinates, like an ordered pair, which correspond to the two numerical values.

Year	Students
2014	92
2015	97
2016	89
2017	99
2018	106
2019	115
2021	112
2021	126



Not every scatterplot will represent a function, but it is important that there is a relationship between the two axis of the graph.

LEARN TOGETHER
What are ways to stay positive and work toward goals?

Try It! 1. List three scenarios that could be graphed on a coordinate plane. What similarities and differences exist about the data in your scenarios?

EXAMPLE 2 Understand a Line Graph

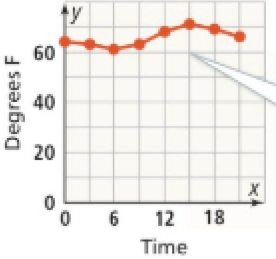
The data shows the temperature in Fahrenheit at St. Petersburg measured every 3 hours on a December day, using military time. What is a good way to display this data? Why add the line connecting the points?

CHOOSE EFFICIENT METHODS
The data is recorded in military time which uses a 24 hour clock. How could this be helpful?

00:00	03:00	06:00	09:00	12:00	15:00	18:00	21:00
64	63	61	63	68	71	69	66

The data is bivariate numerical data, so it should be displayed on a coordinate plane. The data points are discrete values, but they represent temperatures along a continuous time period.

It can be useful to visualize how the temperatures change between each measured point. Use a line graph to visualize the change in data.



There are no points that share an x-coordinate, so the graph represents a function.

So the line graph is used when the domain of the data is continuous and to help visualize the changes in data.

Try It! **2.** A local park has a children's play area, a baseball field, and a dog park. The park is open from sunrise to sunset.

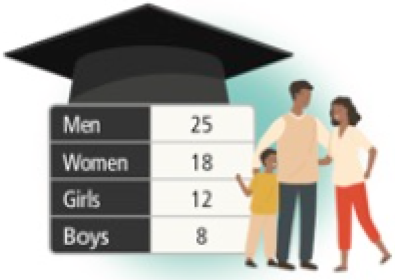
a. Copy and complete the table with possible numbers for people in the park at the various times of day for a Saturday in April.

b. Create a line graph for your data. Be sure to be able to explain any spikes or dips in attendance. Does your graph represent a function?

Time	00:00	03:00	06:00	09:00	12:00	15:00	18:00	21:00
People								

EXAMPLE 3 Create a Two-Way Frequency Table

The demographics of family members attending a high school graduation ceremony are recorded in a list. What is another way to organize and display the data? What are the advantages of changing the display?



Men	25
Women	18
Girls	12
Boys	8

The categories can be measured across two different statistical variables: gender and age. This is **bivariate categorical data**. It can be displayed in a two-way table.

COMMON ERROR
When organizing bivariate categorical data into a two-way table, be sure that your categories do not overlap.

For bivariate categorical data, instead of displaying the data points on a coordinate plane, you can show counts of the data in each category along the two dimensions of a table.

One of the advantages of organizing the data this way, you can easily see information on each variable individually, as well as on the variables paired together.

	Male	Female	Total
Adult	165	224	389
Child	123	148	271
Total	288	372	660

There are 389 adults and 271 children. 288 in attendance are male and 372 are female.

Try It! 3. Another local park has 4 soccer fields and 4 baseball fields. All the fields are busy on a Saturday morning. Each soccer team has 11 players and each baseball team has 9 players. There are 100 people at the park for baseball and 115 for soccer. Copy and complete the two-way table to show how many individuals will be playing and how many will be watching the games.

	Soccer	Baseball	Total
Playing			
Watching			
Total	115	100	215

EXAMPLE 4 Create a Segmented Bar Chart

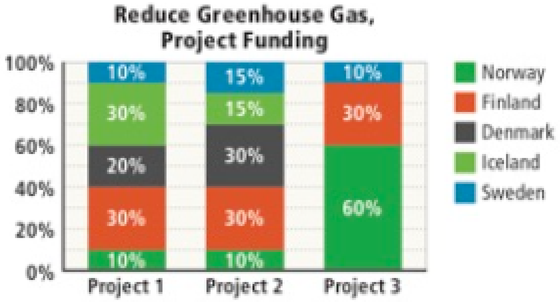
An alliance of 5 countries have agreed to fund 3 projects to reduce greenhouse gas emissions. The two-way frequency table shows the percentages of support contributed to the projects by each country. What is another way to represent the data?

	Project 1	Project 2	Project 3	Total
Norway	10%	10%	60%	80%
Finland	30%	30%	30%	90%
Denmark	20%	30%	■	50%
Iceland	30%	15%	■	45%
Sweden	10%	15%	10%	35%
Total	100%	100%	100%	300%

CHOOSE EFFICIENT METHODS
Could you represent the data using one or more circle graphs?

A drawback to using a frequency table for this data is that the final "Total" column does not make sense. Each project is funded at 100%, but the 3 projects together are not funded at 300%—that would mean that the projects were overfunded, and they are not. Also, the cost of each project is likely not the same, so the horizontal totals are sums of unlike values.

Another way to display the data is a segmented bar chart. This chart works well with percentages or with divisions that represent part of a whole. It is easy to see that each project is fully funded and the height of the bars by country correlates to the portion of their commitment.



The segmented bar chart is a good choice when comparing parts that combine together to make a whole.

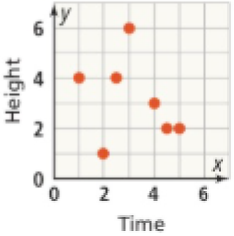
Try It! **4.** Two inner city parks are being renovated. The city is funding 50% of the cost for each park. For Park A, a service club is funding the rest and for Park B, local businesses are funding 20% and a family philanthropic trust is funding the rest. Create a segmented bar chart to represent this information.

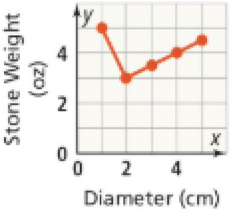
CONCEPT SUMMARY Representing Bivariate Data

Bivariate Numerical Data

Displayed on a coordinate grid as a scatterplot or line grid.

Use data display to identify associations between two variables.



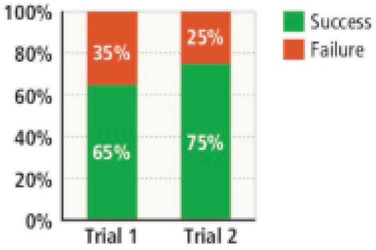


Bivariate Categorical Data

Displayed in a two-way frequency table

Use data display to identify associations between sets of categories

	RSVP	Stand-By	Total
Adult	8	3	11
Child	4	1	5
Total	12	4	16



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What representations may be used to display bivariate data?

2. **Error Analysis** Marta says that a two-way frequency table may be used for numerical data. Do you agree with her? Explain.

3. **Vocabulary** Explain the difference between univariate and bivariate numerical data. Give an example of each.

4. **Represent and Connect** What types of data may be represented with a segmented bar chart?

Do You KNOW HOW?

5. Copy the table and complete the right column and final row for the frequency table about how individuals prefer to give a gift.

	Males	Females	Total
Gift Bags	10	8	■
Wrapped Packages	4	7	■
Total	■	■	■

6. Using the information from item 5, what percent of individuals prefer to use gift bags for wrapping.

7. For a coordinate grid that represents bivariate numerical data, the number of families living in an apartment building is listed on the horizontal axis. What could you label the vertical axis?

PRACTICE & PROBLEM SOLVING

UNDERSTAND

8. **Communicate and Justify** Noemi is curious what is the best utensil for eating rice. She conducts a survey of students at her school, asking each student how often they eat rice, and what utensil they use primarily.

	Fork	Spoon	Chopsticks
Often	9	31	22
Not Often	8	13	1

a. Create segmented bar graphs from the data.

b. What conclusions do you think Noemi may make?

9. **Represent and Connect** The table shows data on ticket and medium popcorn prices at various movie theaters.

Ticket Price ($)	Popcorn Price ($)
7.50	1.25
13.00	4.50
11.25	5.00
9.75	4.25
11.00	5.00
6.00	2.50

a. Would it be appropriate to display this data on a scatter plot? Explain.

b. Would it be appropriate to display this data on a line graph? Explain.

c. **Higher Order Thinking** Would it be appropriate to display this data in a segmented bar graph? Explain.

10. **Choose Efficient Methods** Can circle graphs be used to display the same type of data as a segmented bar graph? Explain.

PRACTICE

Could scenarios in items 11–12 be represented on a coordinate grid? Explain. SEE EXAMPLE 1

11. Data about the number of volunteers compared to the number of sandbags that they fill before a storm.

12. Data about careers with current starting salaries.

13. A local charity feeds the homeless a midday meal. They begin serving at 11 AM and serve the final meals at 2 PM. Create a line graph showing how many meals served as a function of time of day. SEE EXAMPLE 2

Time	Meals Served
11:00	60
11:30	58
12:00	60
12:30	57
1:00	52
1:30	48
2:00	36

14. In one hour, a traveling petting zoo counts boys and girls going into a pen to pet the animals and counts them again as toddlers and school age children as they leave. Make a two-way frequency table for the information. SEE EXAMPLE 3

Toddlers		School Age	
Boys	Girls	Boys	Girls
8	12	8	7

15. In Florida, 21% of the population are immigrants with another 13% of the native-born individuals having at least one immigrant parent. In California, the numbers are 27% and 24% and in New York the numbers are 23% and 18% in the same categories. Create a segmented bar chart to represent this data. SEE EXAMPLE 4

PRACTICE & PROBLEM SOLVING

APPLY

16. **Analyze and Persevere** The International Club put on Dance Night for the school and encouraged students to dress in traditional clothing to be eligible for a doorprize. Of those who came to the dance, 75% of the freshmen, 55% of the sophomores, 20% of the juniors, and no seniors dressed for the chance to win the door prize.

a. Create a representation of the data.

b. Is there enough information to determine if a freshman won the door prize? Explain.

17. **Apply Math Models** A disc-golf course has hole lengths in feet and pars as shown.

a. Create a representation for the data.

b. Why did you choose the representation that you did?

Hole	Length	Par
1	304	3
2	272	3
3	342	4
4	312	3
5	280	3
6	368	4
7	312	3
8	306	3
9	332	4

ASSESSMENT PRACTICE

18. Select all representatives of bivariate categorical data. DP.1.1

☐ Favorite cuisines of a class

☐ Percentages of pets who sleep in cages or crates

☐ Number of animals at a zoo by class (Reptile, Mammalia, Aves, etc)

☐ Distances traveled by time

☐ Athletes characterized by age groups and gender

19. **SAT/ACT** Which choice best completes the following sentence?
A ____________ is used for bivariate numerical data when the dependent variable is _________ of the independent variable at discrete points.

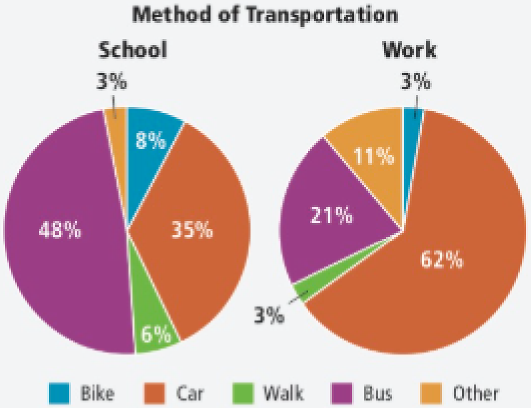
Ⓐ Scatterplot graph; not a function

Ⓑ line graph; a function

Ⓒ scatterplot graph; a function

Ⓓ two-way frequency table; not a function

20. **Performance Task** Yong made graphs showing how his classmates and their parents typically travel to school or to work.



a. Create another way to represent the data.

b. Create a representation that is based on univariate categorical data.

c. What are the advantages and disadvantages between your representation and the given one?

10-4

Analyzing Lines of Fit

I CAN… find the line of best fit for a data set and evaluate its goodness of fit.

VOCABULARY

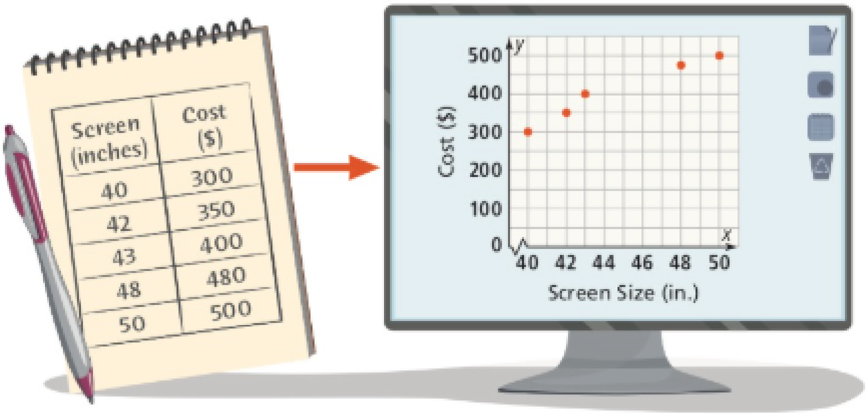
- causation
- extrapolation
- interpolation
- line of best fit
- linear regression
- residual

MA.912.DP.2.4–Fit a linear function to bivariate numerical data that suggests a linear association and interpret the slope and *y*-intercept of the model. Use the model to solve real-world problems in terms of the context of the data.
Also DP.1.3, DP.2.5, DP.2.6

MA.K12.MTR.1.1, MTR.5.1, MTR.6.1

MODEL & DISCUSS

Nicholas plotted data points to represent the relationship between screen size and cost of television sets. Everything about the televisions is the same, except for the screen size.



Screen (inches)	Cost ($)
40	300
42	350
43	400
48	480
50	500

A. Describe any patterns you see.

B. What does this set of points tell you about the relationship of screen size and cost of the television?

C. **Use Patterns and Structure** Where do you think the point for a 46-inch television would be on the graph? How about for a 60–inch TV? Explain.

ESSENTIAL QUESTION

How can you evaluate the goodness of fit of a line of best fit for a paired data set?

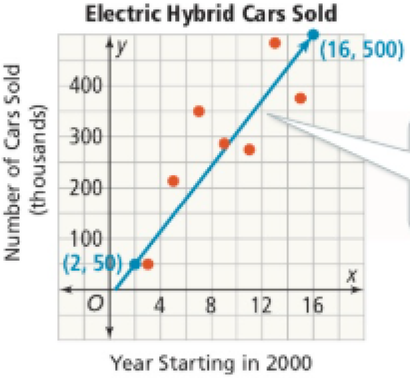
EXAMPLE 1 Write the Equation of a Trend Line

What trend line models the data in the scatterplot? What do the slope and *y*-intercept of the line represent?

A **trend line** models the data in a scatter plot by showing the general direction of the data. A trend line fits the data as closely as possible.

STUDY TIP
Remember that the trend line might not pass through any of the data points.

Step 1 Sketch a trend line for the data.



Draw the trend line so the data points are approximately balanced above and below it.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Step 2 Write the equation of this trend line.

Select two points on the trend line to find the slope.

$$m = \frac{500 - 50}{16 - 2}$$
$$\approx 32.1$$

(2, 50) and (16, 500) are two points on the trend line.

Use point-slope form, then convert the equation to slope-intercept form.

$$y - 50 = 32.1(x - 2)$$
$$y - 50 = 32.1x - 64.2$$
$$y = 32.1x - 14.2$$

Use the point-slope formula.

The trend line that models the data is $y = 32.1x - 14.2$.

This trend line is one of many possible trend lines.

Step 3 Interpret the slope and y-intercept of the trend line.

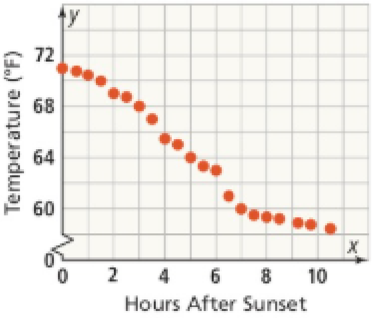
The slope of the trend line, 32.1, represents the year-to-year increase in the number of electric hybrid cars sold per year.

The y-intercept, −14.2, represents the number of electric hybrid cars sold in the year ending at the start of the year 2000; that is, the number of electric hybrid cars sold in 1999. Since a negative value does not make sense in the context of the problem, it is likely that no electric hybrid cars were sold before the year 2000.

CHECK FOR REASONABLENESS
Would it be reasonable for a trendline to have a negative slope in this situation? What would that indicate?

Try It! 1. What trend line models the relationship between the hours after sunset, x, and the temperature, y? What do the slope and y-intercept of the line represent?

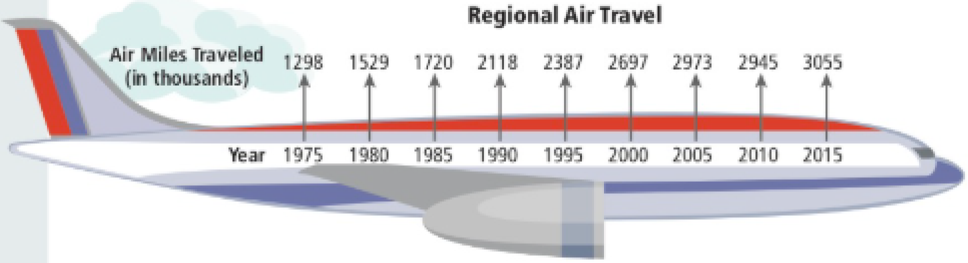
Temperature After Sunset



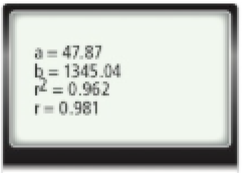
APPLICATION

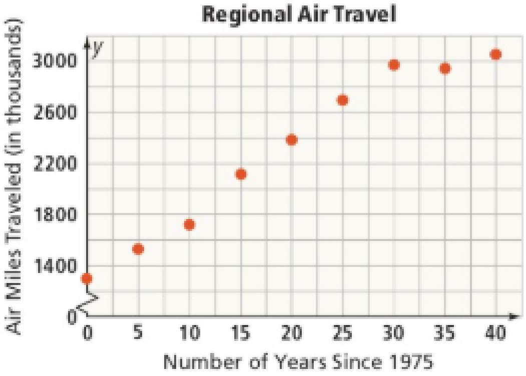
EXAMPLE 2 Interpolate and Extrapolate Using Linear Models

The graphic shows regional air travel data recorded by a domestic airline company. How can you use the data to estimate the number of air miles people flew in 2003? If the trend in air travel continues, what is a reasonable estimate for the number of miles that people will fly in 2030?



Formulate Plot the data points on a scatter plot. Using technology, perform a **linear regression** to determine the line of best fit for the data. For the *x*-values, use number of years since 1975.


a = 47.87
b = 1345.04
$r^2 = 0.962$
r = 0.981



Compute Use the values of *a* and *b* (from the linear regression) to write the line of best fit.

$$y = 47.87x + 1345.04$$

Interpolation OR **Extrapolation**

Interpolation is using a model to estimate a value within the range of known values.

Interpolate to estimate the miles people flew in 2003, or 28 years after 1975.

$$y = 47.87(28) + 1345.04 = 2,685.4$$

Extrapolation is using a model to make a prediction about a value outside the range of known values.

Extrapolate to predict the miles that people will fly in 2030, 55 years after 1975.

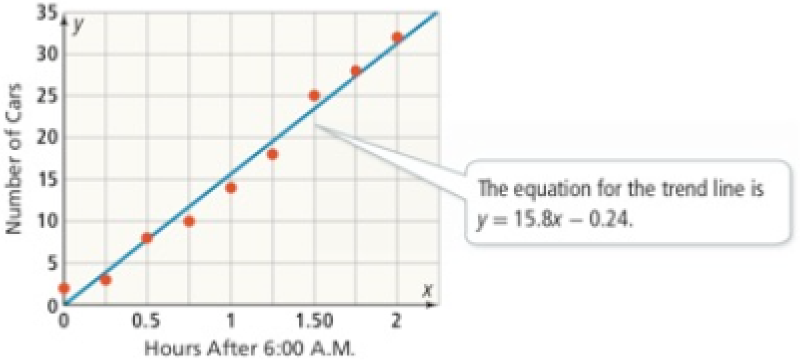
$$y = 47.87(55) + 1345.04 = 3,977.89$$

Interpret The model predicts that people flew a total of 2,685 thousand air miles on the airline in 2003, and that people will fly a total of 3,978 thousand air miles in 2030. This prediction is not as reliable as the estimate for 2003 because the trend may not continue.

Try It! **2.** Using the model from Example 2, estimate the number of miles people flew on the airline in 2012.

EXAMPLE 3 Understand Correlation

The scatterplot shows the number of cars in the student parking lot at the high school from 6:00 A.M. to 8:00 A.M. What is the direction of the correlation?



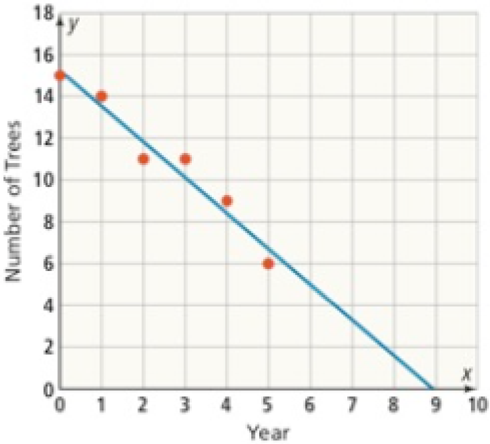
VOCABULARY
Recall that When y-values tend to increase as x-values increase, the two data sets have a *positive association*.

The data suggests a positive association. When data with a positive association are modeled with a line, there is a **positive correlation**.

When data with a negative association are modeled with a line, there is a **negative correlation**. If data do not have an association, they can not be modeled with a linear function.

The positive direction of the correlation in the situation above means that as more time passes from 6:00 A.M. to 8:00 A.M. there are more cars in the student parking lot. The slope of the trendline, 15.8, indicates that the number of cars increases by roughly 16 cars per hour.

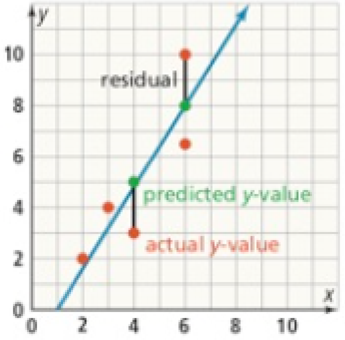
Try It! 3. The scatterplot shows the number of palm trees on a street over the past 6 years. The equation of the line of best fit is $y = -1.7x + 15.3$. Do the data show a positive or negative correlation? What does the direction of the correlation mean in terms of the situation?



CONCEPT Residuals

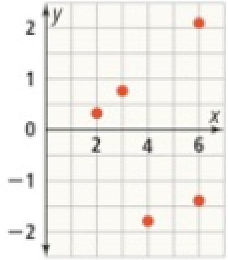
A **residual** is the difference between the y-value of a data point and the corresponding y-value from the line of best fit, or the predicted y-value.

residual = actual y-value − predicted y-value



A residual plot shows how well a linear model fits the data set. If the residuals are randomly distributed on either side of the x-axis and clustered close to the x-axis, then the linear model is likely a good fit.

If the residuals are not randomly distributed, then the pattern might indicate that a non-linear model is more appropriate.

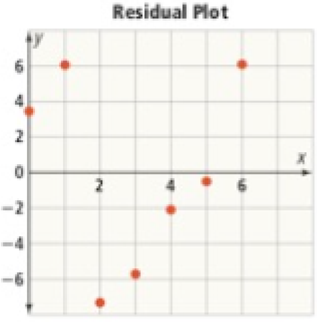


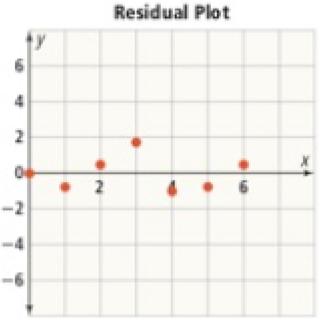
CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Connect Residuals and Strength of Correlation

Rodrigo and Talia each run their own online craft shop. They track the number of sales they make each month starting in April. Based on the given scatter plots, trendlines, and residual plots for their sales, which data set shows a stronger correlation? Explain.

COMMUNICATE AND JUSTIFY
Why is it helpful to use the same horizontal and vertical scales when comparing residual graphs?





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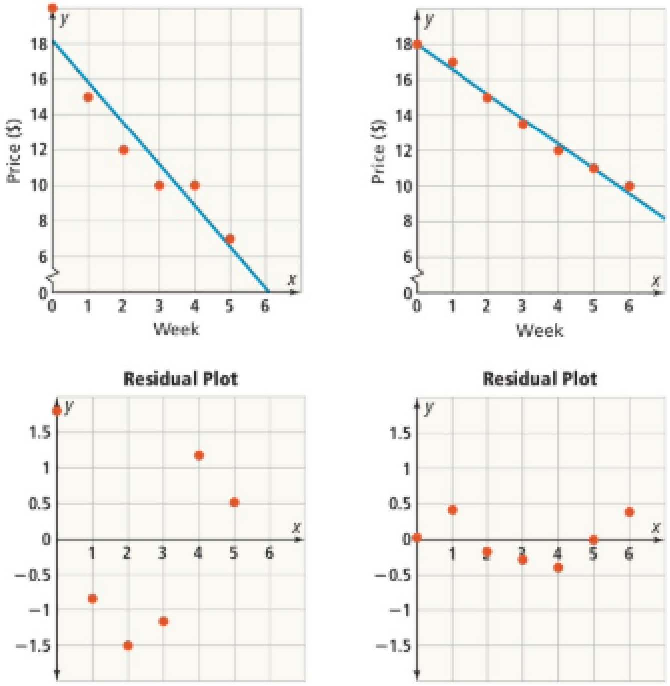
EXAMPLE 4 CONTINUED

Both data sets show a positive correlation. The number of sales increases as the year goes on. Talia's residual plot is more tightly clustered around the x-axis than Rodrigo's residual plot. This means that there are smaller differences between the actual values and the predicted values. Talia's data has a stronger positive correlation than Rodrigo's data.

Try It! **4.** Two stores track the price of a clearance item. They discount it each week. Describe the correlation for each data set. Use the residual plots to explain which correlation is stronger.



APPLICATION

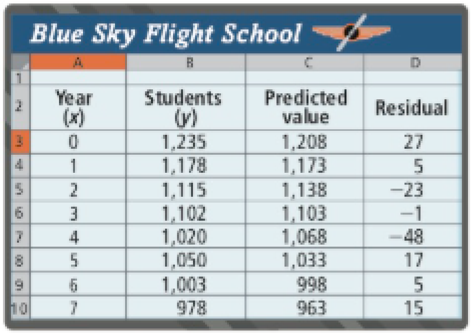
EXAMPLE 5 Interpret Residual Plots

Student enrollment at Blue Sky Flight School over 8 years is shown. The owner used linear regression to determine the line of best fit. The equation for the line of best fit is $y = -35x + 1208$. How well does this linear model fit the data?

Step 1 Evaluate the equation for each x-value to find the predicted y-values.

Step 2 Calculate the differences between the actual and predicted y-values for each x-value.

STUDY TIP
Using a spreadsheet is a good way to organize data and calculations.



Blue Sky Flight School

	A	B	C	D
1				
2	Year (x)	Students (y)	Predicted value	Residual
3	0	1,235	1,208	27
4	1	1,178	1,173	5
5	2	1,115	1,138	−23
6	3	1,102	1,103	−1
7	4	1,020	1,068	−48
8	5	1,050	1,033	17
9	6	1,003	998	5
10	7	978	963	15

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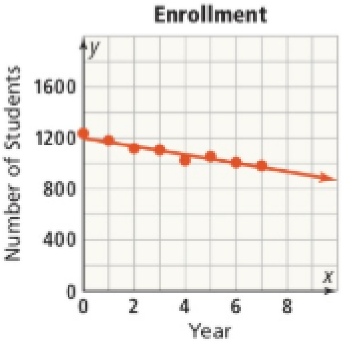
EXAMPLE 5 CONTINUED

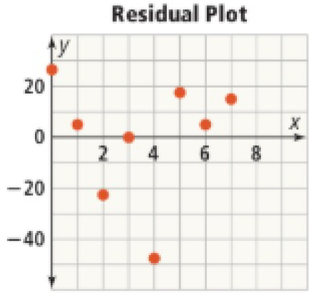
Step 3 Plot the residual for each x-value.

COMMON ERROR

The appearance of a residual plot does not correspond to a positive or negative correlation. The data shown might be misinterpreted as data with no correlation, but it has a negative correlation that is seen when the actual data points are plotted.



The scatter plot with the line of best fit suggests that there is a negative correlation between years and enrollment.



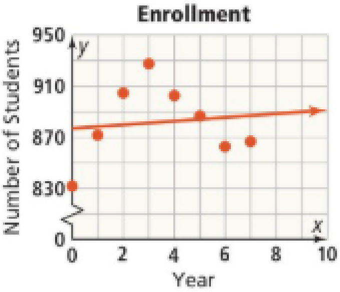
The residual plot shows the residuals randomly distributed. There are 5 positive residuals, 2 negative residuals, and 1 residual of zero value. They are mostly clustered close to the x-axis,

There is one residual of −48, which corresponds to $x = 4$, when the Blue Sky Flight School had 1,020 students instead of the predicted 1,068. This appears to be somewhat of an outlier compared to the other values.

The linear model is likely a good fit for the data.

Try It! **5.** The owner of Horizon Flight School also created a scatter plot and calculated the line of best fit for her enrollment data shown in the table. The equation of the line of best fit is $y = 1.44x + 877$. Find the residuals and plot them to determine how well this linear model fits the data.

Year (x)	0	1	2	3	4	5	6	7
Students (y)	832	872	905	928	903	887	863	867

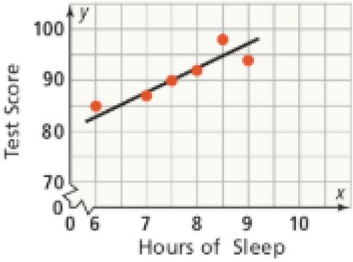


EXAMPLE 6 Correlation and Causation

COMMON ERROR

Be careful not to assume that if a strong correlation exists between two variables, that a change in one causes change in the other. The change could be caused entirely by a third, unknown variable.

A. A student found a positive correlation between the number of hours of sleep his classmates got before a test and their scores on the test. Can he conclude that he will do well on the test if he goes to bed early?

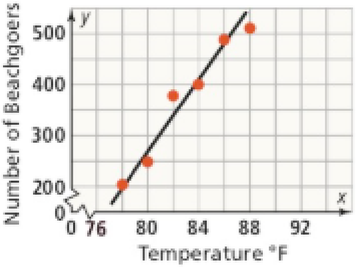


Causation describes a cause-and-effect relationship. A change in the one variable causes a change in the other variable.

To determine whether two variables have a causal relationship, you have to carry out an experiment that can control for other variables that might influence the relationship between the two target variables.

The student cannot conclude that he will do well if he goes to bed early. Other variables, like the time spent studying or proficiency with the content, could affect how well he does on the test.

B. A lifeguard notices that as the outside temperature rises, the number of people coming to the beach increases. Can she conclude that the change in temperature results in more people going to the beach?



She did not carry out an experiment or control for other variables that might affect the relationship. These include weather forecast and time of year.

She cannot conclude that the only reason that more people come to the beach is the outside temperature.

Try It! **6.** The number of cars in a number of cities shows a positive correlation to the population of the respective city. Can it be inferred that an increase of cars in a city leads to an increase in the population? Defend your response.

CONCEPT SUMMARY Linear Models, Lines of Best Fit, and Residuals

WORDS A linear regression is a method for finding the line of best fit, or a linear model, for a bivariate data set.

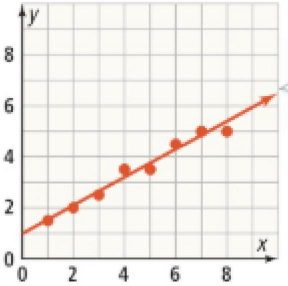
A residual plot reveals how well the linear model fits the data set. If the residuals are randomly distributed around and clustered close to the *x*-axis, the linear model is likely a good fit.

ALGEBRA Use the values of *a* and *b* from the linear regression to write the equation for the line of best fit.

```
LinReg
 y=ax+b
 a=.5416666667
 b=1
 r²=.9688779689
 r=.9843159904
```

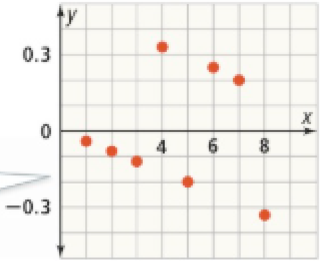
The equation is $y = 0.542x + 1$.

GRAPHS



The equation for the line of best fit is $y = 0.542x + 1$.

This is the residual plot for the data at the left.



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you evaluate the goodness of fit of a line of best fit for a paired data set?

2. **Vocabulary** Describe the difference between *interpolation* and *extrapolation*.

3. **Error Analysis** A student says that a good trend line must pass through at least two points from the data set. Explain the error the student made.

4. **Check for Reasonableness** A student found a strong correlation between the age of people who run marathons and their marathon time. Can the student conclude that young people will run marathons faster than older people? Explain.

Do You KNOW HOW?

Use the table for Exercises 5 and 6.

x	10	20	30	40	50
y	7	11	14	20	22

5. Use technology to determine the equation of the line of best fit for the data.

6. Make a residual plot for the line of best fit and the data in the table. How well does the linear model fit the data?

7. The table shows the number of customers *y* at a store for *x* weeks after the store's grand opening. The equation for the line of best fit is $y = 7.77x + 38.8$. Assuming the trend continues, what is a reasonable prediction of the number of visitors to the store 7 weeks after its opening?

x	1	2	3	4	5	6
y	46	53	65	71	75	86

PRACTICE & PROBLEM SOLVING

UNDERSTAND

8. **Generalize** Can you use any two of the given data points in a scatter plot to write an equation for a trend line? Explain.

9. **Error Analysis** Describe and correct the error a student made in determining the equation for the line of best fit for the data in the table.

x	3	6	9	12	15	18	21
y	4	17	28	40	55	67	72

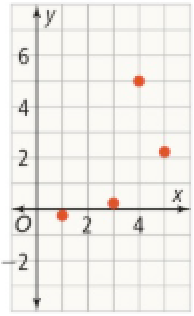
Enter y data in L1 and x data in L2. Then perform a linear regression.
Line of best fit: $y = 0.25x + 1.83$ ✗

10. **Higher Order Thinking** Which is likely to be more accurate: an estimate through interpolation or a prediction through extrapolation? Explain.

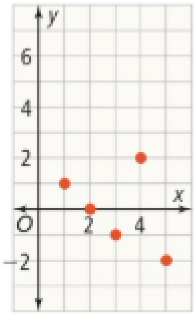
11. **Generalize** Describe how the values of a and b in a linear model $y = ax + b$ are related to the data being modeled.

12. **Use Patterns and Structure** How can you use the graph of the line of best fit to make predictions about future behaviors of the quantities of the data set?

13. **Communicate and Justify** Arthur and Tavon each found trend lines for their last five math tests. Based on the residual plots, Arthur states that his scores are more closely aligned to a linear model than Tavon's scores. Make a mathematical argument to support or refute Arthur's claim.



Arthur's Residual Plot



Tavon's Residual Plot

PRACTICE

For each table, make a scatter plot of the data. Draw a trend line and write its equation. SEE EXAMPLE 1

14.

x	y
2	3
4	6
5	5
7	7
8	9
8	8

15.

x	y
3	9
5	8
5	6
6	5
6	6
8	3

16.

x	y
1	1
2	3
3	5
3	6
5	8
6	9

Use technology to perform a linear regression to determine the equation for the line of best fit for the data. Estimate the value of y when $x = 19$.
SEE EXAMPLE 2

17.

x	y
12	35
14	39
16	41
18	44
20	48

18.

x	y
16	105
20	83
24	62
28	34
32	15

19. For each data set, make a scatterplot and graph the given trendline. Describe the direction of each correlation. Make residual plots for each linear model. Which data set has a stronger correlation? Explain. SEE EXAMPLES 3, 4, AND 5

A: $y = 0.38x + 9.6$

x	y
10	12
15	16
20	20
25	17
30	21

B: $y = -0.58x + 25.2$

x	y
10	19
15	17
20	14
25	10
30	8

Construct an argument for each scenario given.
SEE EXAMPLE 6

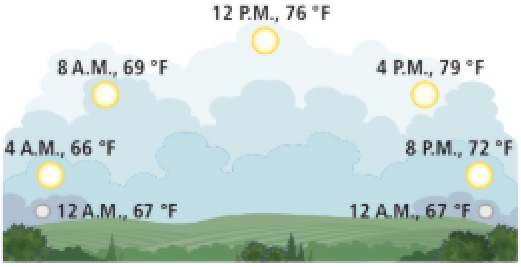
20. The average monthly heating bills for houses in a neighborhood are positively correlated to the number of pets in the house. Can it be inferred that the number of pets in a household causes an increase in average monthly heating bills? Explain.

21. A person's level of education is positively correlated to the salary the person earns. Can it be inferred that a person with a doctorate degree will always earn more than a person with a bachelor's degree? Explain.

PRACTICE & PROBLEM SOLVING

APPLY

22. Analyze and Persevere Temperatures at different times of day are shown. How can you describe the relationship between temperature and time? Would a linear model be a good fit for the data? Explain.



23. Apply Math Models The table shows the number of miles people in the U.S. traveled by car annually from 1975 to 2015. The equation for the line of best fit is $y = 0.048x + 1.345$, where x is the number of years since 1975. What does the slope represent? Estimate the number of miles people in the U.S. traveled in 2007. What is a reasonable prediction for the number of miles people in the U.S. will travel in 2025?

Year	Vehicle-Miles Traveled in U.S. (in trillions)
1975	1.298
1980	1.529
1985	1.720
1990	2.118
1995	2.387
2000	2.697
2005	2.973
2010	2.945
2015	3.055

24. Analyze and Persevere The table shows the file size y in megabytes of photos taken at different resolutions x in megapixels. The equation for the line of best fit is $y = 0.3x$. Use this equation to create a residual plot. What does the residual plot tell you about the data?

x	4	5	6	7	8	10	12
y	1.2	1.5	1.8	2.1	2.4	3.0	3.6

ASSESSMENT PRACTICE

25. Akuti starts her own craft company selling hand-carved animals. The table shows her inventory of animals for weeks 4 through 8. Find a trend line for the data. Identify and interpret the y-intercept and slope in terms of the situation. DP.2.4

Week	4	5	6	7	8
Inventory	21	17	16	12	10

26. SAT/ACT Students who eat breakfast are more likely to do well in school. Which of the following can be inferred from this relationship?

Ⓐ The two events are causally related.

Ⓑ The more often a student eats breakfast, the better the student will do in school.

Ⓒ Without more evidence, it cannot be determined whether the correlation is causal.

Ⓓ Providing free breakfast to all students will close the achievement gap.

27. Performance Task A store records the price of kites in dollars, x, and the number of kites, y, sold at each price.

Kite Style	Price	Quantity
	\$10.00	25
	\$12.00	23
	\$15.00	20
	\$22.00	18
	\$30.00	15

Part A Make a scatter plot of the data. What trend line models the data? What does the slope of the trend line represent?

Part B What factors other than price could influence the number of kites sold? Could you use any of these factors to make another scatter plot? Explain.

10-5

Analyzing Two-Way Frequency Tables

I CAN… organize data in two-way frequency tables and use them to make inferences.

VOCABULARY

- conditional relative frequency
- joint frequency
- joint relative frequency
- marginal frequency
- marginal relative frequency

MA.912.DP.3.1–Construct a two-way frequency table summarizing bivariate categorical data. Interpret joint and marginal frequencies and determine possible associations in terms of a real-world context. Also DP.1.3, DP.3.2, DP.3.3

MA.K12.MTR.2.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

Baseball teams at a high school and a college play at the same stadium. Results for every game last season are given for both teams. There were no ties.

Baseball Season Results at Mountain View Stadium

	HOME	AWAY
WEST MOUNTAIN HIGH SCHOOL	11 OUT OF 16	08 OUT OF 14
BIG MOUNTAIN COLLEGE	18 OUT OF 26	18 OUT OF 30

A. How could you organize the data in table form?

B. **Use Patterns and Structure** How would you analyze the data to determine whether the data support the claim that the team that plays at home is more likely to win?

ESSENTIAL QUESTION

How can you use two-way frequency tables to analyze data?

EXAMPLE 1 Construct and Interpret a Two-Way Frequency Table

Owners of a Mexican food chain are planning to add a vegetarian item to its menu. Customers were asked to choose between fajitas and enchiladas. The owners found that 40 out of 90 teenagers who were customers chose fajitas and 75 out 135 adults who were customers chose fajitas. What trends do the results suggest?

Step 1 Construct a two-way frequency table.

Two categories of data are collected: age range, and food choice. Use the categories to show all of the results.

A joint frequency is where a column and row join.

A marginal frequency is at the margin, or edge, of a column or row.

	Enchiladas	Fajitas	Totals
Teenagers	■	40	90
Adults	■	75	135
Totals	■	■	■

To find unknown frequencies, use the fact that the sum of joint frequencies along a row or column is equal to the marginal frequency.

STUDY TIP
A two-way frequency table can show possible relationships between two sets of categorical data.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Step 2 Complete the table.

	Enchiladas	Fajitas	Totals
Teenagers	50	40	90
Adults	60	75	135
Totals	110	115	225

Joint frequencies indicate the frequency of a single option for one category; for example, the frequency of teenagers choosing a veggie enchilada is 50.

Marginal frequencies indicate the total frequency for each category, such as the total frequency of adult respondents is 135.

The joint frequencies suggest an association between age and food choice. Teenage customers prefer veggie enchiladas over fajitas, while adults prefer veggie fajitas.

Try It! **1. a.** What do the marginal frequencies tell you about customers' food preference?

b. What do the marginal frequencies tell you about the number of adult and teenage customers?

EXAMPLE 2 Calculate Relative Frequencies

What do the survey results reveal about teenage and adult customer preferences for veggie enchiladas?

Joint relative frequency is the ratio, or percent, of the joint frequency to the total.

Marginal relative frequency is the ratio, or percent, of the marginal frequency to the total.

COMMON ERROR
Divide each frequency by the total count, found in the bottom right corner of the two-way frequency table. Express relative frequency as a fraction, decimal, or percent.

	Enchiladas	Fajitas	Totals
Teenagers	$\frac{50}{225} \approx 22\%$	$\frac{40}{225} \approx 18\%$	$\frac{90}{225} \approx 40\%$
Adults	$\frac{60}{225} \approx 27\%$	$\frac{75}{225} \approx 33\%$	$\frac{135}{225} \approx 60\%$
Totals	$\frac{110}{225} \approx 49\%$	$\frac{115}{225} \approx 51\%$	$\frac{225}{225} \approx 100\%$

Of the customers surveyed, about 22% were teenagers who selected veggie enchiladas and about 27% were adults who selected veggie fajitas.

Try It! **2.** How can you tell whether a greater percent of customers surveyed selected veggie enchiladas or veggie fajitas?

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Interpret Conditional Relative Frequencies by Row

Using data from Examples 1 and 2, a marketing team concludes that adults prefer veggie enchiladas more than teenagers do. Do the survey results support this conclusion?

Conditional relative frequency is the ratio of the joint frequency and the related marginal frequency.

Calculating the conditional relative frequency for each row will adjust for differences in the number of teenage and adult customers surveyed.

COMMUNICATE AND JUSTIFY
What do the conditional relative frequencies tell you about associations between gender and menu item choice?

	Enchiladas	Fajitas	Totals
Teenagers	$\frac{50}{90} \approx 56\%$	$\frac{40}{90} \approx 44\%$	$\frac{90}{90} = 100\%$
Adults	$\frac{60}{135} \approx 44\%$	$\frac{75}{135} \approx 56\%$	$\frac{135}{135} = 100\%$

$$\text{Conditional relative frequency} = \frac{\text{joint frequency}}{\text{marginal frequency}}$$

The results do not support this conclusion. The conditional relative frequencies show that while about 56% of the teenagers surveyed prefer veggie enchiladas, only about 44% of the adults prefer them.

Try It! 3. What conclusion could the marketing team make about customers who prefer veggie enchiladas compared with customers who prefer veggie fajitas? Justify your answer.

EXAMPLE 4 Interpret Conditional Relative Frequencies by Column

Using data from Examples 1 and 2, the marketing team also concludes that there is a greater variation between the percent of teenagers and adults who like veggie fajitas than there is for those who prefer veggie enchiladas. Do the survey results support this conclusion?

Calculating the conditional relative frequency for each column allows you to analyze teenagers' and adults' preferences within each food choice category.

USE PATTERNS AND STRUCTURE
The conditional relative frequencies calculated for rows are not the same as those calculated for columns. How are the questions you can answer looking at the table in Example 3 different from the questions you can answer looking at the table in Example 4?

The conclusion is supported by the survey results. Conditional relative frequencies show that of the customers who prefer veggie fajitas, 65% are adult and only 35% are teenager. Of those who prefer veggie enchiladas, 55% are adults and 45% are teenagers.

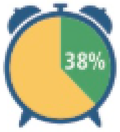
	Enchiladas	Fajitas
Teenagers	$\frac{50}{110} \approx 45\%$	$\frac{40}{115} \approx 35\%$
Adults	$\frac{60}{110} \approx 55\%$	$\frac{75}{115} \approx 65\%$
Totals	$\frac{110}{110} = 100\%$	$\frac{115}{115} = 100\%$

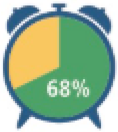
Try It! 4. What conclusion could you draw if the percentages for teenage and adult customers were the same across the rows in this table?

APPLICATION

EXAMPLE 5 Use Marginal and Conditional Relative Frequencies to Determine Relative Frequencies

A college conducts a survey to better understand how well its students are balancing wellness with their studies. After an exam, students in a large freshman math class were asked how much they slept the night before. Overall, 72% of the students passed the exam.





CHECK FOR REASONABLENESS
How do you know 38% and 68% represent conditional relative frequencies?

A. Analyze the information. What do the different percentages mean?

Among students who passed, 38% slept at least 8 hours. This is a conditional relative frequency within the "passed" group. Similarly, 1 – 0.68 = 0.32 or 32% is a conditional relative frequency within the "did not pass" group. The conditional relative frequency table displays the information.

	At least 8 hours	Less than 8 hours	Totals
Pass	0.38	0.62	1.00
Did not pass	0.32	0.68	1.00

The students who passed the exam represent one of the subgroups being compared in the study. So, 72% represents a marginal relative frequency. The incomplete relative frequency table shows the known marginal relative frequencies.

	At least 8 hours	Less than 8 hours	Totals
Pass	■	■	0.72
Did not pass	■	■	0.28
Totals	■	■	1.00

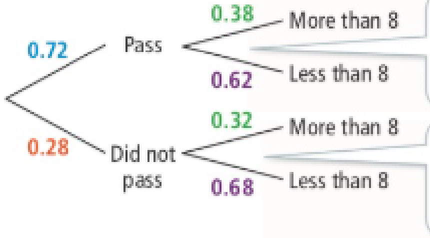
The "pass" and "did not pass" marginal frequencies must sum to 1.00.

B. Construct a two-way relative frequency table for the situation.

Use a tree diagram to find the relative frequencies.

Step 1 Draw one set of branches with the marginal relative frequencies.

Step 2 Draw the next set of branches with the conditional relative frequencies.



Conditional relative frequencies for the "pass" group.

Conditional relative frequencies for the "did not pass" group.

CONTINUED ON THE NEXT PAGE

Step 3 Complete the relative frequency table.

The relative frequencies are equal to the products along the tree diagram.

	At least 8 hours	Less than 8 hours	Totals
Pass	$0.72 \cdot 0.38 \approx 0.27$	$0.72 \cdot 0.62 \approx 0.45$	0.72
Did not pass	$0.28 \cdot 0.32 \approx 0.09$	$0.28 \cdot 0.68 \approx 0.19$	0.28
Totals	0.36	0.64	1.00

HAVE A GROWTH MINDSET
In what ways can you be inquisitive and open to learning new things?

C. Does the survey suggest an association between amount of sleep and exam performance?

Compare the relative frequencies of the group of students who passed within the two sleep groups.

Slept at least 8 hours	Slept less than 8 hours
$\frac{0.27}{0.36} = 0.75$	$\frac{0.45}{0.64} \approx 0.70$

A greater percent of the students who slept at least 8 hours passed the test, than those who slept less than 8 hours. So, the survey suggests an association between amount of sleep and exam performance.

D. Can the college conclude that more sleep will result in better academic performance?

The survey suggests that more sleep is associated with better academic performance, but it cannot conclude that sleep caused better performance. Other factors such as having a job, or number of courses enrolled, can impact students' sleep, as well as academic performance.

Try It! **5.** A hospital uses a test to help determine if a person has a particular disease. About 95% of patients who have the disease test positive, and 10% of patients who do not have the disease test positive. The disease is present in about 0.5% of the general population.

a. Create a two-way relative frequency table for the situation.

b. What percent of the people who test positive are actually expected to have the disease?

CONCEPT SUMMARY Two-Way Frequency Tables

WORDS Two-way frequency tables show relationships between two sets of categorical data. Entries can be frequency counts or relative frequencies. Entries in the body of the table are **joint frequencies** (counts) or **joint relative frequencies** (ratios). Entries in the totals column or row are **marginal frequencies** or **marginal relative frequencies**.

Conditional relative frequencies show the frequency of responses for a given condition, or the ratio of the joint frequencies to the corresponding marginal frequency.

TABLES

Movie Time Preferences

	Afternoon	Evening	Totals
Student	$\frac{90}{200} = 45\%$	$\frac{50}{200} = 25\%$	$\frac{140}{200} = 70\%$
Adult	$\frac{20}{200} = 10\%$	$\frac{40}{200} = 20\%$	$\frac{60}{200} = 30\%$
Totals	$\frac{110}{200} = 55\%$	$\frac{90}{200} = 45\%$	$\frac{200}{200} = 100\%$

20 of the 200 respondents, or 10%, were adults who prefer the afternoon show.

70% of the respondents were students.

Conditional Relative Frequency

	Afternoon	Evening	Totals
Student	$\frac{90}{140} \approx 64\%$	$\frac{50}{140} \approx 36\%$	$\frac{140}{140} = 100\%$
Adult	$\frac{20}{60} \approx 33\%$	$\frac{40}{60} \approx 67\%$	$\frac{60}{60} = 100\%$

Of all of the adult respondents, 33% prefer afternoon shows.

Conditional Relative Frequency

	Afternoon	Evening
Student	$\frac{90}{110} \approx 82\%$	$\frac{50}{90} \approx 56\%$
Adult	$\frac{20}{110} \approx 18\%$	$\frac{40}{90} \approx 44\%$
Totals	$\frac{110}{110} = 100\%$	$\frac{90}{90} = 100\%$

Of all of the respondents that prefer evening shows, 44% were adults.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you use two-way frequency tables to analyze data?

2. **Communicate and Justify** How are joint frequencies and marginal frequencies similar? How are they different?

3. **Represent and Connect** How are conditional relative frequencies related to joint frequencies and marginal frequencies?

4. **Error Analysis** Zhang says that the marginal relative frequency for a given variable is 10. Could Zhang be correct? Explain your reasoning.

Do You KNOW HOW?

In a survey, customers select Item A or Item B. Item A is selected by 20 males and 10 females. Of 20 customers who select Item B, five are males.

5. Make a two-way frequency table to organize the data.

6. Make a two-way relative frequency table to organize the data.

7. Calculate conditional relative frequencies for males and females. Is it reasonable to conclude that males prefer Item A more than females do?

8. Calculate conditional relative frequencies for Item A and Item B. Is it reasonable to conclude that a customer who prefers Item B is more likely to be a female than a male?

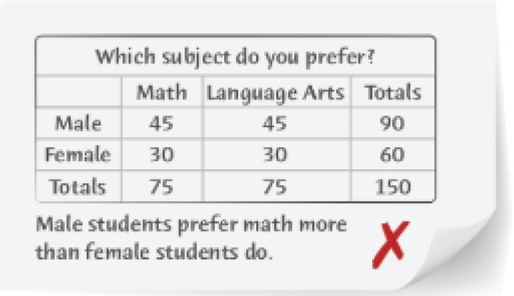
PRACTICE & PROBLEM SOLVING

UNDERSTAND

9. **Check for Reasonableness** An equal number of juniors and seniors were surveyed about whether they prefer lunch item A or B. Is it reasonable to infer from the table that more juniors prefer lunch item B while more seniors prefer lunch item A? Explain.

	Item A	Item B	Totals
Junior	0.1	0.4	0.5
Senior	0.3	0.2	0.5
Totals	0.6	0.4	1.0

10. **Error Analysis** Describe and correct the errors a student made when making a generalization based on a two-way frequency table.



Which subject do you prefer?			
	Math	Language Arts	Totals
Male	45	45	90
Female	30	30	60
Totals	75	75	150

Male students prefer math more than female students do. ✗

11. **Use Patterns and Structure** In a two-way relative frequency table, how are joint relative frequencies and marginal relative frequencies related?

12. **Higher Order Thinking** Students are surveyed to see how long they studied for a test.

- 10% of the students who studied 3 hours or more failed the test.
- 40% of the students who studied less than 3 hours passed the test.
- 2 students who studied 3 hours or more failed the test.
- 4 students who studied less than 3 hours passed the test.

a. Make a two-way frequency table that shows the association between hours spent studying and passing the test.

b. Does the association appear to be significant? Explain.

PRACTICE

In a survey, music club members select their preference between Song A or Song B. Song A is selected by 30 teens and 10 adults. Of 20 members who select Song B, five are teens. SEE EXAMPLES 1–4

Make a two-way frequency table to organize the data.

13. Is it reasonable to say that more people surveyed prefer Song A? Explain.

Calculate conditional relative frequencies.

14. Is it reasonable to say that teens prefer Song A more than adults do? Explain.

15. Is a member who prefers Song B significantly more likely to be an adult than a teen? Explain.

In the two-way frequency table, frequencies are shown on the top of each cell in blue, and relative frequencies are shown at the bottom in red. Most of the frequencies are missing. SEE EXAMPLES 1–5

High School Graduate?	Choice A	Choice B	Totals
Yes	16 0.08	____ ____	____ 0.56
No	____ ____	24 ____	____ ____
Totals	____ ____	____ ____	____ ____

16. Complete the table and calculate the conditional relative frequencies for Yes and No and Choices A and B.

17. Is a high school graduate more likely to prefer Choice A or B? Explain.

18. Is someone who prefers Choice A more likely to be a high school graduate than not? Explain.

19. What does the joint relative frequency $\frac{64}{200}$ represent in this context?

A clinic uses a test to determine if a person has a particular disease. About 95% of patients who have the disease test positive, and 8% of patients who do not have the disease test positive. The disease is present in about 0.4% of the general population. SEE EXAMPLE 5

20. Use a tree diagram to create a two-way relative frequency table for the situation.

21. How likely is it for a person who tests positive to have the disease?

PRACTICE & PROBLEM SOLVING

APPLY

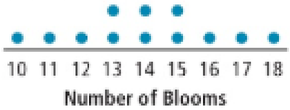
22. **Communicate and Justify** Is there a significant association between income and whether or not a voter supports the referendum? Justify your answer.

Do you support the referendum?

Income	Yes	No	Totals
≤ \$100,000	80	20	100
> \$100,000	40	10	50
Totals	120	30	150

23. **Analyze and Persevere** A gardener is only satisfied when a hydrangea bush has at least 14 blooms. How can you organize the data shown in the line plots into two-way frequency tables to make inferences about the new plant food and the number of blooms?

Hydrangeas Without New Plant Food



Hydrangeas with New Plant Food

24. **Represent and Connect** Based on the survey data below, a marketing team for an airline concludes that someone between 18 and 24 years of age is more likely never to have flown on a commercial airliner than someone 25 years or older. Do you agree with this conclusion? Justify your answer.

Terminal

Have you ever flown on a commercial airline?

	Yes	No	Totals
18–24 yrs	198	81	279
25+ yrs	2,539	448	2,987
Totals	2,737	529	3,266

ASSESSMENT PRACTICE

25. Marco collected data and made a two-way frequency table. Select all the statements that are true about Marco's data. DP.3.1

☐ **A.** The sum of all joint frequencies equals the total frequency.

☐ **B.** The sum of all marginal frequencies equals the total frequency.

☐ **C.** The sum of all marginal frequencies in a row equals the total frequency.

☐ **D.** The sum of all joint frequencies in a column equals the marginal frequency at the bottom of the column.

☐ **E.** A relative frequency is the ratio of a joint frequency and a marginal frequency.

26. **SAT/ACT** In a two-way frequency table, the joint frequency in a cell is 8 and the marginal frequency in the same row is 32. What is the conditional relative frequency for the cell?

Ⓐ 0.12

Ⓑ 0.20

Ⓒ 0.25

Ⓓ 0.40

Ⓔ 0.50

27. **Performance Task** A high school offers a prep course for students who are taking a retest for a college entrance exam.

- Of 25 students who took the prep course, 20 scored at least 50 points higher on the retest than on the original exam.
- Overall, 100 students took the retest and 50 students scored at least 50 points higher on the retest than on the original exam.

Part A Create a two-way frequency table to organize the data.

Part B Funding for the prep course may be cut because more students scored at least 50 points higher on the retest without taking the prep course. Do you agree with this decision? If not, how could you use a two-way frequency table to construct an argument to keep the funding?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.DP.1.1–Given a set of data, select an appropriate method to represent the data, depending on whether it is numerical or categorical data and on whether it is univariate or bivariate. **Also DP.1.2**

MA.K12.MTR.7.1

Text Message

Text messages used to be just that: text only. Now you can send multimedia messages (or MMS) with emojis, images, audio, and videos. Did you know Finland was the first country to offer text messaging to phone customers?

Some people send and receive so many texts that they use textspeak to make typing faster. RU 1 of them? You will see one person keep track of his text messages in this Modeling Mathematics in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Is your refined conjecture between the highs and lows you set up earlier?
8. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC

10

Topic Review

TOPIC ESSENTIAL QUESTION

1. How do you use statistics to model situations and solve problems?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A(n) __________ is the difference between an actual and a predicted data value.
3. __________ indicate the frequency of a single option for one category.
4. A trend line that most closely models the relationship between two variables displayed in a scatter plot is the __________.
5. __________ is the ratio of the joint frequency and the related marginal frequency.

- conditional relative frequency
- joint frequencies
- residual
- line of best fit
- joint relative frequency

Concepts & Skills Review

LESSON 10-1 Representing Numerical Data

Quick Review

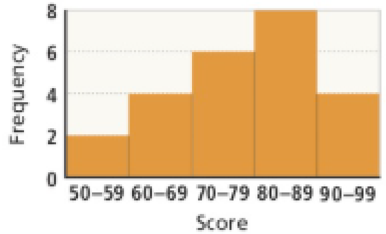
Line plots show counts of values within data sets. **Histograms** show the distribution of values within a data set in ranges or intervals. **Box plots** show the center and spread of a distribution using a five-number summary.

Example

Different types of displays can highlight different aspects of a univariate data set. The table below shows a class's math test scores. Create a histogram of the data.

83	92	56	63	80	91	78	59
75	79	62	85	81	90	82	74
60	95	88	82	77	74	68	82

Break the scores into intervals of 10.



Practice & Problem Solving

6. **Generalize** In what situations would a box plot be the best display for a data set?

7. What does the histogram in the Example tell you about the class's math test scores?

For the data set below, create the data display that best reveals the answer to the question. Explain your reasoning.

8. What is the frequency of the data value 35?

30	33	35	39
37	35	31	36
39	30	35	35

Choose the most appropriate data display to answer each question about a data set. Explain.

9. What are the frequencies for each interval of 2 points?

10. How many data values are less than any given value in the data set?

LESSON 10-2 Representing Categorical Data

Quick Review

Univariate categorical data is data that can be placed into categories and is counted with a single measure.

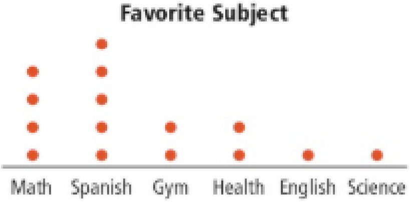
Line plots, bar charts, circle graphs, frequency tables, and relative frequency tables are methods for displaying univariate categorical data. The displays have various strengths and weaknesses. Choosing an appropriate display can help you highlight different aspects of a data set.

Example

Students in Micah's class are asked to state their favorite subject. Which type of display could Micah use to show what the most popular subject or subjects are?

Favorite Subject				
Health	Math	Spanish	Science	Spanish
Math	Gym	Gym	Spanish	English
Math	Math	Spanish	Health	Spanish

Most displays will show the mode of a categorical data set. A line plot is a simple way to do this.



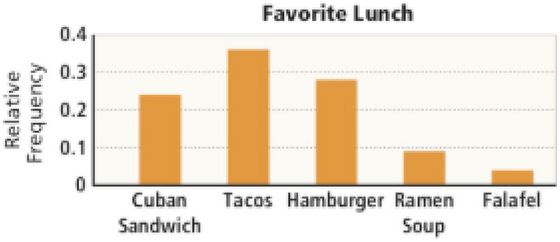
Practice & Problem Solving

For situation, determine which representation is more appropriate to represent the data. Explain your answer. Tell whether the data is categorical or numerical.

11. A technology company surveys a group of 100 people to determine which of three smartphone sizes they prefer: *line plot* or *stem-and-leaf plot*

12. Halona records where her cat sleeps in the house each afternoon. She describes the locations as, *couch, windowsill,* and *rug.* Halona wants to show that her cat sleeps on the windowsill about twice as often as it sleeps on the rug. *circle graph* or *line plot*

May reads a story online about people's favorite lunches. The article includes the graph below. Answer each question about the graph.



13. What type of data is represented? Explain.

14. What information cannot be determined by looking at the graph?

15. The graph highlights the relative popularity of the various lunches. How would you compare the popularity of tacos to the popularity of ramen soup?

16. The height of the bar representing the Cuban sandwich is 0.24. If 135 people were surveyed, roughly how many said that the Cuban sandwich was their favorite lunch?

LESSON 10-3 Representing Bivariate Data

Quick Review

Data formed by two sets of data is called **bivariate**. Bivariate data can be numerical or categorical. Scatter plots and line plots are typically used to display bivariate numerical data. Frequency tables and segmented bar charts can be used to represent bivariate categorical data.

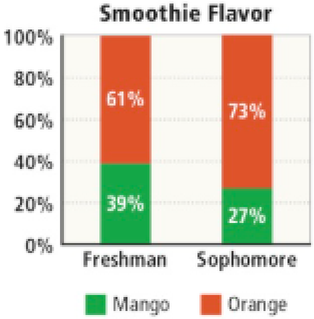
Example

Which display would be more appropriate to represent temperature data measured hourly over a 6-hour period, a *scatterplot* or a *line graph*? Explain.

The data is bivariate numerical so either display could be used. While the data points are discrete, the domain is continuous over the 6-hour interval. Using a line graph can help the viewer visualize how the temperature changes.

Practice & Problem Solving

Terry surveyed freshmen and sophomores in his high school to determine which smoothie flavor they preferred. He made a graph to represent his results. Use the display to answer the following questions.



17. What type of data is represented? Explain.
18. What information cannot be determined by looking at the graph?
19. Is there a difference in the preferences of the freshmen and the sophomores? Use the graph to support your argument.
20. **Represent and Connect** How is segmented bar graph similar to a circle graph? How is it different?

LESSON 10-4 Analyze Lines of Fit

Quick Review

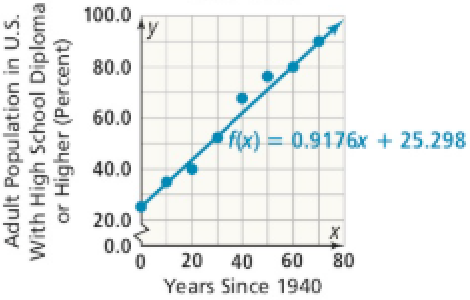
You can model data that suggests a linear association by drawing a **trendline** or using technology to calculate the **line of best fit**. Once there is a linear model for data with a positive or negative association there is a **positive** or **negative correlation**.

Residuals are the differences between the actual values and the predicted values based on the linear model. Residuals can be analyzed to help determine the strength of the correlation.

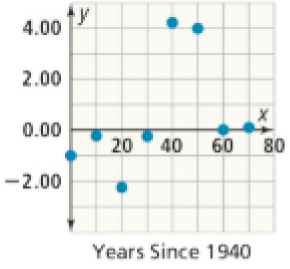
Example

The scatter plot shows the percentage of American adults with a high school diploma or higher from 1940 to 2010. Based on the residual plot below the scatter plot, how appropriate is the linear model for the data?

Educational Attainment 1940–2010



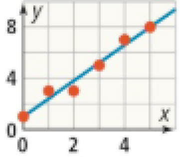
Residual Plot

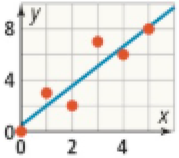


The residual plot shows the residuals distributed above and below the x-axis and clustered somewhat close to the x-axis. The linear model is likely a good fit for the data.

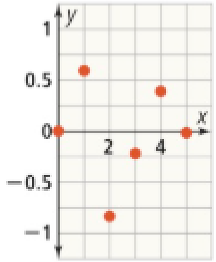
Practice & Problem Solving

21. **Two data sets and their lines of best fit are shown below. Describe the correlation for each data set. Use the residual plots to explain which correlation is stronger.**

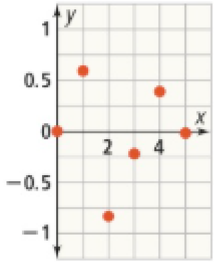




Residual Plot



Residual Plot



22. **Apply Math Models** The table shows the winning times for the 100-meter run in the Olympics since 1928. What is the equation of the line of best fit for the data? What do the slope and y-intercept represent? Estimate the winning time in 2010, and predict the winning time in 2020.

Year	Time (s)	Year	Time (s)
1928	10.80	1980	10.25
1932	10.30	1984	9.99
1936	10.30	1988	9.92
1948	10.30	1992	9.96
1952	10.40	1996	9.84
1956	10.50	2000	9.87
1960	10.20	2004	9.85
1964	10.00	2008	9.69
1968	9.95	2012	9.63
1972	10.14	2016	9.81
1976	10.06		

LESSON 10-5 Analyzing Frequency Tables

Quick Review

Two-way frequency tables show relationships between two sets of categorical data. **Joint frequencies** indicate the frequency of one category. **Marginal frequencies** indicate the total frequency for each category.

Joint relative frequency is the ratio, or percent, of the joint frequency to the total. **Marginal relative frequency** is the ratio, or percent, of the marginal frequency to the total. **Conditional relative frequency** is the ratio of the joint frequency and the related marginal frequency.

Example

A teacher asked her 30 students to choose between the museum or the zoo for a class trip. Out of 12 male students, 5 chose to go to the museum. Out of the 13 students who chose the zoo, 6 were female.

Construct a two-way frequency table. What trends do the results suggest?

	Museum	Zoo	Totals
Male	5	■	12
Female	■	6	■
Totals	■	13	30

$12 - 5 = 7$, 7 male students chose the zoo.

$30 - 12 = 17$; 17 students chose the museum.

$17 - 5 = 12$; 12 female students chose the museum.

$12 + 6 = 18$; 18 students are female.

	Museum	Zoo	Totals
Male	5	7	12
Female	12	6	18
Totals	17	13	30

The joint frequencies suggest that males prefer the zoo and females prefer the museum. The marginal frequencies suggest that all respondents showed a slight preference for going to the museum.

Practice & Problem Solving

In a survey, TV viewers can choose between two movies. 40 men and 10 women choose the action movie that is featured. Of the 30 people who chose the comedy, 20 are women and 10 are men.

23. Make a two-way frequency table to organize the data. Is it reasonable to say that more people surveyed prefer action movies? Explain.

A city planner surveyed residents to see if there is a relationship between bicycle ownership and bus ridership. The survey found the following:

- 72% of those surveyed who owned a bicycle rode the bus less than once a week.
- 68% of those surveyed who did not own a bicycle rode the bus at least once a week.
- Overall, 76% of those surveyed owned a bicycle.

24. Copy and complete the relative frequency table.

	Owns a bicycle	Does not own a bicycle	Totals
$\geq$ 1 time per week	■	■	1.00
$<$ 1 time per week	■	■	1.00

25. Residents who owned a bicycle are one of the subgroups in the study. So, 76% represents a marginal relative frequency. Copy and complete the table.

	Owns a bicycle	Does not own a bicycle	Totals
$\geq$ 1 time per week	■	■	0.76
$<$ 1 time per week	■	■	■
Totals	■	■	1.00

26. **Communicate and Justify** Does the survey suggest an association between bicycle ownership and bus ridership? Explain.

Visual Glossary

English

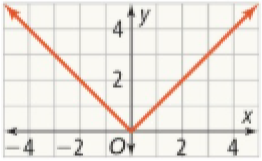
Spanish

A

Absolute value function $f(x) = |x|$

Función de valor absoluto $f(x) = |x|$

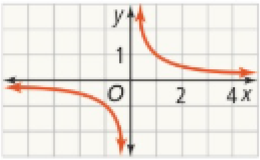
Example



Asymptote A line that the graph of a function gets closer to as x or y gets larger in absolute value.

Asintota Linea recta a la que la gráfica de una función se acerca indefinidamente, mientras el valor absoluto de x o y aumenta.

Example

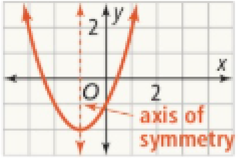


The y-axis is a vertical asymptote for $y = \frac{1}{x}$. The x-axis is a horizontal asymptote for $y = \frac{1}{x}$.

Axis of symmetry The line that intersects the vertex, and divides the graph into two congruent halves that are reflections of each other.

Eje de simetría El eje de simetría es la línea que corta el vértice y divide la gráfica en dos mitades congruentes que son reflexiones una de la otra.

Example



C

Causation When a change in one quantity causes a change in a second quantity. A correlation between quantities does not always imply causation.

Causalidad Cuando un cambio en una cantidad causa un cambio en una segunda cantidad. Una correlación entre las cantidades no implica siempre la causalidad.

Closure property A set of numbers is closed under an operation when the result of the operation is also part of the same set of numbers.

Propiedad de cerradura Un conjunto de números está cerrado bajo una operación cuando el resultado de la operación también forma parte del mismo conjunto de números.

Example The set of integers is closed under addition because the sum of two integers is always an integer.

English

Spanish

Common difference The difference between consecutive terms of an arithmetic sequence.

Diferencia común La diferencia común es la diferencia entre los términos consecutivos de una progresión aritmética.

Example The common difference is 3 in the arithmetic sequence 4, 7, 10, 13, ...

Completing the square The process of adding $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to form a perfect-square trinomial.

Completar el cuadrado El proceso de sumar $\left(\frac{b}{2}\right)^2$ a $x^2 + bx$ para formar un trinomio cuadrado perfecto.

Example $x^2 + 6x - 7 = 9$ is rewritten as $(x + 3)^2 = 25$ by completing the square.

Compound inequalities Two inequalities that are joined by *and* or *or*.

Desigualdades compuestas Dos desigualdades que están enlazadas por medio de una *y* o una *o*.

Examples $5 < x$ and $x < 10$
$14 < x$ or $x \leq -3$

Compound interest Interest paid on both the principal and the interest that has already been paid.

Interés compuesto Interés calculado tanto sobre el capital como sobre los intereses ya pagados.

Example For an initial deposit of \$1,000 at a 6% interest rate with interest compounded quarterly, the function $y = 1000\left(\frac{0.06}{4}\right)^{4x}$ gives the account balance y after x years.

Conditional relative frequency The ratio of the joint frequency and the related marginal frequency.

Frecuencia relativa condicional La razón de la frecuencia conjunta y la frecuencia marginal relacionada.

Example

	Afternoon	Evening	Totals
Student	$\frac{90}{140} = 64\%$	$\frac{50}{140} = 36\%$	$\frac{140}{140} = 100\%$
Adult	$\frac{20}{60} = 33\%$	$\frac{40}{60} = 67\%$	$\frac{60}{60} = 100\%$

Constant ratio The number that an exponential function repeatedly multiplies an initial amount by.

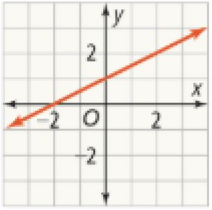
Razón constante El número por el que una función exponencial multiplica repetidamente a una cantidad inicial.

Example In an exponential function of the form $f(x) = ab^x$, b is the constant ratio.

Continuous A graph that is unbroken.

Continua Una gráfica continua es una gráfica ininterrumpida.

Example



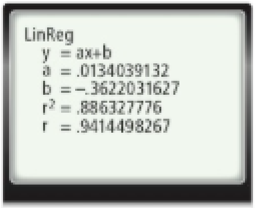
English

Spanish

Correlation coefficient A number from –1 to 1 that tells you how closely the equation of the line of best fit models the data. It is represented by the variable, r.

Coeficiente de correlación Número de –1 a 1 que indica con cuánta exactitud la línea de mejor encaje representa los datos. Se representa con la variable r.

Example



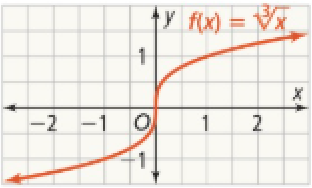
```
LinReg
  y  = ax+b
  a  = .0134039132
  b  = –.3622031627
  r² = .886327776
  r  = .9414498267
```

The correlation coefficient is approximately 0.94.

Cube root function $f(x) = \sqrt[3]{x}$

Función de raíz cúbica $f(x) = \sqrt[3]{x}$

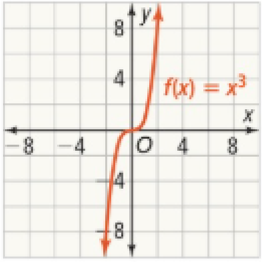
Example



Cubic function $f(x) = x^3$

Función cúbica $f(x) = x^3$

Example



D

Decay factor 1 minus the decay rate in an exponential function when $0 < b < 1$.

Factor de decremento 1 menos la tasa de decremento en una función exponencial si $0 < b < 1$.

Example The decay factor of the function $y = 5(0.3)^x$ is 0.3.

Degree of a monomial The sum of the exponents of the variables of a monomial.

Grado de un monomio La suma de los exponentes de las variables de un monomio.

Example $-4x^3y^2$ is a monomial of degree 5.

Degree of a polynomial The highest degree of any term of the polynomial.

Grado de un polinomio El grado de un polinomio es el grado mayor de cualquier término del polinomio.

Example The polynomial $P(x) = x^6 + 2x^3 - 3$ has degree 6.

English

Spanish

Difference of two squares A difference of two squares is an expression of the form $a^2 - b^2$. It can be factored as $(a + b)(a - b)$.

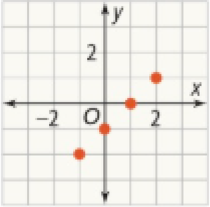
Diferencia de dos cuadrados La diferencia de dos cuadrados es una expresión de la forma $a^2 - b^2$. Se puede factorizar como $(a + b)(a - b)$.

Examples $25a^2 - 4 = (5a + 2)(5a - 2)$
$m^6 - 1 = (m^3 + 1)(m^3 - 1)$

Discrete A graph composed of isolated points.

Discreta Una gráfica discreta es compuesta de puntos aislados.

Example



Discriminant The discriminant of a quadratic equation of the form $ax^2 + bx + c = 0$ is $b^2 - 4ac$. The value of the discriminant determines the number of solutions of the equation.

Discriminante El discriminante de una ecuación cuadrática $ax^2 + bx + c = 0$ es $b^2 - 4ac$. El valor del discriminante determina el número de soluciones de la ecuación.

Example The discriminant of $2x^2 + 9x - 2 = 0$ is 97.

Domain (of a relation or function) The possible values for the input of a relation or function.

Dominio (de una relación o función) Posibles valores de entrada de una relación o función.

Example In the function $f(x) = x + 22$, the domain is all real numbers.

E

Elements (of a set) Members of a set.

Elementos Partes integrantes de un conjunto.

Example Cats and dogs are elements of the set of mammals.

Exponential decay A situation modeled with a function of the form $y = ab^x$, where $a > 0$ and $0 < b < 1$.

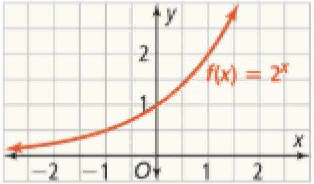
Decremento exponencial Para $a > 0$ y $0 < b < 1$, la función $y = ab^x$ representa el decremento exponencial.

Example $y = 5(0.1)^x$

Exponential function The function $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

Función exponencial La función $f(x) = b^x$, donde $b > 0$ y $b \neq 1$.

Example



Exponential growth A situation modeled with a function of the form $y = ab^x$, where $a > 0$ and $b > 1$.

Incremento exponencial Para $a > 0$ y $b > 1$, la función $y = ab^x$ representa el incremento exponencial.

Example $y = 100(2)^x$

English

Spanish

Extrapolation The process of predicting a value outside the range of known values.

Extrapolación Proceso que se usa para predecir un valor por fuera del ámbito de los valores dados.

F

Family of functions A group of functions that use the same common operation in their equation forms.

Familia de funciones Un grupo de funciones que usan la misma operación común en su forma de ecuación.

Example $f(x) = 3x + 7$ and $f(x) = \frac{2}{3}x - 9$ are members of the linear family of functions.

Formula An equation that states a relationship among quantities.

Fórmula Ecuación que establece una relación entre cantidades.

Example The formula for the volume V of a cylinder is $V = \pi r^2 h$, where r is the radius of the cylinder and h is its height.

Function A relation in which each element of the domain corresponds with exactly one element in the range.

Función Una relación en la cual cada elemento del dominio se corresponde con exactamente un elemento del rango.

Example Earned income is a function of the number of hours worked. If you earn $4.50/h, then your income is expressed by the function $f(h) = 4.5h$.

Function notation A method for writing variables as a function of other variables.

Notación de una función Un método para escribir variables como una función de otras variables.

Example $f(x) = 3x - 8$ is in function notation.

G

Growth factor 1 plus the growth rate in an exponential function when $b > 1$.

Factor incremental 1 más la tasa de incremento en una función exponencial si $b > 1$.

Example The growth factor of $y = 7(1.3)^x$ is 1.3.

I

Identity An equation that is true for every value.

Identidad Una ecuación que es verdadera para todos los valores.

Example $5 - 14x = 5\left(1 - \frac{14}{5}x\right)$ is an identity because it is true for any value of x.

Interpolation The process of estimating a value between two known quantities.

Interpolación Proceso que se usa para estimar el valor entre dos cantidades dadas.

English

Spanish

J

Joint frequency The frequency of a single option for one category.

Frecuencia conjunta La frecuencia de una única opción por categoría.

Example

	Afternoon	Evening	Totals
Student	90	50	140
Adult	20	40	60
Totals	110	90	200

90, 50, 20, and 40 are joint frequencies.

Joint relative frequency The ratio, or percent, of the joint frequency to the total.

Frecuencia relativa conjunta La razón, o porcentaje, de la frecuencia conjunta al total.

Example

	Afternoon	Evening	Totals
Student	$\frac{90}{200} = 45\%$	$\frac{50}{200} = 25\%$	$\frac{140}{200} = 70\%$
Adult	$\frac{20}{200} = 10\%$	$\frac{40}{200} = 20\%$	$\frac{60}{200} = 30\%$
Totals	$\frac{110}{200} = 55\%$	$\frac{90}{200} = 45\%$	$\frac{200}{200} = 100\%$

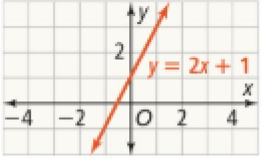
45%, 25%, 10%, and 20% are joint relative frequencies.

L

Linear function A function whose graph is a line is a linear function. You can represent a linear function with a linear equation.

Función lineal Una función cuya gráfica es una recta es una función lineal. La función lineal se representa con una ecuación lineal.

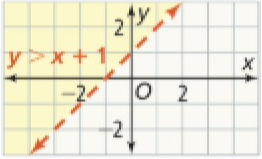
Example



Linear inequality in two variables An inequality in two variables whose graph is a region of the coordinate plane that is bounded by a line. Each point in the region is a solution of the inequality.

Desigualdad lineal con dos variables Una desigualdad lineal es una desigualdad de dos variables cuya gráfica es una región del plano de coordenadas delimitado por una recta. Cada punto de la región es una solución de la desigualdad.

Example



Linear regression A method used to calculate the line of best fit.

Regresión lineal Método que se utiliza para calcular la línea de mejor ajuste.

English

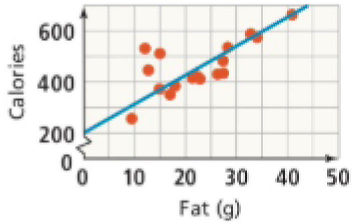
Spanish

Line of best fit The most accurate trend line on a scatter plot showing the relationship between two sets of data.

Recta de mayor aproximación La línea de tendencia en un diagrama de puntos que más se acerca a los puntos que representan la relación entre dos conjuntos de datos.

Example

Calories and Fat for Fast Food Meals



Literal equation An equation expressed in variables.

Ecuación literal Ecuación que se expresa con variables.

Example $4x + 2y = 18$ is a literal equation.

M

Marginal frequency The total frequency for each option or category.

Frecuencia marginal La frecuencia total para cada opción o categoría.

Example

	Afternoon	Evening	Totals
Student	90	50	140
Adult	20	40	60
Totals	110	90	200

140, 60, 110, and 90 are marginal frequencies.

Marginal relative frequency The ratio, or percent, of the marginal frequency to the total.

Frecuencia relativa marginal La razón, o porcentaje, de la frecuencia marginal al total.

Example

	Afternoon	Evening	Totals
Student	$\frac{90}{200} = 45\%$	$\frac{50}{200} = 25\%$	$\frac{140}{200} = 70\%$
Adult	$\frac{20}{200} = 10\%$	$\frac{40}{200} = 20\%$	$\frac{60}{200} = 30\%$
Totals	$\frac{110}{200} = 55\%$	$\frac{90}{200} = 45\%$	$\frac{200}{200} = 100\%$

70%, 30%, 55%, and 45% are marginal relative frequencies.

Monomial A real number, a variable, or a product of a real number and one or more variables with whole-number exponents.

Monomio Número real, variable o el producto de un número real y una o más variables con números enteros como exponentes.

Example 9, n, and $-5xy^2$ are examples of monomials.

English

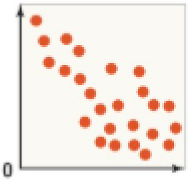
Spanish

N

Negative association When y-values tend to decrease as x-values increase, the two data sets have a negative association.

Asociación negativa Cuando los valores de y tienden a disminuir a medida que los valores de x aumentan, los dos conjuntos de datos tienen una asociación negativa.

Example



Negative correlation When data with a negative association are modeled with a line, there is a negative correlation.

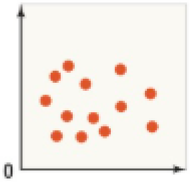
Correlación negativa Cuando los datos que tienen una asociación negativa se representan con una linea, hay una correlación negativa.

Example

No association When there is no general relationship between x-values and y-values, the two data sets have no association.

Sin asociación Cuando no existe ninguna relación general entre los valores de x y los valores de y, los dos conjuntos de datos no tienen ninguna asociación.

Example

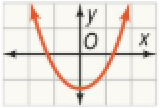


P

Parabola The graph of a quadratic function.

Parábola La gráfica de una función cuadrática.

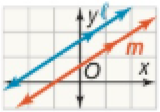
Example



Parallel lines Two lines in the same plane that never intersect. Parallel lines have the same slope.

Rectas paralelas Dos rectas situadas en el mismo plano que nunca se cortan. Las rectas paralelas tienen la misma pendiente.

Example



English

Spanish

Perfect-square trinomial Any trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$. It is the result when a binomial is squared.

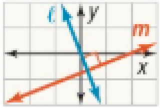
Trinomio cuadrado perfecto Todo trinomio de la forma $a^2 + 2ab + b^2$ ó $a^2 - 2ab + b^2$. Es el resultado cuando un binomio se eleva al cuadrado.

Example $(x + 3)^2 = x^2 + 6x + 9$

Perpendicular lines Lines that intersect to form right angles. Two lines are perpendicular if the product of their slopes is –1.

Rectas perpendiculares Rectas que forman ángulos rectos en su intersección. Dos rectas son perpendiculares si el producto de sus pendientes es –1.

Example



Point-slope form A linear equation of a nonvertical line written as $y - y_1 = m(x - x_1)$. The line passes through the point (x_1, y_1) with slope m.

Forma punto-pendiente La ecuación lineal de una recta no vertical que pasa por el punto (x_1, y_1) con pendiente m está dada por $y - y_1 = m(x - x_1)$.

Example An equation with a slope of $-\frac{1}{2}$ passing through (2, –1) would be written $y + 1 = -\frac{1}{2}(x - 2)$ in point-slope form.

Polynomial A monomial or the sum or difference of two or more monomials.

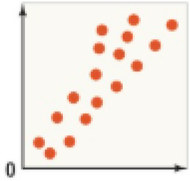
Polinomio Un monomio o la suma o diferencia de dos o más monomios.

Example $2x^2$, $3x + 7$, 28, and $-7x^3 - 2x^2 + 9$ are all polynomials.

Positive association When y-values tend to increase as x-values increase, the two data sets have a positive association.

Asociación positiva Cuando los valores de y tienden a aumentar a medida que los valores de x aumentan, los dos conjuntos de datos tienen una asociación positiva.

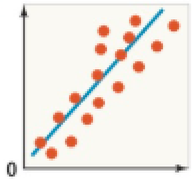
Example



Positive correlation When data with a positive association are modeled with a line, there is a positive correlation.

Correlación positiva Cuando los datos que tienen una asociación positiva se representan con una línea, hay una correlación positiva.

Example



English	Spanish
Product Property of Cube Roots $\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b}$	**Propiedad del producto de las raíces cúbicas** $\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b}$
Example $\sqrt[3]{27 \cdot 64} = \sqrt[3]{27} \cdot \sqrt[3]{64}$	
Product Property of Square Roots $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, when both a and b are greater than or equal to 0.	**Propiedad del producto de las raíces cuadradas** $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, cuando tanto a como b son mayores que o iguales a 0.
Example $\sqrt{16 \cdot 25} = \sqrt{16} \cdot \sqrt{25}$	
Q	
Quadratic equation An equation of the second degree.	**Ecuación cuadrática** Una ecuación de segundo grado.
Example $4x^2 + 9x - 5 = 0$	
Quadratic formula If $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.	**Fórmula cuadrática** Si $ax^2 + bx + c = 0$ y $a \neq 0$, entonces $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
Example $2x^2 + 10x + 12 = 0$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-10 \pm \sqrt{10^2 - 4(2)(12)}}{2(2)}$ $x = \frac{-10 \pm \sqrt{4}}{4}$ $x = \frac{-10 + 2}{4}$ or $\frac{-10 - 2}{4}$ $x = -2$ or -3	
Quadratic function A function of the form $y = ax^2 + bx + c$, where $a \neq 0$. The graph of a quadratic function is a parabola, a U-shaped curve that opens up or down.	**Función cuadrática** La función $y = ax^2 + bx + c$, en la que $a \neq 0$. La gráfica de una función cuadrática es una parábola, o curva en forma de U que se abre hacia arriba o hacia abajo.
Example $y = 5x^2 - 2x + 1$ is a quadratic function.	
Quadratic parent function The simplest quadratic function $f(x) = x^2$ or $y = x^2$.	**Función cuadrática madre** La función cuadrática más simple $f(x) = x^2$ ó $y = x^2$.
Example $y = x^2$ is the parent function for the family of quadratic equations of the form $y = ax^2 + bx + c$.	
Quadratic regression A method used to find the quadratic function that best fits a data set.	**Regresión cuadrática** Método que se utiliza para hallar la función cuadrática que se ajusta mejor a un conjunto de datos.
R	
Range (of a relation or function) The possible values of the output, or dependent variable, of a relation or function.	**Rango (de una relación o función)** El conjunto de todos los valores posibles de la salida, o variable dependiente, de una relación o función.
Example In the function $y = \|x\|$, the range is the set of all nonnegative numbers.	
Rational exponent Another way to express radicals.	**Exponente racional** Otra forma de expresar los radicales.
Example $\sqrt[3]{x} = x^{\frac{1}{3}}$ $\frac{1}{3}$ is the rational exponent.	

English

Spanish

Reciprocal The reciprocal of a number is 1 divided by that number.

Reciproco El reciproco de un número es 1 dividido entre ese número.

Example $\frac{2}{5}$ and $\frac{5}{2}$ are reciprocals because
$1 \div \frac{2}{5} = \frac{5}{2}$.

Residual The difference between the *y*-value of a data point and the corresponding *y*-value of a model for the data set.

Residuo La diferencia entre el valor de *y* de un punto y el valor de *y* correspondiente a ese punto en el modelo del conjunto de datos.

Root The input values for which the related function is zero.

Ráiz Los valores de entrada para los cuales la función relacionada es cero.

S

Set A well-defined collection of elements.

Conjunto Un grupo bien definido de elementos.

Example The set of integers:
$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Set-builder notation Set-builder notation uses a verbal description or an inequality to describe the numbers in a set.

Notación conjuntista La notación conjuntista usa una descripción verbal o una desigualdad para describir los números.

Example $\{x \mid x \text{ is a real number}\}$
$\{x \mid x > 3\}$

Simple interest Interest paid only on the principal.

Interés simple Intéres basado en el capital solamente.

Example The interest on \$1,000 at 6% for 5 years is \$1,000(0.06)5 = \$300.

Simplest form of a radical expression A radical expression where each radicand is greater than 1 and has no other perfect square factors under a square root or perfect cube factors under a cube root. Additionally, there should be no radicals in the denominator of any fraction.

Mínima expresión de una expresión radical. Una expresión radical donde cada radicando es mayor que 1 y no tiene otros factores que son cuadrados perfectos bajo una raíz cuadrada ni factores que son cubos perfectos bajo una raíz cúbica. Asimismo, no tiene radicales en el denominador de ninguna fracción.

Example Use properties of square and cube roots to reduce the expression $\frac{\sqrt{3} \cdot \sqrt{6} \cdot \sqrt[3]{54}}{\sqrt{20}}$.

$$\frac{\sqrt{3} \cdot \sqrt{6} \cdot \sqrt[3]{54}}{\sqrt{20}} = \frac{\sqrt{3 \cdot 6} \cdot \sqrt[3]{2 \cdot 27}}{\sqrt{4 \cdot 5}}$$
$$= \frac{\sqrt{9 \cdot 2} \cdot \sqrt[3]{2}\sqrt[3]{27}}{\sqrt{4}\sqrt{5}}$$
$$= \frac{3\sqrt{2} \cdot 3\sqrt[3]{2}}{2\sqrt{5}}$$
$$= \frac{9\sqrt{2} \cdot \sqrt[3]{2} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$$
$$= \frac{9\sqrt{10} \cdot \sqrt[3]{2}}{10}$$

Slope-intercept form The slope-intercept form of a linear equation is $y = mx + b$, where *m* is the slope of the line and *b* is the *y*-intercept.

Forma pendiente-intercepto La forma pendiente-intercepto es la ecuación lineal $y = mx + b$, en la que *m* es la pendiente de la recta y *b* es el punto de intersección de esa recta con el eje *y*.

Example $y = 8x - 2$

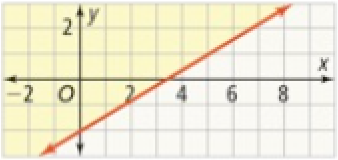
English

Spanish

Solution of an inequality in two variables Any ordered pair that makes the inequality true.

Solución de una desigualdad con dos variables Cualquier par ordenado que haga verdadera la desigualdad.

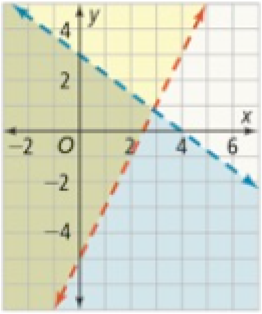
Example Each ordered pair in the yellow area and on the solid red line is a solution of $3x - 5y \leq 10$.



Solution of a system of linear inequalities Any ordered pair that makes all of the inequalities in the system true.

Solución de un sistema de desigualdades lineales Todo par ordenado que hace verdaderas todas las desigualdades del sistema.

Example



The shaded green area shows the solution of the system $\begin{matrix} y > 2x - 5 \\ 3x + 4y < 12 \end{matrix}$.

Square root function A function that contains the independent variable in the radicand.

Función de raíz cuadrada Una función que contiene la variable independiente en el radicando.

Example $y = \sqrt{2x}$ is a square root function.

Standard form of a linear equation The standard form of a linear equation is $Ax + By = C$, where A, B, and C are real numbers and A and B are not both zero.

Forma normal de una ecuación lineal La forma normal de una ecuación lineal es $Ax + By = C$, donde A, B y C son números reales, y donde A y B no son iguales a cero.

Example $6x - y = 12$

Standard form of a polynomial The form of a polynomial that places the terms in descending order by degree.

Forma normal de un polinomio Cuando el grado de los términos de un polinomio disminuye de izquierda a derecha, está en forma normal, o en orden descendente.

Example $15x^3 + x^2 + 3x + 9$

Standard form of a quadratic equation The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$.

Forma normal de una ecuación cuadrática Cuando una ecuación cuadrática se expresa de forma $ax^2 + bx + c = 0$.

Example $-x^2 + 2x - 9 = 0$

English

Spanish

Standard form of a quadratic function The standard form of a quadratic function is $f(x) = ax^2 + bx + C$, where $a \neq 0$.

Forma normal de una función cuadrática La forma normal de una función cuadrática es $f(x) = ax^2 + bx + C$, donde $a \neq 0$.

Example $f(x) = 2x^2 - 5x + 2$

Subset A subset of a set consists of elements from the given set.

Subconjunto Un subconjunto de un conjunto consiste en elementos del conjunto dado.

Example If $B = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{1, 2, 5\}$, then A is a subset of B.

System of linear inequalities Two or more linear inequalities using the same variables.

Sistema de desigualdades lineales Dos o más desigualdades lineales que usen las mismas variables.

Example $y \leq x + 11$
$y < 5x$

T

Transformation A transformation of a function maps each point of its graph to a new location.

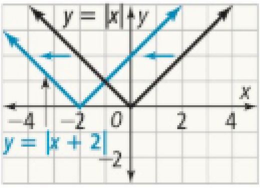
Transformación Una transformación de una función desplaza cada punto de su gráfica a una ubicación nueva.

Example Transformations can be translations, rotations, reflections, or dilations.

Translation A transformation that shifts the graph of a function the same distance horizontally, vertically, or both.

Translación Proceso de mover una gráfica horizontalmente, verticalmente o en ambos sentidos.

Example

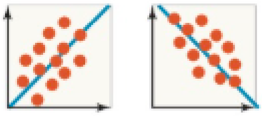


$y = |x + 2|$ is a translation of $y = |x|$.

Trend line A line that models the data in a scatter plot by showing the general direction of the data.

Línea de tendencia Una línea que representa los datos en un diagrama de puntos y muestra la dirección general de los datos.

Example



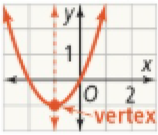
Positive Negative

V

Vertex The highest or lowest point on the graph of a function.

Vértice El punto más alto o más bajo de la gráfica de una función.

Example



English	Spanish
Vertex form of a quadratic function The function $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The vertex of the graph is at (h, k).	**Forma canónica de una función cuadrática** La función $f(x) = a(x - h)^2 + k$, donde $a \neq 0$. El vértice de la gráfica está en (h, k).

Example If the vertex form of a function is $f(x) = 5(x + 3)^2 + 7$, the vertex of the graph is $(-3, 7)$.

English	Spanish
Vertical motion model The vertical motion model is the quadratic function $h(t) = -16t^2 + v_0t + h_0$. The variable h represents the height of an object, in feet, t seconds after it is launched into the air. The term v_0 is the object's initial velocity and h_0 is its initial height.	**Modelo de movimiento vertical** El modelo de movimiento vertical es la función cuadrática $h(t) = -16t^2 + v_0t + h_0$. La variable h representa la altura en pies de un objeto t segundos después de lanzarlo al aire. El término v_0 es la velocidad inicial del objeto y h_0 es su altura inicial.

Example If an object is launched from a height of 10 ft with an initial velocity of 8 ft/s, then the equation of the object's height over time is $h(t) = -16t^2 + 8t + 10$.

Y

English	Spanish
y-intercept The y-coordinate of a point where a graph crosses the y-axis.	**Intercepto en y** Coordenada y por donde la gráfica cruza el eje de las y.

Example The y-intercept of $y = 5x + 2$ is 2.

Z

English	Spanish
Zero-Product Property For all real numbers a and b, if $ab = 0$, then $a = 0$ or $b = 0$.	**Propiedad del producto cero** Para todos los números reales a y b, si $ab = 0$, entonces $a = 0$ ó $b = 0$.

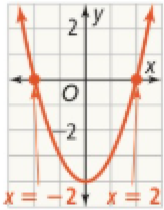
Example $x(x + 3) = 0$

$x = 0$ or $x + 3 = 0$

$x = 0$ or $x = -3$

English	Spanish
Zero of a function An x-intercept of the graph of a function.	**Cero de una función** Intercepto x de la gráfica de una función.

Example The zeros of $y = x^2 - 4$ are ± 2.



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