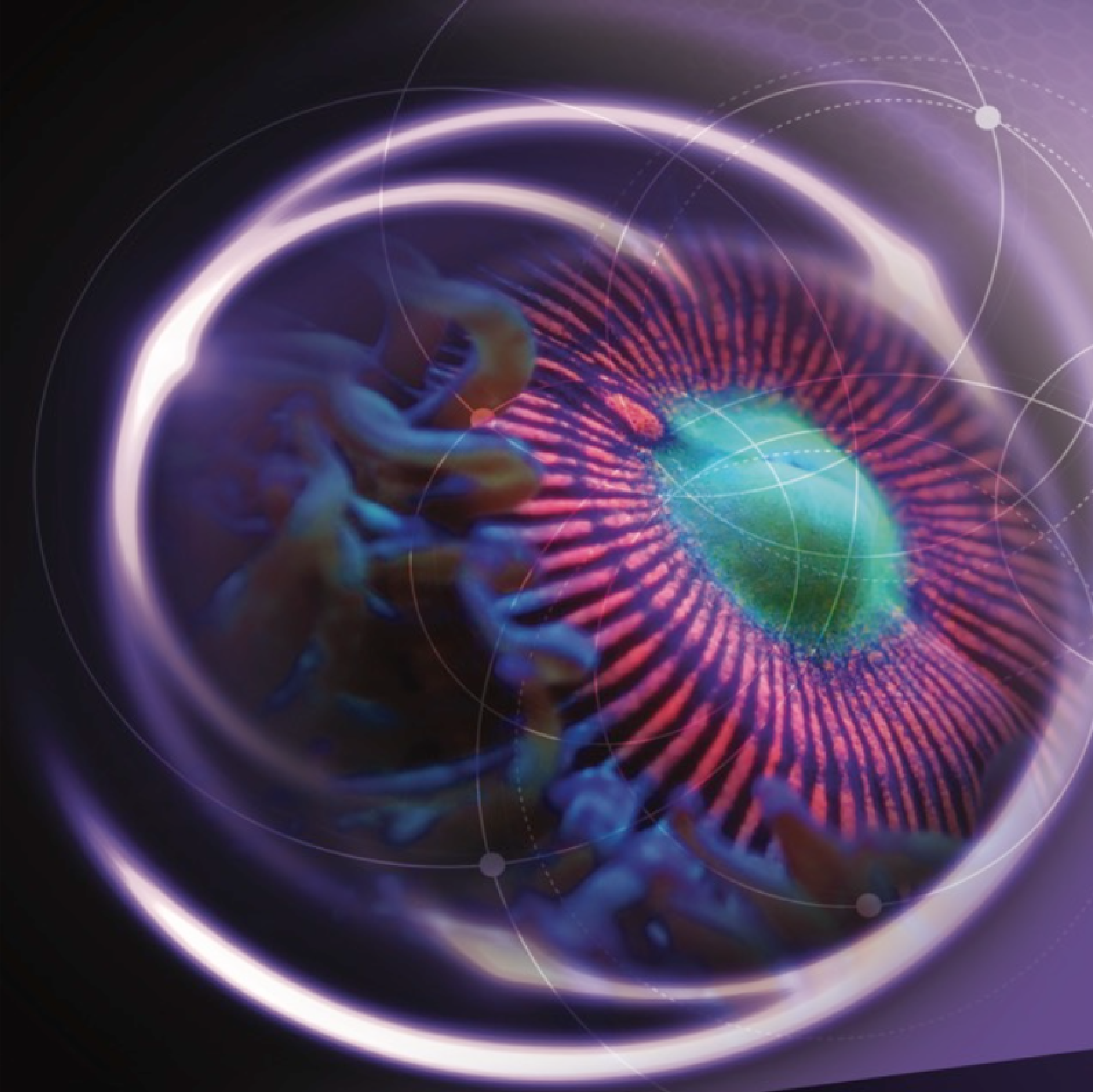

# enVision® Florida

## B.E.S.T. ALGEBRA 2

Student Edition

# enVision® Florida

## B.E.S.T. ALGEBRA 2

SAVVAS
LEARNING COMPANY

Savvas Learning Company LLC, 15 East Midland Avenue, Paramus, NJ 07652



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LEARNING COMPANY

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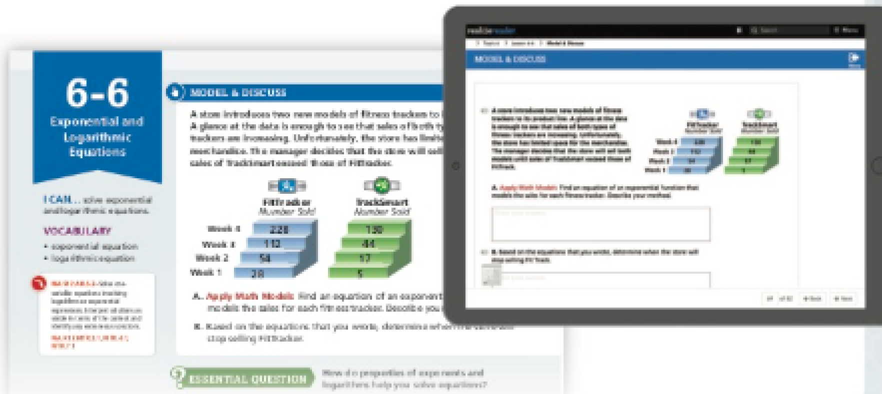
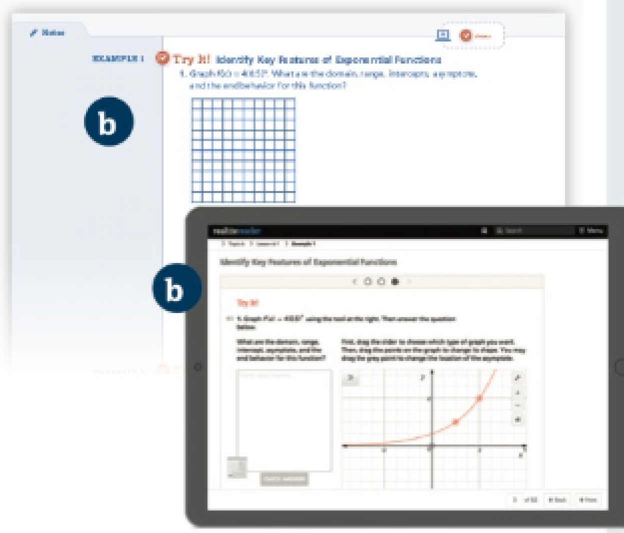
# About enVision Florida B.E.S.T. ALGEBRA 2

**enVision Florida B.E.S.T. Algebra 2 offers a carefully constructed lesson design to help you succeed in math.**

**Step 1** At the start of each lesson, you and your classmates will work together to come up with a solution strategy for the problem or task posed. After a class discussion, you'll be asked to reflect back on the processes and strategies you used in solving the problem.



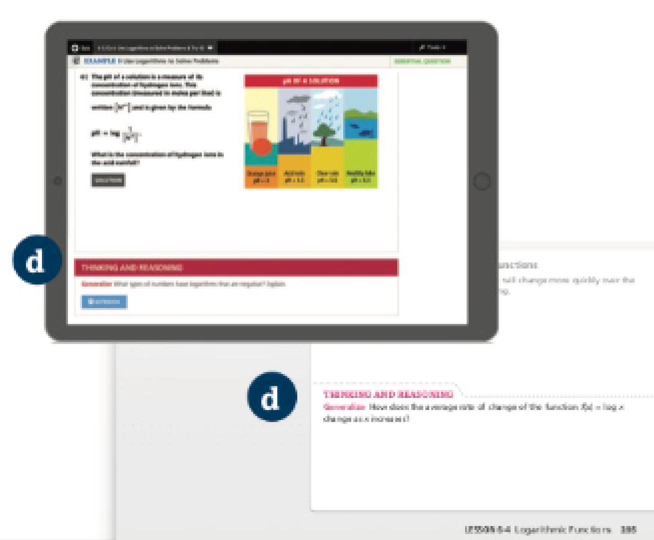

**Step 2** Next, your teacher will guide you through new concepts and skills for the lesson.

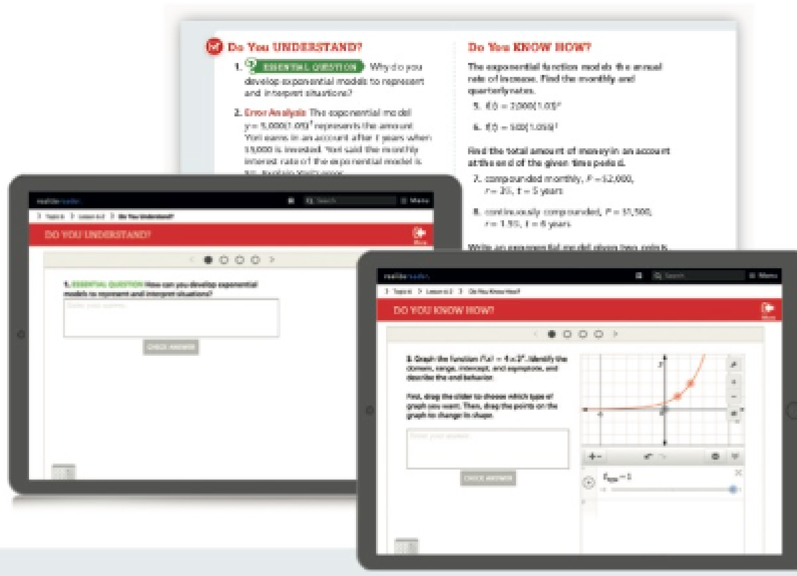

After each Example **a**, you work out a problem called the **Try It!** **b** to solidify your understanding of these concepts.

Side notes **c** help you with study tips, suggestions for avoiding common errors, and questions that support learning together and having a growth mindset.

In addition, you will periodically answer **Thinking and Reasoning** **d** questions to refine your thinking and problem-solving skills.

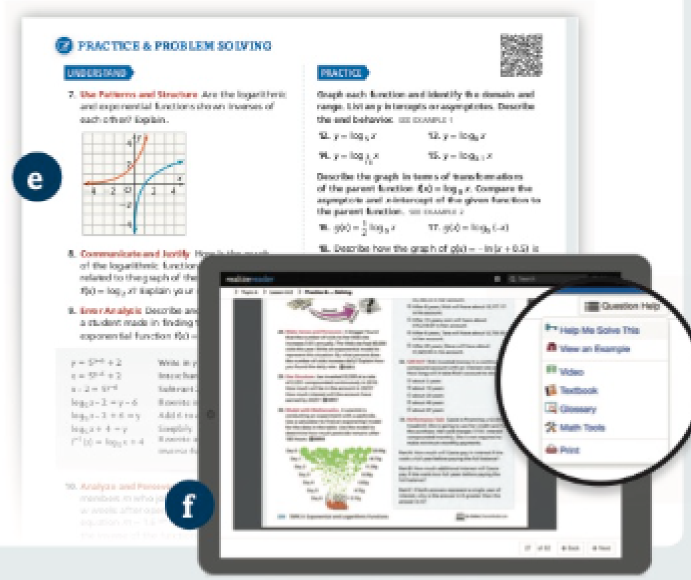


**Step 2 cont.** This part of the lesson concludes with a Lesson Check that helps you to know how well you are understanding the new content presented in the lesson. With the exercises in the **Do You Understand?** and **Do You Know How?**, you can gauge your understanding of the lesson concepts.

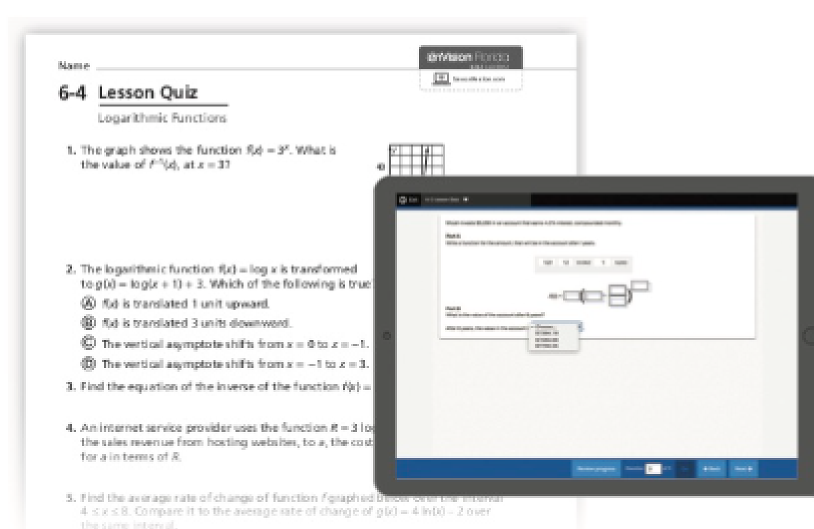



**Step 3** In Step 3, you will find a balanced exercise set with **Understand** exercises that focus on conceptual understanding, **Practice** exercises that target procedural fluency, and **Apply** exercises for which you apply concept and skills to real-world situations **e**.

The **Assessment and Practice** **f** exercises offer practice for high stakes assessments. Your teacher may have you complete the assignment in your Student Edition, Student Companion, or online at SavvasRealize.com.




**Step 4** Your teacher may have you take the Lesson Quiz after each lesson. You can take the quiz online or in print. To do your best on the quiz, review the lesson problems in that lesson.

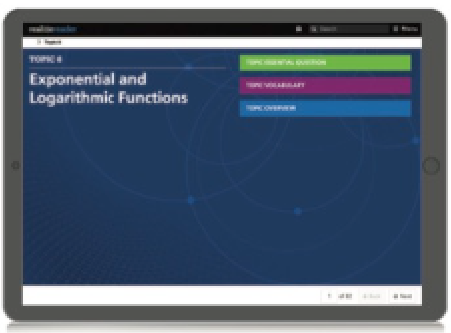


# Digital Resources

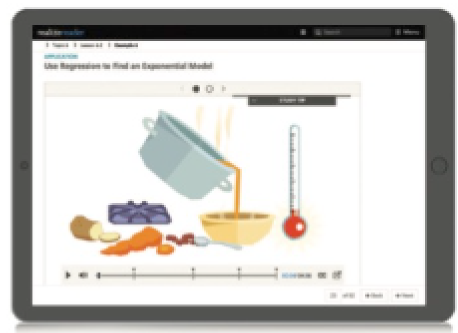
**Everything you need for math, anytime, anywhere.**

SavvasRealize.com is your gateway to all of the digital resources for **enVision® Florida B.E.S.T. Algebra 2.**

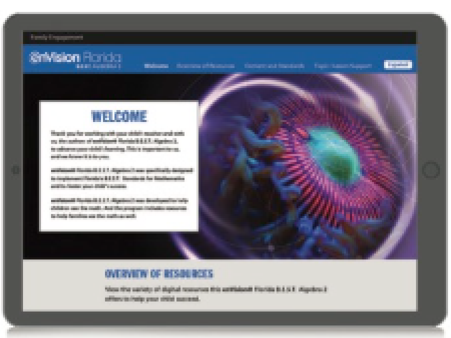
**INTERACTIVE STUDENT EDITION**
Log in to access your interactive student edition, called Realize Reader.



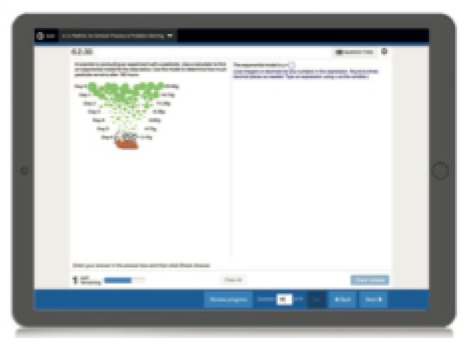

**ANIMATION** View and interact with real-world applications.



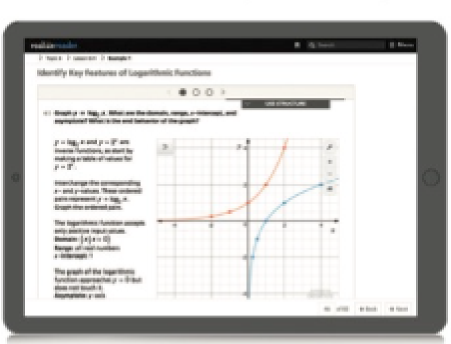
**FAMILY ENGAGEMENT**
Involve family in your learning.



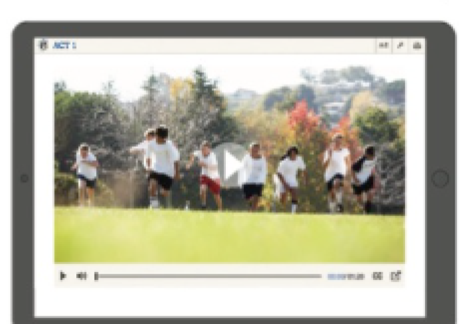

**PRACTICE**
Practice what you've learned.



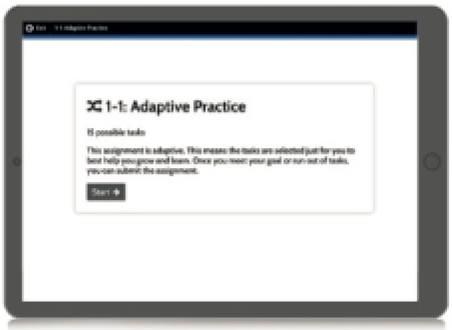
**ACTIVITIES** Complete *Explore & Reason, Model & Discuss, Critique & Explain* activities. Interact with *Examples* and *Try Its.*



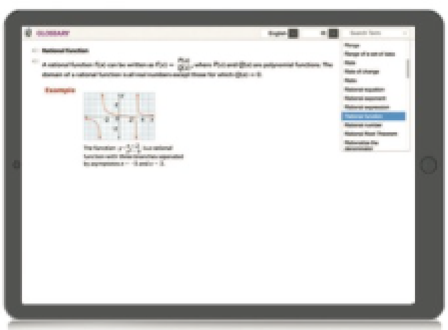
**VIDEOS** Watch clips to support Mathematical Modeling in 3 Acts Lessons and **enVision®** STEM Projects.



**ADAPTIVE PRACTICE**
Practice that is *just right* and *just for you.*

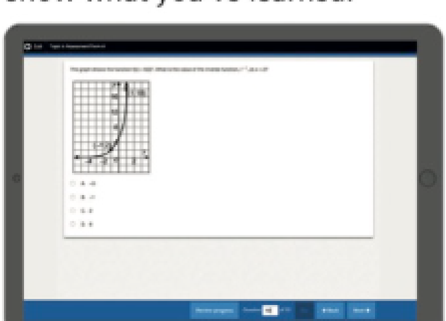



**GLOSSARY** Read and listen to English and Spanish definitions.

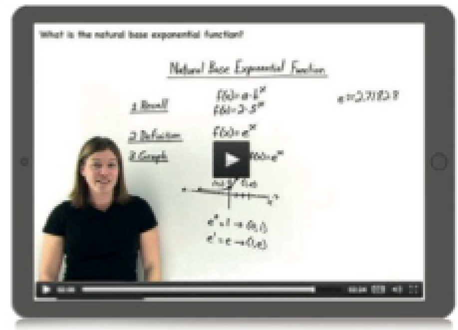



**CONCEPT SUMMARY** Review key lesson content through multiple representations.


**ASSESSMENT**
Show what you've learned.

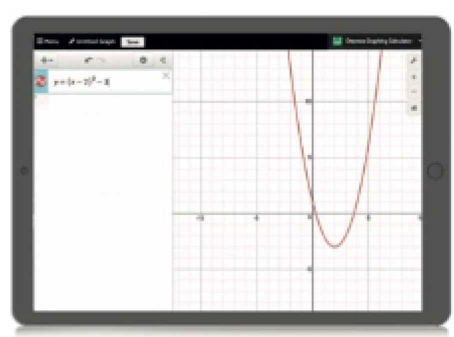


**TUTORIALS** Get help from Virtual Nerd, right when you need it.

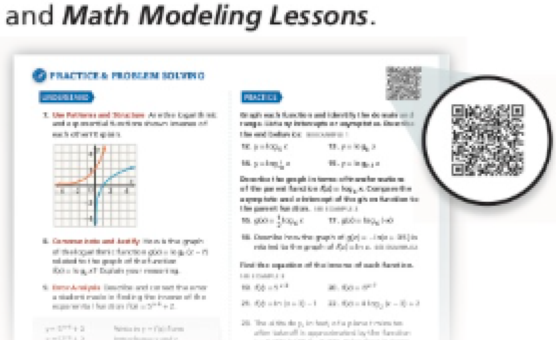



**MATH TOOLS** Explore math with digital tools and manipulatives.


**DESMOS** Use Anytime and as embedded Interactives in Lesson content.



**QR CODES** Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.



## Number Sense and Operations

**MA.912.NSO.1 Generate equivalent expressions and perform operations with expressions involving exponents, radicals or logarithms.**

MA.912.NSO.1.3 Generate equivalent algebraic expressions involving radicals or rational exponents using the properties of exponents.

*Clarification 1*: Within the Algebra 2 course, radicands are limited to monomial algebraic expressions.

MA.912.NSO.1.5 Add, subtract, multiply and divide algebraic expressions involving radicals.

*Clarification 1*: Within the Algebra 2 course, radicands are limited to monomial algebraic expressions.

MA.912.NSO.1.6 Given a numerical logarithmic expression, evaluate and generate equivalent numerical expressions using the properties of logarithms or exponents.

*Clarification 1*: Within the Mathematics for Data and Financial Literacy Honors course, problem types focus on money and business.

MA.912.NSO.1.7 Given an algebraic logarithmic expression, generate an equivalent algebraic expression using the properties of logarithms or exponents.

*Clarification 1*: Within the Mathematics for Data and Financial Literacy Honors course, problem types focus on money and business.

**MA.912.NSO.2 Represent and perform operations with expressions within the complex number system.**

MA.912.NSO.2.1 Extend previous understanding of the real number system to include the complex number system. Add, subtract, multiply and divide complex numbers.

**MA.912.NSO.4 Represent and perform operations with matrices.**

MA.912.NSO.4.1 Given a mathematical or real-world context, represent and manipulate data using matrices.

MA.912.NSO.4.2 Given a mathematical or real-world context, represent and solve a system of two- or three-variable linear equations using matrices.

MA.912.NSO.4.3 Solve mathematical and real-world problems involving addition, subtraction and multiplication of matrices.

*Clarification 1:* Instruction includes identifying and using the additive and multiplicative identities for matrices.

MA.912.NSO.4.4 Solve mathematical and real-world problems using the inverse and determinant of matrices.

## Algebraic Reasoning

**MA.912.AR.1 Interpret and rewrite algebraic expressions and equations in equivalent forms.**

MA.912.AR.1.1 Identify and interpret parts of an equation or expression that represent a quantity in terms of a mathematical or real-world context, including viewing one or more of its parts as a single entity.

*Algebra 1 Example:* Derrick is using the formula $P = 1000(1 + .1)^t$ to make a prediction about the camel population in Australia. He identifies the growth factor as $(1 + .1)$, or 1.1, and states that the camel population will grow at an annual rate of 10% per year.

*Example:* The expression $1.15^t$ can be rewritten as $\left(1.15^{\frac{1}{12}}\right)^{12t}$ which is approximately equivalent to $1.012^{12t}$. This latter expression reveals the approximate equivalent monthly interest rate of 1.2% if the annual rate is 15%.

*Clarification 1:* Parts of an expression include factors, terms, constants, coefficients and variables.

*Clarification 2:* Within the Mathematics for Data and Financial Literacy course, problem types focus on money and business.

**MA.912.AR.1.3** Add, subtract and multiply polynomial expressions with rational number coefficients.

*Clarification 1:* Instruction includes an understanding that when any of these operations are performed with polynomials the result is also a polynomial.

*Clarification 2:* Within the Algebra 1 course, polynomial expressions are limited to 3 or fewer terms.

**MA.912.AR.1.5** Divide polynomial expressions using long division, synthetic division or algebraic manipulation.

**MA.912.AR.1.6** Solve mathematical and real-world problems involving addition, subtraction, multiplication or division of polynomials.

**MA.912.AR.1.8** Rewrite a polynomial expression as a product of polynomials over the real or complex number system.

*Clarification 1:* Instruction includes factoring a sum or difference of squares and a sum or difference of cubes.

**MA.912.AR.1.9** Apply previous understanding of rational number operations to add, subtract, multiply and divide rational algebraic expressions.

*Clarification 1*: Instruction includes the connection to fractions and common denominators.

**MA.912.AR.1.11** Apply the Binomial Theorem to create equivalent polynomial expressions.

*Clarification 1*: Instruction includes the connection to Pascal's Triangle and to combinations.

**MA.912.AR.3 Write, solve and graph quadratic equations, functions and inequalities in one and two variables.**

**MA.912.AR.3.2** Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real and complex number systems.

*Clarification 1*: Within this benchmark, the expectation is to solve by factoring techniques, taking square roots, the quadratic formula and completing the square.

**MA.912.AR.3.3** Given a mathematical or real-world context, write and solve one-variable quadratic inequalities over the real number system. Represent solutions algebraically or graphically.

**MA.912.AR.3.4** Write a quadratic function to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.

*Algebra I Example:* Given the table of values below from a quadratic function, write an equation of that function.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $2$ | $-1$ | $-2$ | $-1$ | $2$ |

*Clarification 1*: Within the Algebra 1 course, a graph, written description or table of values must include the vertex and two points that are equidistant from the vertex.

*Clarification 2*: Instruction includes the use of standard form, factored form and vertex form.

*Clarification 3*: Within the Algebra 2 course, one of the given points must be the vertex or an $x$-intercept.

**MA.912.AR.3.8** Solve and graph mathematical and real-world problems that are modeled with quadratic functions. Interpret key features and determine constraints in terms of the context.

*Algebra 1 Example:* The value of a classic car produced in 1972 can be modeled by the function $V(t) = 19.25t^2 - 440t + 3500$, where $t$ is the number of years since 1972. In what year does the car's value start to increase?

*Clarification 1*: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; vertex; and symmetry.

*Clarification 2*: Instruction includes the use of standard form, factored form and vertex form.

*Clarification 3*: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

*Clarification 4*: Within the Algebra 1 course, notations for domain, range and constraints are limited to inequality and set-builder.

# Florida's B.E.S.T. Standards and Benchmarks

MA.912.AR.3.9 Given a mathematical or real-world context, write two-variable quadratic inequalities to represent relationships between quantities from a graph or a written description.

*Clarification 1:* Instruction includes the use of standard form, factored form and vertex form where any inequality symbol can be represented.

MA.912.AR.3.10 Given a mathematical or real-world context, graph the solution set to a two-variable quadratic inequality.

*Clarification 1*: Instruction includes the use of standard form, factored form and vertex form where any inequality symbol can be represented.

MA.912.AR.4 **Write, solve and graph absolute value equations, functions and inequalities in one and two variables.**

MA.912.AR.4.2 Given a mathematical or real-world context, write and solve one-variable absolute value inequalities. Represent solutions algebraically or graphically.

MA.912.AR.4.4 Solve and graph mathematical and real-world problems that are modeled with absolute value functions. Interpret key features and determine constraints in terms of the context.

*Clarification 1*: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; vertex; end behavior and symmetry.

*Clarification 2*: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

MA.912.AR.5 **Write, solve and graph exponential and logarithmic equations and functions in one and two variables.**

MA.912.AR.5.2 Solve one-variable equations involving logarithms or exponential expressions. Interpret solutions as viable in terms of the context and identify any extraneous solutions.

MA.912.AR.5.4 Write an exponential function to represent a relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context.

*Clarification 1:* Within the Algebra 1 course, exponential functions are limited to the forms $f(x) = ab^x$, where $b$ is a whole number greater than 1 or a unit fraction, or $f(x) = a(1 \pm r)^x$, where $0 < r < 1$.

*Clarification 2:* Within the Algebra 1 course, tables are limited to having successive nonnegative integer inputs so that the function may be determined by finding ratios between successive outputs.

MA.912.AR.5.5 Given an expression or equation representing an exponential function, reveal the constant percent rate of change per unit interval using the properties of exponents. Interpret the constant percent rate of change in terms of a real-world context.

MA.912.AR.5.7 Solve and graph mathematical and real-world problems that are modeled with exponential functions. Interpret key features and determine constraints in terms of the context.

*Example:* The graph of the function $f(t) = e^{5t+2}$ can be transformed into the straight line $y = 5t + 2$ by taking the natural logarithm of the function's outputs.

*Clarification 1*: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; constant percent rate of change; end behavior and asymptotes.

*Clarification 2*: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

*Clarification 3*: Instruction includes understanding that when the logarithm of the dependent variable is taken and graphed, the exponential function will be transformed into a linear function.

*Clarification 4*: Within the Mathematics for Data and Financial Literacy course, problem types focus on money and business.

MA.912.AR.5.8 Given a table, equation or written description of a logarithmic function, graph that function and determine its key features.

*Clarification 1:* Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; and asymptotes.

*Clarification 2:* Instruction includes representing the domain and range inequality notation, interval notation or set-builder notation.

MA.912.AR.5.9 Solve and graph mathematical and real-world problems that are modeled with logarithmic functions. Interpret key features and determine constraints in terms of the context.

*Clarification 1*: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; and asymptotes.

*Clarification 2*: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

MA.912.AR.6 **Solve and graph polynomial equations and functions in one and two variables.**

MA.912.AR.6.1 Given a mathematical or real-world context, when suitable factorization is possible, solve one-variable polynomial equations of degree 3 or higher over the real and complex number systems.

MA.912.AR.6.2 Explain and apply the Remainder Theorem to solve mathematical and real-world problems.

MA.912.AR.6.5 Sketch a rough graph of a polynomial function of degree 3 or higher using zeros, multiplicity and knowledge of end behavior.

MA.912.AR.7 **Solve and graph radical equations and functions in one and two variables.**

MA.912.AR.7.1 Solve one-variable radical equations. Interpret solutions as viable in terms of context and identify any extraneous solutions.

MA.912.AR.7.2 Given a table, equation or written description of a square root or cube root function, graph that function and determine its key features.

*Clarification 1:* Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; and relative maximums and minimums.

*Clarification 2:* Instruction includes representing the domain and range inequality notation, interval notation or set-builder notation.

MA.912.AR.7.3 Solve and graph mathematical and real-world problems that are modeled with square root or cube root functions. Interpret key features and determine constraints in terms of the context.

*Clarification 1*: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; and relative maximums and minimums.

*Clarification 2*: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

MA.912.AR.8 **Solve and graph radical equations and functions in one and two variables.**

MA.912.AR.8.1 Write and solve one-variable rational equations. Interpret solutions as viable in terms of the context and identify any extraneous solutions.

*Clarification 1*: Within the Algebra 2 course, numerators and denominators are limited to linear and quadratic expressions.

MA.912.AR.8.2 Given a table, equation or written description of a rational function, graph that function and determine its key features.

*Clarification 1*: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; and asymptotes.

*Clarification 2*: Instruction includes representing the domain and range with inequality notation, interval notation or set-builder notation.

*Clarification 3*: Within the Algebra 2 course, numerators and denominators are limited to linear and quadratic expressions.

# Florida's B.E.S.T. Standards and Benchmarks

MA.912.AR.8.3 Solve and graph mathematical and real-world problems that are modeled with rational functions. Interpret key features and determine constraints in terms of the context.

*Clarification 1*: Key features are limited to domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior; and asymptotes.

*Clarification 2*: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

*Clarification 3*: Instruction includes using rational functions to represent inverse proportional relationships.

*Clarification 4*: Within the Algebra 2 course, numerators and denominators are limited to linear and quadratic expressions.

MA.912.AR.9 **Write and solve a system of two- and three-variable equations and inequalities that describe quantities or relationships.**

MA.912.AR.9.2 Given a mathematical or real-world context, solve a system consisting of a two-variable linear equation and a non-linear equation algebraically or graphically.

MA.912.AR.9.3 Given a mathematical or real-world context, solve a system consisting of two-variable linear or non-linear equations algebraically or graphically.

*Clarification 1*: Within the Algebra 2 course, non-linear equations are limited to quadratic equations.

MA.912.AR.9.5 Graph the solution set of a system of two-variable inequalities.

*Clarification 1:* Within the Algebra 2 course, two-variable inequalities are limited to linear and quadratic.

MA.912.AR.9.7 Given a real-world context, represent constraints as systems of linear and non-linear equations or inequalities. Interpret solutions to problems as viable or non-viable options.

*Clarification 1*: Instruction focuses on analyzing a given function that models a real-world situation and writing constraints that are represented as non-linear equations or non-linear inequalities.

*Clarification 2*: Within the Algebra 2 course, non-linear equations and inequalities are limited to quadratic.

MA.912.AR.9.10 Solve and graph mathematical and real-world problems that are modeled with piecewise functions. Interpret key features and determine constraints in terms of the context.

*Example:* A mechanic wants to place an ad in his local newspaper. The cost, in dollars, of an ad $x$ inches long is given by the following piecewise function. Find the cost of an ad that would be 16 inches long.

$$C(x) = \begin{cases} 12x, & x < 5 \\ 60 + 8(x - 5), & x \geq 5 \end{cases}$$

*Clarification 1*: Key features are limited to domain, range, intercepts, asymptotes and end behavior.

*Clarification 2*: Instruction includes representing the domain, range and constraints with inequality notation, interval notation or set-builder notation.

MA.912.AR.10 **Solve problems involving sequences and series.**

MA.912.AR.10.1 Given a mathematical or real-world context, write and solve problems involving arithmetic sequences.

*Example:* Tara is saving money to move out of her parent's house. She opens the account with $250 and puts $100 into a savings account every month after that. Write the total amount of money she has in her account after each month as a sequence. In how many months will she have at least $3,000?

MA.912.AR.10.2 Given a mathematical or real-world context, write and solve problems involving geometric sequences.

*Example:* A bacteria in a Petri dish initially covers 2 square centimeters. The bacteria grows at a rate of 2.6% every day. Determine the geometric sequence that describes the area covered by the bacteria after 0, 1, 2, 3 . . . days. Determine using technology, how many days it would take the bacteria to cover 10 square centimeters.

# Functions

MA.912.F.1 **Understand, compare and analyze properties of functions.**

MA.912.F.1.1 Given an equation or graph that defines a function, determine the function type. Given an input-output table, determine a function type that could represent it.

*Clarification 1:* Within the Algebra 1 course, functions represented as tables are limited to linear, quadratic and exponential.

*Clarification 2:* Within the Algebra 1 course, functions represented as equations or graphs are limited to vertical or horizontal translations or reflections over the x-axis of the following parent functions: $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) = |x|$, $f(x) = 2^x$, and $f(x) = \left(\frac{1}{2}\right)^x$.

MA.912.F.1.7 Compare key features of two functions each represented algebraically, graphically, in tables or written descriptions.

*Clarification 1*: Key features include domain; range; intercepts; intervals where the function is increasing, decreasing, positive or negative; end behavior and asymptotes.

MA.912.F.1.9 Determine whether a function is even, odd or neither when represented algebraically, graphically or in a table.

MA.912.F.2 **Identify and describe the effects of transformations on functions. Create new functions given transformations.**

MA.912.F.2.2 Identify the effect on the graph of a given function of two or more transformations defined by adding a real number to the *x*- or *y*-values or multiplying the *x*- or *y*- values by a real number.

MA.912.F.2.3 Given the graph or table of $f(x)$ and the graph or table of $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$, state the type of transformation and find the value of the real number $k$.

*Clarification 1:* Within the Algebra 1 course, functions are limited to linear, quadratic and absolute value.

MA.912.F.2.5 Given a table, equation or graph that represents a function, create a corresponding table, equation or graph of the transformed function defined by adding a real number to the *x*- or *y*-values or multiplying the *x*- or *y*-values by a real number.

MA.912.F.3 **Create new functions from existing functions.**

MA.912.F.3.2 Given a mathematical or real-world context, combine two or more functions, limited to linear, quadratic, exponential and polynomial, using arithmetic operations. When appropriate, include domain restrictions for the new function.

*Clarification 1*: Instruction includes representing domain restrictions with inequality notation, interval notation or set-builder notation.

*Clarification 2*: Within the Mathematics for Data and Financial Literacy course, problem types focus on money and business.

MA.912.F.3.4 Represent the composition of two functions algebraically or in a table. Determine the domain and range of the composite function.

MA.912.F.3.6 Determine whether an inverse function exists by analyzing tables, graphs and equations.

MA.912.F.3.7 Represent the inverse of a function algebraically, graphically or in a table. Use composition of functions to verify that one function is the inverse of the other.

*Clarification 1*: Instruction includes the understanding that a logarithmic function is the inverse of an exponential function.

# Financial Literacy

MA.912.FL.3 **Describe the advantages and disadvantages of short-term and long-term purchases.**

MA.912.FL.3.1 Compare simple, compound and continuously compounded interest over time.

*Clarification 1*: Instruction includes taking into consideration the annual percentage rate (APR) when comparing simple and compound interest.

MA.912.FL.3.2 Solve real-world problems involving simple, compound and continuously compounded interest.

*Example:* Find the amount of money on deposit at the end of 5 years if you started with $500 and it was compounded quarterly at 6% interest per year.

*Example:* Joe won $25,000 on a lottery scratch-off ticket. How many years will it take at 6% interest compounded yearly for his money to double?

*Clarification 1*: Within the Algebra 1 course, interest is limited to simple and compound.

MA.912.FL.3.4 Explain the relationship between simple interest and linear growth. Explain the relationship between compound interest and exponential growth and the relationship between continuously compounded interest and exponential growth.

*Clarification 1*: Within the Algebra 1 course, exponential growth is limited to compound interest.

## Data Analysis and Probability

MA.912.DP.2 **Solve problems involving univariate and bivariate numerical data.**

MA.912.DP.2.8 Fit a quadratic function to bivariate numerical data that suggests a quadratic association and interpret any intercepts or the vertex of the model. Use the model to solve real-world problems in terms of the context of the data.

*Clarification 1*: Problems include making a prediction or extrapolation, inside and outside the range of the data, based on the equation of the line of fit.

MA.912.DP.2.9 Fit an exponential function to bivariate numerical data that suggests an exponential association. Use the model to solve real-world problems in terms of the context of the data.

*Clarification 1*: Instruction focuses on determining whether an exponential model is appropriate by taking the logarithm of the dependent variable using spreadsheets and other technology.

*Clarification 2*: Instruction includes determining whether the transformed scatterplot has an appropriate line of best fit, and interpreting the $y$-intercept and slope of the line of best fit.

*Clarification 3*: Problems include making a prediction or extrapolation, inside and outside the range of the data, based on the equation of the line of fit.

MA.912.DP.4 **Use and interpret independence and probability.**

MA.912.DP.4.1 Describe events as subsets of a sample space using characteristics, or categories, of the outcomes, or as unions, intersections or complements of other events.

MA.912.DP.4.2 Determine if events A and B are independent by calculating the product of their probabilities.

MA.912.DP.4.3 Calculate the conditional probability of two events and interpret the result in terms of its context.

MA.912.DP.4.4 Interpret the independence of two events using conditional probability.

MA.912.DP.4.9 Apply the addition and multiplication rules for counting to solve mathematical and real-world problems, including problems involving probability.

MA.912.DP.4.10 Given a mathematical or real-world situation, calculate the appropriate permutation or combination.

# Mathematical Thinking and Reasoning Standards

**MA.K12.MTR.1.1** Actively participate in effortful learning both individually and collectively.

Mathematicians who participate in effortful learning both individually and with others:

- Analyze the problem in a way that makes sense given the task.
- Ask questions that will help with solving the task.
- Build perseverance by modifying methods as needed while solving a challenging task.
- Stay engaged and maintain a positive mindset when working to solve tasks.
- Help and support each other when attempting a new method or approach.

**MA.K12.MTR.2.1** Demonstrate understanding by representing problems in multiple ways.

Mathematicians who demonstrate understanding by representing problems in multiple ways:

- Build understanding through modeling and using manipulatives.
- Represent solutions to problems in multiple ways using objects, drawings, tables, graphs and equations.
- Progress from modeling problems with objects and drawings to using algorithms and equations.
- Express connections between concepts and representations.
- Choose a representation based on the given context or purpose.

**MA.K12.MTR.3.1** Complete tasks with mathematical fluency.

Mathematicians who complete tasks with mathematical fluency:

- Select efficient and appropriate methods for solving problems within the given context.
- Maintain flexibility and accuracy while performing procedures and mental calculations.
- Complete tasks accurately and with confidence.
- Adapt procedures to apply them to a new context.
- Use feedback to improve efficiency when performing calculations.

**MA.K12.MTR.4.1** Engage in discussions that reflect on the mathematical thinking of self and others.

Mathematicians who engage in discussions that reflect on the mathematical thinking of self and others:

- Communicate mathematical ideas, vocabulary and methods effectively.
- Analyze the mathematical thinking of others.
- Compare the efficiency of a method to those expressed by others.
- Recognize errors and suggest how to correctly solve the task.
- Justify results by explaining methods and processes.
- Construct possible arguments based on evidence.

**MA.K12.MTR.5.1** Use patterns and structure to help understand and connect mathematical concepts.

Mathematicians who use patterns and structure to help understand and connect mathematical concepts:

- Focus on relevant details within a problem.
- Create plans and procedures to logically order events, steps or ideas to solve problems.
- Decompose a complex problem into manageable parts.
- Relate previously learned concepts to new concepts.
- Look for similarities among problems.
- Connect solutions of problems to more complicated large-scale situations.

**MA.K12.MTR.6.1** Assess the reasonableness of solutions.

Mathematicians who assess the reasonableness of solutions:

- Estimate to discover possible solutions.
- Use benchmark quantities to determine if a solution makes sense.
- Check calculations when solving problems.
- Verify possible solutions by explaining the methods used.
- Evaluate results based on the given context.

MA.K12.MTR.7.1 Apply mathematics to real-world contexts.

Mathematicians who apply mathematics to real-world contexts:

- Connect mathematical concepts to everyday experiences.
- Use models and methods to understand, represent and solve problems.
- Perform investigations to gather data or determine if a method is appropriate.
- Redesign models and methods to improve accuracy or efficiency.

# ELA Expectations

ELA.K12.EE.1.1 Cite evidence to explain and justify reasoning.

**Clarifications:**
K-1 Students include textual evidence in their oral communication with guidance and support from adults. The evidence can consist of details from the text without naming the text. During 1st grade, students learn how to incorporate the evidence in their writing.

2-3 Students include relevant textual evidence in their written and oral communication. Students should name the text when they refer to it. In 3rd grade, students should use a combination of direct and indirect citations.

4-5 Students continue with previous skills and reference comments made by speakers and peers. Students cite texts that they've directly quoted, paraphrased, or used for information. When writing, students will use the form of citation dictated by the instructor or the style guide referenced by the instructor.

6-8 Students continue with previous skills and use a style guide to create a proper citation.

9-12 Students continue with previous skills and should be aware of existing style guides and the ways in which they differ.

ELA.K12.EE.2.1 Read and comprehend grade-level complex texts proficiently.

**Clarifications:**
See Text Complexity for grade-level complexity bands and a text complexity rubric.

ELA.K12.EE.3.1 Make inferences to support comprehension.

**Clarifications:**
Students will make inferences before the words infer or inference are introduced. Kindergarten students will answer questions like "Why is the girl smiling?" or make predictions about what will happen based on the title page. Students will use the terms and apply them in 2nd grade and beyond.

ELA.K12.EE.4.1 Use appropriate collaborative techniques and active listening skills when engaging in discussions in a variety of situations.

**Clarifications:**
In kindergarten, students learn to listen to one another respectfully.

In grades 1-2, students build upon these skills by justifying what they are thinking. For Example: "I think ________ because _______." The collaborative conversations are becoming academic conversations.

In grades 3-12, students engage in academic conversations discussing claims and justifying their reasoning, refining and applying skills. Students build on ideas, propel the conversation, and support claims and counterclaims with evidence.

ELA.K12.EE.5.1 Use the accepted rules governing a specific format to create quality work.

**Clarifications:**
Students will incorporate skills learned into work products to produce quality work. For students to incorporate these skills appropriately, they must receive instruction. A 3rd grade student creating a poster board display must have instruction in how to effectively present information to do quality work.

ELA.K12.EE.6.1 Use appropriate voice and tone when speaking or writing.

**Clarifications:**
In kindergarten and 1st grade, students learn the difference between formal and informal language. For example, the way we talk to our friends differs from the way we speak to adults. In 2nd grade and beyond, students practice appropriate social and academic language to discuss texts.

# English Language Development for English Language Learners

ELD.K12.ELL.MA.1 English language learners communicate information, ideas and concepts necessary for academic success in the content area of Mathematics.

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# 1

# Functions, Inequalities, and Systems


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TOPIC 1

# Functions, Inequalities, and Systems

## TOPIC ESSENTIAL QUESTION

**What are the ways in which functions can be used to represent and solve problems involving quantities?**

## Topic Overview

**enVision® STEM Project:**
Fuel Efficiency

**1-1** Key Features of Functions
AR.3.8, AR.4.4, F.1.1, MTR.4.1, MTR.5.1, MTR.7.1

**1-2** Transformations of Functions
AR.4.4, F.1.1, F.1.7, F.2.2, F.2.3, F.2.5, MTR.2.1, MTR.5.1, MTR.6.1

**1-3** Piecewise-Defined Functions
AR.9.10, MTR.1.1, MTR.5.1, MTR.7.1

**1-4** Arithmetic Sequences
AR.10.1, MTR.3.1, MTR.5.1, MTR.7.1

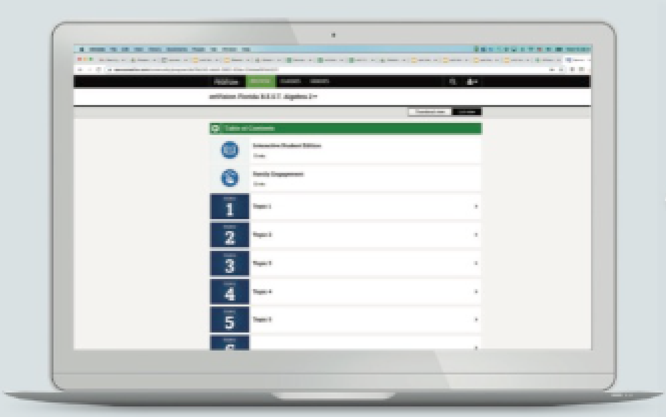
**1-5** Quadratic and Absolute Value Inequalities
AR.3.3, AR.3.8, AR.4.2, MTR.2.1, MTR.3.1, MTR.5.1

**1-6** Linear Systems
AR.9.2, AR.9.3, AR.9.5, AR.9.7, MTR.2.1, MTR.3.1, MTR.5.1

**Mathematical Modeling in 3 Acts:**
Current Events
AR.9.3, AR.9.7, MTR.7.1

## Topic Vocabulary

- arithmetic sequence
- common difference
- compression
- explicit definition
- interval notation
- maximum
- minimum
- piecewise-defined function
- recursive definition
- reflection
- sequence
- set-builder notation
- solution of a system of linear equations
- step function
- stretch
- system of linear equations
- system of linear inequalities
- transformation
- translation
- zero of a function



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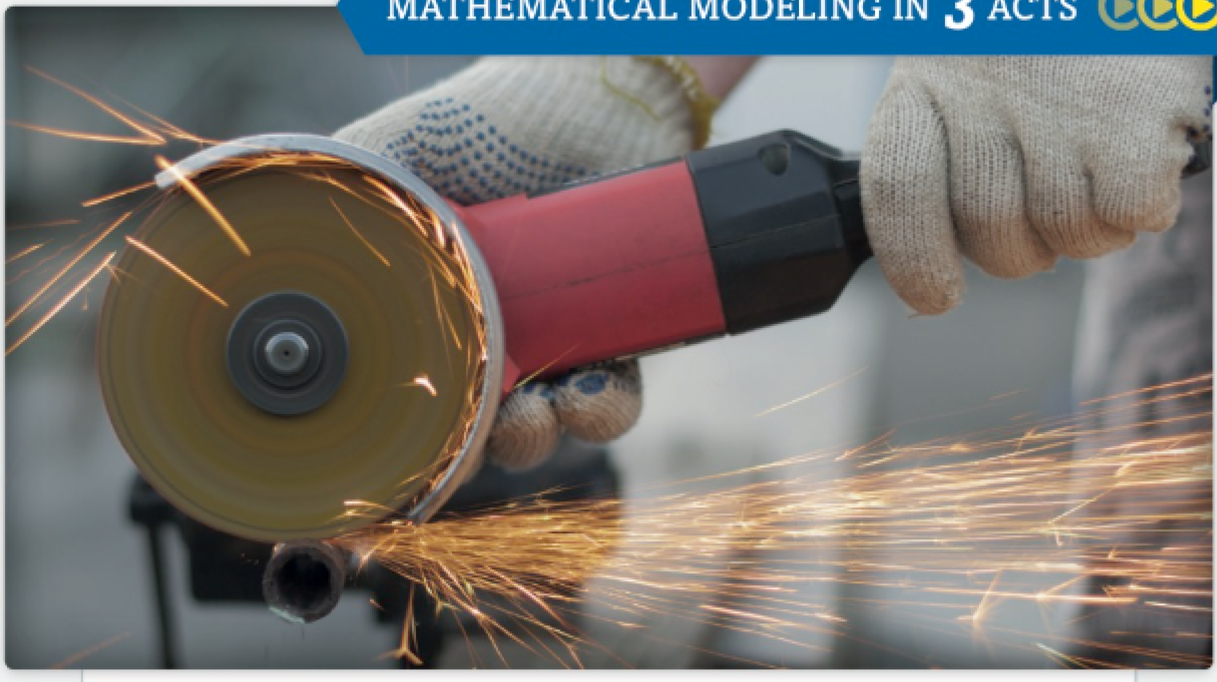
**ACTIVITIES** Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its***.

**ANIMATION** View and interact with real-world applications.

**PRACTICE** Practice what you've learned.



MATHEMATICAL MODELING IN 3 ACTS

## Current Events

You might say that someone who loses their temper has "blown a fuse." However, it's rare to hear about electrical fuses blowing these days. That's because most fuses have been replaced by circuit breakers. A fuse must be replaced once it's blown, but a circuit breaker can be reset.

Ask for permission to look at the electrical panel in your home. If there is a series of switches inside, each of those is a circuit breaker, designed to interrupt the circuit when the electrical current inside is too dangerous. How much electricity does it take to trip a circuit breaker? Think about this question during the Mathematical Modeling in 3-Acts lesson.

**VIDEOS** Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

**ADAPTIVE PRACTICE** Practice that is *just right* and *just for you*.

**GLOSSARY** Read and listen to English and Spanish definitions.

**CONCEPT SUMMARY** Review key lesson content through multiple representations.

**ASSESSMENT** Show what you've learned.

**TUTORIALS** Get help from ***Virtual Nerd***, right when you need it.

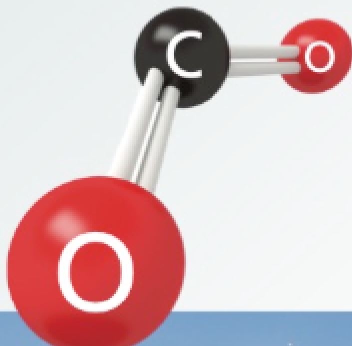
**MATH TOOLS** Explore math with digital tools and manipulatives.

**DESMOS** Use Anytime and as embedded Interactives in Lesson content.

**QR CODES** Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

## Did You Know?

**Carbon dioxide** ($CO_2$) is composed of 1 atom of carbon and 2 atoms of oxygen. A gas that occurs naturally on Earth, $CO_2$ also produced by burning fossil fuels. In its solid form, $CO_2$ is commonly called "dry ice."



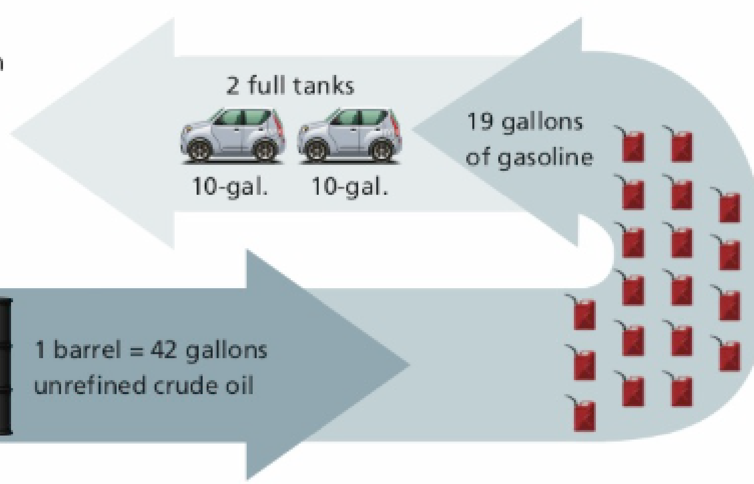


This hybrid hatchback averages **59 mpg** in the city.

This large SUV averages **10 mpg** in the city.

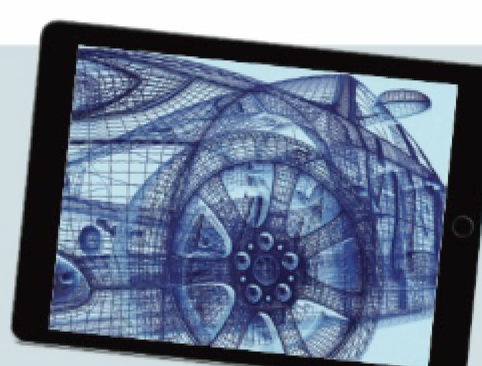
In the United States, each state determines the **tax rate** on gasoline, so the state in which you buy your gas determines how much it costs to fill your tank.

How much **crude oil** do you use when filling up your car?




## Your Task: Fuel Efficiency

You and your classmates will analyze cars' fuel efficiency. If you were designing a car to come out in 2024, what gas mileage would you target?



# 1-1

## Key Features of Functions

**I CAN…** interpret key features of linear, quadratic, and absolute value functions given an equation or a graph.

### VOCABULARY

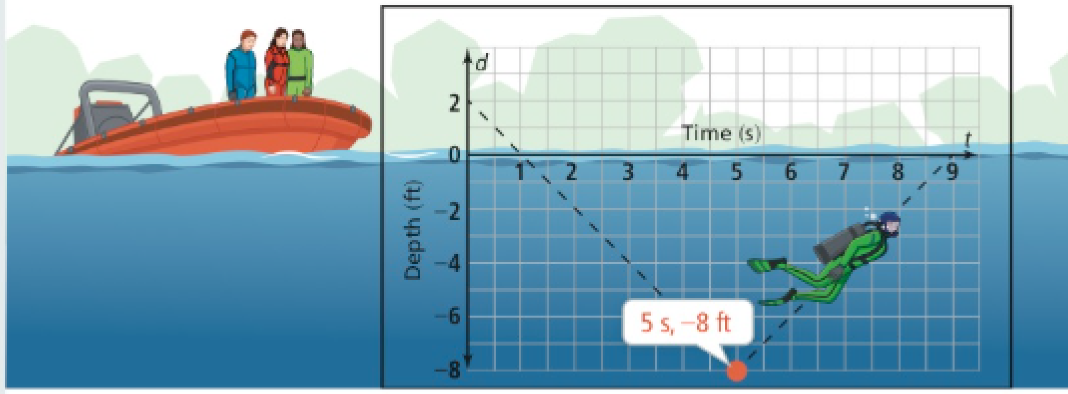
- interval notation
- maximum
- minimum
- set-builder notation
- zero of a function

**MA.912.AR.4.4**–Solve and graph mathematical and real-world problems that are modeled with absolute value functions. Interpret key features and determine constraints in terms of the context. **Also AR.3.8, F.1.1**

**MA.K12.MTR.4.1, MTR.5.1, MTR.7.1**

## EXPLORE & REASON

**A diver is going through ocean search-and-rescue training. The graph shows the relationship between her depth and the time in seconds since starting her dive.**




**A.** What details can you determine about the dive from the coordinates of the point (5, −8)?

**B.** What is the average speed of the diver in the water? How can you tell from the graph?

**C.** Which point on the graph shows the starting location of the diver? Explain.

**D.** **Represent and Connect** What does the V-shape of the graph tell you about the dive? What information does it not tell you about the dive?

## ESSENTIAL QUESTION

**How do graphs and equations reveal information about a relationship between two quantities?**

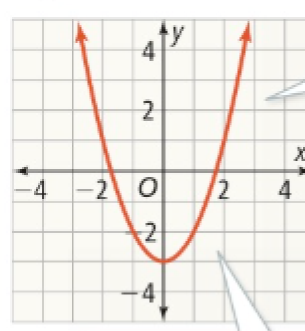
## EXAMPLE 1 Understand Domain and Range

**A. What are the domain and range of the function defined by $y = x^2 - 3$?**

The set of all possible inputs for a relation is called the *domain*.

You can square any real number, so any number can be input for $x$.

The square of a real number is greater than or equal to 0. So the minimum value of $y = x^2 - 3$ is $0 - 3$, or $-3$.




This graph represents a *function* because each input has exactly one output. The function is quadratic.

The set of all possible outputs for a relation is called the *range*.

There are two notations used to represent intervals of numbers like domain and range.

**Set-Builder Notation** uses a verbal description or an inequality to describe the numbers.

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### COMMUNICATE AND JUSTIFY

Consider this explanation of the function's minimum value. How do you know that the function has no maximum value?

EXAMPLE 1 CONTINUED

Using set-builder notation, the domain of this function is $\{x \mid x \text{ is a real number}\}$.

This is read *"The set of all x such that x is a real number."*

This is read *"The set of all y such that y is greater than or equal to −3."*

Using set-builder notation, the range of the function is $\{y \mid y \geq -3\}$.

### STUDY TIP
An interval with excluded boundary points is called "open" and is represented by open circle endpoints on the graph. An interval with included boundary points is called "closed" and is represented by solid endpoints.

**Interval notation** represents a set of real numbers by the pair of values that are its left (minimum) and right (maximum) boundaries. Using interval notation, the domain of the function is $(-\infty, \infty)$.

Using interval notation, the range is $[-3, \infty)$.

To summarize the ways in which we can indicate intervals of numbers, refer to the table below.

| Interval Notation | Words | Set Notation |
|---|---|---|
| $[3, 4]$ | All real numbers that are greater or equal to 3 and less than or equal to 4 | $\{x \mid 3 \leq x \leq 4\}$ |
| $(3, 4]$ | All real numbers that are greater than 3 and less than or equal to 4 | $\{x \mid 3 < x \leq 4\}$ |
| $[3, 4)$ | All real numbers that are greater than or equal to 3 and less than 4 | $\{x \mid 3 \leq x < 4\}$ |
| $(3, 4)$ | All real numbers that are greater than 3 and less than 4 | $\{x \mid 3 < x < 4\}$ |
| $[3, \infty)$ | All real numbers greater than or equal to 3 | $\{x \mid 3 \leq x\}$ |
| $(-\infty, 3]$ | All real numbers less than or equal to 3 | $\{x \mid x \leq 3\}$ |
| $(-\infty, \infty)$ | All real numbers | $\{x \mid -\infty < x < \infty\}$ |

**B. An airtanker flies over forest fires and drops water at a constant rate until its tank is empty. What are the domain and range of the function that represents the volume of water the airtanker can drop in *x* seconds?**


The function is $f(x) = 400x$. The airtanker cannot drop water for a negative number of seconds, so $x \geq 0$. The tanker can drop water for a maximum of $\frac{8{,}000}{400} = 20$ s before running out of water, so $x \leq 20$.

The domain is $\{x \mid 0 \leq x \leq 20\}$, or $[0, 20]$.

The airtanker cannot drop a negative number of gallons, and its maximum capacity is 8,000 gal.

The range is $\{y \mid 0 \leq y \leq 8{,}000\}$, or $[0, 8{,}000]$.

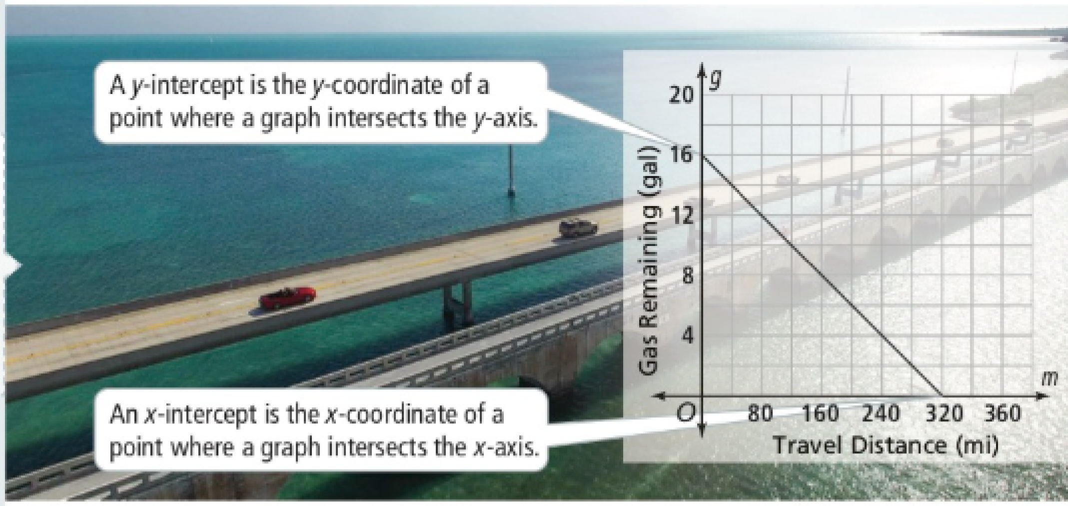
**Try It!** **1.** Graph each function. Write the domain and range in set-builder notation and interval notation.

**a.** $y = |x - 4|$ **b.** $y = 6x - 2x^2$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Zeros of a Function

**A. A car starts a journey with a full tank of gas. The equation $y = 16 - 0.05x$ relates the number of gallons of gas, $y$, left in the tank to the number of miles the car has traveled, $x$. What are the $x$- and $y$-intercepts of the graph of this equation, and what do they represent about the situation?**




**STUDY TIP**
Depending on the situation modeled by a function, the intercept(s) may not be in the domain, and may not represent anything important in the situation.

The graph above intersects the $x$-axis at (320, 0), so the $x$-intercept is 320. This means that the car can travel 320 mi before it runs out of gas.

The graph intersects the $y$-axis at (0, 16), so the $y$-intercept is 16. This means the car has 16 gal of gas when it starts its trip.

**B. What are the $x$- and $y$-intercepts of the graph of $y = |x| - 3$?**

Find the $x$-intercept(s) algebraically:

$$y = |x| - 3$$
$$0 = |x| - 3$$
$$3 = |x|$$
$$x = \pm 3$$

The $y$-coordinate of an $x$-intercept is 0.

Both 3 and −3 are 3 units away from 0, so they both have an absolute value of 3.

The $x$-intercepts of the graph of $y = |x| - 3$ are −3 and 3. The $x$-intercepts are also the **zeros of the function** because they are the input values that result in a function output value of 0.

Find the $y$-intercept algebraically:

$$\mathbf{y = |x| - 3}$$
$$y = |0| - 3$$
$$y = 0 - 3$$
$$y = -3$$

The $x$-coordinate of the $y$-intercept is 0.

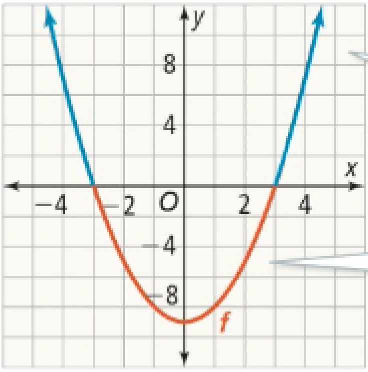
The $y$-intercept of the graph of $y = |x| - 3$ is −3.

**Try It!** **2.** Graph $g(x) = 4 - x^2$. What are the $x$- and $y$-intercepts?

## EXAMPLE 3 Identify Positive or Negative Intervals

**For what intervals is $f(x) = x^2 - 9$ positive? For what intervals is the function negative?**

Use technology to graph the function:




**COMMON ERROR**
Be careful not to confuse a positive function value and a positive rate of change. A positive rate of change means the $y$-values of the function are increasing but are not necessarily greater than 0.

The function is positive on $(-\infty, -3)$ and $(3, \infty)$.

Parentheses indicate that a boundary point is not included.

The function is negative on $(-3, 3)$.

The function is neither positive nor negative at the $x$-intercepts of –3 and 3.

**Try It!** **3. a.** For what interval(s) is $h(x) = -|x| + 5$ positive?

**b.** For what interval(s) is the function negative?

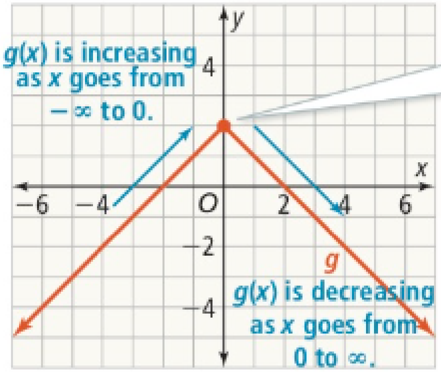
## EXAMPLE 4 Identify Where a Function Increases or Decreases

**For what values of $x$ is $g(x) = 2 - |x|$ increasing? For what values is it decreasing?**

**LEARN TOGETHER**
Do you seek help when needed? Do you offer help and support others?

Construct a table and sketch a graph to represent the absolute value function.

| $x$ | $g(x)$ |
|---|---|
| –3 | –1 |
| –2 | 0 |
| –1 | 1 |
| 0 | 2 |
| 1 | 1 |
| 2 | 0 |
| 3 | –1 |




The greatest value a function attains is the ***maximum*** of the function.

The least value a function attains is the ***minimum***.

The values of $g(x)$ are increasing on the interval $(-\infty, 0)$.

The values of $g(x)$ are decreasing on the interval $(0, \infty)$.

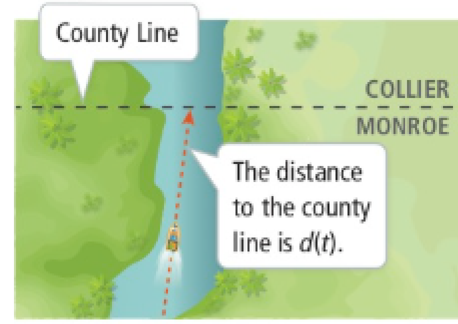
**Try It!** **4.** Graph the functions. For what values of $x$ is each function increasing? For what values of $x$ is it decreasing?

**a.** $f(x) = x^2 - 4x$ **b.** $f(x) = -2x - 3$

APPLICATION

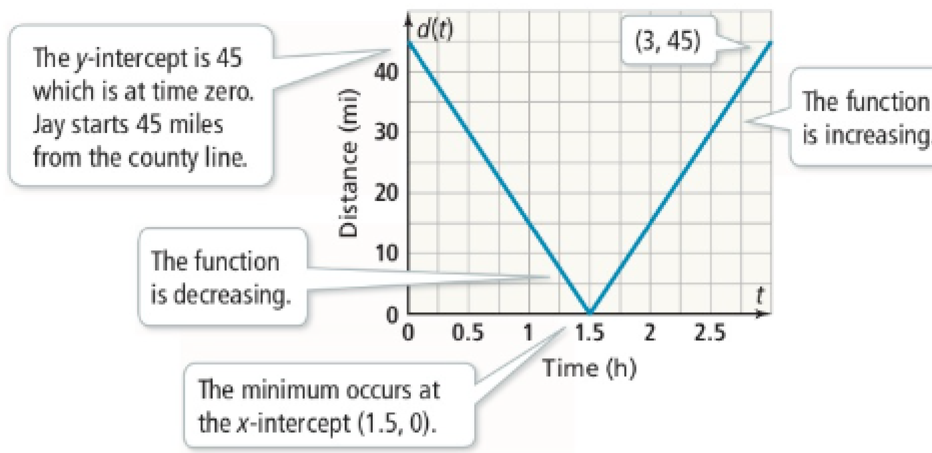
## EXAMPLE 5 Interpret the Graph of a Function

**Jay rides in a boat from his home to his friend's home in a neighboring county. His home and his friend's home are the same distance from the county line. The function $d(t) = 30|t - 1.5|$ gives the distance of the boat in miles from the county line at $t$ hours.**




**A. What is a reasonable domain for the function? What does it represent?**

Graph the function and identify its key features.




Jay starts his trip 45 miles from the county line, so he also ends 45 miles from the county line. The domain of the function is where the distance is less than or equal to 45 miles, or $\{t | 0 \leq t \leq 3\}$. The domain indicates that Jay's trip lasted 3 hours.

**B. What do the intervals where the function is decreasing or increasing, and the minimum of the function represent?**

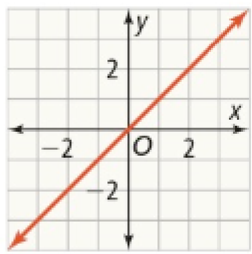
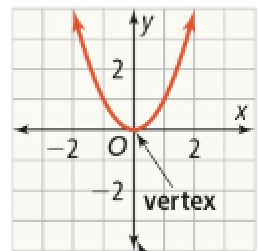
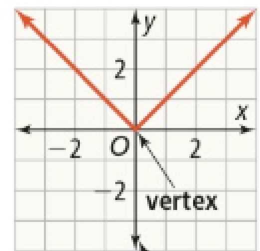
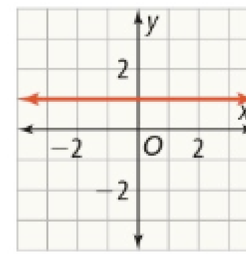
The function is decreasing as Jay's distance to Collier County decreases. The minimum of the function represents when Jay reaches the county line at $t = 1.5$ hours. When the function is increasing, Jay is traveling away from the county line.

**COMMON ERROR**

Remember that a graph representing the time and distance traveled by an object is not necessarily a picture of its path.

**Try It!** 5. A cyclist competing in a race rides past a water station. The graph of the function $d(t) = \frac{1}{3}|t - 60|$ represents her distance in kilometers from the water station at $t$ minutes. What does the graph of the function tell you about her race? Hint: Construct a table of values to help you graph the function.

## CONCEPT SUMMARY Some Functions With Key Features

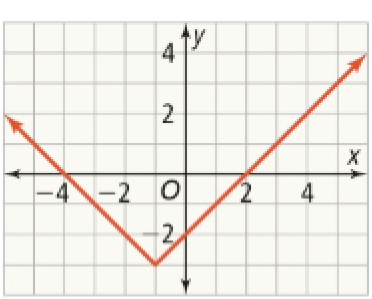
| | | | | |
|---|---|---|---|---|
| **FUNCTION** | Linear<br>$y = x$ | Quadratic<br>$y = x^2$ | Absolute Value<br>$y = \|x\|$ | Constant<br>$y = 1$ |
| **GRAPH** |  |   |   |  |
| **KEY FEATURES** | Domain: $(-\infty, \infty)$<br>Range: $(-\infty, \infty)$<br>Increasing: $(-\infty, \infty)$ | Domain: $(-\infty, \infty)$<br>Range: $[0, \infty)$<br>Increasing: $(0, \infty)$<br>Decreasing: $(-\infty, 0)$ | Domain: $(-\infty, \infty)$<br>Range: $[0, \infty)$<br>Increasing: $(0, \infty)$<br>Decreasing: $(-\infty, 0)$ | Domain: $(-\infty, \infty)$<br>Range: $\{y \mid y = 1\}$ |
| **INTERCEPTS** | The $x$-intercept is 0.<br>The $y$-intercept is 0. | The $x$-intercept is 0.<br>The $y$-intercept is 0. | The $x$-intercept is 0.<br>The $y$-intercept is 0. | There is no $x$-intercept.<br>The $y$-intercept is 1. |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do graphs and equations reveal information about a relationship between two quantities?

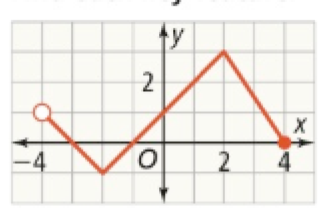
2. **Vocabulary** Define the term *zero of a function* in your own words.

3. **Error Analysis** Lonzell said the function shown in the graph is positive on the interval $(-1, 5)$ and negative on the interval $(-5, -1)$. Identify and correct Lonzell's error.



## Do You KNOW HOW?

**Find each key feature.**



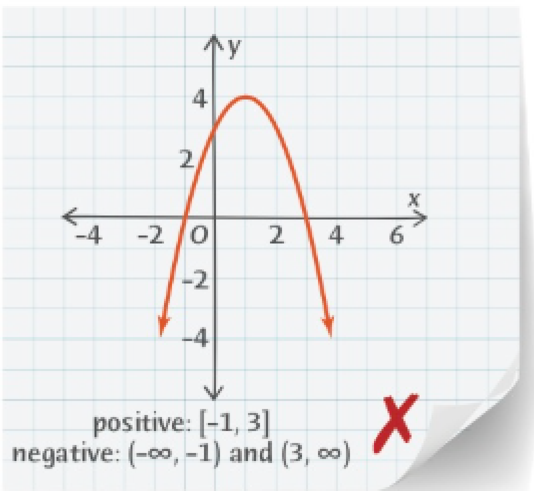
4. domain
5. range
6. $x$-intercept(s)
7. $y$-intercept(s)
8. interval(s) where the graph is positive
9. interval(s) where the graph is decreasing
10. interval(s) where the graph is increasing

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

11. **Analyze and Persevere** The function of $y = -\frac{1}{2}x + 2$ is negative over the interval $(4, \infty)$ and positive over the interval $(-\infty, 4)$. What happens on the graph when $x = 4$? Explain.

12. **Error Analysis** Describe and correct the error a student made in finding the interval(s) over which the function is positive and negative.




13. **Use Patterns and Structure** Sketch a graph of a function with the following key features.

| | |
|---|---|
| domain: $(-4, 4)$ | range: $(-4, 6]$ |
| increasing: $(-4, 1)$ | decreasing: $(1, 4)$ |
| $x$-intercepts: $(-2, 0)$, $(3, 0)$ | $y$-intercept: $(0, 4)$ |
| negative: $(-4, -2)$ and $(3, 4)$ | positive: $(-2, 3)$ |

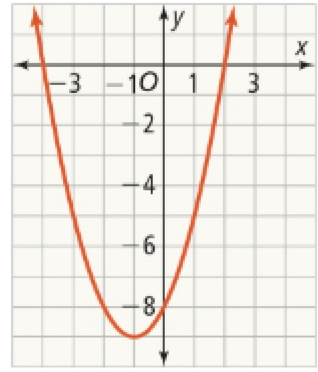
14. **Communicate and Justify** A student says that all linear functions are either increasing or decreasing. Do you agree? Explain.

15. **Higher Order Thinking** A relative maximum of a function occurs at the highest point on a graph over a certain interval. A relative minimum of a function occurs at the lowest point on a graph over a certain interval. Explain how to identify a relative maximum and a relative minimum of a function using key features.

16. **Apply Math Models** For a graph of speed in miles per hour as a function of time in hours, what does it mean when the function is increasing? Decreasing?

### PRACTICE

**Use the graph of the function for Exercises 17–20.**



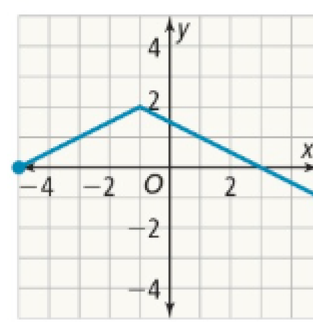

17. Identify the domain and range of the function. SEE EXAMPLE 1

18. Identify the $x$- and $y$-intercepts of the function. SEE EXAMPLE 2

19. On what intervals is the function positive? On what intervals is it negative? SEE EXAMPLE 3

20. On what intervals is the function increasing? On what intervals is it decreasing? SEE EXAMPLE 4

**Use the graph of the function for Exercises 21–24.**




21. Identify the domain and range of the function. SEE EXAMPLE 1

22. Identify the $x$- and $y$-intercepts of the function. SEE EXAMPLE 2

23. Determine over what interval the function is positive or negative. SEE EXAMPLE 3

24. Determine over what interval the function is increasing or decreasing. SEE EXAMPLE 4

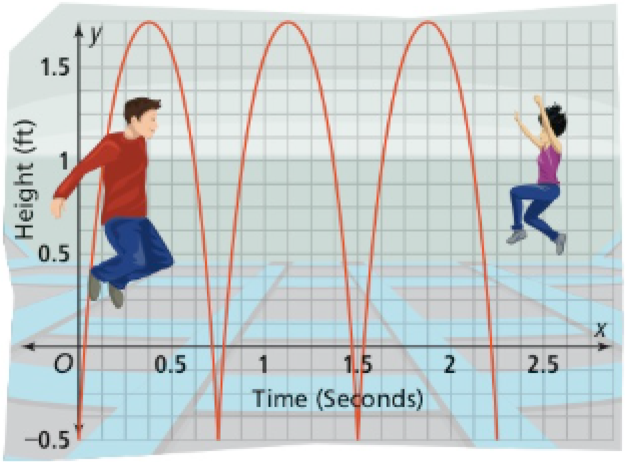
25. Oscar jogs to the park and back, taking the same route both ways. His distance from the park in miles, t minutes after he begins, is represented by the function $f(t) = \frac{1}{12}|t - 6|$. What is a reasonable domain for the function? Explain. SEE EXAMPLE 5

## PRACTICE & PROBLEM SOLVING

**APPLY**

26. **Represent and Connect** Kathryn is filling an empty 100 $ft^3$ container with sand at a rate of 1.25 $ft^3$/min. Describe and interpret the key features of the graph of the amount of sand inside the container.

27. **Analyze and Persevere** The graph shows a jumper's height, $y$, in feet $x$ seconds after starting to jump on a trampoline.




a. What are the $x$- and $y$-intercepts? Explain what the $x$- and $y$-intercepts represent.

b. Over what intervals is the graph positive? Explain what the positive intervals represent.

c. Over what intervals is the graph negative? Explain what the negative intervals represent.

28. **Apply Math Models** Bailey starts playing a game on her cell phone with the battery fully charged, and plays until the phone battery dies. While playing the game, the charge in Bailey's battery decreases by half a percent per minute.

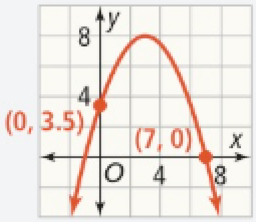
a. Write a function for the percent charge in the battery while Bailey is playing the game.

b. What is the domain and range of the function?

c. How long can Bailey play the game?

**ASSESSMENT PRACTICE**

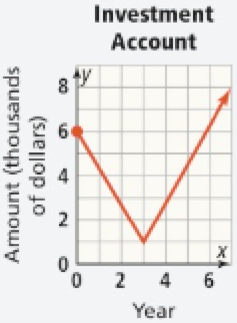
29. A portion of the graph represents the path of water leaving a fountain 3.5 feet above ground. The water hits the ground 7 feet away from the fountain. AR.3.8




a. What does the vertex of the graph represent?

b. If the vertex of the graph is (3, 8), on what interval is the water going up in its path? On what interval is it going down?

30. **SAT/ACT** The graph shows the amount of money in an investment account. Which statement is true?



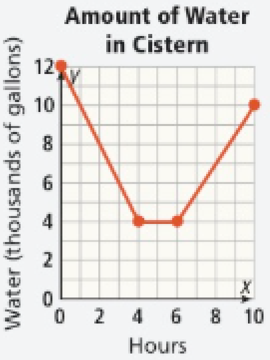

Ⓐ $6,000 was initially invested in the account.

Ⓑ $1,000 was initially invested in the account.

Ⓒ At Year 3, there was $0 in the account.

Ⓓ At Year 7, there was $0 in the account.

31. **Performance Task** The graph shows the amount of water in a cistern over several hours.




**Part A** Over what interval is the function increasing? Over what interval is it decreasing? What is happening with the water in the cistern during these times?

**Part B** What is a possible explanation for what occurred between 4 and 6 h?

**Part C** What are the minimum and maximum values of the function? What do they tell you about the capacity of the cistern? Explain.

# 1-2 Transformations of Functions

**I CAN…** apply transformations to graph functions and write equations.

**VOCABULARY**

- compression
- reflection
- stretch
- transformation
- translation

MA.912.F.2.2–Identify the effect on the graph of a given function of two or more transformations defined by adding a real number to the $x$- or $y$-values or multiplying the $x$- or $y$-values by a real number.
AR.4.4, F.1.1, F.1.7, F.2.3, F.2.5
MA.K12.MTR.2.1, MTR.5.1, MTR.6.1

## EXPLORE & REASON

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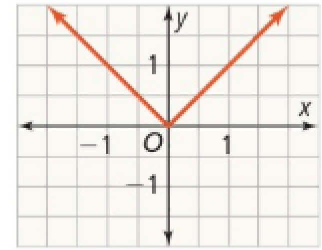
**The graph of the function $f(x) = |x|$ is shown.**




**A.** Graph the function $g(x) = |x + c|$ several times with different values for $c$ (any value from −5 to 5).

**B.** **Use Patterns and Structure** Predict what will happen to the graph if $c$ is a number greater than 100. What if $c$ is a number between 0 and $\frac{1}{2}$?

## ESSENTIAL QUESTION

**What do the differences between the equation of a function and the equation of its parent function tell you about the differences in the graphs of the two functions?**

## EXAMPLE 1 Translate a Function

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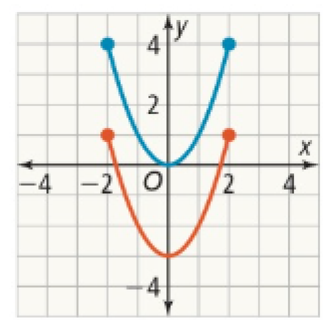
**A. Graph the function $f(x) = x^2$ for the domain [−2, 2]. The graph of $g$ is the graph of $f$ after a translation of 3 units down. How are the equations, domains, and ranges of $f$ and $g$ related?**

Every point on the graph of $g$ is 3 units below a corresponding point on the graph of $f$.




$$(x, f(x)) \rightarrow (x, f(x) - 3) = (x, g(x))$$

$$g(x) = f(x) - 3$$

The domains of $f$ and $g$ are the same: [−2, 2]. The range values of $g$ are 3 units less than the range values of $f$. The range of $f$ is [0, 4] and the range of $g$ is [−3, 1].

A **translation** like this one is a particular kind of transformation of a function, one that shifts each point on a graph the same distance and direction.

Other kinds of **transformations** of a function may reflect its graph across an axis, or stretch or compress its graph.

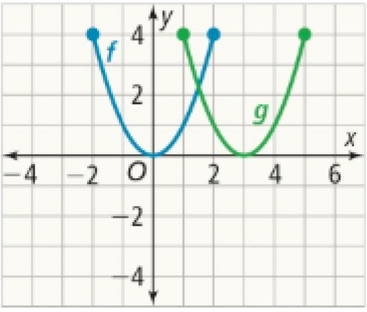
In general, if $g(x) = f(x) + k$, then the graph of $g$ is a vertical translation of the graph of $f$ by $k$ units.

CONTINUED ON THE NEXT PAGE

**GENERALIZE**
This type of transformation slides the graph up or down. You could perform a transformation like this to a nonvertical line and produce a parallel line.

EXAMPLE 1 CONTINUED

**B. Graph the function $f(x) = x^2$ for the domain $[-2, 2]$. The graph of the function $g$ is the graph of $f$ after a translation 3 units to the right. How are the equations, domains, and ranges of $f$ and $g$ related?**




Every point on the graph of $g$ is 3 units to the right of the corresponding point on the graph of $f$.

$$(x, f(x)) \rightarrow (x, g(x + 3))$$

The translation of the graph of $f$ to the graph of $g$ can be described as $g(x + 3) = f(x)$, or $g(x) = f(x - 3)$.

The range values of $f$ and $g$ are the same: $[0, 4]$. The corresponding domain values of $g$ are 3 units more than the domain values of $f$. The domain of $f$ is $[-2, 2]$ and the domain of $g$ is $[1, 5]$.

In general, if $g(x) = f(x - h)$, then the graph of $g$ is a horizontal translation of the graph of $f$ by $h$ units.

**Try It!** **1. a.** How did the transformation of $f$ to $g$ in part (a) affect the intercepts?

**b.** How did the transformation of $f$ to $g$ in part (b) affect the intercepts?

## EXAMPLE 2 Reflect a Function Across the *x*- or *y*-Axis

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**VOCABULARY**
Recall that a **reflection** is a transformation that maps each point to a new point across a given line, called the *line of reflection*. The line of reflection is the perpendicular bisector of the segment between the point and its image.

**A. Graph $f(x) = 2x - 6$ and the function $g$, whose graph is the reflection of the graph of $f$ across the $x$-axis. How are their equations related?**

Graph $f$. Then graph $g$ by reflecting each point of the graph of $f$ across the $x$-axis. For each point $(x, y)$ on the graph of $f$, plot the point $(x, -y)$ to get the graph of $g$.

Since the y-values of the new function have the opposite sign, $g(x) = -f(x)$.

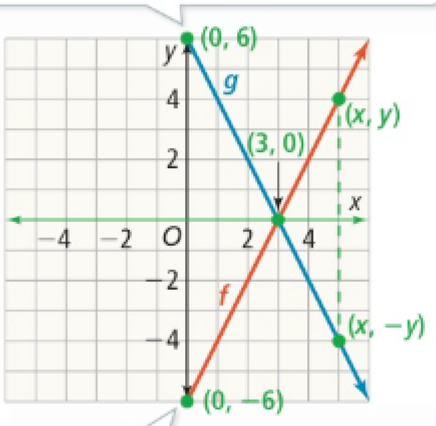
From the graph, you can see that $g(x) = -2x + 6$.

You can check that $g(x) = -f(x)$ by substituting for $f(x)$.

$$g(x) = -f(x)$$
$$= -(2x - 6)$$
$$g(x) = -2x + 6$$

The expression that defines $g$ is the opposite of the expression that defines $f$.

The $y$-intercept of $g$, 6, is the opposite of the $y$-intercept of $f$, $-6$.




The slope of the graph of $f$ is 2, while the slope of the graph of $g$ is $-2$.

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EXAMPLE 2 CONTINUED

**B. Graph $f(x) = 2x - 6$ and the function $h$, whose graph is the reflection of the graph of $f$ across the $y$-axis. How are their equations related?**

Graph $f$. Then reflect every point on the graph of $f$ over the $y$-axis to produce the graph of $h$.

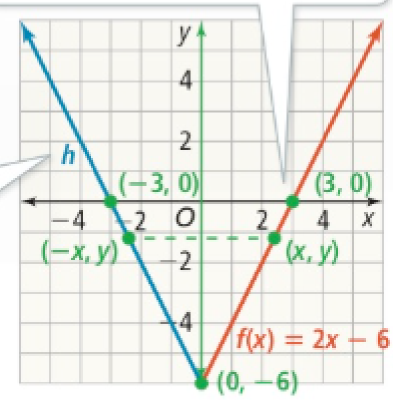
From the graph, you can see that $h(x) = -2x - 6$.

You can check that $h(x) = f(-x)$ by substituting for $f(-x)$.

$$h(x) = f(-x)$$
$$= 2(-x) - 6$$
$$h(x) = -2x - 6$$

The slope of the graph of $f$ is 2, and the slope of the graph of $h$ is $-2$.

For any point $(x, y)$ on the graph of $f$, there is a reflected point $(-x, y)$ on the graph of $h$, so $h(x) = f(-x)$. The $x$-intercept of $h$, $-3$, is the opposite of the $x$-intercept of $f$, 3.




The function $h$ has a slope that is the opposite of the slope of $f$ but with the same $y$-intercept.

**USE PATTERNS AND STRUCTURE**
Why does $g(x) = -f(x)$ affect the $y$-coordinate of each point and $g(x) = f(-x)$ affect the $x$-coordinate of each point?

**Try It!** 2. What is an equation for the reflected graph? Check by graphing.

**a.** the graph of $f(x) = x^2 - 2$ reflected across the $x$-axis

**b.** the graph of $f(x) = x^2 - 2$ reflected across the $y$-axis

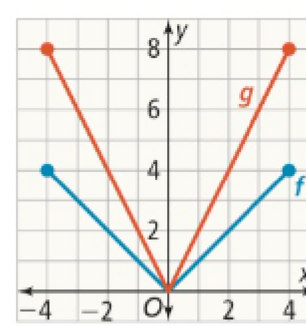
CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Understand Stretches and Compressions

**A. Graph $f(x) = |x|$ with domain $[-4, 4]$ and $g(x) = 2 \bullet f(x)$. How are the domains and ranges related?**

Use a table to find points on the graph of $g$.

| $x$ | $f(x)$ | $g(x) = 2 \bullet f(x)$ | $(x, g(x))$ |
|---|---|---|---|
| −4 | 4 | $2 \bullet f(-4) = 2(4) = 8$ | (−4, 8) |
| −2 | 2 | $2 \bullet f(-2) = 2(2) = 4$ | (−2, 4) |
| 0 | 0 | $2 \bullet f(0) = 2(0) = 0$ | (0, 0) |
| 2 | 2 | $2 \bullet f(2) = 2(2) = 4$ | (2, 4) |
| 4 | 4 | $2 \bullet f(4) = 2(4) = 8$ | (4, 8) |




The domains of $f$ and $g$ are the same. Each $y$-value is multiplied by 2, so for the function with the given domain, the values in the range of $f$, [0, 4], are doubled for $g$ to [0, 8].

A transformation that increases the distance between the points of a graph and a given line by the same factor is called a **stretch**. The graph of $g$ is a vertical stretch of the graph of $f$ by a factor of 2.

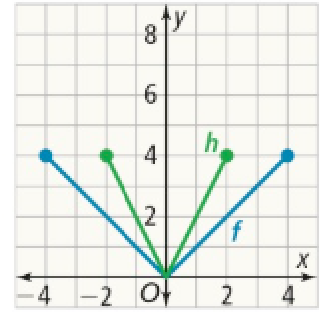
CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

**B. Graph $f(x) = |x|$ with domain $[-4, 4]$ and $h(x) = f(2x)$. How are the domains and ranges related?**

Use a table to find points on the graph of $h$.

| $x$ | $f(x)$ | $h = f(2x)$ | $(x, h(x))$ |
|---|---|---|---|
| $-2$ | 2 | $f(2(-2)) = f(-4) = 4$ | $(-2, 4)$ |
| $-1$ | 1 | $f(2(-1)) = f(-2) = 2$ | $(-1, 2)$ |
| 0 | 0 | $f(2(0)) = f(0) = 0$ | $(0, 0)$ |
| 1 | 1 | $f(2(1)) = f(2) = 2$ | $(1, 2)$ |
| 2 | 2 | $f(2(2)) = f(4) = 4$ | $(2, 4)$ |




**COMMON ERROR**
Be careful not to assume that the domain of a transformed function is the same as the domain of the original function. Notice that $h(-4)$ would equal $f(-8)$, which is outside the domain of $f$, $[-4, 4]$.

For each corresponding output, the value of the input for $h$ is half the value of the input for $f$. The two functions have the same range, but the values in the domain of $h$, $[-2, 2]$, are half as large as the values in the domain of $f$, $[-4, 4]$.

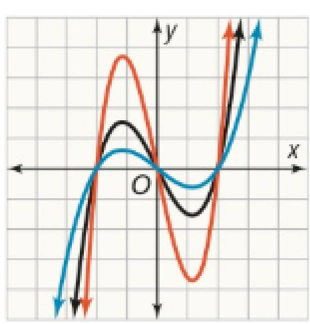
A transformation that decreases the distance between the points of a graph and a given line by the same factor is called a **compression**. The graph of $h$ is a horizontal compression of the graph of $f$ by a factor of 2.

**Try It!** 3. Show that $j(x) = f\left(\frac{1}{2}x\right)$ is a horizontal stretch of the graph of $f$.

## CONCEPT Stretches and Compressions

**Vertical**

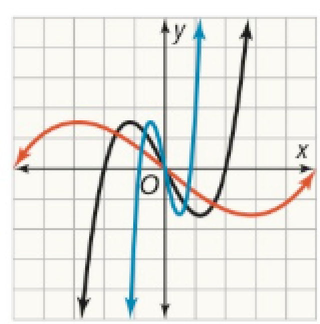
$g(x) = k \cdot f(x)$




- stretch when $k > 1$
- compression when $0 < k < 1$

**Horizontal**

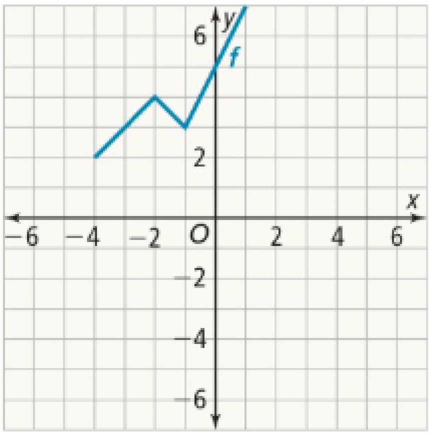
$h(x) = f(kx)$




- stretch when $0 < k < 1$
- compression when $k > 1$

## EXAMPLE 4 Graph a Combination of Transformations

**The graph represents $y = f(x)$. Using $y = f(x)$, how can you graph a combination of transformations?**

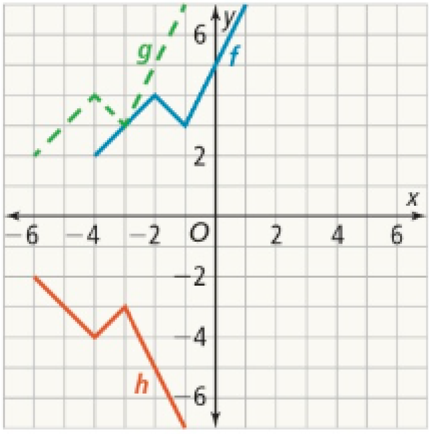


**A. Graph $y = -f(x + 2)$.**

Graph $g(x) = f(x + 2)$, which is a translation of $f$ left 2 units.

Graph $h(x) = -f(x + 2)$, which is a reflection of $g$ across the $x$-axis.

The graph of $h$ is a translation of $f$ left 2 units followed by a reflection across the $x$-axis.



**STUDY TIP**

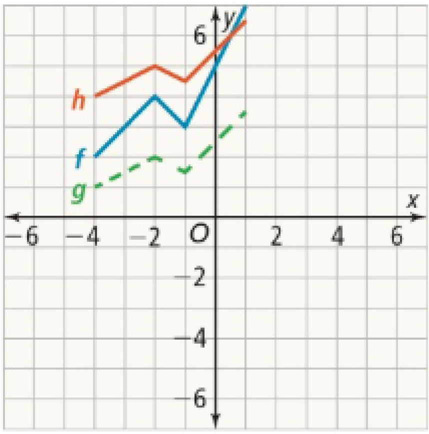
It is easier to perform one transformation at a time. Thinking about order of operations will help determine which transformation would be better to do first. Trying to perform both transformations at the same time will often result in an incorrect graph.

**B. Graph $y = \frac{1}{2}f(x) + 3$.**

Graph $g(x) = \frac{1}{2}f(x)$, which is a vertical compression of $f$ by the factor $\frac{1}{2}$.

Graph $h(x) = \frac{1}{2}f(x) + 3$, which is a translation of $g$ up 3 units.

The graph of $h$ is a vertical compression of $f$ by the factor $\frac{1}{2}$ followed by a translation 3 units up.



**Try It!** **4.** Using the graph of $f$ above, graph each equation.

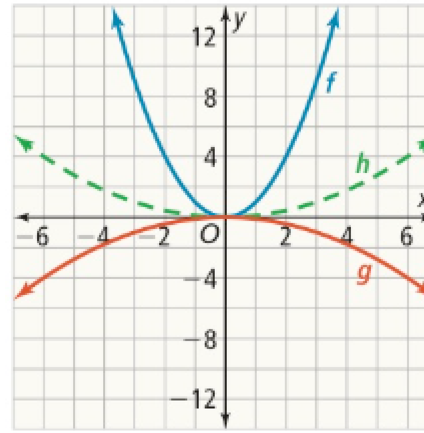
**a.** $y = f(2x) - 4$

**b.** $y = f(2x - 3) - 2$

## EXAMPLE 5 Identify Transformations From an Equation

**What transformations of $f(x) = x^2$ result in the graph of the function $g$?**

**A.** $g(x) = -\left(\frac{1}{3}x\right)^2$



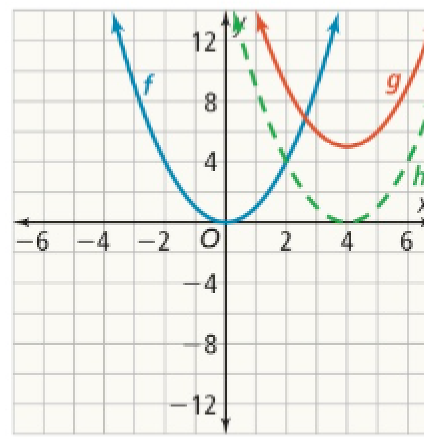

$h(x) = f\left(\frac{1}{3}x\right) = \left(\frac{1}{3}x\right)^2$ represents a horizontal stretch of the graph of $f$ by the factor $\frac{1}{3}$.

$g(x) = -h(x) = -f\left(\frac{1}{3}x\right) = -\left(\frac{1}{3}x\right)^2$ represents a reflection across the $x$-axis of $f\left(\frac{1}{3}x\right)$.

The graph of $g$ is a horizontal stretch by the factor $\frac{1}{3}$ and a reflection across the $x$-axis of the graph of $f$.

**CHECK FOR REASONABLENESS**
You can use graphing technology to graph the original and transformed equations to check that the transformations you have identified are correct.

**B.** $g(x) = (x - 4)^2 + 5$




$h(x) = f(x - 4) = (x - 4)^2$ represents a translation 4 units to the right of the graph of $f$.

$g(x) = h(x) + 5 = f(x - 4) + 5 = (x - 4)^2 + 5$ represents a translation 5 units up of the graph of $f(x - 4)$.

The graph of $g$ is a translation 4 units right and 5 units up of the graph of $f$.

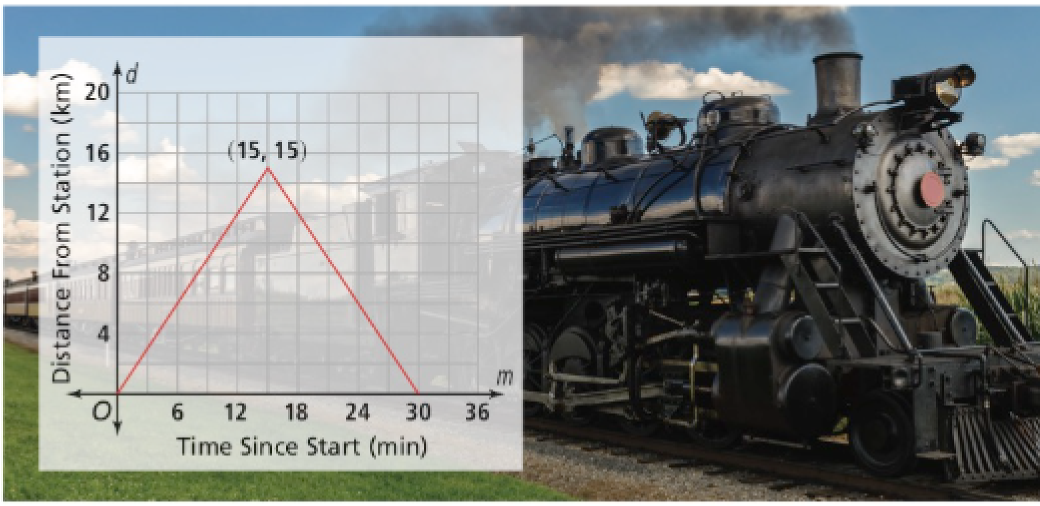
**Try It!** 5. What transformations of the graph of $f(x) = |x|$ are applied to graph the function $g$?

**a.** $g(x) = \frac{1}{2}|x + 3|$ **b.** $g(x) = -|x| + 2$

APPLICATION

## EXAMPLE 6 Write an Equation From a Graph

**A scenic train ride makes trips on an old mining line. The graph shows the distance $y$ in kilometers of the train from the station $x$ minutes after the ride begins. What equation represents the distance from the station as a function of time? What is its domain?**




The graph shows a reflection of an absolute value graph across the $x$-axis and a translation upward and to the right. The general form of this absolute value function is $y = -a|x - h| + k$, where the point $(h, k)$ represents the vertex and $-a$ indicates that the graph opens downward.

Substituting the point of the vertex (15, 15) for $(h, k)$ gives the equation $y = -a|x - 15| + 15$.

To solve for $a$, you can use any point on the graph. Using the point (0, 0) to substitute for $(x, y)$ in the equation simplifies the computation:

$$y = -a|x - 15| + 15$$

$$0 = -a|0 - 15| + 15$$

$$0 = -15a + 15$$

$$-15 = -15a$$

$$a = 1$$

**STUDY TIP**
Solving algebraically is only one method for determining a stretch or compression factor.

Now you can write the equation for distance as a function of time:

$$y = -|x - 15| + 15.$$

According to the graph, the train returns to its station after 30 minutes, so the function's domain is [0, 30].

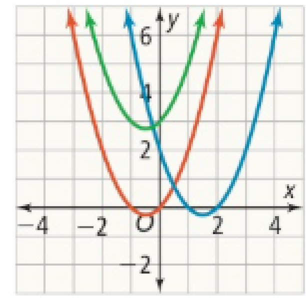
**Try It!** **6.** How would the graph and equation be affected if the train traveled twice as far in the same amount of time?

## CONCEPT SUMMARY Transformations of Functions

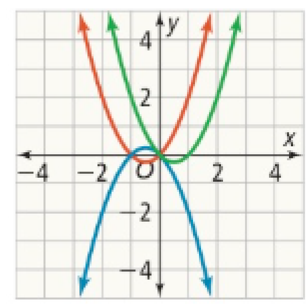
For a function $f(x)$, the graph of $f(x) = a \cdot f[b(x - h)] + k$ represents a transformation of the graph of that function by translation, reflection, or stretching.

| WORDS | EQUATIONS |
|---|---|
| **Horizontal translation of *f* right 2 units (altering *h*)** | $f(x)$ becomes $g(x) = f(x - 2)$<br>$f(x) = x^2 + x$<br>$g(x) = (x - 2)^2 + (x - 2) = x^2 - 3x + 2$ |
| **Vertical translation of *f* up 3 units (altering *k*)** | $f(x)$ becomes $h(x) = f(x) + 3$<br>$f(x) = x^2 + x$<br>$h(x) = x^2 + x + 3$ |

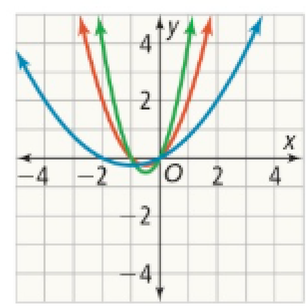
**GRAPHS**




| WORDS | EQUATIONS |
|---|---|
| **Reflection of *f* across the *x*-axis (altering *a*)** | $f(x)$ becomes $-f(x)$<br>$f(x) = x^2 + x$<br>$-f(x) = -(x^2 + x) = -x^2 - x$ |
| **Reflection of *f* across the *y*-axis (altering *b*)** | $f(x)$ becomes $f(-x)$<br>$f(x) = x^2 + x$<br>$f(-x) = (-x)^2 + (-x) = x^2 - x$ |




| WORDS | EQUATIONS |
|---|---|
| **Horizontal stretch of *f* by the factor $\frac{1}{2}$ (altering *b*)** | $f(x)$ becomes $f\left(\frac{1}{2}x\right)$<br>$f(x) = x^2 + x$<br>$f\left(\frac{1}{2}x\right) = \left(\frac{1}{2}x\right)^2 + \frac{1}{2}x = \frac{1}{4}x^2 + \frac{1}{2}x$ |
| **Vertical stretch of *f* by the factor 2 (altering *a*)** | $f(x)$ becomes $2f(x)$<br>$f(x) = x^2 + x$<br>$2f(x) = 2(x^2 + x) = 2x^2 + 2x$ |




## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What do the differences between the equation of a function and the equation of its parent function tell you about the differences in the graphs of the two functions?

2. **Generalize** Do $k$ and $h$ affect the input or output for $g(x) = f(x) + k$ and $g(x) = f(x - h)$? Explain.

3. **Error Analysis** Margo is comparing the functions $f(x) = |x|$ and $g(x) = |x + 1| - 5$. She said the graph of $g$ is a vertical translation of the graph of $f$ 5 units down and a horizontal translation of the graph of $f$ 1 unit right. What is Margo's error?

## Do You KNOW HOW?

**Graph each function and its parent function.**

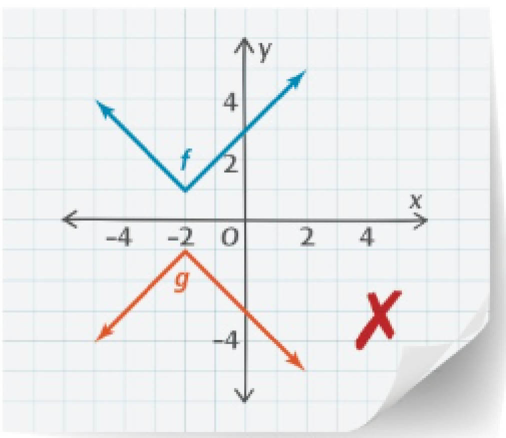
4. $g(x) = |x| - 1$
5. $g(x) = (x - 3)^2$
6. $g(x) = -|x|$
7. $g(x) = -x$
8. $g(x) = x^2 - 2$
9. $g(x) = \frac{1}{2}|x|$
10. $g(x) = 4x$
11. $g(x) = |5x|$
12. Which types of transformations in Exercises 4–11 do not change the shape of a graph? Which types of transformations change the shape of a graph? Explain.

## PRACTICE & PROBLEM SOLVING

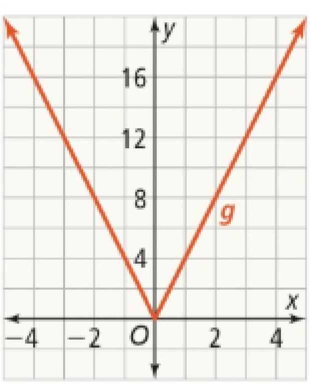
### UNDERSTAND

**13.** **Use Patterns and Structure** Write a function $g$ with the parent function $f(x) = x^2$ that has a vertex at $(3, -6)$.

**14.** **Error Analysis** Describe and correct the error a student made in graphing $g(x) = f(-x)$ as a reflection across the $y$-axis of the graph of $f(x) = |x + 2| + 1$.




**15.** **Higher Order Thinking** Describe the transformation $g$ of $f(x) = |x|$ as a stretch and as a compression. Then write two equations to represent the function. What can you conclude? Explain.




**16.** **Generalize** The graph of the parent function $f(x) = x^2$ is reflected across the $y$-axis. Write an equation for the function $g$ after the reflection. Show your work. Based on your equation, what happens to the graph? Explain.

**17.** **Error Analysis** Monisha is comparing $f(x) = |x|$ and $g(x) = |2x - 4|$. She said the graph of $g$ is a horizontal translation of the graph of $f$ 4 units to the right and a horizontal compression of the graph of $f$ by a factor of 2. What is Monisha's error?

### PRACTICE

**Graph each function as a translation of its parent function, $f$. How did the transformation affect the domain and range?** SEE EXAMPLE 1

**18.** $g(x) = |x| - 5$

**19.** $g(x) = (x + 1)^2$

**20.** $g(x) = |x - 3|$

**21.** $g(x) = x^2 + 2$

**What is the equation for the image graph? Check by graphing.** SEE EXAMPLE 2

**22.** Reflect $f(x) = x^2 + 1$ across the $x$-axis.

**23.** Reflect $f(x) = x^2 + 1$ across the $y$-axis.

**Graph each function as a vertical stretch or compression of its parent function.**
SEE EXAMPLE 3

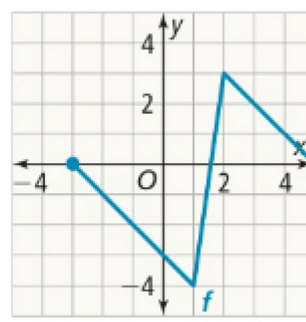
**24.** $g(x) = 0.25|x|$

**25.** $g(x) = 3x^2$

**26.** $g(x) = 1.5|x|$

**27.** $g(x) = 0.75x^2$

**28.** Use the graph of $f(x)$ to graph $y = f(x + 1) + 2$. SEE EXAMPLE 4




**What transformations of $f(x) = x^2$ are applied to the function $g$?** SEE EXAMPLE 5

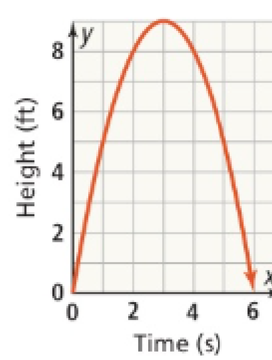
**29.** $g(x) = 2(x + 1)^2$

**30.** $g(x) = (x - 3)^2 + 5$

**31.** $g(x) = -x^2 - 6$

**32.** $g(x) = 4(x - 7)^2 - 9$

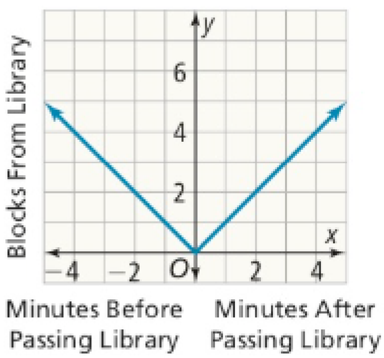
**33.** The graph shows the height $y$ in feet of a flying insect $x$ seconds after taking off from the ground. Write an equation that represents the height of the insect as a function of time.
SEE EXAMPLE 6



# PRACTICE & PROBLEM SOLVING

## APPLY

**34. Apply Math Models** Chiang walks to school each day. She passes the library halfway on her walk to school. She walks at a rate of 1 block per minute. The graph shows the distance Chiang is from the library as she walks to school.

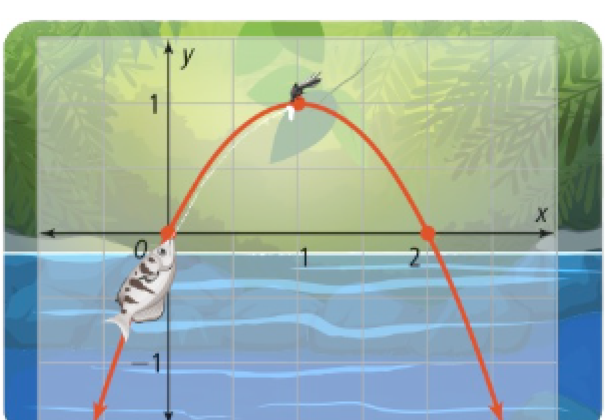



a. Write a function, $f$, to model the distance Chiang is from the library when she walks to school.

b. If Chiang jogs to school, she travels at a rate of 2.5 blocks per minute. Write a function, $g$, to model the distance Chiang is from the library when she jogs to school.

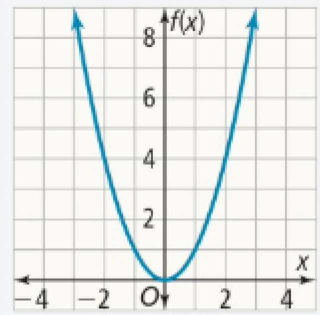
c. Graph the function, $g$, that models the distance Chiang is from the library when she jogs to school.

**35. Represent and Connect** The archer fish spits water at flying insects to knock them into the water. The path of the water is shown with $x$ and $y$ distances in feet. Write an equation to represent the path of the water in relation to the coordinate grid. Then determine the coordinates of the point of maximum height of the water.




## ASSESSMENT PRACTICE

**36.** The graph shows $f(x)$. Which statements about the graph of $f(2x) + 3$ are true? F.2.2




Ⓐ The vertex of $f(2x) + 3$ is $(0, 0)$.

Ⓑ The vertex of $f(2x) + 3$ is $(0, 3)$.

Ⓒ The graph of $f(2x) + 3$ passes through $(-1, 7)$.

Ⓓ The graph of $f(2x) + 3$ is wider than the graph of $f(x)$.

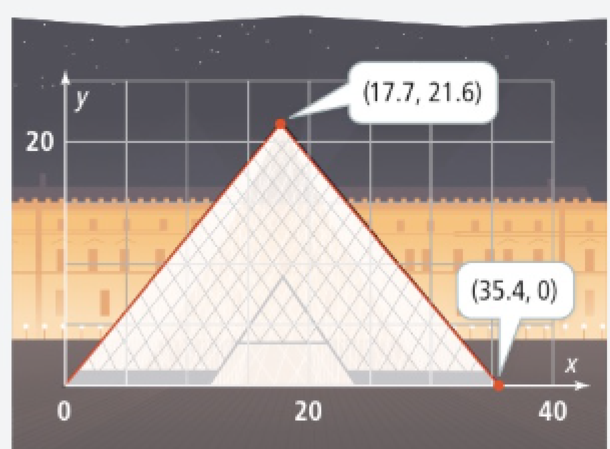
Ⓔ The graph of $f(2x) + 3$ is narrower than and translated up from the graph of $f(x)$.

**37. SAT/ACT** Which translation is part of transforming $f(x) = x^2$ into $h(x) = (x + 4)^2 - 2$?

Ⓐ left 4 units    Ⓒ right 2 units

Ⓑ left 2 units    Ⓓ right 4 units

**38. Performance Task** The Louvre Pyramid in Paris is shown on the coordinate grid, where $x$ and $y$ are measured in meters and the ground is represented by the $x$-axis.




**Part A** The outline of the Pyramid is a transformation of the function $f(x) = |x|$. Write a function $g$ to model the outline of the Pyramid.

**Part B** What is the domain and range of the function that models the outline of the Pyramid? What do the domain and range represent?

# 1-3

## Piecewise-Defined Functions

**I CAN…** graph and interpret piecewise-defined functions.

**VOCABULARY**

- piecewise-defined function
- step function

MA.912.AR.9.10–Solve and graph mathematical and real-world problems that are modeled with piecewise functions. Interpret key features and determine constraints in terms of the context.

MA.K12.MTR.1.1, MTR.5.1, MTR.7.1

## MODEL & DISCUSS

**A music teacher needs to buy guitar strings for her class. At store A, the guitar strings cost \$6 each. At store B, the guitar strings are \$20 for a pack of 4.**


**A.** Make graphs that show the income each store receives if the teacher needs 1–20 guitar strings

**B.** Describe the shape of the graph for store A. Describe the shape of the graph for store B. Why are the graphs different?

**C.** Communicate and Justify Compare the graphs for stores A and B. For what numbers of guitar strings is it cheaper to buy from store B? Explain how you know.

## ESSENTIAL QUESTION

**How do you model a situation in which a function behaves differently over different parts of its domain?**

CONCEPTUAL UNDERSTANDING

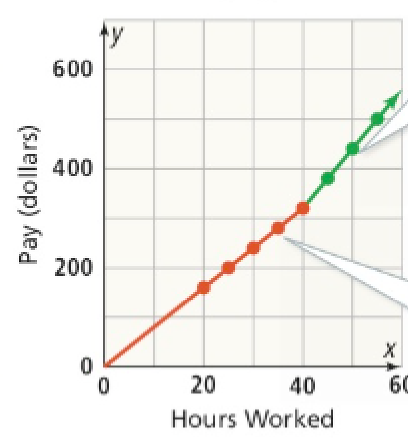
## EXAMPLE 1 Model With a Piecewise-Defined Function

**Alani has a summer job as a lifeguard. She makes \$8/h for up to 40 h each week. If she works more than 40 h, she makes 1.5 times her hourly pay, or \$12/h, for each hour over 40 h. How could you make a graph and write a function that shows Alani's weekly earnings based on the number of hours she worked?**

**STUDY TIP**
Remember that Alani makes \$8/h for the first 40 h and \$12/h for any additional hours after that.

**Step 1** Make a table of values and a graph.

| Hours Worked | Pay |
|---|---|
| 20 | 160 |
| 25 | 200 |
| 30 | 240 |
| 35 | 280 |
| 40 | 320 |
| 45 | 380 |
| 50 | 440 |
| 55 | 500 |




When $x > 40$, Alani's pay is $P(x) = (\$8)(40) + (\$12)(x - 40)$, or $P(x) = 12x - 160$.

When $0 \le x \le 40$, Alani's pay $P(x)$ is \$8/h times the number of hours worked, or $8x$.

**Step 2** Notice that the plot contains two linear segments with a slope that changes slightly at $x = 40$. A function that has different rules for different parts of its domain is called a **piecewise-defined function**.

**USE PATTERNS AND STRUCTURE**
This notation is used for piecewise-defined functions to indicate the different functions at different parts of the domain.

**Step 3** Write an equation for each piece of the graph.

$$P(x) = \begin{cases} 8x, & 0 \le x \le 40 \\ 12x - 160, & x > 40 \end{cases}$$

CONTINUED ON THE NEXT PAGE

**Try It!** 1. How much will Alani earn if she works:

a. 37 hours?

b. 43 hours?

## EXAMPLE 2 Graph a Piecewise-Defined Function

**How do you graph a piecewise defined function?**

$$f(x) = \begin{cases} 4x + 11, & -10 \le x < -2 \\ x^2 - 1, & -2 \le x \le 2 \\ x + 1, & 2 < x \le 10 \end{cases}$$

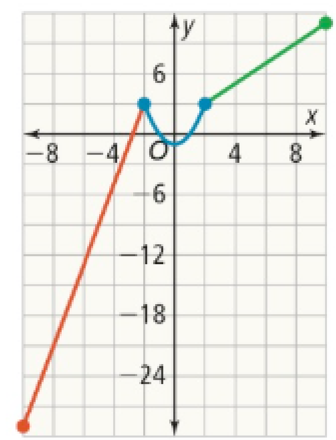
**What are the domain and range? Over what intervals is the function increasing or decreasing?**

**COMMON ERROR**
The values of −2 and 2 are only included in one piece of the graph. If they were included in more than one piece and had different values for different pieces, this would not be a function.

Sketch the graph of $y = 4x + 11$ for values of $x$ between −10 and −2.

Sketch the graph of $y = x^2 - 1$ for values of $x$ between −2 and 2.

Sketch the graph of $y = x + 1$ for values between 2 and 10.




To determine the range, calculate the $y$-values that correspond to the minimum and maximum $x$-values on the graph. For this graph, these values occur at the endpoints of the domain of the piecewise function, $-10 \le x \le 10$.

Evaluate $y = 4x + 11$ for $x = -10$

$y = 4(-10) + 11$

$y = -29$

Evaluate $y = x + 1$ for $x = 10$

$y = 10 + 1$

$y = 11$

The range is $-29 \le y \le 11$.

The domain is $\{x \mid -10 \le x \le 10\}$. The range is $\{y \mid -29 \le y \le 11\}$. The function is increasing when $-10 < x < -2$ and $0 < x < 10$. The function is decreasing when $-2 < x < 0$.

**Try It!** 2. Graph the piecewise-defined function. What are the domain and range? Over what intervals is the function increasing or decreasing?

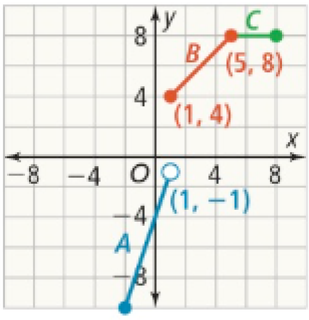
a. $f(x) = \begin{cases} 2x + 5, & -6 \le x \le -2 \\ 2x^2 - 7, & -2 < x < 1 \\ -4 - x, & 1 \le x \le 3 \end{cases}$

b. $f(x) = \begin{cases} 3, & -4 < x \le -1 \\ -x, & 0 \le x \le 2 \\ 3 - x, & 2 < x < 4 \end{cases}$

## EXAMPLE 3 Write a Piecewise-Defined Rule From a Graph

**STUDY TIP**
A closed circle on the graph means the coordinates of the point are included in the domain and range of the function. An open circle indicates they are not included.

**What is the rule that describes the piecewise-defined function shown in the graph?**




**Step 1** Notice three separate linear pieces that make up the function.

**Step 2** Determine the domain of each segment.

**Step 3** For each segment, use the graph to locate points on the line and to find the slope.

**Step 4** You can use the slope-intercept form of a linear function, $f(x) = mx + b$, to define the function for each segment.

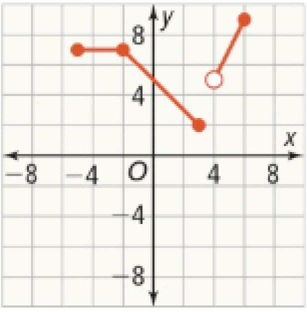
| Segment A | Segment B | Segment C |
|---|---|---|
| Domain: $-2 \le x < 1$ | Domain: $1 \le x \le 5$ | Domain: $5 < x \le 8$ |
| $(1, -1)$, slope $= 3$ | $(1, 4)$, slope $= 1$ | $(5, 8)$, slope $= 0$ |
| $y = mx + b$<br>$-1 = (3)(1) + b$<br>$b = -4$ | $y = mx + b$<br>$4 = (1)(1) + b$<br>$b = 3$ | $y = mx + b$<br>$8 = (0)(5) + b$<br>$b = 8$ |
| $f(x) = 3x - 4$ | $f(x) = x + 3$ | $f(x) = 8$ |

The rule for this function is:

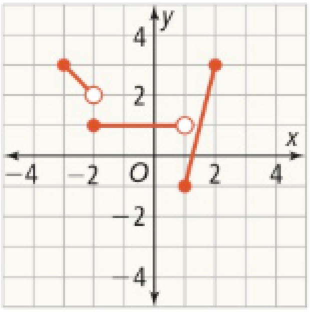
$$f(x) = \begin{cases} 3x - 4, & -2 \le x < 1 \\ x + 3, & 1 \le x \le 5 \\ 8, & 5 < x \le 8 \end{cases}$$

**Try It!** 3. What rule defines the function in each of the following graphs?

a.




b.



## EXAMPLE 4 Write a Rule for an Absolute Value Function

**How can you rewrite the function $f(x) = |6x + 18|$ as a piecewise-defined fuction?**

Step 1 Write the function in the form $f(x) = a|x - h| + k$ to find the vertex of the function.

$$f(x) = |6x + 18|$$
$$= |6(x + 3)|$$
$$= 6|x - \underbrace{(-3)}_{h = -3}| + \underbrace{0}_{k = 0}$$

The vertex is $(h, k) = (-3, 0)$. The graph has two linear pieces, one to the left of $x = -3$, and one to the right of $x = -3$.

**Step 2** Determine the slope and equation of each piece of the function by testing $x$-values on either side of $-3$.

| | Choose a point so that $x < -3$: let $x = -4$ | Choose a point so that $x > -3$: let $x = 0$ |
|---|---|---|
| **Point** | $(-4, 6)$ | $(0, 18)$ |
| **Slope to $(-3, 0)$** | $-6$ | $6$ |
| **Equation** | $y = -6x - 18$ | $y = 6x + 18$ |

**GENERALIZE**
The parent absolute value function $f(x) = |x|$ is a piecewise-defined function:

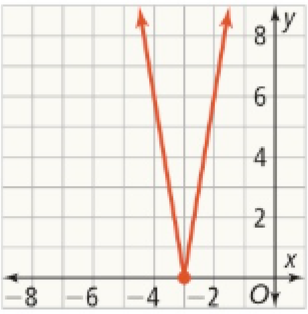
$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

**Step 3** Write the piecewise-defined function.

The absolute value function $f(x) = |6x + 18|$ can be written as the piecewise-defined function:

$$f(x) = \begin{cases} -6x - 18, & x < -3 \\ 6x + 18, & x \geq -3 \end{cases}$$

**Step 4** Confirm by graphing.




The vertex is $(-3, 0)$.

**Try It!** 4. How can you rewrite each function as a piecewise-defined function?

a. $f(x) = |-5x - 10|$

b. $f(x) = -|x| + 3$

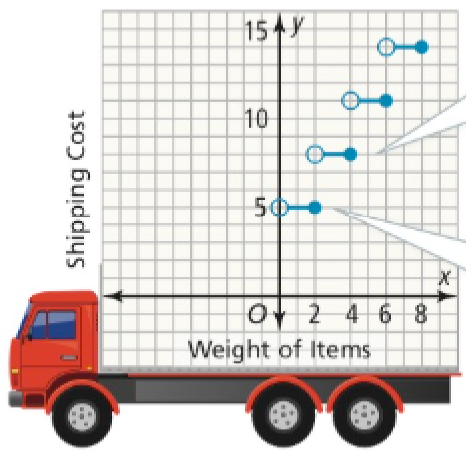
APPLICATION

## EXAMPLE 5 Graph a Step Function

**The shipping cost of items purchased from an online store is dependent on the weight of the items. The table represents shipping costs $y$ based on the weight $x$. Graph the function. What are the domain and range of the function, and what do they represent? What are the maximum and minimum values?**

| Weight of Items | $0 < x \leq 2$ lb | $2 < x \leq 4$ lb | $4 < x \leq 6$ lb | $6 < x \leq 8$ lb |
|---|---|---|---|---|
| Shipping Cost | \$5 | \$8 | \$11 | \$14 |

The graph of the function looks like the steps of a staircase. This is called a **step function** since it pairs every input in an interval with the same output value.




Each $y$-value goes with a set interval of $x$-values. For example, $y = 5$ for $0 < x \leq 2$ and $y = 8$ for $2 < x \leq 4$.

A closed circle means that the $x$-value is included in that interval, and the $\leq$ or $\geq$ sign is used. An open circle means it is not included.

**COMMON ERROR**
You might think that the range of this function would be the interval [5, 14], but only the values 5, 8, 11, and 14 are possible outputs.

Domain: $\{x \mid 0 < x \leq 8\}$

Range: $\{5, 8, 11, 14\}$

The domain of this function represents all possible weights of the items shipped. The store can ship items that weigh 8 pounds or less. Negative numbers and 0 are not part of the domain because an item must weigh something and cannot have a negative weight.

The range represents all possible costs that a customer could pay to ship the items.

This function has a minimum of 5 and a maximum of 14.

**Try It!** 5. The table below represents fees for a parking lot. Graph the function. What are the domain and range of the function, and what do they represent? What are the maximum and minimum values?

| Time | $0 < t \leq 3$h | $3 < t \leq 6$h | $6 < t \leq 9$h | $9 < t \leq 12$h |
|---|---|---|---|---|
| Cost | \$10 | \$15 | \$20 | \$25 |

## CONCEPT SUMMARY Piecewise-Defined Functions

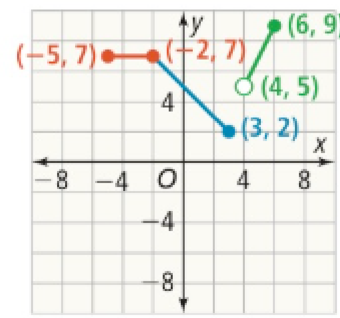
**WORDS**

A piecewise function has different rules for different parts of its domain.

**ALGEBRA**

$$f(x) = \begin{cases} 7, & -5 \le x \le -2 \\ 5 - x, & -2 < x \le 3 \\ 2x - 3, & 4 < x \le 6 \end{cases}$$

**GRAPH**




## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do you model a situation in which a function behaves differently over different parts of its domain?

2. **Vocabulary** How do piecewise-defined functions differ from step functions?

3. **Error Analysis** Given the function
$$f(x) = \begin{cases} 2x + 5, & -2 < x \le 4 \\ -4x - 7, & 4 < x \le 9 \end{cases}$$
Rebecca says there is an open circle at $x = 4$ for both pieces of the function. Explain her error.

4. **Analyze and Persevere** What steps do you follow when graphing a piecewise-defined function?

5. **Use Patterns and Structure** Is the relation defined by the following piecewise rule a function? Explain.
$$y = \begin{cases} 7x - 4, & x < 2 \\ -x + 5, & x \ge -2 \end{cases}$$

## Do You KNOW HOW?

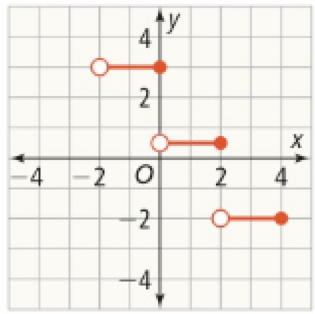
**Graph the function.**

6. $f(x) = \begin{cases} -x + 1, & -10 \le x < -3 \\ x^2 - 9, & -3 \le x \le 3 \\ 2x + 1, & 3 < x < 5 \end{cases}$

7. $g(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3, & 2 \le x < 4 \\ 5, & 4 \le x < 6 \\ 7, & 6 \le x < 8 \end{cases}$

8. Given the function
$$f(x) = \begin{cases} -2x + 4, & 0 \le x < 8 \\ -5x + 11, & x \ge 8 \end{cases}$$
is the function increasing or decreasing over the interval [2, 7]? Find the rate of change over this interval.

9. What is the rule that defines the function shown in the graph?

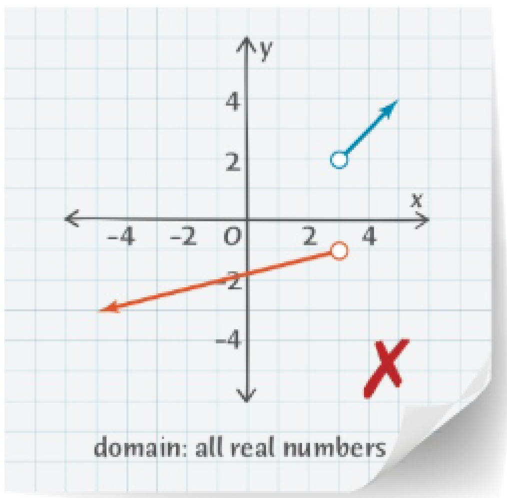


## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

10. **Generalize** What do closed circles and open circles on the graph of a step function indicate?

11. **Error Analysis** What error did Damian make when defining the domain of the graph? Explain.




12. **Communicate and Justify** For what values of $x$ is the function $f(x) = \begin{cases} -3x + 4, & -2 < x \leq 3 \\ 2x + 1, & 4 \leq x < 9 \end{cases}$ defined?

13. **Mathematical Connections** For the piecewise-defined function $f(x) = \begin{cases} 7, & x > 3 \\ 5x - 3, & x \leq 3 \end{cases}$ find two $x$-values that have the same $y$-value and the sum of the $x$-values is 10.

14. **Higher Order Thinking** The function $f(x) = \lfloor x \rfloor$ is called the greatest integer function because the output returned is the greatest integer less than or equal to $x$. For example, $f(3.2) = \lfloor 3.2 \rfloor = 3$ and $f(0.975) = \lfloor 0.975 \rfloor = 0$. Graph the function $f(x) = \lfloor x \rfloor$. What type of graph does this look like?

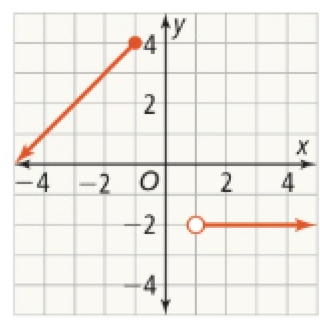
### PRACTICE

15. A phone company offers a monthly cellular phone plan for \$25. The plan includes 250 anytime minutes, and charges \$0.20 per minute above 250 min. Write a piecewise-defined function for $C(x)$, the cost for using $x$ minutes in a month. SEE EXAMPLE 1

16. Graph the piecewise-defined function. State the domain and range. Identify whether the function is increasing, constant, or decreasing on each interval of the domain. SEE EXAMPLE 2

$$f(x) = \begin{cases} \frac{1}{4}x + 3, & -2 < x \leq 0 \\ 2, & 0 < x \leq 4 \\ 3 - x, & 4 < x \leq 7 \end{cases}$$

17. Write the rule that defines the function in the following graph. SEE EXAMPLE 3




**Write each absolute value function as a piecewise-defined function.** SEE EXAMPLE 4

18. $f(x) = |3x + 1|$

19. $g(x) = |-2x - 6|$

**Graph the step function.** SEE EXAMPLE 5

20. $f(x) = \begin{cases} 2, & -3 \leq x < 1 \\ 5, & 1 \leq x < 4 \\ 8, & 4 \leq x < 6 \\ 9, & 6 \leq x < 10 \end{cases}$

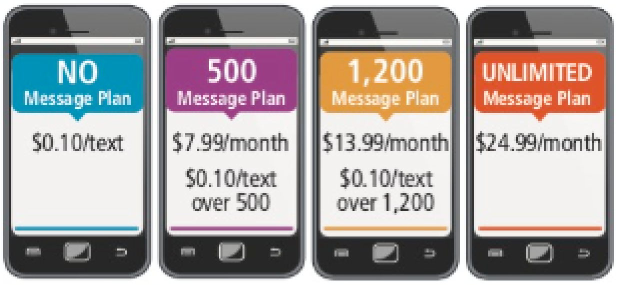
21. The parking rates for a parking garage are shown. Graph the function for the cost of parking rates at the garage. SEE EXAMPLE 5

## PRACTICE & PROBLEM SOLVING

### APPLY

22. **Apply Math Models** If Kyle works more than 40 h per week, his hourly wage for the extra hour(s) is 1.5 times the normal hourly wage of \$10 per hour. Write a piecewise-defined function that gives Kyle's weekly pay $P$ in terms of the number $h$ of hours he works. Determine how much Kyle will get paid if he works 45 h.

23. **Apply Math Models** Text message plans offered at a phone company, along with overage charges, are shown.




a. Write a function for each plan where $x$ is the number of texts and $f(x)$ is the total monthly cost.

b. Sarah uses approximately 1,500 texts per month. What is the monthly cost under each text message plan?

c. Write an interval for the number of text messages that would make each plan the best one to purchase.

24. **Represent and Connect** The cost $C$ (in dollars) of sending next-day mail depends on the weight $x$ (in ounces) of a package. The cost of packages, up to 5 lb, is given by the function below.

$$C(x) = \begin{cases} 12.25, & 0 < x \leq 8 \\ 16.75, & 8 < x \leq 32 \\ 19.50, & 32 < x \leq 48 \\ 23.50, & 48 < x \leq 64 \\ 25.25, & 64 < x \leq 80 \end{cases}$$

a. What are the domain and range of the function? What do they represent?

b. How much will it cost to ship a package that weighs 32.6 ounces.

### ASSESSMENT PRACTICE

25. Use the functions below to construct a piecewise-defined function $f$ on the intervals $(-\infty, -1)$, $[-1, 2]$, and $(2, \infty)$ that has a graph that is completely connected. Then draw the graph.

AR.9.10

$$f(x) = \begin{cases} \underline{\quad ? \quad}, & \text{if } x < -1 \\ \underline{\quad ? \quad}, & \text{if } -1 \leq x \leq 2 \\ \underline{\quad ? \quad}, & \text{if } x > 2 \end{cases}$$

$g_1(x) = x$ $\qquad g_2(x) = -x$ $\qquad g_3(x) = x^2$

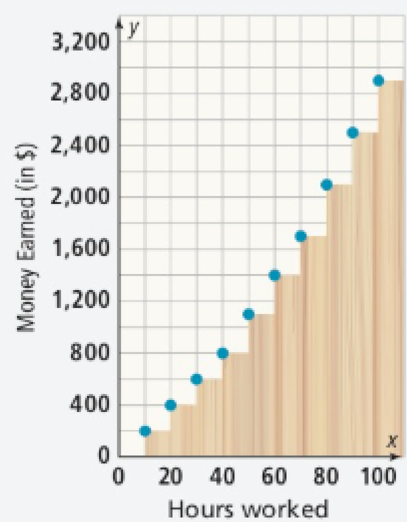
$g_4(x) = -x^2$ $\qquad g_5(x) = 4$

26. **SAT/ACT** What is the vertex of the absolute value function $f(x) = -|x - a| + b$ where $a$ and $b$ are real numbers?

Ⓐ $(a, b)$ Ⓒ $(a, -b)$

Ⓑ $(-a, b)$ Ⓓ $(-a, -b)$

27. **Performance Task** Yama works a varying number of hours per month for a construction company. The following scatter plot shows how much money he earns for each number of hours he works. Write the piecewise-defined function that represents Yama's earnings as a function of his hours worked.



# 1-4 Arithmetic Sequences

**I CAN…** interpret arithmetic sequences.

**VOCABULARY**

- arithmetic sequence
- common difference
- explicit definition
- recursive definition
- sequence

MA.912.AR.10.1–Given a mathematical or real-world context, write and solve problems involving arithmetic sequences.
MA.K12.MTR.3.1, MTR.5.1, MTR.7.1

## CRITIQUE & EXPLAIN

**Yumiko and Hugo are looking at the table of data about the number of families attending a party and how many items are needed to fill piñatas.**

| Attendees | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Items in Pinata | 1 | 5 | 9 | 13 | 17 |

Yumiko writes $f(1) = 1 + 4 = 5$,

$$f(2) = f(1) + 4 = 5 + 4 = 9,$$
$$f(3) = f(2) + 4 = 9 + 4 = 13,$$
$$f(4) = f(3) + 4 = 13 + 4 = 17.$$

Hugo writes $g(x) = 1 + 4x$.

**A.** Describe the pattern Yumiko found for finding items needed for the piñatas.

**B.** Describe the pattern Hugo found for finding items needed for the piñatas.

**C.** **Choose Efficient Methods** Compare the two methods. Which method would be more useful in finding the items needed when the 100th family attends? Why?

## ESSENTIAL QUESTION

**What is an arithmetic sequence, and how do you represent and find its terms?**

CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Understand Arithmetic Sequences

**A. Is the sequence arithmetic? If so, what is the common difference? What is the next term in the sequence?**

3, 8, 13, 18, 23, …

This is a **sequence**, a function whose domain is the Natural numbers.

Create a table that shows the term number, or domain, and the term, or range.

| Term Number | Term |
|---|---|
| 1 | 3 |
| 2 | 8 |
| 3 | 13 |
| 4 | 18 |
| 5 | 23 |
| 6 | ? |

+5 +5 +5 +5 +5

The difference between consecutive numbers in the range is 5.

**COMMON ERROR**
The common difference is always calculated by subtracting a term from the next term; $d = a_n - a_{n-1}$.

An **arithmetic sequence** is a sequence with a constant difference between consecutive terms. This difference is known as the **common difference**, or $d$.

This sequence is an arithmetic sequence with the common difference, $d = 5$. The next term in the sequence is 23 + 5, or 28.

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EXAMPLE 1 CONTINUED

**B. How could you write a formula for finding the next term in the sequence?**

**STUDY TIP**
An arithmetic sequence is a function, so you can write the terms using function notation.

Each term can be represented by $f(n)$ where $n$ represents the number of the term.

So, for $n = 1$, $f(1) = 3$.

If $n > 1$, each term is the sum of the previous term and 5.

$$f(2) = f(1) + 5$$

$$f(3) = f(2) + 5$$

$$f(n) = f(n - 1) + 5$$

Write the general rule for an arithmetic sequence as a piecewise-defined function:

$$f(n) = \begin{cases} f(1), \; n = 1 \\ f(n-1) + d, \; n > 1 \end{cases}$$

This is the **recursive definition** for an arithmetic sequence. Each term is defined by operations on the previous term.

Another way to write a recursive definition is to use subscript notation.

$$a_n = \begin{cases} a_1, \; n = 1 \\ a_{n-1} + d, \; n > 1 \end{cases}$$

With the notation, the subscript shows the number of the term.

**C. Is the sequence 4, 7, 10, 13, 16, … arithmetic? If so, write the recursive definition for the sequence.**

$a_1 = 4$

4, 7, 10, 13, 16

The common difference, $d$, is 3, so this is an arithmetic sequence.

The recursive definition for this sequence is

$$a_n = \begin{cases} 4, \; n = 1 \\ a_{n-1} + 3, \; n > 1. \end{cases}$$

**Try It!** 1. Are the following sequences arithmetic? If so, what is the recursive definition, and what is the next term in the sequence?

**a.** 25, 20, 15, 10, …

**b.** 2, 4, 7, 12, 13, …

## EXAMPLE 2 Translate Between Recursive and Explicit Forms

**A. Given the recursive definition $a_n = \begin{cases} 3, n = 1 \\ a_{n-1} + 0.5, n > 1 \end{cases}$ what is an explicit definition for the sequence?**

An **explicit definition**, also written as $a_n = a_1 + d(n - 1)$, allows you to find any term in the sequence without knowing the previous term.

Use the recursive definition to find a pattern:

$a_1 = 3$

$a_2 = 3 + 0.5$

$a_3 = a_2 + 0.5 = [3 + 0.5] + 0.5 = 3 + 2(0.5)$

$a_4 = a_3 + 0.5 = [3 + 2(0.5)] + 0.5 = 3 + 3(0.5)$

The first term has 0 common differences added. The second term has 1 common difference added to the first term. The third term has 2 common differences added, and so on.

So the explicit definition is $a_n = 3 + (n - 1)(0.5)$.

**USE PATTERNS AND STRUCTURE**
Since $a_2 = a_1 + 0.5$, use substitution to simplify the expression.

In general, the explicit definition of an arithmetic sequence is $a_n = a_1 + d(n - 1)$.

**B. Given the explicit definition $a_n = 16 - 3(n - 1)$, what is the recursive definition for the arithmetic sequence?**

The common difference $d$ is $-3$ and $a_1 = 16$.

The recursive definition is $a_n = \begin{cases} 16, n = 1 \\ a_{n-1} - 3, n > 1. \end{cases}$

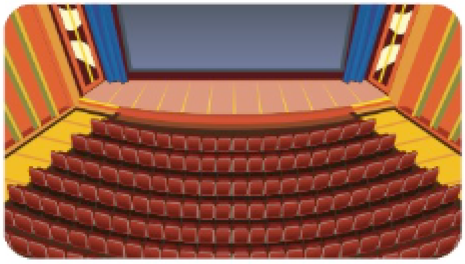
**Try It!** 2. a. For the recursive definition $a_n = \begin{cases} 45, n = 1 \\ a_{n-1} - 2, n > 1, \end{cases}$ what is the explicit definition?

b. For the explicit definition $a_n = 1 + 7(n - 1)$, what is the recursive definition?

APPLICATION

## EXAMPLE 3 Solve Problems With Arithmetic Sequences

**A high school auditorium has 18 seats in the first row and 26 seats in the fifth row. The number of seats in each row forms an arithmetic sequence.**



**A. What is the explicit definition for the sequence?**

The problem states that $a_1 = 18$, $n = 5$, and $a_5 = 26$.

| | |
|---|---|
| $a_n = a_1 + d(n - 1)$ | Write the general explicit formula. |
| $26 = 18 + d(5 - 1)$ | Substitute. |
| $26 = 18 + 4d$ | Simplify. |
| $8 = 4d$ | Simplify. |
| $2 = d$ | Solve. |

Each row has two more seats than the previous row.

The explicit definition is $a_n = 18 + 2(n - 1)$.

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EXAMPLE 3 CONTINUED

**B. How many seats are in the twelfth row?**

$a_n = 18 + 2(n - 1)$ ........ Write the explicit formula.

$a_{12} = 18 + 2(12 - 1)$ ........ Substitute 12 for $n$.

$a_{12} = 40$ ........ Simplify.

The twelfth row has 40 seats.

**Try It!** **3.** Samantha is training for a race. The distances of her training runs form an arithmetic sequence. She runs 1 mi the first day and 2 mi the seventh day.

**a.** What is the explicit definition for this sequence?

**b.** How far does she run on day 19?

APPLICATION

## EXAMPLE 4 Solve Problems Missing the Initial Value

**Emma has a school lunch account, and every day the school deducts the same amount for her lunch. When Emma checked her balance on the morning of the 10th day of school, it was $57.25. On the 16th day, it was $28.75.**

**A. What is the explicit definition for this sequence?**

First, find the common difference.

$$d = \frac{28.75 - 57.25}{16 - 10}$$

Emma's account changed from 57.25 when $n = 10$ to 28.50 when $n = 16$.

$$= -\frac{28.50}{6}$$

$$= -4.75$$

Emma's account balance decreases by $4.75 each day.

Next, find the initial value.

$a_n = a_1 + d(n - 1)$ ........ Write the general explicit formula.

$57.25 = a_1 + -4.75(10 - 1)$ ........ Substitute.

$57.25 = a_1 - 42.75$ ........ Simplify.

$100 = a_1$ ........ Solve.

The explicit definition is $a_n = 100 - 4.75(n - 1)$.

**B. When will Emma need to add money to her account?**

Emma will need to add money to her account when the balance is less than $4.75.

$a_n = 100 - 4.75(n - 1)$

$4.75 = 100 - 4.75(n - 1)$ ........ Substitute 4.75 for $a_n$.

$4.75 = -4.75n + 104.75$ ........ Simplify.

$21.05 \approx n$ ........ Solve.

On the 21st day of school, Emma will have more than $4.75 in her account, so she will be able to buy lunch. But on the 22nd day, Emma will have less than $4.75, so she will need to add money to her account.

CONTINUED ON THE NEXT PAGE

**STUDY TIP**
Recall that the term number is always a natural number. A decimal result means that Emma's account balance will not equal $4.75 exactly.

EXAMPLE 4 CONTINUED

**Try It!** 4. Eli rides his bicycle to the visitor center of a park, and then chooses a bike path to ride along. Eli then rides the same path once a week, always covering the same distance but he no longer has time for other rides. He noticed on his fitness app that, after his bike ride on the 7th week, his total distance was 119.4 miles. After his ride on the 10th week, his total distance was 144 miles.

a. Write an explicit definition for this sequence.

b. When will Eli reach 160 miles?

## CONCEPT SUMMARY Arithmetic Sequences

In an arithmetic sequence, each term is equal to the previous term plus a constant $d$, the common difference.

| | Recursive Formula | Explicit Formula |
|---|---|---|
| ALGEBRA | $a_n = \begin{cases} a_1, & n = 1 \\ a_{n-1} + d, & n > 1 \end{cases}$ | $a_n = a_1 + d(n-1)$ |
| NUMBERS | For $a_1 = 1$ and $d = 7$,<br>$a_2 = 1 + 7 = 8$<br>$a_3 = 8 + 7 = 15$<br>$a_4 = 15 + 7 = 22$<br>....and so on | For $a_1 = 90$ and $d = -4$,<br>$a_2 = 90 + 1(-4) = 86$<br>$a_3 = 90 + 2(-4) = 82$<br>$a_4 = 90 + 3(-4) = 78$<br>....and so on |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What is an arithmetic sequence, and how do you represent and find its terms?

2. **Vocabulary** How does the recursive definition of an arithmetic sequence differ from its explicit definition?

3. **Error Analysis** A student claims the sequence 0,1,3, 6, ... is an arithmetic sequence, and the next number is 10. What error did the student make?

4. **Analyze and Persevere** How would you tell someone how to find the common difference of an arithmetic sequence when given 2 specific terms of the sequence?

## Do You KNOW HOW?

**Find the common difference and the next three terms of each arithmetic sequence.**

5. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \ldots$

6. 6, 1, −4, −9, −14, . . .

7. 215, 227, 239, 251, . . .

8. −4, −5, −6, −7, . . .

9. 4.1, 6.3, 8.5, 10.7, . . .

10. −17, −9, −1, 7, 15, . . .

11. In June, you start a holiday savings account with a deposit of \$30. You then deposit \$44 each month until the end of the year. How much money will you have saved by the end of December?

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

12. **Represent and Connect** Write an arithmetic sequence with at least four terms, and describe it using both an explicit and recursive definition.

13. **Error Analysis** Alex says the common difference for an arithmetic sequence is always negative because of the definition of *difference*. Why is he wrong? Write an arithmetic sequence to show he is wrong.

14. **Use Patterns and Structure** A company will pay Becky \$120 for her first sale. For each sale after that, they will pay an extra \$31.50 per sale. So, she will make \$151.50 for the second sale, \$183 for the third sale, and so on. How many sales will Becky have to make to earn at least \$2,000?

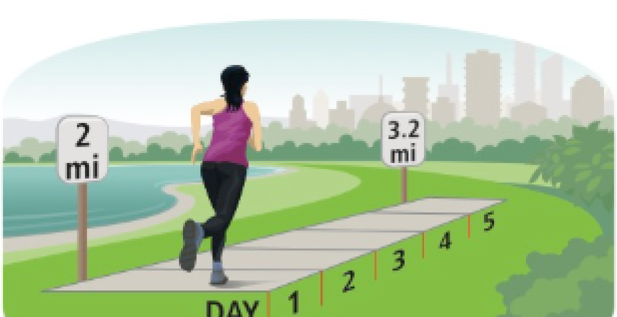
15. **Higher Order Thinking** Felipe and Gregory are given the arithmetic sequence −1, 6, 13, . . . . Gregory wrote the explicit definition $a_n = -1 + 7(n - 1)$ for the sequence. Felipe wrote the definition as $a_n = 7n - 8$. Which one of them is correct? Explain.

16. **Apply Math Models** Suppose you are building 10 steps with 8 concrete blocks in the top step and 80 blocks in the bottom step. If the number of blocks in each step forms an arithmetic sequence, find the common difference.

17. **Apply Math Models** With her half-marathon quickly approaching, Talisa decides to increase her training by the same amount every day up to the day of the race. She plans to run 2 mi the first day and 3.2 mi the fifth day.

    a. What is the explicit definition for this sequence?

    b. Which day of training will she run the distance of a half-marathon (13.1 mi)?




### PRACTICE

**Are the following sequences arithmetic? If so, what is the common difference? What is the next term in the sequence?** SEE EXAMPLE 1

18. 10, 20, 30, 40, . . .

19. 97, 86, 75, 64, . . .

20. 1, 4, 9, 16, . . .

21. 3, 7, 11, 15, . . .

**Translate between the recursive and explicit definitions for each sequence.** SEE EXAMPLE 2

22. $a_n = \begin{cases} 2, & n = 1 \\ a_{n-1} + 2, & n > 1 \end{cases}$

23. $a_n = -2 + 7(n - 1)$

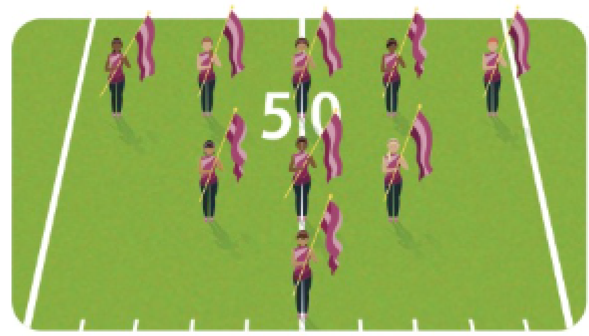
24. $a_n = \frac{1}{8}(n - 1)$

25. $a_n = \begin{cases} -4, & n = 1 \\ a_{n-1} - 4, & n > 1 \end{cases}$

26. The members of a school's color guard begin their performance in a pyramid formation. The first row has 1 member, and the third row has 5 members. SEE EXAMPLE 3

    a. What is the explicit definition for this sequence?

    b. How many members are in the eighth row?




**Find the explicit definition for the arithmetic sequence with the given terms.** SEE EXAMPLE 4

27. $a_5 = 31$, $a_{12} = 80$

28. $a_7 = -41$, $a_{11} = -65$

29. $a_4 = 10$, $a_{19} = 15$

30. $a_8 = -17.6$, $a_{14} = -15.2$

31. The number of seats in each row of an auditorium increases as you go back from the stage. The second row has 29 seats, and the third row has 34 seats. What is the first row to have more than 200 seats in it? SEE EXAMPLE 4

## PRACTICE & PROBLEM SOLVING

### APPLY

32. **Represent and Connect** A piece of tile artwork is in the shape of a triangle. The second row has 3 tiles, and the third row has 5 tiles. If the last row of the artwork has 27 tiles, how many rows are there?

33. **Apply Math Models** A race car driver travels 34 ft in the first second of a race. If the driver travels 3.5 additional feet each subsequent second, how long will it take the drive to reach a speed of 50 ft/s?

34. **Communicate and Justify** A school board committee has decided to spend its annual technology budget this year on 90 student laptops and plans to buy 40 new laptops each year from now on.

    a. The school board decided that each student in the school should have access to a laptop in the next ten years. If there are 500 students, will the technology coordinator meet this goal? Explain.

    b. What are some pros and cons of buying student laptops in this manner? If you could change the plan, would you? If so, how would you change it?

35. **Analyze and Persevere** On October 1, Nadia starts a push-up challenge by doing 18 push-ups. On October 2, she does 21 push-ups. On October 3, she does 24 push-ups. She continues until October 16, when she does the final push-ups in the challenge.

    a. Write an explicit definition to model the number of push-ups Nadia does each day.

    b. Write a recursive definition to model the number of push-ups Nadia does each day.

    c. How many push-ups will Nadia do on October 16?


### ASSESSMENT PRACTICE

36. Prior to 1994, the Winter Olympic Games were held in the same year as the Summer Olympic Games. The 1994 Winter Olympics were held in Lillehammer, Norway, and have taken place every four years since.

    A recursive definition for a sequence that models the year of the Winter Olympics is given.

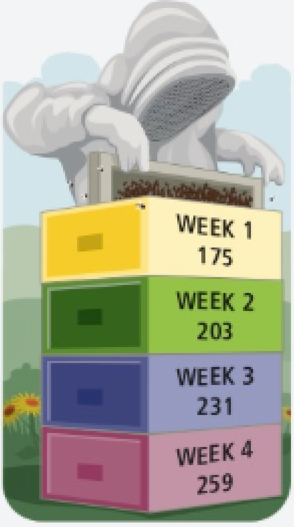
    $$a_n = \begin{cases} 1994, & n = 1 \\ a_{n-1} + 4, & n > 1 \end{cases}$$

    Write an explicit formula for the same sequence. AR.10.1

37. **SAT/ACT** Tamika is selling magazines door to door. On her first day, she sells 12 magazines, and she intends to sell 5 more magazines per day than on the previous day. If she meets her goal and sells magazines for a total of 10 days, how many magazines would she sell on the tenth day?

    Ⓐ 45 Ⓑ 55 Ⓒ 57 Ⓓ 62 Ⓔ 65

38. **Performance Task** The chart shows the population of Edgar's beehive over the first four weeks. Assume the population will continue to grow at the same rate.




    **Part A** Write an explicit definition for the sequence.

    **Part B** If Edgar's bees have a mass of 1.5 g each, what will the total mass of all his bees be in 12 wk?

    **Part C** When the colony reaches 1,015 bees, Edgar's beehive will not be big enough for all of them. In how many weeks will the bee population be too large?

# 1-5

## Quadratic and Absolute Value Inequalities

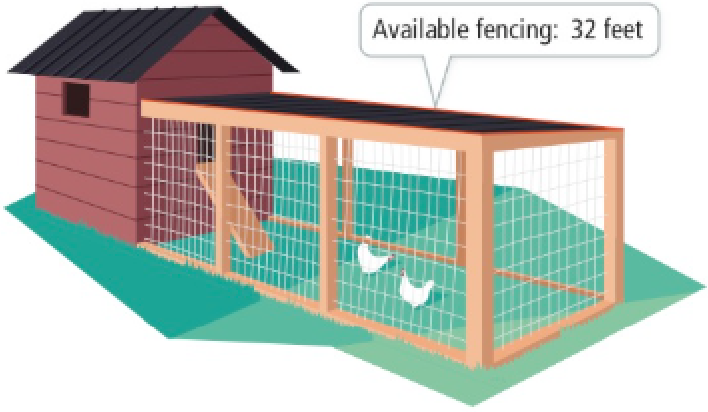
**I CAN…** use graphs or algebra to solve quadratic inequalities.

MA.912.AR.3.3–Given a mathematical or real-world context, write and solve one variable quadratic inequalities over the real number system. Represent solutions algebraically or graphically.
Also AR.3.8, AR.4.2

MA.K12.MTR.2.1, MTR.3.1, MTR.5.1

## MODEL & DISCUSS

**A homeowner has 32 feet of fencing to build three sides of a rectangular chicken run.**




**A.** Make a table of values for the length, width, and area of different rectangular chicken runs that will utilize 32 feet of fencing. Then write a function for the area, in terms of width, of a rectangular run using this much fencing.

**B.** Graph your function.

**C.** **Represent and Connect** Explain the meaning of the part of the graph that lies above the line $y = 80$.

## ESSENTIAL QUESTION

**How can you solve quadratic and absolute value inequalities?**

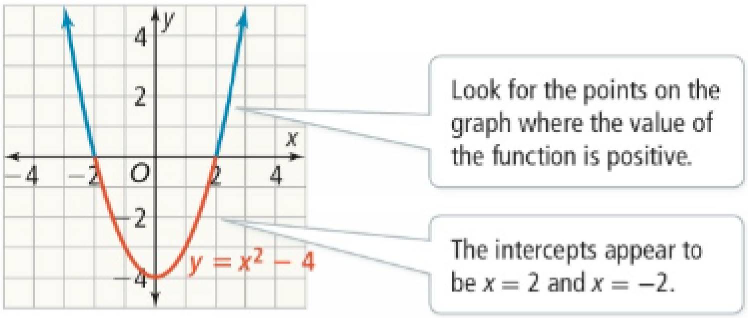
CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Solve a Quadratic Inequality

**Use a graph to solve the inequality $x^2 - 4 > 0$.**

To solve the inequality, identify the values of $x$ that make the value of the expression $x^2 - 4$ greater than 0.

Graph the equation $y = x^2 - 4$ by translating the parent function $y = x^2$ down 4 units.




**COMMON ERROR**

You may think that you need to find interval(s) where $x > 0$. However, you need to find where $x^2 - 4 > 0$, so you are looking for interval(s) where $y > 0$.

The graph of the function is positive over the intervals $(-\infty, -2)$ and $(2, \infty)$. So $x^2 - 4 > 0$ when $x < -2$ or $x > 2$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

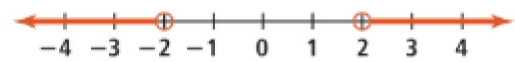
**USE PATTERNS AND STRUCTURE**
Quadratic expressions in factored form have similar patterns. You can use those patterns to efficiently analyze when the product of the factors is positive, negative, or 0.

You can also solve the inequality algebraically. Since $x^2 - 4$ is a difference of two squares, factor:

$$x^2 - 4 = (x - 2)(x + 2).$$

So, solving $x^2 - 4 > 0$ is equivalent to solving $(x - 2)(x + 2) > 0$. This inequality is true when both $x - 2$ and $x + 2$ are positive, or both are negative. Use a table to analyze when $x - 2$, $x + 2$, and $(x - 2)(x + 2)$ are positive, negative, or 0.

| | $x < -2$ | $x = -2$ | $-2 < x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|---|---|
| $(x - 2)$ | $-$ | $-$ | $-$ | 0 | $+$ |
| $(x + 2)$ | $-$ | 0 | $+$ | $+$ | $+$ |
| $(x - 2)(x + 2)$ | $+$ | 0 | $-$ | 0 | $+$ |




The product $(x - 2)(x + 2)$ is positive over the intervals $(-\infty, -2)$ and $(2, \infty)$, which agrees with where the graph of $y = x^2 - 4$ is above the $x$-axis. Therefore, $x^2 - 4 > 0$ when $x < -2$ or $x > 2$, which you can represent with a number line.

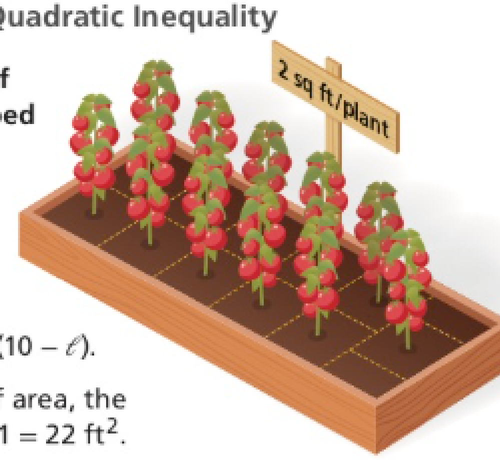
**Try It!** **1.** Solve each inequality. Represent each solution on a number line.

**a.** $x^2 - 25 \geq 0$ **b.** $x^2 - 16 < 0$

APPLICATION

## EXAMPLE 2 Write and Solve a Quadratic Inequality

**La Escuelita High School uses 20 feet of lumber to build a rectangular garden bed for 11 tomato plants. What are the possible lengths for the garden bed?**

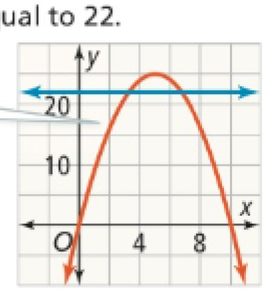



The perimeter of the garden bed is 20 ft. Let $\ell$ be the length of the bed. If $w$ is the width, then $2\ell + 2w = 20$, so $w = 10 - \ell$. The area can be written as $\ell(10 - \ell)$.

Since each tomato needs about 2 ft$^2$ of area, the garden bed area must be at least $2 \times 11 = 22$ ft$^2$.

The inequality $\ell(10 - \ell) \geq 22$ represents the size of the garden. To solve, use technology to graph the function $y = \ell(10 - \ell)$ and the line $y = 22$. Then determine where the function is greater than or equal to 22.

$y = \ell(10 - \ell)$ intersects the line $y = 22$ at approximately (3.3, 22) and (6.7, 22).




The garden bed area will be at least 22 ft$^2$ when $3.3 \leq \ell \leq 6.7$. Therefore, the garden bed can have any length between 3.3 ft and 6.7 ft.

**Try It!** **2.** What are the possible lengths of a rectangle with a perimeter of 24 inches, and an area of at least 16 square inches?

## EXAMPLE 3 Solve an Inequality with Two Quadratic Expressions

**Solve $x^2 - 6x \leq -x^2 + 12x - 36$.**

**Step 1** Rewrite the inequality so that a single quadratic expression is compared with 0.

$$x^2 - 6x \leq -x^2 + 12x - 36$$

$$(x^2 - 6x) - (-x^2 + 12x - 36) \leq 0 \quad \text{Subtract.}$$

$$2x^2 - 18x + 36 \leq 0 \quad \text{Simplify.}$$

$$x^2 - 9x + 18 \leq 0 \quad \text{Divide by 2.}$$

$$(x - 3)(x - 6) \leq 0 \quad \text{Factor.}$$

**Step 2** Use a table to determine when $(x - 3)(x - 6)$ is positive, negative, or 0.

| | $x < 3$ | $x = 3$ | $3 < x < 6$ | $x = 6$ | $x > 6$ |
|---|---|---|---|---|---|
| $(x - 3)$ | – | 0 | + | + | + |
| $(x - 6)$ | – | – | – | 0 | + |
| $(x - 3)(x - 6)$ | + | 0 | – | 0 | + |

**Step 3** Solve the inequality. The expression $(x - 3)(x - 6)$ is either 0 or negative over the interval $[3, 6]$. So $x^2 - 6x \leq -x^2 + 12x - 36$ when $3 \leq x \leq 6$.

**Try It!** **3.** Solve each inequality.

**a.** $x^2 - 3 \leq -x^2 + 4x - 3$

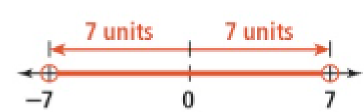
**b.** $x^2 + 4x + 5 \geq x^2 - 2x + 1$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 4 Understand Absolute Value Inequalities

**What are the solutions of an absolute value inequality?**

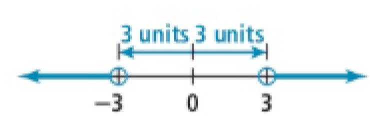
**A.** $|x| - 5 < 2$




Combine like terms and the inequality becomes $|x| < 7$. The distance between $x$ and 0 must be less than 7, so all values within 7 units to the right and 7 units to the left of 0 are solutions.

$|x| < 7$ is equivalent to the compound inequality $x < 7$ *and* $x > -7$, which can also be written as $-7 < x < 7$.

**B.** $2|x| - 0.7 > 0.8$




Isolate the absolute value of $x$, so the equation becomes $|x| > 3$. The distance between $x$ and 0 must be greater than 3. Positive values of $x$ must be more than 3 units to the right of 0, and negative values of $x$ must be more than 3 units to the left of 0.

$|x| > 3$ is equivalent to the compound inequality $x < -3$ *or* $x > 3$.

**COMMON ERROR**
Solutions to inequalities of the form $|x| > k$ usually consist of two separate intervals. Remember to write solutions as separate inequalities, $x < -k$ or $x > k$. It is incorrect to write $-k > x < k$.

**Try It!** **4.** Solve and graph the solutions of each inequality.

**a.** $|x| + 2.5 > 13.8$

**b.** $\frac{1}{2}|x| - 4 \leq 1$

APPLICATION

## EXAMPLE 5 Write an Absolute Value Inequality

**Members of the debate team are traveling to a tournament, where they will stay in a hotel for 3 nights. The total cost for each member must be within \$35 of \$205. Which of the hotels shown can they consider?**

**Hotel Room Costs***


*Per person per night

**Formulate** Write an absolute value inequality to represent the situation.

Let $x$ be the cost per night of a hotel room.

The difference between total cost and \$205 is less than or equal to \$35.

$$|3x - 205| \leq 35$$

**Compute** Solve the inequality to find the maximum and minimum hotel cost for each team member.

**Maximum Cost**

$$3x - 205 \leq 35$$
$$3x - 205 + 205 \leq 35 + 205$$
$$3x \leq 240$$
$$\frac{3x}{3} \leq \frac{240}{3}$$
$$x \leq 80$$

**Minimum Cost**

$$3x - 205 \geq -35$$
$$3x - 205 + 205 \geq -35 + 205$$
$$3x \geq 180$$
$$\frac{3x}{3} \geq \frac{180}{3}$$
$$x \geq 60$$

**Interpret** The cost of the hotel room can be between \$60 and \$80, inclusive.

The debate team can consider Hotel A or Hotel B.

**Try It!** **5.** If the debate team increased their limit to \$240 plus or minus \$20, would they be able to afford Hotel D at \$90 per night? Explain.

## CONCEPT SUMMARY Quadratic and Absolute Value Inequalities

| | QUADRATIC | ABSOLUTE VALUE |
|---|---|---|
| | Solve $x^2 - 8x + 12 < 0$. | Solve $\lvert 3x - 5 \rvert \geq 8$. |

### ALGEBRA

**Quadratic**

Factor.

$$x^2 - 8x + 12 < 0$$

$$(x - 2)(x - 6) < 0$$

Use a table.

| | $x < 2$ | $2 < x < 6$ | $x > 6$ |
|---|---|---|---|
| $(x - 2)$ | − | + | + |
| $(x - 6)$ | − | − | + |
| $(x - 2)(x - 6)$ | + | − | + |

Write the solution:

$2 < x < 6$

**Absolute Value**

Write a compound inequality.

$3x - 5 \geq 8$ or $3x - 5 \leq 8$

Solve each inequality.

$$3x - 5 \geq 8 \quad \text{or} \quad 3x - 5 \leq -8$$

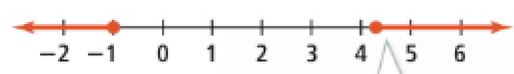
$$x \geq \frac{13}{3} \qquad\qquad x \leq -1$$

Write the solution.

$x \geq \frac{13}{3}$ or $x \leq -1$

### GRAPHS


All numbers greater than 2 and less than 6 satisfy $x^2 - 8x + 12 < 0$.




All numbers greater than or equal to $\frac{13}{3}$, or less than or equal to −1, satisfy $\lvert 3x - 5 \rvert > -8$ .

## Do You UNDERSTAND?

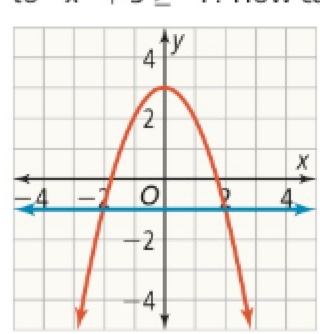
1. **ESSENTIAL QUESTION** How can you solve quadratic and absolute value inequalities?

2. **Choose Efficient Methods** What is an advantage of solving a quadratic inequality graphically?

3. **Error Analysis** Ben said the graph of the inequality $\lvert x - 17 \rvert < 13$ is made up of two disconnected intervals. Is Ben correct? Explain.

## Do You KNOW HOW?

4. Using the graph below, what is the solution to $-x^2 + 3 \geq -1$? How can you tell?

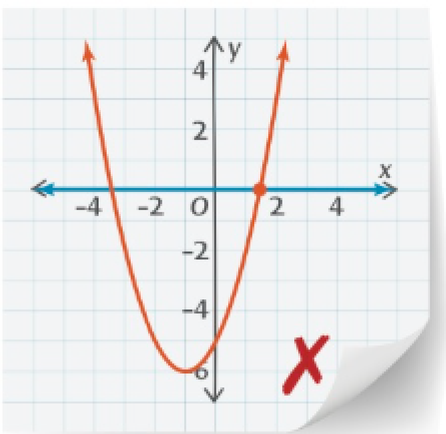


# PRACTICE & PROBLEM SOLVING

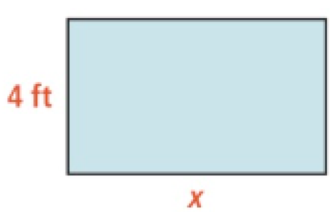
## UNDERSTAND

5. **Communicate and Justify** Use a graph to solve the inequality $x^2 - x - 2 \geq 0$. How can you use algebra to confirm that your graph shows the correct solution?

6. **Error Analysis** Victor used a graph to solve the inequality $x^2 + 2x - 5 < 0$. He used the INTERSECT feature on his graphing calculator to find the approximate $x$-intercept 1.449. Victor said the solution to the inequality is $x < 1.449$. Describe and correct the error Victor made.




7. **Mathematical Connections** Raul wants to model the perimeter of the rectangle below knowing that $1.5 \leq x \leq 6$ feet.




Because there is a range of values, Raul decides to use an absolute value inequality for his model. Do you agree with his decision? Explain your reasoning.

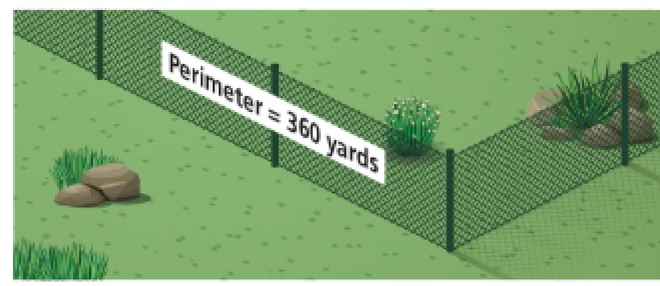
8. **Higher Order Thinking** Let $a$, $b$, $c$ and $x$ be real numbers. How is solving $|ax| + b \leq c$ different from solving $|ax + b| \leq c$?

## PRACTICE

**Solve each inequality. Graph each solution on a number line.** SEE EXAMPLES 1 AND 2

9. $x^2 - 16 < 0$
10. $-x^2 + 25 \leq 0$
11. $-x^2 - 49 > 0$
12. $x^2 + 2x - 8 > -5$
13. $-2x^2 + 5x \geq -3$
14. $-x^2 + 5 \leq 2$

15. A city is constructing a rectangular field for recreation at Woods Park. The field should be at least 8000 square yards. The city's budget can afford 360 yards of fencing to surround the field. Write an inequality to represent the situation. Then find the possible lengths for the field.




**Solve each inequality.** SEE EXAMPLE 3

16. $2x^2 + 9x - 17 < -x^2 + 3x + 7$
17. $3x^2 - 51x + 99 \geq x^2 - 15x - 55$
18. $x^2 + 10x - 24 < 10x - 15$
19. $x^2 + 5x + 8 > x^2 - 2x + 1$

**Solve each absolute value inequality. Graph the solution.** SEE EXAMPLES 4 AND 5

20. $|x| + 8 \geq 16$
21. $|x| + 7.6 < 5.1$
22. $\left|\frac{2}{3}x + \frac{7}{6}\right| \geq \frac{1}{4}$
23. $\left|\frac{2}{3}x - \frac{7}{6}\right| < \frac{1}{4}$
24. $-3|x + 12| \leq -9$
25. $-3|3x - 12| + 18 > -9$

**Match each absolute value inequality to thegraph that represents its solution. Explain yourreasoning.** SEE EXAMPLES 4 AND 5

26. $5|x - 3| - 7 \leq 8$

A. 


27. $3|x - 3| - 1 < 8$

B. 


28. $5\left|\frac{1}{3}x - 1\right| + 7 > 12$

C. 

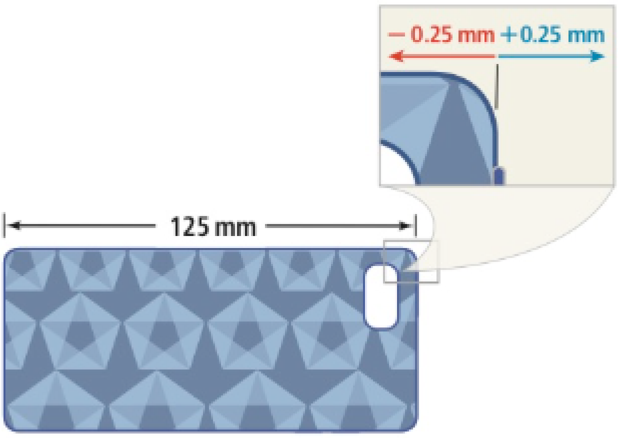

29. $3\left|\frac{2}{3}x - 2\right| - 11 \geq -5$

D. 

## PRACTICE & PROBLEM SOLVING

### APPLY

**30. Represent and Connect** A company manufactures cell phone cases. The length of a certain case must be within 0.25 mm of 125 mm, as shown (figure is not to scale). All cases with lengths outside of this range are removed from the inventory. How could you use an absolute value inequality to represent the lengths of all the cases that should be removed? Explain.




**31. Analyze and Persevere** Ashton is hosting a banquet. He plans to spend \$400, plus or minus \$50, at a cost of \$25 per guest. Solve an absolute value inequality to find the possible numbers of guests within his budget. If there can be up to 5 people at each table, what number of tables should Ashton reserve so that everyone is guaranteed to have a seat?

**32. Analyze and Persevere** The amount, in millions of dollars, that a company earns in revenue for selling $x$ items, in thousands, is $R = -2x^2 + 18x - 2$. The expenses, in millions of dollars, for selling $x$ items, in thousands, is $E = -0.25x + 6$.

**a.** The profit in millions of dollars, for selling $x$ items, in thousands, is the difference between the revenues and the expenses. Write an inequality that models the company earning a profit.

**b.** Use graphing technology to find the least number of items the company can sell while earning a profit. Round to the nearest item.

### ASSESSMENT PRACTICE

**33.** Kenzie finished recording the last song for her new album. The song was 4 minutes and 47 seconds long. The song must be within 30 seconds of 250 seconds long to fit in the recording space left on the album. Write an absolute value inequality to represent the situation and solve. Will the last song fit on the album? AR.3.3

**34. SAT/ACT** Solve $-2(x - 5)(x + 4) \leq 0$.

Ⓐ $x \leq -5$ or $x \geq 4$

Ⓑ $-5 \leq x \leq 4$

Ⓒ $-4 \leq x \leq 5$

Ⓓ $x \leq 4$ or $x \geq 5$

Ⓔ $x \leq -4$ or $x \geq 5$

**35. Performance Task** A road sign shows a vehicle's speed as the vehicle passes.


**Part A** The sign blinks for vehicles traveling within 5 mi/h of the speed limit. Write and solve an absolute value inequality to find the range of speeds of an oncoming vehicle that will cause the sign to blink.

**Part B** Another sign blinks when it detects a vehicle traveling within 2 mi/h of a 35 mi/h speed limit. Write and solve an absolute value inequality to represent the speeds of the vehicles that cause the sign to blink.

**Part C** The sign is programmed to blink using absolute value inequalities of the form $|x - a| \leq b$ and $|x - a| \geq b$. Which of these formulas is used to program the sign for cars traveling more than 5 mi/h above or below the 20 mi/h speed limit? What are the values of $a$ and $b$? Explain.

# 1-6
## Linear Systems

**I CAN…** use a variety of tools to solve systems of linear equations and inequalities.

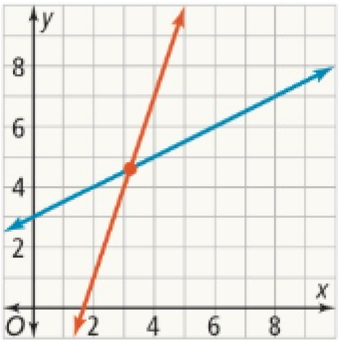
**VOCABULARY**
- solution of a system of linear equations
- system of linear equations
- system of linear inequalities

MA.912.AR.9.3–Given a mathematical or real-world context, solve a system consisting of two-variable linear or non-linear equations algebraically or graphically. Also AR.9.2, AR.9.5, AR.9.7

MA.K12.MTR.2.1, MTR.3.1, MTR.5.1

### EXPLORE & REASON

**The graph shows two lines that intersect at one point.**




**A.** What are the approximate coordinates of the point of intersection?

**B.** How could you verify whether the coordinates you estimated are, in fact, the solution? Is the point the solution to the equations of both lines?

**C.** **Analyze and Persevere** Use your result to refine your approximation, and try again. Can you find the point of intersection this way? Is there a more efficient way?

**ESSENTIAL QUESTION** How can you find and represent solutions of systems of linear equations and inequalities?

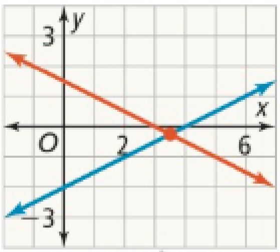
### EXAMPLE 1 Solve a System of Linear Equations

**What is the solution of the system of linear equations $\begin{cases} x + 2y = 3 \\ x - 2y = 4 \end{cases}$?**

A **system of linear equations** is a set of two or more equations using the same variables. The **solution of a system of linear equations** is the set of all ordered coordinates that simultaneously make all equations in the system true.

CONCEPTUAL UNDERSTANDING

Sketch the graph of each equation to estimate the solutions. Then solve algebraically.




The $x$-coordinate of the solution is between 3 and 4, and the $y$-coordinate of the solution is between –1 and 0.

$x + 2y = 3 \Rightarrow x = 3 - 2y$

$x - 2y = 4 \Rightarrow x = 4 + 2y$

$\Downarrow$

Substitute for $x$ in both equations and solve.

$$3 - 2y = 4 + 2y$$
$$-1 = 4y$$
$$-\frac{1}{4} = y$$

Substitute the value for $y$ into either original equation to find the value of $x$.

$$x = 3 - 2\left(-\frac{1}{4}\right)$$
$$x = \frac{7}{2}$$

The solution is $\left(\frac{7}{2}, -\frac{1}{4}\right)$. These values are close to the estimate made from the graph. You can check to confirm that these values satisfy both equations.

**CHOOSE EFFICIENT METHODS**
Another way to solve this system is to add the two equations together. This would eliminate the $y$ variable, yielding the equation $2x = 7$. How would you finish solving the system?

**Try It!** 1. Solve each system of equations.

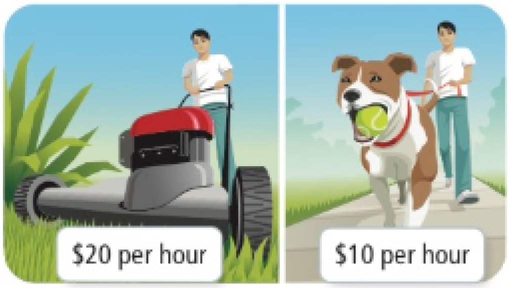
a. $\begin{cases} 2x + y = -1 \\ 5y - 6x = 7 \end{cases}$

b. $\begin{cases} 3x + 2y = 5 \\ 6x + 4y = 3 \end{cases}$

APPLICATION

## EXAMPLE 2 Solve a System of Linear Inequalities

**Malcolm earns \$20 per hour mowing lawns and \$10 per hour walking dogs. His goal is to earn at least \$200 each week, but he can work a maximum of 20 hour per week. Malcolm must spend at least 5 hour per week walking his neighbors' dogs. For how many hours should Malcolm work at each job in order to meet his goals?**




A **system of linear inequalities** is a set of two or more inequalities using the same variables.

**HAVE A GROWTH MINDSET**
When it takes time to learn something new, how do you stick with it?

**Step 1** Define the variables.

$x$ = number of hours spent mowing lawns

$y$ = number of hours spent walking dogs

**Step 2** Write inequalities to model the constraints.

Malcolm wants to earn at least \$200 each week at \$20 per hour mowing lawns and \$10 per hour walking dogs: $20x + 10y \geq 200$.

Malcolm cannot work more than 20 h each week: $x + y \leq 20$.

Malcolm must spend at least 5 h walking dogs each week: $y \geq 5$.

**Step 3** Solve each inequality for y, then graph the inequalities on the same coordinate plane.

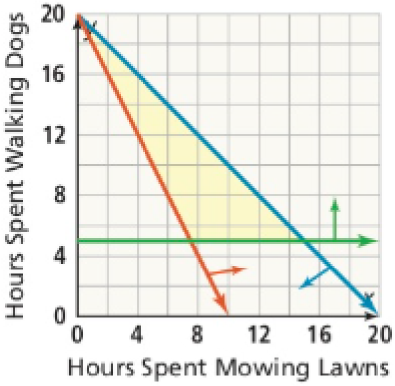
$y \geq 20 - 2x$

$y \leq 20 - x$

$y \geq 5$

Use arrows to show the region of the graph that satisfies each inequality.

Shade the region that satisfies all three inequalities.




Any point in the shaded region, such as (12, 7), is a solution to the system of inequalities. So if Malcolm spends 12 h mowing lawns and 7 h walking dogs, he will have met his goals.

**Try It!** 2. Sketch the graph of the set of all points that solve this system of linear inequalities.

$$\begin{cases} 2x + y \leq 14 \\ x + 2y \leq 10 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

CONCEPTUAL UNDERSTANDING

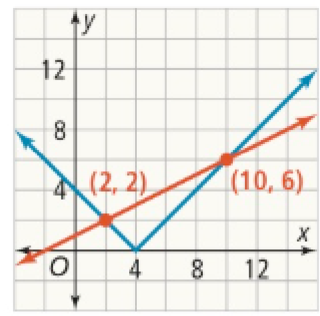
## EXAMPLE 3 Solve a System With a Nonlinear Equation

**How can you solve the system of equations?**

$$\begin{cases} y = \frac{1}{2}x + 1 \\ y = |x - 4| \end{cases}$$

**Method 1** Solve the system graphically.

Graph both equations.




**GENERALIZE**
Not every system of equations is easy to solve algebraically. However, you may use a graphing calculator and find points of intersection.

The graphs of the two equations intersect at (2, 2) and (10, 6).

**Method 2** Solve the system algebraically.

Use substitution to rewrite the system as a single equation. Then solve for $x$.

$$|x - 4| = \frac{1}{2}x + 1$$

The solutions to $|x| = a$ are $a$ and $-a$.

$$x - 4 = \frac{1}{2}x + 1 \quad \text{or} \quad x - 4 = -\left(\frac{1}{2}x + 1\right)$$

$$\frac{1}{2}x = 5 \quad \text{or} \quad \frac{3}{2}x = 3$$

$$x = 10 \quad \text{or} \quad x = 2$$

Substitute the values back into the original equations:

$$y = |10 - 4| \quad \text{or} \quad y = |2 - 4|$$

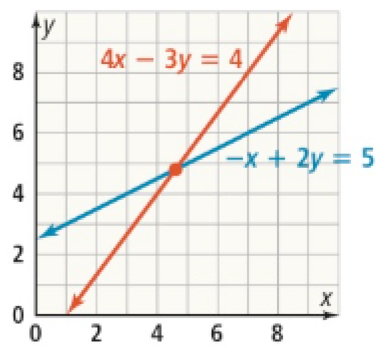
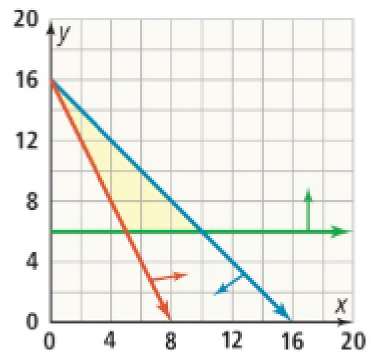
$$y = 6 \quad \text{or} \quad y = 2$$

The solutions are (10, 6) and (2, 2)

**Try It!** 3. Solve the system of equations algebraically and graphically.

$$\begin{cases} y = -\frac{1}{2}x - 2 \\ y = -|x - 2| \end{cases}$$

## CONCEPT SUMMARY Linear Systems

| | System of linear equations | System of linear inequalities |
|---|---|---|
| **WORDS** | a set of two or more equations using the same variables | a set of two or more inequalities using the same variables |
| **ALGEBRA** | $\begin{cases} 4x - 3y = 4 \\ -x + 2y = 5 \end{cases}$ | $\begin{cases} y \geq 16 - 2x \\ y \leq 16 - x \\ y \geq 6 \end{cases}$ |
| **GRAPHS** |  |  |





## Do You UNDERSTAND?

1. ESSENTIAL QUESTION How can you find and represent solutions of systems of linear equations and inequalities?

2. **Error Analysis** Shandra said the solution of the system of equations $\begin{cases} 2x + y = 3 \\ -x + 4y = -6 \end{cases}$ is $(-1, 2)$. Is she correct? Explain.

3. **Choose Efficient Methods** Why is a system of linear inequalities often solved graphically?

4. **Analyze and Persevere** How might knowing how to solve a system of linear equations help you to solve a system of equations where one equation is not linear?

5. **Vocabulary** What is the difference between a system of linear equations and a system of linear inequalities?

## Do You KNOW HOW?

6. Solve the following system of equations.
$$\begin{cases} 2x + 2y = 10 \\ x + 5y = 13 \end{cases}$$

7. Graph the following system of inequalities.
$$\begin{cases} -x + 2y < 1 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

8. Solve the following system of equations.
$$2x - y + z = 3$$
$$3x + y + 3z = 10$$
$$x - 2y - 2z = 3$$

9. Equations with two variables that are raised only to the first power represent lines. There are three possible outcomes for the intersections of two lines. Describe the outcomes.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

10. **Communicate and Justify** Consider a point that lies on the border of the shaded region of the graph of a system of linear inequalities. Under what conditions is that point a solution to the system?

11. **Error Analysis** Describe and correct the error a student made in solving the system of equations.

$$\begin{aligned} 2x + 4y = 0 \quad &\Rightarrow \quad 2x + 4y = 0 \\ 3x - 2y = -24 \quad &\Rightarrow \quad 6x - 4y = -24 \end{aligned}$$

$$\Downarrow$$

$$8x = -24$$

$$x = -3$$

$$\Downarrow$$

$$2(-3) + 4y = 0$$

$$-6 + 4y = 0$$

$$4y = 6$$

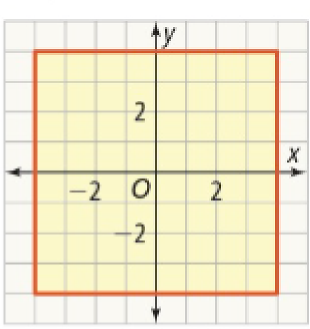
$$y = \frac{3}{2} \quad ✗$$

12. **Higher Order Thinking** Suppose that solving a system of two linear equations algebraically gives the following result: $0 = 0$

    What does this mean about the graphs of the two equations?

13. **Represent and Connect** A system of equations has three solutions. What must be true about the graphs of the equations?

14. **Analyze and Persevere** Write a system of inequalities for the shaded region.




15. **Mathematical Connections** Consider the following system of equations.

    $\begin{cases} x = 5 - 3y \\ y = -2x \end{cases}$

    Write a system of inequalities whose solution includes the solution to the system of equations above.

### PRACTICE

**Solve the following systems of equations.**
SEE EXAMPLE 1

16. $\begin{cases} x = 2y - 5 \\ 3x - y = 5 \end{cases}$

17. $\begin{cases} y = 2x + 3 \\ 2y - x = 12 \end{cases}$

18. $\begin{cases} x - 3y = 1 \\ 2x - y = 7 \end{cases}$

19. $\begin{cases} x + 2y = -4 \\ 3x - y = -5 \end{cases}$

**Write a system of linear equations that has the solution shown.**

20. $(-3, 5)$

21. $(10, -1)$

**Sketch the graph of the set of all points that solve each system of linear inequalities.** SEE EXAMPLE 2

22. $\begin{cases} 0 < x \leq 125 \\ x \geq 2y > 0 \\ 2x + 2y \leq 300 \end{cases}$

23. $\begin{cases} y + 2x < 10 \\ x - 2y < 8 \\ x > 0 \\ y > 0 \end{cases}$

24. $\begin{cases} y \leq -2x + 19 \\ y \geq \frac{3}{7}x + 2 \\ x \leq 7 \end{cases}$

25. $\begin{cases} y < \frac{3}{2}x \\ 3x + 2y < 36 \\ 3 < y < 6 \end{cases}$

26. Charles has a collection of dimes and quarters worth \$1.25. He has 8 coins. Write a system of equations to represent this situation. Then solve the system to determine how many dimes and how many quarters Charles has. SEE EXAMPLE 2

27. A set of triangular and square tiles contains 50 pieces and 170 sides. Write a system of equations to represent this situation. Then solve the system to determine how many triangular and how many square tiles there are. SEE EXAMPLE 2

**Solve the following systems of equations.**
SEE EXAMPLE 3

28. $\begin{cases} y = 2x - 20 \\ y = -4|x - 1| \end{cases}$

29. $\begin{cases} y = \frac{1}{4}x - 3 \\ y = \frac{1}{2}|x - 6| - 3 \end{cases}$

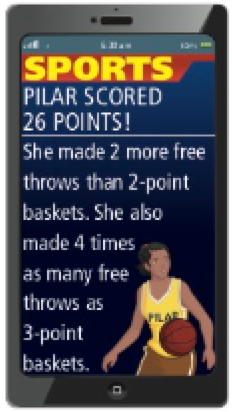
30. $\begin{cases} y = 6 \\ y = -|x + 5| + 4 \end{cases}$

31. $\begin{cases} y = -\frac{1}{3}x + 7 \\ y = |x + 8| + 7 \end{cases}$

# PRACTICE & PROBLEM SOLVING

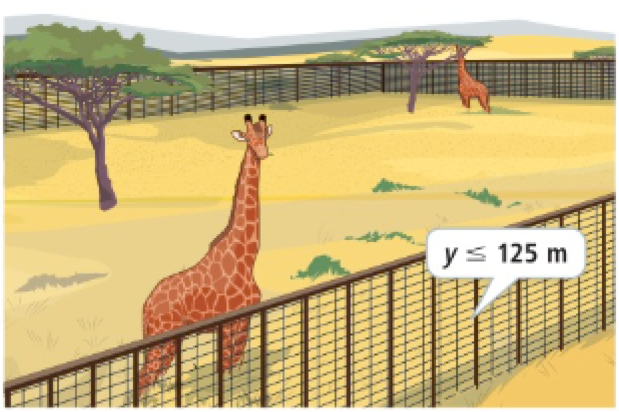
## APPLY

**32.** **Apply Math Models** In basketball, a successful free throw is worth 1 point, a basket made from inside the 3-point arc is worth 2 points, and a basket made from outside the 3-point arc basket is worth 3 points. How many of each type of basket did Pilar make?




**33.** **Apply Math Models** Raul is paid \$75 per week plus \$5 for each new gym membership he sells. He may switch to a gym that pays \$50 per week and \$7.50 for each new membership. How many memberships per week does Raul have to sell for the new gym to be a better deal for him?

**34.** **Represent and Connect** Keisha is designing a rectangular giraffe enclosure with a length of at most 125 m. The animal sanctuary can afford at most 300 m of fencing, and the length of the enclosure must be at least double the width.




a. Write inequalities to represent each constraint where $x$ = width and $y$ = length.

b. Graph and solve the linear system of inequalities.

c. What does the solution mean?

**35.** **Analyze and Persevere** Ramona needs 10 mL of a 30% saline solution. She has a 50% saline solution and a 25% saline solution. How many milliliters of each solution does she need to create the 30% solution?

## ASSESSMENT PRACTICE

**36.** One equation in a system of equations with one solution is $4x + 2y = 14$. Select all equations that could be the second equation in the system. AR.9.3

☐ **A.** $2x + y = 7$
☐ **B.** $3x - 6y = -12$
☐ **C.** $2x + 6y = 32$
☐ **D.** $-3x + 10y = 1$
☐ **E.** $2x + y = 5$

**37.** **SAT/ACT** What value of $a$ gives $(-1, 1)$ as the solution of the system $\begin{cases} 3x + 5y = 2 \\ ax + 8y = 14 \end{cases}$?

Ⓐ $-22$ Ⓑ $-6$ Ⓒ 0 Ⓓ 6 Ⓔ 22

**38.** **Performance Task** Each Sophomore and Junior at a high school collected aluminum cans and plastic bottles. The table shows the average number of cans and bottles collected per student, by grade level during a 2 week recyling drive.


| | Sophomores | Juniors |
|---|---|---|
| Week 1 | 1 | 4 |
| Week 2 | 4 | 2 |

**Part A** Write a system of equations to represent the situation.

**Part B** Find the solution of the system of equations you found in Part A.

**Part C** What does your solution to part B represent in terms of this scenario?

# MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.9.7–Given a real-world context, represent constraints as systems of linear and non-linear equations or inequalities. Interpret solutions to problems as viable or non-viable options.
Also MA.912.AR.9.3
MTR.7.1

## Current Events

You might say that someone who loses their temper has "blown a fuse." However, it's rare to hear about electrical fuses blowing these days. That's because most fuses have been replaced by circuit breakers. A fuse must be replaced once it's blown, but a circuit breaker can be reset.

Ask for permission to look at the electrical panel in your home. If there is a series of switches inside, each of those is a circuit breaker, designed to interrupt the circuit when the electrical current inside is too dangerous. How much electricity does it take to trip a circuit breaker? Think about this question during the Mathematical Modeling in 3-Acts lesson.

### ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

### ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

### ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC 1

# Topic Review

## TOPIC ESSENTIAL QUESTION

1. What are different ways in which functions can be used to represent and solve problems involving quantities?

## Vocabulary Review

**Choose the correct term to complete each sentence.**

2. A __________ pairs every input in an interval with the same output value.
3. The point at which a function changes from increasing to decreasing is the __________ of the function.
4. A __________ of a function $y = af(x - h) + k$ is a change made to at least one of the values $a$, $h$, and $k$.
5. A __________ is the value of $x$ when $y = 0$.
6. A __________ is defined by two or more functions, each over a different interval.

- step function
- piecewise-defined function
- minimum
- maximum
- system of linear equations
- transformation
- zero of the function

## Concepts & Skills Review

### LESSON 1-1 Key Features of Functions

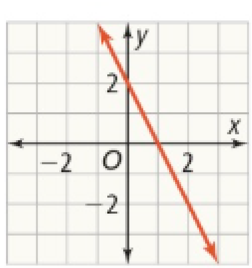
#### Quick Review

The domain of a function is the set of input values, or $x$-values. The range of a function is the set of output values, or $y$-values. These sets can be described using **interval notation** or **set-builder notation.**

A $y$-intercept is a point on the graph of a function where $x = 0$. An $x$-intercept is a point on the graph where $y = 0$. An $x$-intercept is also a **zero of a function.**

#### Example

**Find the zeros of the function. Then determine over what domain the function is positive or negative.**

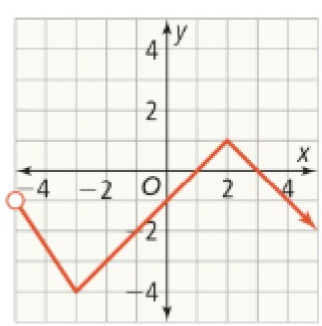



The point where the line crosses the $x$-axis is (1, 0), so $x = 1$ is a zero of the function. The function is positive on the interval $(-\infty, 1)$ and negative on the interval $(1, \infty)$.

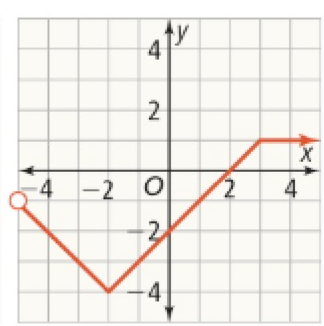
#### Practice & Problem Solving

**Identify the domain and range of the function in interval notation. Find the zeros of the function. Then determine for which values of $x$ the function is positive and for which it is negative.**

7.




8.




9. **Use Structure** Sketch a graph given the following key features.

   domain: (−5, 5); decreasing: (−3, 1); $x$-intercepts: −4, −2; positive: (−4, −2)

10. **Communicate Precisely** Jeffrey is emptying a 50 $ft^3$ container filled with water at a rate of 0.5 $ft^3$/min. Write the equation and interpret its key features for this situation.

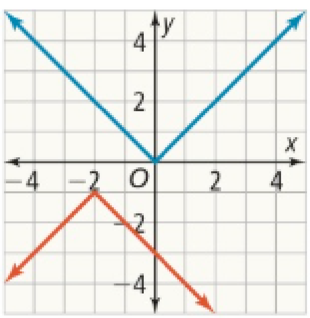
## LESSON 1-2 Transformations of Functions

### Quick Review

There are different types of **transformations** that change the graph of the parent function. A **translation** shifts each point on a graph the same distance and direction. A **reflection** maps each point to a new point across a given line. A **stretch** or a **compression** increases or decreases the distance between the points of a graph and a given line by the same factor.

### Example

**Graph the parent function $f(x) = |x|$ and $g(x) = -|x + 2| - 1$. Describe the transformation.**




Multiplying the absolute value expression by −1 indicates a reflection over the $x$-axis.

Adding 2 to $x$ indicates a translation 2 units to the left and subtracting 1 from the absolute value expression indicates a translation 1 unit down.

So the graph of $g$ is a reflection of the graph of the parent function $f$ over the $x$-axis, and then a translation 2 units left and 1 unit down.

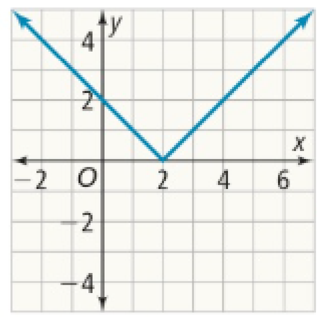
### Practice & Problem Solving

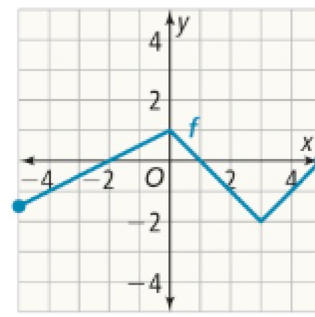
**Graph each function as a translation of its parent function, *f*.**

**11.** $g(x) = |x| - 7$ **12.** $g(x) = x^2 + 5$

**Graph the function, *g*, as a reflection of the graph of *f* across the given axis.**

**13.** across the $x$-axis **14.** across the $y$-axis




**15.** **Look for Relationships** Describe the effect of a vertical stretch by a factor greater than 1 on the graph of the absolute value function. How is that different from the effect of a horizontal stretch by the same factor?

**16.** **Use Structure** Graph the function that is a vertical stretch by a factor of 3.5 of the parent function $f(x) = |x|$.

**17.** **Use Structure** Graph the function that is a horizontal translation 1 unit to the right of the parent function $f(x) = x^2$.

# LESSON 1-3 Piecewise-Defined Functions

## Quick Review

A **piecewise-defined function** is a function defined by two or more function rules over different intervals. A **step function** pairs every number in an interval with a single value.

## Example

**Graph the function.**

$$y = \begin{cases} -3, \text{ if } -5 \leq x < -2 \\ x + 1, \text{ if } -2 < x < 2 \\ -x + 2, \text{ if } 2 \leq x < 5 \end{cases}$$

**State the domain and range. Determine where the function is increasing, constant, or decreasing.**

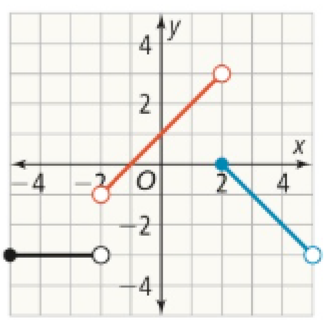
Domain: $-5 \leq x < -2$ and $-2 < x < 5$

Range: $-3 \leq y < 3$

Increasing: $-2 < x < 3$

Constant: $-5 < x < -2$

Decreasing: $2 < x < 5$

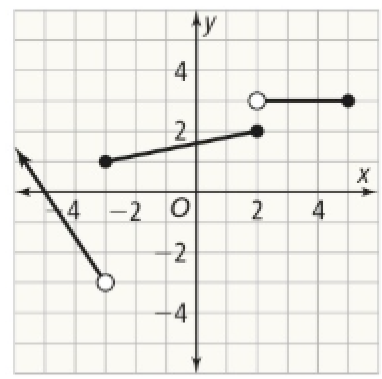



## Practice & Problem Solving

**18.** Graph the function.

$$y = \begin{cases} -3, \text{ if } -4 \leq x < -2 \\ -1, \text{ if } -2 \leq x < 0 \\ 1, \text{ if } 0 \leq x < 2 \\ 3, \text{ if } 2 \leq x < 4 \end{cases}$$

**19.** What rule defines the function in the following graph?




**20.** **Generalize** Can every transformation of the absolute value function also be written as a piecewise-defined function? Explain.

# LESSON 1-4 Arithmetic Sequences

## Quick Review

An **arithmetic sequence** is a sequence with a constant difference between consecutive terms.

**recursive definition:** $a_n = \begin{cases} a_1, \text{ if } n = 1 \\ a_{n-1} + d, \text{ if } n > 1 \end{cases}$

**explicit definition:** $a_n = a_1 + (n - 1)d$

## Example

**Given the sequence 22, 17, 12, 7, ..., write the explicit formula. Then find the 6th term.**

$d = -5$ ........ Find the common difference.

$a_n = 22 + (n - 1)(-5)$ ........ Substitute 22 for $a_1$ and $-5$ for $d$.

$a_n = 22 - 5(n - 1)$ ........ Simplify.

$a_6 = 22 - 5(6 - 1)$ ........ Substitute 6 for $n$.

$a_6 = -3$ ........ Solve for the 6th term.

## Practice & Problem Solving

**What are the common difference, the 5th term, and the recursive and explicit functions for each arithmetic sequence?**

**21.** 3, 15, 27, 39, ...

**22.** 19, 13, 7, 1, ...

**22.** $a_1 = -5$, $a_7 = 22$

**23.** $a_4 = 20$, $a_{13} = 17$

**24.** Given the recursive definition

$$a_n = \begin{cases} 15, \text{ if } n = 1 \\ a_{n-1} - 3, \text{ if } n > 1 \end{cases}$$

what is an explicit definition for the sequence?

**25.** Given the explicit definition $a_n = 1 + \frac{1}{2}(n - 1)$, what is a recursive definition for the sequence?

**26.** **Make Sense and Persevere** Cubes are stacked in the shape of a pyramid. The top row has 1 cube, the second row has 3, and the third row has 5. If the bottom row has 17 cubes, how many rows does the pyramid have?

## LESSON 1-5 Quadratic and Absolute Value Inequalities

### Quick Review

To solve a quadratic inequality by graphing, write two new equations by setting $y$ equal to each expression in the inequality. Find the coordinates of any points of intersection. The solution to the inequality is the set of $x$-values where one graph is above or below the other graph. You can also solve a quadratic inequality algebraically.

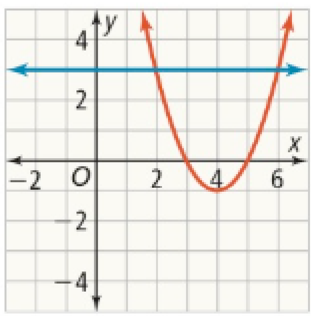
### Example

**Solve $x^2 - 8x + 15 \le 3$ by graphing.**

Graph $y = x^2 - 8x + 15$ and $y = 3$.

The graphs intersect at $x = 2$ and $x = 6$.

The function at $y = x^2 - 8x + 15$ is less than or equal to 3 when $2 \le x \le 6$.




### Practice & Problem Solving

**Solve each inequality.**

27. $-x^2 + 1 < 0$

28. $x^2 - 12x + 25 \le 2$

30. $2x^2 - 3x \ge x^2 + x - 3$

31. $x^2 - 5 \le -x^2 + 2x$

32. $2|x - 3| < 10$

33. $|3x - 4| \ge 8$

34. **Choose Efficient Methods** Is graphing always the most convenient method for solving a quadratic inequality? Why or why not?

35. Hugo is pumping diesel gas into his truck. Diesel costs \$2.75 per gallon. Hugo expects that to fill the truck, it will cost \$30, plus or minus \$1. Write and solve an absolute value inequality to represent how many gallons of gas he needs to fill his truck.

## LESSON 1-6 Linear Systems

### Quick Review

A **system of linear equations** is a set of two or more equations using the same variables. The **solution of a system of linear equations** is the set of all ordered coordinates that simultaneously make all equations in the system true. A **system of linear inequalities** is a set of two or more inequalities using the same variables.

### Example

**Solve the system.** $\begin{cases} -4x + 4y = 16 \\ -x + 2y = 10 \end{cases}$

$x = 2y - 10$ ······ Solve the second equation for $x$.

$-4(2y - 10) + 4y = 16$ ······ Substitute $2y - 10$ for $x$. Solve for $y$.
$y = 6$

$x = 2(6) - 10$ ······ Substitute 6 for $y$ in the equation $x = 2y - 10$.
$x = 2$

### Practice & Problem Solving

**Solve each system of equations.**

36. $\begin{cases} y = 2x + 5 \\ 2x + 4y = 10 \end{cases}$

37. $\begin{cases} y = 2x - 6 \\ 6x + y = 10 \end{cases}$

38. **Use Structure** Write a linear system in two variables that has infinitely many solutions.

39. **Model With Mathematics** It takes Leo 12 hours to make a table and 20 hours to make a chair. In 8 weeks, Leo wants to make at least 5 tables and 8 chairs to display in his new shop. Leo works 40 hours a week. Write a system of linear inequalities relating the number of tables $x$ and the number of chairs $y$ Leo will be able to make. List two different combinations of tables and chairs Leo could have to display at the opening of his new shop.

TOPIC

# 2

# Quadratic Functions and Equations

## TOPIC ESSENTIAL QUESTION

**How do you use quadratic functions to model situations and solve problems?**

## Topic Overview

**enVision® STEM Project:**
Hit a Home Run

**2-1** Vertex Form of a Quadratic Function
AR.3.4, AR.3.8, F.1.1, F.1.7, F.2.2, F.2.3, F.2.5, MTR.1.1, MTR.4.1, MTR.5.1

**2-2** Standard Form of a Quadratic Function
DP.2.8, AR.3.4, AR.3.8, F.1.1, MTR.1.1, MTR.4.1, MTR.5.1

**2-3** Factored Form of a Quadratic Function
AR.1.1, AR.3.2, AR.3.4, AR.3.8, MTR.1.1, MTR.2.1, MTR.5.1

**2-4** Complex Numbers and Operations
AR.3.2, NSO.2.1, MTR.6.1, MTR.7.1

**Mathematical Modeling in 3 Acts:** Swift Kick
AR.3.4, AR.3.8, MTR.7.1

**2-5** Completing the Square
AR.3.2, AR.3.4, AR.3.8, MTR.1.1, MTR.2.1, MTR.6.1

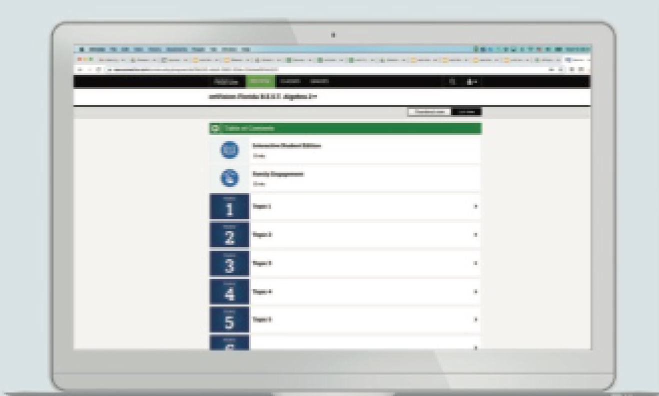
**2-6** The Quadratic Formula
AR.3.2, AR.3.8, MTR.3.1, MTR.4.1

**2-7** Quadratic Inequalities
AR.3.9, AR.3.10, MTR.2.1, MTR.6.1, MTR.7.1

**2-8** Systems Involving Quadratic Equations and Inequalities
AR.9.2, AR.9.5, MTR.1.1, MTR.2.1, MTR.4.1

## Topic Vocabulary

- completing the square
- complex conjugates
- complex number
- discriminant
- imaginary number
- imaginary unit *i*
- parabola
- Quadratic Formula
- quadratic function
- standard form of a quadratic function
- vertex form of a quadratic function
- Zero Product Property



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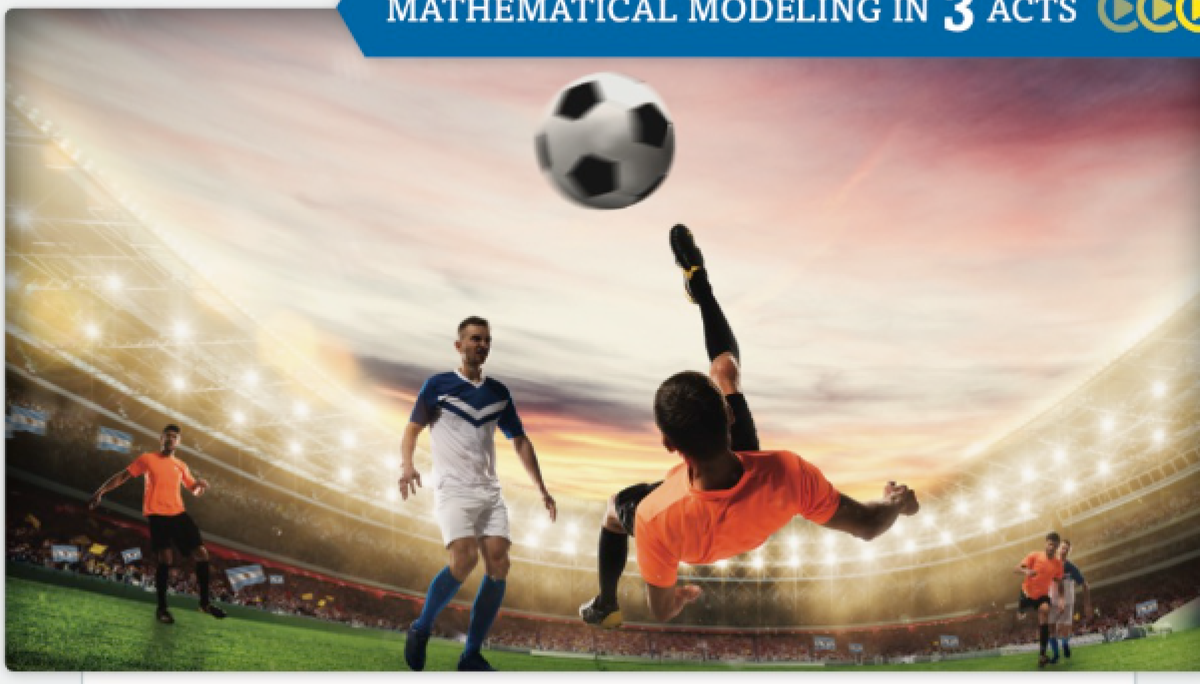
**ACTIVITIES** Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its***.

**ANIMATION** View and interact with real-world applications.

**PRACTICE** Practice what you've learned.

## MATHEMATICAL MODELING IN 3 ACTS



### Swift Kick

Whether you call it soccer, football, or fùtbol, it's the most popular sport in the world by far. Even if you don't play soccer, you probably know several people who do.

There are many ways to kick a soccer ball: you can use any part of either foot. If you want the ball to end up in the goal, you also need to try different amounts of spin and power. You'll see one person's effort in the Mathematical Modeling in 3-Acts lesson.

**VIDEOS** Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

**ADAPTIVE PRACTICE** Practice that is *just right* and *just for you.*

**GLOSSARY** Read and listen to English and Spanish definitions.

**CONCEPT SUMMARY** Review key lesson content through multiple representations.

**ASSESSMENT** Show what you've learned.

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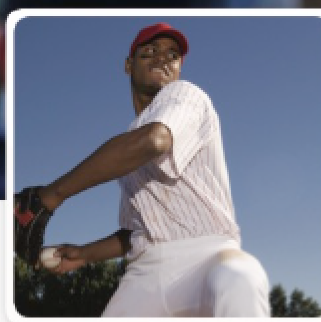
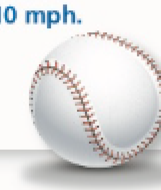
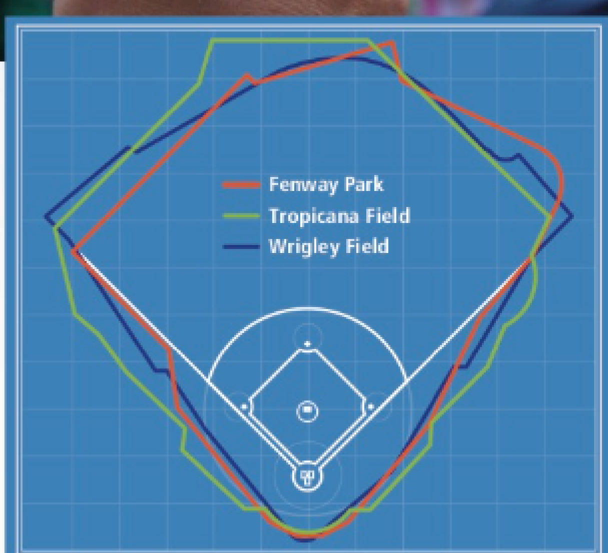
## Did You Know?

Cameras and RADAR precisely track everything that happens in a professional baseball game, including the **speed** and **launch angle** of every batted ball.

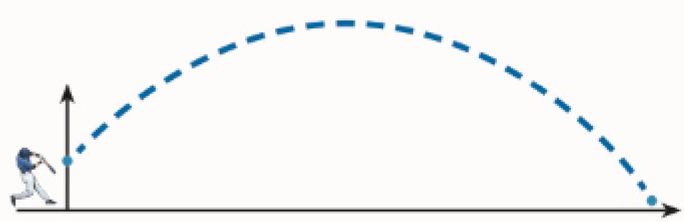



If a baseball player hits a **90-mph** pitch with more than **8,000 pounds of force,** the ball leaves the bat at a **speed of 110 mph.**






Each baseball park has **unique features** that help determine whether a hit will be a home run.



You can model the flight of a hit baseball with a **parabola.** The initial vertical and horizontal speed of the ball can be found using **right triangle trigonometry** and the **launch angle of the hit.**

## Your Task: Hit a Home Run

You and your classmates will design a ballpark and determine what it would take to hit a home run at that park.

# 2-1

## Vertex Form of a Quadratic Function

**I CAN…** identify key features of quadratic functions.

**VOCABULARY**

- parabola
- quadratic function
- vertex form of a quadratic function

MA.912.AR.3.4–Write a quadratic function to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. Also AR.3.8, AR.4.4, F.1.1, F.1.7, F.2.2, F.2.3, F.2.5

MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

### EXPLORE & REASON

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**The table represents $A(x)$, the area of a square as a function of side length $x$ units, where $x$ is a positive real number.**

| Side Length (units) | $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Model | $x$ by $x$ | | | | |
| Area (sq. units) | $A(x)$ | 1 | 4 | 9 | 16 |

**A.** Consider the function where the areas in the table are doubled. Write the equation of a function that represents this.

**B.** **Use Patterns and Structure** Graph the ordered pairs for both $A(x)$ and your new function. How would you describe the differences in the locations of these points?

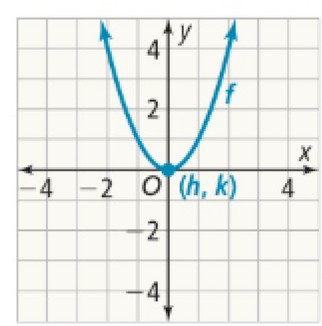
**C.** Find the equation for a function whose $x$-values are the same as $A(x)$ but whose $y$-values are 2 units greater than each $y$-value in $A(x)$.

### ESSENTIAL QUESTION

**How does the equation of a quadratic function in vertex form highlight key features of the function's graph?**

### CONCEPT Representations of Quadratic Functions

A function is a **quadratic function** if its equation can be written in the form $f(x) = ax^2 + bx + c$, with $a \neq 0$.




All quadratic functions are transformations of the parent function defined by $f(x) = x^2$.

The graph of a quadratic function is called a **parabola**.

The **vertex form of a quadratic function** is $f(x) = a(x - h)^2 + k$ where $(h, k)$ is the vertex of the parabola. Vertex form is useful because it highlights the vertex of the graph of the quadratic function.

CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Transform a Quadratic Function

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**How are transformations of the graph of $f(x) = x^2$ related to an equation representing another quadratic function?**

Vertex form shows three different ways in which the graph of the function $f(x) = x^2$ may be transformed.

$$f(x) = a(x - h)^2 + k$$

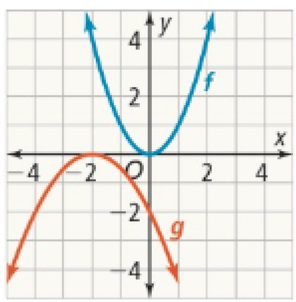
The value of $a$ determines the direction the parabola opens and whether the graph is stretched or compressed.

The value of $h$ determines the horizontal translation.

The value of $k$ determines the vertical translation.

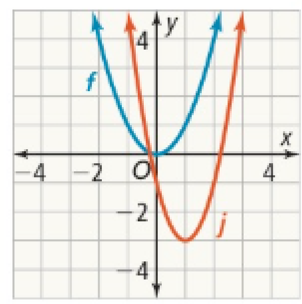
**A.** $g(x) = -\frac{1}{2}(x + 2)^2$

The equation shows that the graph is to be translated 2 units left, will open downward, and will be vertically compressed.




**B.** $j(x) = 2(x - 1)^2 - 3$

The equation shows that the graph is to be translated 1 unit right, vertically stretched, and translated down 3 units. It will open upward.




**STUDY TIP**

Recall that a vertical stretch makes the graph narrower and that a vertical compression makes the graph wider. To see the effects easily, use the same axes or units for all graphs.

When $a > 0$ the parabola opens upward. When $a < 0$, the parabola opens downward. When $|a| > 1$, the graph is stretched, and when $0 < |a| < 1$, the graph is compressed.

**Try It!** **1.** Describe the transformations of the parent function $f(x) = x^2$. Then graph the function.

**a.** $g(x) = 0.4(x - 3)^2$ **b.** $g(x) = -3(x - 1)^2 + 2$

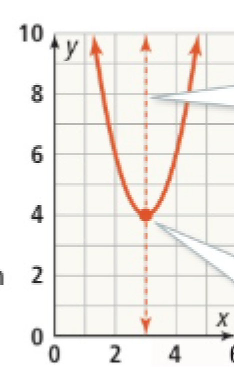
## EXAMPLE 2 Determine Key Features of a Quadratic Function

**What are the key features of the quadratic function $f(x) = 2(x - 3)^2 + 4$?**

The graph represents $f(x) = 2(x - 3)^2 + 4$.

The 2 indicates that the graph opens upward and is vertically stretched.

The range is $y \geq 4$, and 4 is the minimum value. There are no restrictions on the value of $x$, so the domain is all real numbers.




The axis of symmetry is the line $x = h$ or $x = 3$.

The vertex is the ordered pair $(h, k)$, or $(3, 4)$.

**Try It!** **2.** Identify the vertex, axis of symmetry, minimum or maximum, domain, and range of the function $f(x) = -(x + 4)^2 - 5$.

## EXAMPLE 3 Write an Equation of a Parabola

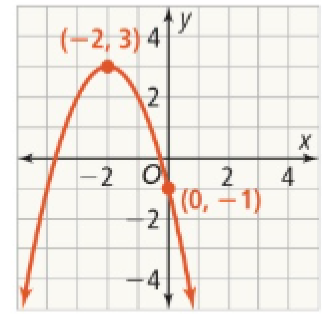
**What is the equation of a quadratic function with vertex (−2, 3) and *y*-intercept −1?**

**Step 1** Substitute the coordinates of the vertex for $h$ and $k$ in the vertex form of a quadratic function.

$$(h, k) = (-2, 3), \text{ so } y = a(x - (-2))^2 + 3$$

**Step 2** Substitute the values of $x$ and $y$ from the $y$-intercept, and then solve for $a$.

$$(x, y) = (0, -1), \text{ so } -1 = a(0 + 2)^2 + 3$$
$$-4 = a(2)^2$$
$$-4 = 4a$$
$$a = -1$$




**COMMON ERROR**
Be careful to not switch the coordinate values when substituting them into the equation.

**Step 3** Substitute the value of $a$ into the vertex form of a quadratic function.

$$a = -1 \text{ so } y = -(x + 2)^2 + 3$$

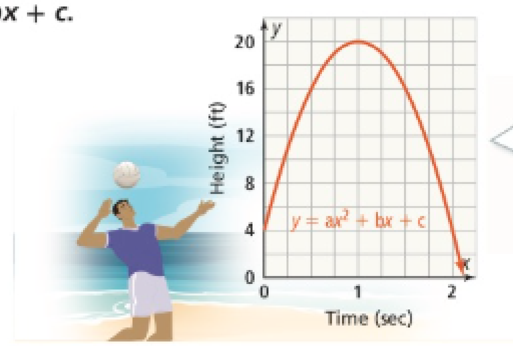
The equation of the parabola is $y = -(x + 2)^2 + 3$.

**Try It!** 3. What is the equation of a parabola with a vertex of (1, −4) and which passes through (−2, −1)?

**APPLICATION**

## EXAMPLE 4 Write an Equation of a Parabola Given the Graph

**The height of a thrown ball is a quadratic function of the time it has been in the air. The ball is thrown with an initial height of 4 feet and after 1 second reaches its vertex at 20 feet. What is an expression that defines this function? Write the quadratic equation in vertex form and in the form $y = ax^2 + bx + c$.**




The initial height is the $y$-intercept. The maximum height occurring at one second is the vertex.

**USE PATTERNS AND STRUCTURE**
By converting vertex form into standard form, you can see how $h$ and $k$ relate to the coefficients of the equation.

$$y = a(x - h)^2 + k$$
$$4 = a(0 - 1)^2 + 20$$
$$4 = a(-1)^2 + 20$$
$$4 = a + 20$$
$$-16 = a$$

Find a by substituting the vertex and a given point.

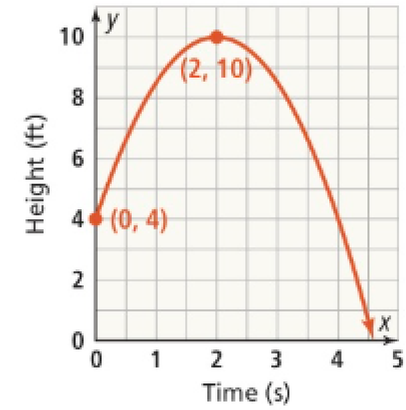
$$y = -16(x - 1)^2 + 20$$
$$y = -16(x^2 - 2x + 1) + 20$$
$$y = -16x^2 + 32x + 4$$

The equation of the parabola in vertex form is $y = -16(x - 1)^2 + 20$.

In the form $y = ax^2 + bx + c$, the equation is $y = -16x^2 + 32x + 4$.

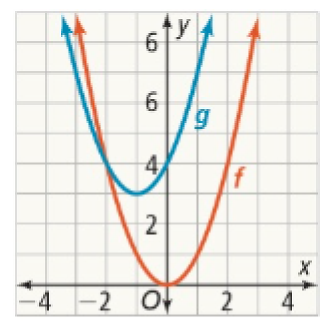
CONTINUED ON THE NEXT PAGE

**Try It!** **4.** The graph shows the height of a ball with respect to time. What is the equation of the function? Write the equation in vertex form. Then write the equation in the form $y = ax^2 + bx + c$.




## EXAMPLE 5 Write an Equation of a Transformed Function

**The function *g* is a translation of the parent function *f* 1 unit left and 3 units up. What is the equation of *g*? Write the quadratic equation in vertex form and in the form $f(x) = ax^2 + bx + c$.**




Translate the graph of $f(x)$ left 1 unit to locate the graph of $f(x + 1)$, then translate the graph of $f(x + 1)$ up 3 units to locate the graph of $f(x + 1) + 3$.

$$g(x) = f(x + 1) + 3$$
$$g(x) = a(x + 1)^2 + 3$$

From the graph, the point (0, 4) appears to be on $g$. Use the point (0, 4) to find $a$.

$$4 = a(0 + 1)^2 + 3$$
$$4 = a + 3$$
$$1 = a$$

Substituting $a = 1$, the equation is

$$g(x) = a(x + 1)^2 + 3$$
$$g(x) = (x + 1)^2 + 3$$
$$g(x) = x^2 + 2x + 1 + 3$$
$$g(x) = x^2 + 2x + 4$$

**STUDY TIP**
You can confirm your equation by picking a point that is on the graph and checking to make sure it satisfies your equation.

In vertex form, $g(x) = (x + 1)^2 + 3$ and in the form $y = ax^2 + bx + c$, the equation is $g(x) = x^2 + 2x + 4$.

**Try It!** **5.** What is the equation of $j$? Write the equation in vertex form and in the form $y = ax^2 + bx + c$.

**a.** Let $j$ be a quadratic function whose graph is a translation 2 units right and 5 units down of the graph of $f$.

**b.** Let $j$ be a quadratic function whose graph is a reflection of the graph of $f$ in the $x$-axis followed by a translation 1 unit down.

## CONCEPT SUMMARY Vertex Form of a Quadratic Function

**WORDS** The graph of a quadratic function is called a parabola.

A quadratic function can be represented by an equation in vertex form $y = a(x - h)^2 + k$. Vertex form shows the different ways in which the graph of the parent function $f(x) = x^2$ can be transformed. When $a < 0$, the graph opens downward or is reflected over the $x$-axis.

**ALGEBRA**

$f(x) = x^2$
vertex (0, 0)
axis of symmetry $x = 0$
opens upward
minimum $y = 0$
domain $(-\infty, \infty)$
range $[0, \infty)$

$y = a(x - h)^2 + k$
$a \neq 0$
vertex $(h, k)$
axis of symmetry $x = h$
domain: $(-\infty, \infty)$

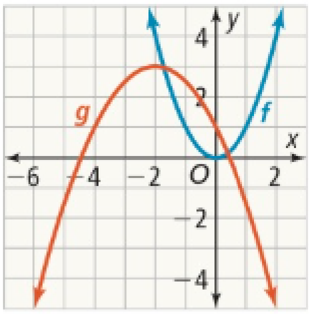
If $a > 0$:
opens upward
minimum value is $k$
range: $[k, \infty)$

if $a < 0$:
opens downward
maximum value is $k$
range: $(-\infty, k]$

**NUMBERS**

$g(x) = -\frac{1}{2}(x + 2)^2 + 3$
vertex (−2, 3)
axis of symmetry $x = -2$
opens downward
maximum $y = 3$
domain $(-\infty, \infty)$
range $(-\infty, 3]$

**GRAPH**




## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does the equation of a quadratic function in vertex form highlight key features of the function's graph?

2. **Error Analysis** Given the function $g(x) = (x + 3)^2$, Martin says the graph should be translated right 3 units from the parent graph $f(x) = x^2$. Explain his error.

3. **Vocabulary** What shape does a quadratic function have when graphed?

4. **Communicate and Justify** How are the graphs of $f(x) = x^2$ and $g(x) = -(x + 2)^2 - 4$ related?

## Do You KNOW HOW?

**Describe the transformation of the parent function $f(x) = x^2$.**

5. $g(x) = -(x + 5)^2 + 2$

6. $h(x) = (x + 2)^2 - 7$

**Write the equation of each parabola in vertex form.**

7. Vertex: (−3, 7); Point: (−2, −5)

8. Vertex: (1, 3); Point: (2, 5)

9. Vertex: (−4, 6); Point: (−2, −2)

10. Vertex: (7, 4); Point: (5, 16)

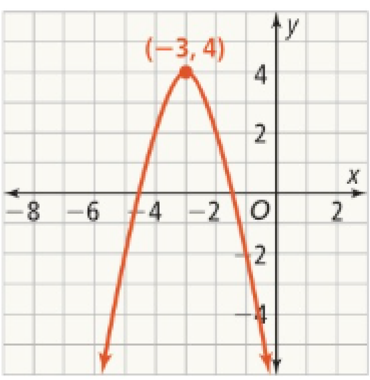
# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

**11.** **Use Patterns and Structure** The graph of the function $f(x) = x^2$ will be translated 3 units up and 1 unit left. What is the resulting function $g(x)$?

**12.** **Error Analysis** A classmate said that the vertex of $g(x) = -5(x + 2)^2 - 4$ is (2, 4). Is your classmate correct? If not, what is the correct vertex?

**13.** **Higher Order Thinking** The graph below is a transformation of the graph of the parent function. Write the quadratic function to model the graph.




**14.** **Communicate and Justify** Explain why the graph of the equation $g(x) = -(x + 1)^2 - 3$ would be a parabola opening downward.

**15.** **Use Patterns and Structure** Amaya is standing 30 ft from a volleyball net. The net is 8 ft high. Amaya serves the ball. The path of the ball is modeled by the equation $y = -0.02(x - 18)^2 + 12$, where $x$ is the ball's horizontal distance in feet from Amaya's position and $y$ is the distance in feet from the ground to the ball.

a. How far away is the ball from Amaya when it is at its maximum height? Explain.

b. Describe how you would find the ball's height when it crosses the net at $x = 30$.

## PRACTICE

**Describe the transformation of the parent function $f(x) = x^2$. Then graph the transformed function.** SEE EXAMPLE 1

**16.** $f(x) = (x - 1)^2 + 3$

**17.** $y = (x + 1)^2 - 3$

**18.** $g(x) = 2x^2$

**19.** $f(x) = -(x - 1)^2 + 7$

**20.** $y = -2(x + 1)^2 + 1$

**21.** $f(x) = \frac{1}{2}(x - 2)^2 + 3$

**Identify the vertex, axis of symmetry, maximum or minimum, domain, and range of each function.** SEE EXAMPLE 2

**22.** $y = 2(x - 2)^2 + 5$

**23.** $f(x) = -(x - 1)^2 + 2$

**24.** $g(x) = -(x + 4)^2$

**25.** $y = \frac{1}{3}(x + 2)^2 - 1$

**Write the equation of each parabola in vertex form.** SEE EXAMPLE 3

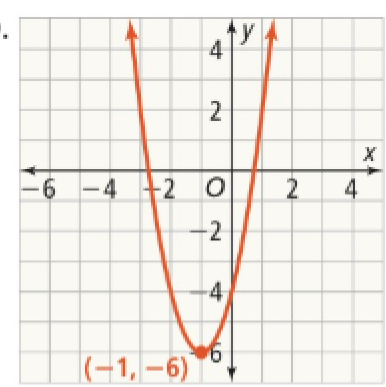
**26.** Vertex: (1, 2); Point: (2, −5)

**27.** Vertex: (3, 6); $y$-intercept: 2

**28.** Vertex: (0, 5); Point: (1, −2)

**Write the equation of the function represented by the parabola in vertex form and in the form $y = ax^2 + bx + c$.** SEE EXAMPLE 4

**29.**




**Write the equation $g(x)$ in vertex form of a quadratic function for the transformations given the function $f(x) = x^2$.** SEE EXAMPLE 5

**30.** Let $g(x)$ be the function whose graph is a translation 4 units left and 1 unit up of the graph of $f(x)$.

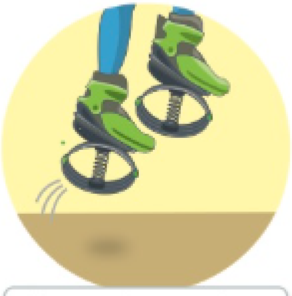
**31.** Let $g(x)$ be the function whose graph is a reflection in the $x$-axis and translated 3 units right of the graph of $f(x)$.

## PRACTICE & PROBLEM SOLVING

### APPLY

**32. Use Patterns and Structure** The height, in inches, that a person can jump while wearing a pair of jumping shoes is based on the time, $x$, in seconds, from the start of the jump. Beth is testing out Max Jumps and Jumpsters to determine which shoes she likes better. Compare the maximum heights on the two sets of shoes.

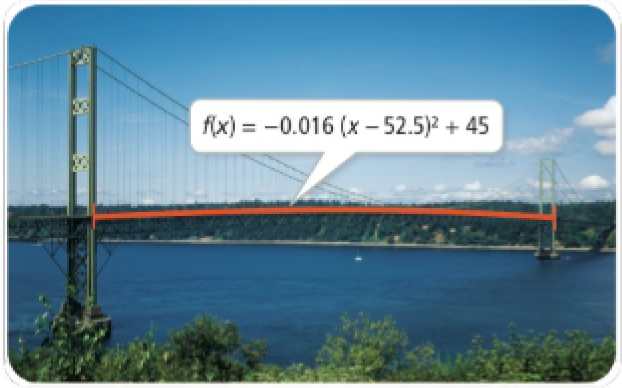
$f(x) = -192(x - 0.289)^2 + 16$



$g(x) = -192(x - 0.445)^2 + 38$

**33. Analyze and Persevere** Find three additional points on the parabola that has vertex (1, –2) and passes through (0, –5).

**34. Analyze and Persevere** The curvature of the Tacoma Narrows Bridge in Washington is in the shape of a parabola.




In the given function, $x$ represents the horizontal distance (in meters) from the arch's left end and $f(x)$ represents the distance (in meters) from the base of the arch. What is the width of the arch?

**35. Apply Math Models** An object is thrown from a height of 5 in. After 2 s, the object reaches a maximum height of 9 in., and then it lands back on the ground 5 s after it was thrown. Write the vertex form of the quadratic equation that models the object's height, and draw the graph.

### ASSESSMENT PRACTICE

**36.** The graph of $g(x) = 3(x - 2)^2$ is a transformation of the graph of $f(x) = x^2$. Describe in words the sequence of transformations that takes the graph of $f$ to the graph of $g$. F.2.2

**37. SAT/ACT** Which of the following functions represents a parabola with a vertex at (–3, 4) and that passes through the point (–1, –4)?

Ⓐ $f(x) = x^2 - 5$

Ⓑ $f(x) = -2(x + 3)^2 + 4$

Ⓒ $f(x) = 2(x + 1)^2 - 4$

Ⓓ $f(x) = 2(x - 3)^2 - 32$

**38. Performance Task** The Bluebird Bakery sells more alfajores when it lowers its prices, but this also changes profits.


The profit function for the alfajores is $f(x) = -500(x - 0.45)^2 + 400$. This function represents the profit earned when the price of an alfajor is $x$ dollars. The bakery wants to maximize their profits.

**Part A** What is the domain of the function?

**Part B** Find the daily profits for selling alfajores for $0.40 each and for $0.75 each.

**Part C** What price should the bakery charge to maximize their profits from selling alfajores?

**Part D** What is the maximum profit?

# 2-2

## Standard Form of a Quadratic Function

**I CAN…** write and graph quadratic functions in standard form.

**VOCABULARY**

- standard form of a quadratic function

**MA.912.AR.3.4**–Write a quadratic function to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. **Also AR.3.8, F.1.1, DP.2.8**
**MA.K12.MTR.1.1, MTR.5.1, MTR.7.1**

### CRITIQUE & EXPLAIN

Jordan and Emery are rewriting the vertex form of the parabola $y = 2(x - 4)^2 + 5$ in the form $y = ax^2 + bx + c$.

| Jordan | Emery |
|---|---|
| $y = 2(x-4)^2 + 5$ | $y = 2(x-4)^2 + 5$ |
| $= (2x-8)^2 + 5$ | $= 2(x^2 - 16) + 5$ |
| $= 4x^2 - 32x + 64 + 5$ | $= 2x^2 - 32 + 5$ |
| $= 4x^2 - 32x + 69$ | $= 2x^2 - 27$ |

A. **Communicate and Justify** Did Jordan rewrite the equation correctly? Did Emery? Explain.

B. Without rewriting the equation, how could you prove that Jordan or Emery's equations are not equivalent to the original?

### ESSENTIAL QUESTION

**What key features can you determine about a quadratic function from an equation in standard form?**

### EXAMPLE 1 Find the Vertex of a Quadratic Function in Standard Form

**CONCEPTUAL UNDERSTANDING**

**How can you find the vertex of a quadratic function written in standard form?**

A. What is the $x$-coordinate of the vertex of $f(x) = ax^2 + bx + c$?

The **standard form of a quadratic function** is $y = ax^2 + bx + c$ where $a$, $b$, and $c$ are real numbers, and $a \neq 0$. Use vertex form to derive standard form.

$y = a(x - h)^2 + k$ ……… Write the vertex form of a quadratic equation.

$y = a(x^2 - 2xh + h^2) + k$ ……… Square the binomial.

$y = ax^2 - 2ahx + ah^2 + k$ ……… Simplify.

The equation $y = ax^2 - 2ahx + ah^2 + k$ is a *quadratic function* in standard form with $a = a$, $b = -2ah$, and $c = ah^2 + k$.

The vertex of a quadratic function is $(h, k)$, so to determine the $x$-coordinate of the vertex, solve $b = -2ah$ for $h$.

$$b = -2ah$$

$$-\frac{b}{2a} = h$$

Since $h$ is the $x$-coordinate of the vertex, you can use this value to find the $y$-value, $k$, of the vertex.

**USE PATTERNS AND STRUCTURE**
By converting vertex form into standard form, you can see how $h$ and $k$ relate to the coefficients of the standard form equation.

B. What is the vertex of the function $f(x) = x^2 - 6x + 10$?

**Step 1 Identify the coefficients $a$, $b$, and $c$.**

$a = 1$, $b = -6$, and $c = 10$

**Step 2 Solve for $h$, the $x$-coordinate of the vertex.**

$$h = -\frac{b}{2a} = -\frac{(-6)}{2(1)} = 3$$

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

**Step 3 Substitute the value of $h$ into the equation for $x$ to find $k$, the $y$-coordinate of the vertex.**

$$f(3) = (3)^2 - 6(3) + 10$$
$$= 9 - 18 + 10$$
$$= 1$$

The vertex of the function is $(h, k) = (3, 1)$.

**Try It!** **1.** What is the vertex of the graph of the function $f(x) = x^2 - 8x + 5$?

## EXAMPLE 2 Graph a Quadratic Function in Standard Form

**How can you use key features to graph $f(x) = x^2 - 4x + 8$?**

For $f(x)$, identify $a$, $b$, and $c$: $a = 1$, $b = -4$, and $c = 8$.

**Step 1 Find the vertex and the axis of symmetry of the quadratic function.**

The $x$-coordinate of the vertex and the axis of symmetry can be determined by:

$$h = -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2$$

Substitute the value of $h$ for $x$ into the equation to find the $y$-coordinate of the vertex, $k$:

$$f(2) = (2)^2 - 4(2) + 8 = 4$$

The vertex is $(2, 4)$, and the axis of symmetry is $x = 2$.

**USE PATTERNS AND STRUCTURE**
The $y$-intercept of a quadratic function in standard form is given by the ordered pair $(0, c)$. Verify this by substituting $x = 0$ into the standard form equation.

**Step 2 Find the y-intercept of the quadratic function.**

The $y$-intercept occurs at

$$f(0) = (0)^2 - 4(0) + 8 = 8.$$

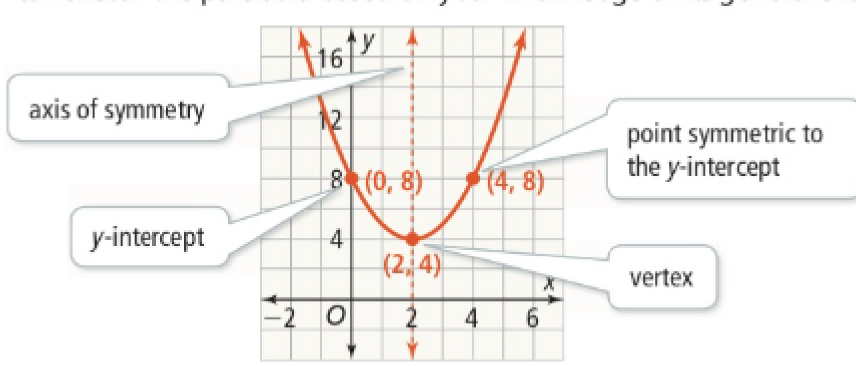
If the $y$-intercept is the same as the vertex, choose a different point here.

**Step 3 Find a point symmetric to the $y$-intercept across the axis of symmetry.**

Since $(0, 8)$ is a point on the parabola 2 units to the left of the axis of symmetry, $x = 2$, $(4, 8)$ will be a point on the parabola 2 units to the right of the axis of symmetry.

**Step 4 Sketch the graph.**

Once you have three points associated with the quadratic function, you can sketch the parabola based on your knowledge of its general shape.

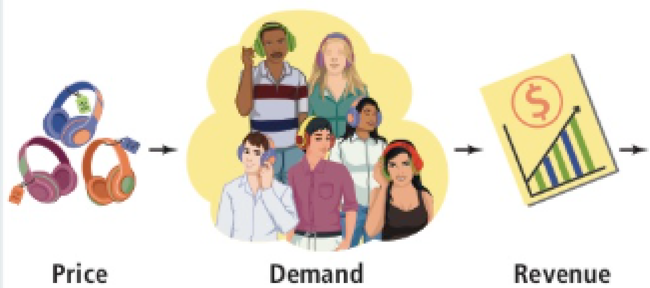



**Try It!** **2.** Use the key features to graph the function $f(x) = x^2 - 6x - 1$.

APPLICATION

## EXAMPLE 3 Interpret the Graph of a Quadratic Function

**The graph of the function $f(x) = -10x^2 + 700x - 6{,}000$ shows the profit a company earns for selling headphones at different prices. What is the maximum profit the company can expect to earn?**





**Formulate** The $x$-axis shows the selling price and the $y$-axis shows the profit. The maximum $y$-value of the profit function occurs at the vertex of its parabola. Find the vertex of the parabola.

**Compute** Use the function to find the $x$- and $y$-coordinates of the vertex.

Find the $x$-coordinate of the vertex.

$h = -\frac{b}{2a}$ ······ Use the formula to find the $x$-coordinate of the vertex.

$h = -\frac{700}{2(-10)}$ ······ Substitute –10 for $a$ and 700 for $b$.

$h = 35$ ······ Simplify.

**COMMON ERROR**
Be careful with the negative signs; there is a negative in the formula and a negative value for $a$.

Find the $y$-coordinate of the vertex.

$y = -10x^2 + 700x - 6{,}000$ ······ Write the original function.

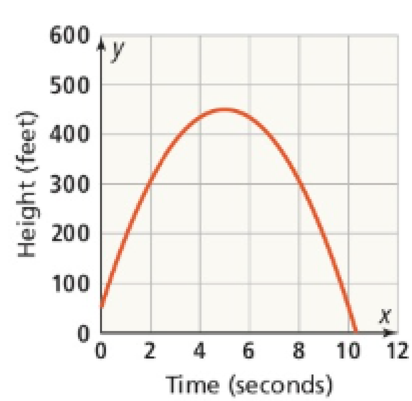
$y = -10(35)^2 + 700(35) - 6{,}000$ ······ Substitute 35 for $x$.

$y = 6{,}250$ ······ Simplify.

The vertex is (35, 6,250).

**Interpret** The selling price of \$35 per item gives the maximum profit of \$6,250.

**Try It!** 3. A flare was fired from a window. The height of the flare over time can be modeled by the function $y = -16x^2 + 160x + 50$. What was the maximum height of the flare after it was fired?



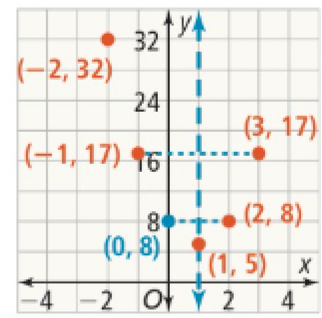
## Write the Equation of a Parabola Given Points

**What is the equation of a parabola that passes through the following points? Write the equation in standard form.**

| $x$ | −2 | −1 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $y$ | 32 | 17 | 5 | 8 | 17 |

**STUDY TIP**
Parabolas are symmetric about the vertical line that passes through the vertex. The line of symmetry also contains the midpoints of all horizontal segments that have both endpoints on the graph.

**Step 1** Graph the points and use symmetry to determine features of the parabola.
The axis of symmetry is the line $x = 1$.
The vertex, (1, 5), is on the axis of symmetry. Reflect (2, 8) across the axis of symmetry. One unit to the left of the axis is (0, 8), therefore the $y$-intercept is 8.




**Step 2** Find the coefficients $a$, $b$, and $c$ for $y = ax^2 + bx + c$.

**CHOOSE EFFICIENT METHODS**
Which method to do you prefer? Both methods build on prior skills.

**Method 1**

Substitute into vertex form.

$$y = a(x - 1)^2 + 5$$

Substitute another given point and find $a$.

$$17 = a(3 - 1)^2 + 5$$
$$12 = 4a$$
$$a = 3$$

Write equation in standard form.

$$y = 3(x - 1)^2 + 5$$
$$y = 3(x^2 - 2x + 1) + 5$$
$$y = 3x^2 - 6x + 3 + 5$$
$$y = 3x^2 - 6x + 8$$

**OR**

**Method 2**

The axis of symmetry is $x = 1$, so $-\frac{b}{2a} = 1$ and $b = -2a$. The $y$-intercept is 8, so $c = 8$.

Substitute in standard form.

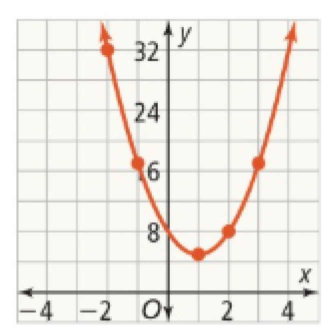
$$y = ax^2 + bx + c$$
$$y = ax^2 - 2ax + 8$$

Using a given point

$$17 = a(3)^2 - 2a(3) + 8$$
$$17 = 9a - 6a + 8$$
$$9 = 3a$$
$$a = 3$$

So, $y = 3x^2 - 6x + 8$

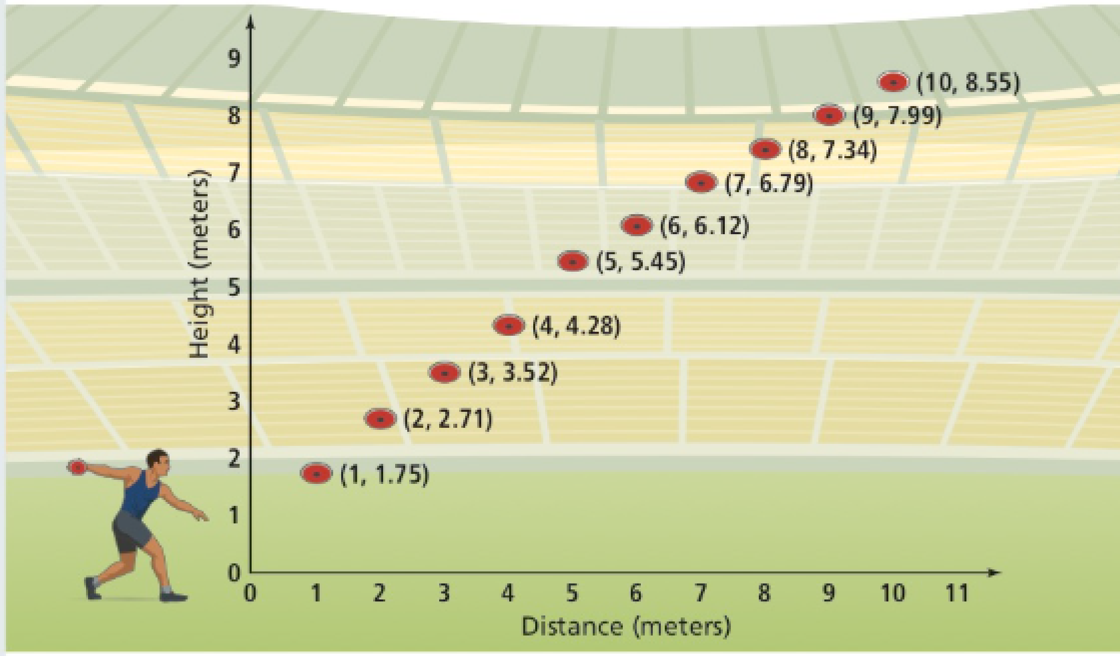
**Step 3** Confirm that the graph of the equation passes through the given points.




**Try It!** 4. What is the equation of a parabola that passes through the points (10, −422), (4, −98), (2, −38), (−2, 10), and (−6, −38)? Write the equation in standard form.

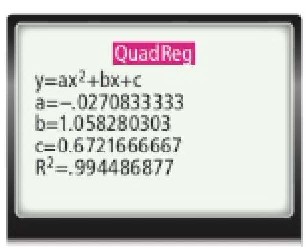
## EXAMPLE 5 Use Quadratic Regression

**Esteban is training for the discus throw. His coach recorded the horizontal distance and height of one of Esteban's discus throws. The graph shows the horizontal distance the discus traveled, in meters, and the height of the discus, in meters.**




**A.** What will be the height of the discus when it has traveled 15 meters from Esteban?

Use graphing technology to perform quadratic regression with the data.




The data show the discus only rising, but the model will resemble a parabola as the discus returns to the ground.

**COMMON ERROR**
Remember that if you are using a regression equation, you are approximating values. Regression is used to make predictions, not to find exact values of variables.

$y \approx -0.027x^2 + 1.058x + 0.672$ ........ Write the regression model.

$y \approx -0.027(15)^2 + 1.058(15) + 0.672$ ........ Substitute 15 for $x$.

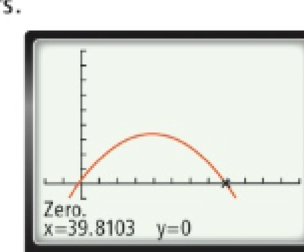
$y \approx 10.467$ ........ Simplify.

Based on this model, when the discus is 15 meters away from Esteban, it will be at a height of approximately 10.5 meters.

**B.** Predict how far Esteban throws the discus.

Use technology to graph the quadratic regression and find the right side zero.

Based on the model, Esteban throws the discus about 40 meters.




**Try It!** **5.** Predict the maximum height reached by the discus during Esteban's throw.

## CONCEPT SUMMARY Standard Form of a Quadratic Function

**STANDARD FORM**

$y = ax^2 + bx + c$ $\qquad$ $y = -2x^2 - 8x + 1$

**KEY FEATURES**

**Vertex** $x$-coordinate of vertex: $h = -\frac{b}{2a}$ $\qquad$ $h = -\frac{(-8)}{2(-2)} = -2$

Substitute $h$ for $x$ and solve for $y$ to find the $y$-coordinate of the vertex.

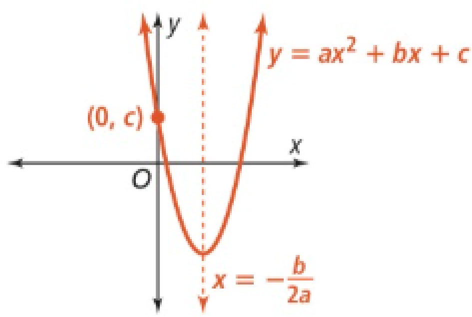
$$y = -2(-2)^2 - 8(-2) + 1$$
$$= -8 + 16 + 1$$
$$= 9$$

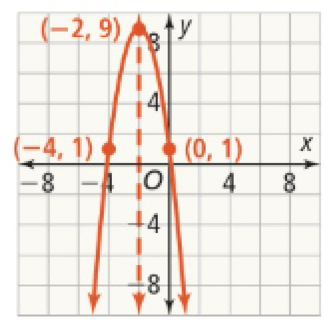
The vertex is $(-2, 9)$.

**Axis of Symmetry** $x = -\frac{b}{2a}$ $\qquad$ $x = -2$

**$y$-intercept** $(0, c)$ $\qquad$ $(0, 1)$

**GRAPHS**




## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What key features can you determine about a quadratic function from an equation in standard form?

2. **Error Analysis** Cameron said that the $y$-intercept of a quadratic function always tells the maximum value of that function. Explain Cameron's error.

3. **Vocabulary** Write a quadratic function in standard form.

4. **Analyze and Persevere** Why do you need to know either the vertex or the zeros to find an equation for a parabola?

## Do You KNOW HOW?

**Find the vertex and $y$-intercept of the quadratic function.**

5. $y = 3x^2 - 12x + 40$ $\qquad$ 6. $y = -x^2 + 4x + 7$

**For 7 and 8, find the maximum or minimum of the parabola.**

7. $y = -2x^2 - 16x + 20$ $\qquad$ 8. $y = x^2 + 12x - 15$

9. Find the equation in standard form of the parabola that passes through the points $(0, 6)$, $(-3, 15)$, and $(-6, 6)$.

**Graph the parabola.**

10. $y = 3x^2 + 6x - 2$

11. $y = -2x^2 + 4x + 1$

# PRACTICE & PROBLEM SOLVING

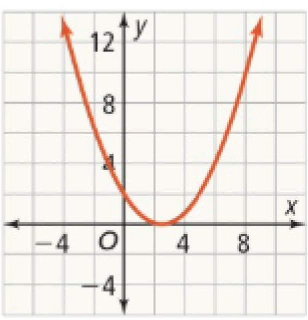
## UNDERSTAND

12. **Communicate and Justify** Devin found the parabola that fits the three points in the table to be $y = 0.345x^2 - 0.57x - 2.78$. Is Devin correct? Explain.

| $x$ | $-4$ | $0.6$ | $9$ |
|---|---|---|---|
| $y$ | $5$ | $-3$ | $20$ |

13. **Generalize** How can you find the maximum or minimum value of a quadratic function?

14. **Higher Order Thinking** The quadratic function whose graph is shown represents the interior of a cereal bowl. Its equation is $y = 0.32x^2 - 1.6x + 2$. Describe how you could use the function to find the inner diameter of the cereal bowl if you know its depth.




15. **Error Analysis** Micah found the vertex for the function $y = -9.5x^2 - 47.5x + 63$ as shown.

$$x = -\frac{b}{2a}$$

$$x = -\frac{47.5}{2(-9.5)}$$

$$x = -\frac{47.5}{-19}$$

$$x = -(-2.5)$$

$$x = 2.5$$

$$y = -9.5(2.5)^2 - 47.5(2.5) + 63$$

$$y = -59.375 - 118.75 + 63$$

$$y = -115.125$$

Find and correct Micah's error.

## PRACTICE

**Find the vertex of each parabola.** SEE EXAMPLE 1

16. $y = -x^2 + 6x + 30$

17. $y = 3x^2 + 12x - 5$

**Find the vertex and $y$-intercept of the quadratic function, and use them to graph the function.** SEE EXAMPLES 1 AND 2

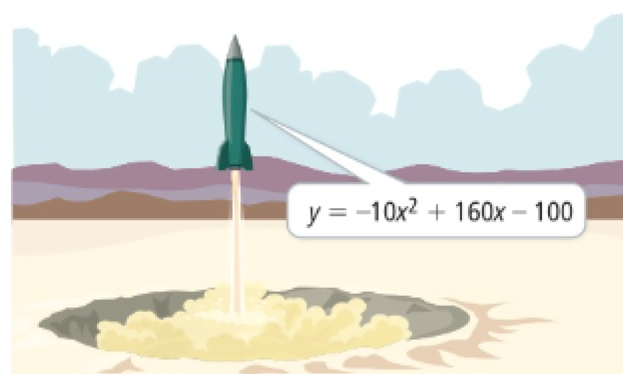
18. $y = -x^2 + 6x - 8$

19. $y = x^2 - 8x + 11$

20. $y = 3x^2 + 18x + 10$

21. $y = -2x^2 - 12x - 5$

22. A rocket is launched into the air. The height in feet of the rocket is modeled by the equation $y = -10x^2 + 160x - 100$ after $x$ seconds. What is the maximum height reached by the rocket, in feet? SEE EXAMPLE 3




**Write the equation of a quadratic function in standard form for the parabola that passes through the given points.** SEE EXAMPLE 4

23. $(-1, 5)$, $(4, 0)$, $(5, -7)$, $(1, 9)$, $(3, 5)$, $(0, 8)$

24. $(-2, 2)$, $(1, 8)$, $(4, 50)$, $(-1, 0)$, $(0, 2)$

**Use quadratic regression to find the equation of a quadratic function that fits the given points.** SEE EXAMPLE 5

25. A fan threw a souvenir football from the top of the bleachers. The table shows the height of the football, in feet, above the ground at various times, in seconds. If the football was not touched by anyone on its way to the ground, about how long did it take the football to reach the ground after it was thrown?

| Time (s) | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| Height (ft) | 10 | 11.76 | 12.24 | 11.44 | 9.36 | 6.0 |

## PRACTICE & PROBLEM SOLVING

### APPLY

**26. Apply Math Models** The height of Imani's mid-section was measured three times during a long jump.

| Time in seconds, $x$ | 0 | 0.5 | 1 |
|---|---|---|---|
| Height in meters, $y$ | 0.7 | 1.5 | 0.55 |


Write the equation of a quadratic function that describes Imani's height as a function of time.

**27. Analyze and Persevere** A college's business office found the relationship between the number of admissions counselors they employ and the college's profit from tuition could be modeled by the function $y = -10x^2 + 1{,}500x - 35{,}000$.

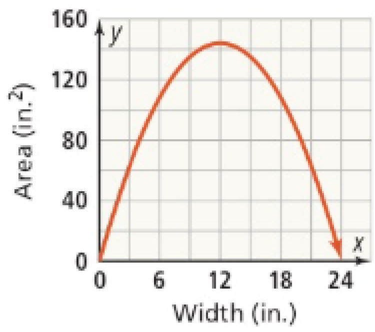
a. Graph the function.

b. How many admissions counselors should the college employ to maximize its profit?

c. What is the maximum amount of profit the college can make?

**28. Mathematical Connections** A rectangular tile has a perimeter of 48 inches.

a. The graph shows the relationship between the **width** of the tile and the **area** of the tile. What function describes this relationship?




b. What is the maximum area? What length and width give the maximum area?

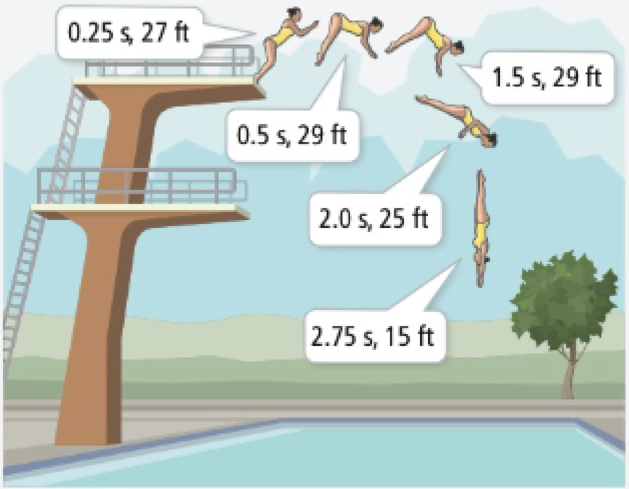
### ASSESSMENT PRACTICE

**29.** The height above the surface of the Earth (in meters) of a rock thrown into the air at 10 m/s after $x$ seconds is given by $f(x) = -4.9x^2 + 10x + 1.5$. On the surface of the moon, the height is given by $g(x) = -0.8x^2 + 10x + 1.5$. How much higher does the rock travel on the moon than on Earth? AR.3.8

**30. SAT/ACT** Which quadratic equation contains the three points (–4, 12), (2, 42), and (3, 40)?

Ⓐ $y = -x^2 + 3x + 42$
Ⓑ $y = 1.7x^2 - 10x - 55.2$
Ⓒ $y = -1.7x^2 + 10x + 55.2$
Ⓓ $y = x^2 - 3x - 40$
Ⓔ $y = -x^2 + 3x + 40$

**31. Performance Task** A diver jumped from a diving platform. The image shows her height above the water at several different times after leaving the platform.




**Part A** Find the equation of the quadratic function that describes the relationship between the diver's time and height. Round to the nearest tenth.

**Part B** How high is the platform the diver jumped from? What is the maximum height reached?

**Part C** From the maximum height, how long does it take the diver to get halfway down? Which part of the dive is faster, from the top to the halfway point, or from the halfway point to the water? Explain.

# 2-3

## Factored Form of a Quadratic Function

**I CAN…** find the zeros of quadratic functions.

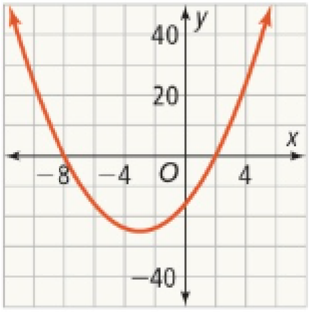
**VOCABULARY**

- Zero Product Property

MA.912.AR.3.2–Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real and complex number systems.
Also AR.3.4, AR.3.8, AR.1.1
MA.K12.MTR.1.1, MTR.2.1, MTR.5.1

### CRITIQUE & EXPLAIN

**Corey wrote an equation in factored form, $y = (x + 8)(x - 2)$, to represent a quadratic function. Kimberly wrote the equation $y = x^2 + 6x - 16$, and Joshua wrote the equation $y = (x + 3)^2 - 25$.**




A. Represent and Connect Do all three equations represent the same function? If not, whose is different? Explain algebraically.

B. How else could you determine if all three equations represent the same function?

C. What information can Corey's form help you find that is more difficult to find using Kimberly's or Joshua's form?

### ESSENTIAL QUESTION

**How is the factored form helpful in solving quadratic equations?**

### EXAMPLE 1 Factor a Quadratic Expression

**Factor the expression.**

**A. $x^2 + 7x + 12$**

Recall that using the Distributive Property, $(x + m)(x + n) = x^2 + (m + n)x + mn$.

$$x^2 + \underbrace{7}_{m+n}x + \underbrace{12}_{mn}$$

Add factor pairs of 12 to find the numbers that add to 7. The numbers 3 and 4 have a product of 12 and a sum of 7. Therefore, the factored form of the expression $x^2 + 7x + 12$ is $(x + 3)(x + 4)$.

**STUDY TIP**
You can check your work by multiplying the factors using the Distributive Property.

**B. $2x^2 - 5x - 3$**

When the leading coefficient is not 1, multiply the leading coefficient and the constant. Look for factors of this product that add to the middle coefficient. Rewrite the middle term using these factors, then factor by grouping.

$2x^2 - 5x - 3$ The factors of $-6$ that have a sum of $-5$ are 1 and $-6$.

$2x^2 + x - 6x - 3$ Rewrite $-5x$ as $x - 6x$.

$x(2x + 1) - 3(2x + 1)$

$(2x + 1)(x - 3)$

The factored form of the expression $2x^2 - 5x - 3$ is $(2x + 1)(x - 3)$.

**Try It!** 1. Factor the expression.

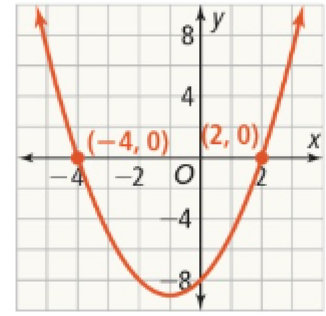
a. $x^2 - 9$

b. $3x^2 - 7x + 2$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Relate Factors to Zeros of a Function

**The graph shows the function defined by $y = x^2 + 2x - 8$. How do the zeros of the function relate to the factors of the expression $x^2 + 2x - 8$?**




The expression $x^2 + 2x - 8$ can be represented as a product of two factors. The factors of −8 that have a sum of 2 are 4 and −2.

$$y = x^2 + 2x - 8 \quad \rightarrow \quad y = (x + 4)(x - 2)$$

The $x$-intercepts of the graph are −4 and 2, so the zeros of the function are $x = -4$ and $x = 2$.

Substitute $x = -4$ and $x = 2$ in to the factored form of the equation.

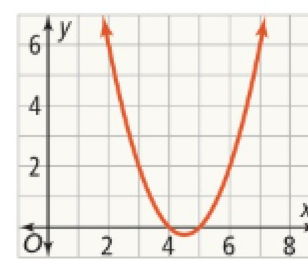
$$y = (-4 + 4)(-4 - 2) = 0(-6) = 0$$

$$y = (2 + 4)(2 - 2) = 6(0) = 0$$

The factors, $(x + 4)$ and $(x - 2)$, are related to the zeros $x = -4$ and $x = 2$ since each of the zeros makes one of the factors 0.

**VOCABULARY**
Recall that a zero of a function is a number, $z$, for which $f(z) = 0$. A zero of $f$ is also called an $x$-intercept of the graph since the graph of $f$ passes through the point $(z, 0)$.

**Try It!** 2. The graph shows the function $y = x^2 - 9x + 20$. Identify the zeros of the function. How do the zeros relate to the factors of $x^2 - 9x + 20$?




## CONCEPT Zero Product Property

The **Zero Product Property** states that if a product of real-number factors is 0, then at least one of the factors must be 0.

In the case of two factors, if $ab = 0$, then either $a = 0$ or $b = 0$, or both.

To use the Zero Product Property, rewrite the equation so that it is an expression equal to 0, then factor and solve.

## EXAMPLE 3 Solve Quadratic Equations by Factoring

**Solve the equation.**

**A.** $x^2 + x = 42$

| | | | |
|---|---|---|---|
| $x^2 + x - 42 = 0$ | | | Set equation equal to 0. |
| $(x + 7)(x - 6) = 0$ | | | Factor. |
| $x + 7 = 0$ | or | $x - 6 = 0$ | Use the Zero Product Property. |
| $x = -7$ | or | $x = 6$ | Solve. |

**ANALYZE AND PERSEVERE**
If you can write an expression in factored form, you can find the value of the variable that makes each factor 0. These values are the zeros of the function.

CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

**B.** $2x^2 = -9x + 5$

$$2x^2 + 9x - 5 = 0$$ Rewrite as equation equal to 0.

$$2x^2 - x + 10x - 5 = 0$$ Factor by grouping.

$$x(2x - 1) + 5(2x - 1) = 0$$

$$(x + 5)(2x - 1) = 0$$

$$x + 5 = 0 \quad \text{or} \quad 2x - 1 = 0$$ Use the Zero Product Property.

$$x = -5 \quad \text{or} \quad x = \frac{1}{2}$$ Solve.

**STUDY TIP**

Check your work algebraically, by plugging the solutions in to the original equation. Or check graphically by confirming that your solutions are the $x$-intercepts of the graph.

**Try It!** 3. Solve the equation by factoring.

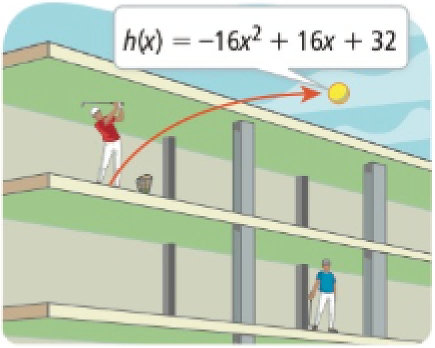
a. $x^2 + 8x = 20$

b. $2x^2 = 3x + 2$

APPLICATION

## EXAMPLE 4 Find the Zeros of a Quadratic Function

**A multilevel driving range has three levels. Marco hits golf balls from the second level, which is 32 ft high. The height of a ball $x$ seconds after Marco hits it is modeled by the function $h(x) = -16x^2 + 16x + 32$. When does the ball hit the ground?**




The ball hits the ground when the height, $h(x)$, is 0.

$$0 = -16x^2 + 16x + 32$$ Substitute 0 for $h(x)$.

$$0 = -16(x^2 - x - 2)$$ Factor out the GCF, −16.

$$0 = -16(x + 1)(x - 2)$$ Factor.

$$x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$ Use the Zero Product Property.

$$x = -1 \quad \text{or} \quad x = 2$$ Solve.

The zeros of the function are at $x = -1$ and $x = 2$.

Since time has to be positive, $x = 2$ is the only solution that makes sense.

This means that after 2 seconds, the golf ball will hit the ground.

**GENERALIZE**

Zeros of the function $h(x) = -16x^2 + 16x + 32$ are solutions of the equation $0 = -16x^2 + 16x + 32$.

**Try It!** 4. A baseball is thrown from the upper deck of a stadium, 128 ft above the ground. The function $h(x) = -16x^2 + 32x + 128$ gives the height of the ball $x$ seconds after it is thrown. How long will it take the ball to reach the ground?

## EXAMPLE 5 Determine Positive or Negative Intervals

**Identify the interval(s) on which the function $y = x^2 - 2x - 3$ is positive.**

The $y$-values of a quadratic function can only turn from positive to negative or from negative to positive when the graph crosses the $x$-axis. Find the zeros of the function to identify these points.

$0 = x^2 - 2x - 3$ ········ Set expression equal to 0.

$0 = (x - 3)(x + 1)$ ········ Factor.

$x - 3 = 0$ or $x + 1 = 0$ ········ Zero Product Property

$x = 3$ or $x = -1$ ········ Solve.

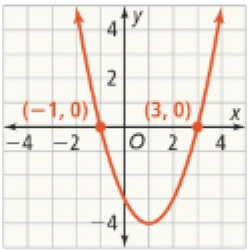
Two zeros create three intervals. Choose an $x$-value to test in each interval. Substitute the $x$-value into the original expression to determine if the corresponding $y$-value is positive or negative.

**USE PATTERNS AND STRUCTURE**
The sign of the $y$-value of the test point is the same as for the $y$-value of any other point over the entire interval you are testing.

| $x < -1$ | $-1 < x < 3$ | $x > 3$ |
|---|---|---|
| Choose $x = -3$.<br>$(-3)^2 - 2(-3) - 3$<br>$= 9 + 6 - 3$<br>$= 12$ | Choose $x = 1$.<br>$(1)^2 - 2(1) - 3$<br>$= 1 - 2 - 3$<br>$= -4$ | Choose $x = 6$.<br>$(6)^2 - 2(6) - 3$<br>$= 36 - 12 - 3$<br>$= 21$ |
| Positive | Negative | Positive |

Graph the function to verify where the function is positive or negative.

The function is positive when the graph is above the $x$-axis, or on the intervals $x < -1$ and $x > 3$.




**Try It!** 5. Identify the interval(s) on which the function $y = x^2 - 4x - 21$ is negative.

## EXAMPLE 6 Write the Equation of a Parabola in Factored Form

**Write an equation of a parabola with $x$-intercepts at $(-2, 0)$ and $(-1, 0)$ and which passes through the point $(-3, 20)$.**

$y = a(x - p)(x - q)$ ········ Write the general form of a factored equation.

$y = a(x - (-2))(x - (-1))$ ········ Substitute −1 and −2 for zeros.

$y = a(x + 2)(x + 1)$ ········ Simplify.

$20 = a(-3 + 2)(-3 + 1)$ ········ Substitute −3 for $x$ and 20 for $y$.

$20 = 2a$ ········ Simplify.

$10 = a$ ········ Solve.

$y = 10(x + 2)(x + 1)$ ········ Subsitute 10 for $a$.

**COMMON ERROR**
If $x = -2$ is an $x$-intercept, then $x + 2$ is the factor, not $x - 2$.

**Try It!** 6. Write an equation of a parabola with $x$-intercepts at $(3, 0)$ and $(-3, 0)$ and which passes through the point $(1, 2)$.

## CONCEPT SUMMARY Factored Form of a Quadratic Function

**FACTORED FORM** $y = ax^2 + bx + c$ can be written as $0 = a(x - p)(x - q)$, where $p$ and $q$ are the zeros of the function. The $x$-intercepts of the graph correspond to the zeros of the function. Two zeros denote 3 intervals of $x$ values.

**GRAPH** For the function $y = 2x^2 + 3x - 14$, write the equation $0 = 2x^2 + 3x - 14$ in factored form to identify the zeros.

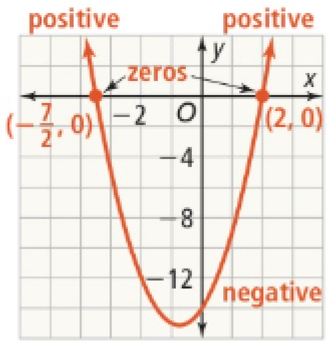
$0 = 2x^2 + 3x - 14$

$0 = (2x + 7)(x - 2)$

The zeros of the function are $x = -\frac{7}{2}$ and $x = 2$.

intervals where function values are positive: $x < -\frac{7}{2}$, and $x > 2$

interval where function values are negative: $-\frac{7}{2} < x < 2$




## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is the factored form helpful in solving quadratic equations?

2. **Error Analysis** Amir says the graph of $y = x^2 + 16$ has $-4$ as a zero. Is Amir correct? Explain.

3. **Vocabulary** How does the factored form of a quadratic equation relate to the Zero Product Property?

4. **Generalize** How does knowing the zeros of a function help determine where a function is positive?

## Do You KNOW HOW?

**Factor each expression.**

5. $x^2 - 5x - 24$

6. $5x^2 + 3x - 2$

**Solve each equation.**

7. $x^2 = 12x - 20$

8. $4x^2 - 5x = 6$

9. The height, in feet, of a t-shirt launched from a t-shirt cannon high in the stands at a football stadium is given by $h(x) = -16x^2 + 64x + 80$, where $x$ is the time in seconds after the t-shirt is launched. How long will it take before the t-shirt reaches the ground?

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

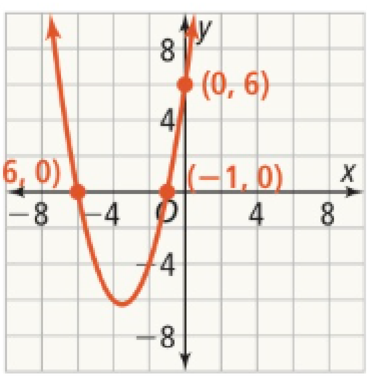
10. **Generalize** Can you write the equation of a quadratic function knowing its zeros and its non-zero $y$-intercept? If so, describe the process. If not, explain why.

11. **Error Analysis** Describe and correct the error a student made in solving a quadratic equation.

$$0 = 2x^2 + 7x + 5$$
$$0 = 2x^2 + 2x + 5x + 5$$
$$0 = 2x(x + 1) + 5(x + 1)$$
$$0 = 2x,\ 0 = x + 1,\ 0 \neq 5$$
$$0 = x,\ -1 = x$$

✗

12. **Represent and Connect** Use the graph of the function to write the equation in factored form.




13. **Generalize** For what values of $x$ is the expression $(x - 4)^2 > 0$?

14. **Error Analysis** A student says that the zeros of $y = (x - 2)(x + 7)$ are $-2$ and 7. Is the student correct? If not, describe and correct the error the student made.

15. **Communicate and Justify** Explain why $x^2 + 25$ is not equal to $(x + 5)^2$.

16. **Mathematical Connections** Describe how factoring can help you find the $x$-intercepts of the graph of the quadratic function $y = x^2 - 4x + 3$.

### PRACTICE

**Factor each quadratic expression.** SEE EXAMPLE 1

17. $x^2 - 3x - 10$
18. $3x^2 - 5x - 12$
19. $x^2 + 15x + 56$
20. $2x^2 + 7x - 15$
21. $3x^2 - 18x - 48$
22. $4x^2 - 11x - 3$

23. What are the zeros of the quadratic function $y = 3(x - 5)(x + 4)$? SEE EXAMPLE 2

**Solve each quadratic equation.** SEE EXAMPLE 3

24. $x^2 - 5x - 14 = 0$
25. $x^2 = 5x - 6$
26. $3x^2 - 60 = 3x$
27. $5x^2 + 12x = 9$
28. $4x^2 + 3x - 7 = 0$
29. $6x^2 = 5x + 6$

30. A penny is dropped from the top of a new building. Its height in feet can be modeled by the equation $y = 256 - 16x^2$, where $x$ is the time in seconds since the penny was dropped. How long does it take for the penny to reach the ground? SEE EXAMPLE 4

**Identify the interval(s) on which each quadratic function is positive.** SEE EXAMPLE 5

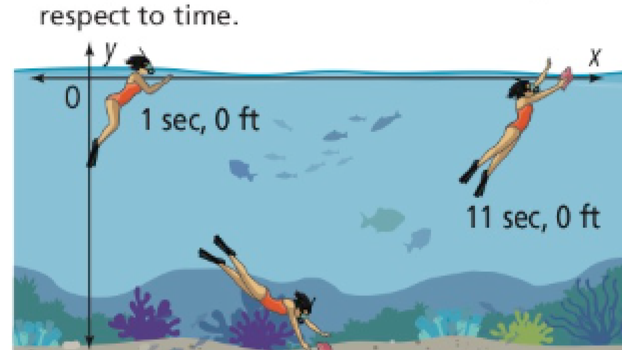
31. $y = x^2 + 9x + 18$
32. $y = x^2 + 2x - 8$
33. $y = x^2 - 5x - 24$
34. $y = -x^2 + 4x + 12$
35. $y = 2x^2 + 12x + 18$
36. $y = 5x^2 - 3x - 8$

**Write an equation for each parabola.** SEE EXAMPLE 6

37. A parabola with $x$-intercepts at $(-1, 0)$ and $(3, 0)$ which passes through the point $(1, -8)$

38. A parabola with $x$-intercepts at 0 and 1 and which passes through the point $(2, -2)$

39. A snorkeler dives for a shell on a reef. She is at the surface at 1 second and again at 11 seconds. At 2 seconds she is at a depth of $\frac{11}{3}$ ft. Write an equation that models the diver's depth with respect to time.

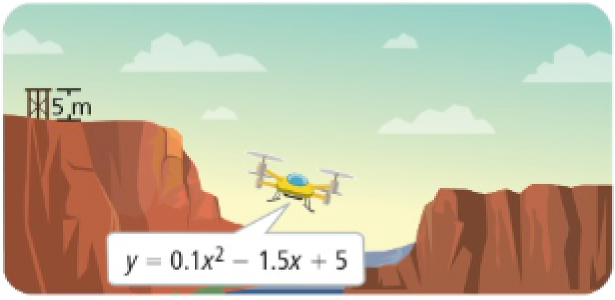


## PRACTICE & PROBLEM SOLVING

### APPLY

40. **Analyze and Persevere** Rectangular apartments are 12 ft longer than they are wide. Each apartment has 1,053 ft$^2$ of floor space. What are the dimensions of an apartment? Explain.

41. **Use Patterns and Structure** The height of a drone, in meters, above its launching platform that is 5 m above the ground, is modeled by $y = 0.1x^2 - 1.5x + 5$, where $x$ is the time in seconds. The drone leaves the launch pad, flies down into a canyon, and then it flies back up again.




a. What is the factored form of the equation for the height of the drone?

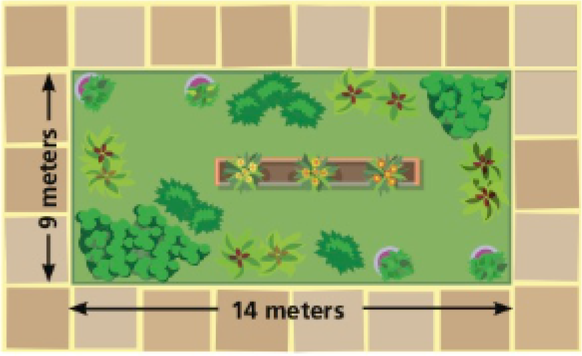
b. After how many seconds will the drone be at ground level?

c. After how many seconds will the drone come back to the height of its platform?

42. **Higher Order Thinking** LaTanya is designing a rectangular garden with a uniform walkway around its border. LaTanya has 140 m$^2$ of material to build the walkway.

a. Write an equation for the dimensions of the garden and the surrounding walkway.

b. How wide is the walkway? Explain.



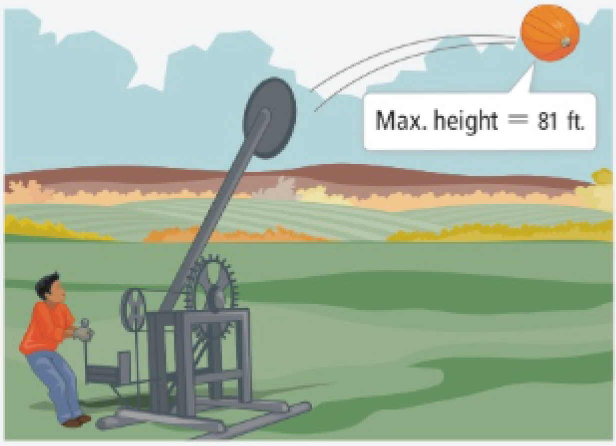

### ASSESSMENT PRACTICE

43. A factory can use either of two processes to manufacture a product. The costs, in dollars, for each process are given by the functions $f(x) = 6x$ and $g(x) = x^2 + 5$, where $x$ represents the number of units, in thousands. Solve the equation $f(x) = g(x)$. When are the costs for both processes the same? AR.3.2

44. **SAT/ACT** What is the sum of the zeros of the function $y = x^2 - 9x - 10$?

Ⓐ −10 Ⓑ −9 Ⓒ 0 Ⓓ 9 Ⓔ 10

45. **Performance Task** A pumpkin is launched from the ground into the air and lands 4.5 s later.




**Part A** Write a quadratic function that models the height, in feet, of the pumpkin $x$ seconds after it is launched. Explain how you found the function.

**Part B** A second pumpkin is launched from the ground. After 1 second, it is 64 feet high. The pumpkin lands after 5 seconds. What is the maximum height of the pumpkin? Explain.

# 2-4 Complex Numbers and Operations

**I CAN…** solve problems with complex numbers.

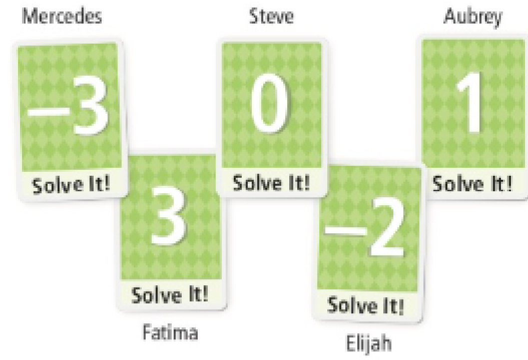
**VOCABULARY**

- complex conjugates
- complex number
- imaginary number
- imaginary unit $i$

MA.912.NSO.2.1–Extend previous understanding of the real number system to include the complex number system. Add, subtract, multiply and divide complex numbers. Also AR.3.2
MA.K12.MTR.6.1, MTR.7.1

## EXPLORE & REASON

**A math class played a game called "Solve It, You're Out." At the start of each round, students chose a card from a deck marked with integers from −5 to 5. When an equation is shown, any student whose card states the solution to the equation is eliminated. Five students remain.**




**A.** The next equation presented was $x^2 = 9$. Which student(s) was eliminated? Explain.

**B.** **Check for Reasonableness** In the next round, the equation presented was $x^2 = -4$. Elijah thought he was eliminated, but this is not the case. Explain why Elijah was incorrect.

**C.** What is true about solutions to $x^2 = a$ when $a$ is a positive number? When $a$ is a negative number? What about when $a = 0$?

## ESSENTIAL QUESTION

**How can you represent and operate on numbers that are not real numbers?**

## EXAMPLE 1 Solve a Quadratic Equation Using Square Roots

**How can you use square roots to solve each equation?**

**A.** $x^2 = 16$

Notice that each side of the equation involves a perfect square.

$$x^2 = 16$$
$$x = \pm\sqrt{16}$$
$$= \pm 4$$

What numbers can you square that result in 16?

The solutions of the equation $x^2 = 16$ are 4 and −4.

**STUDY TIP**
You can also solve this equation by subtracting 16 from both sides, then factoring the expression.

**B.** $x^2 = -9$

There are no real numbers that you can square that result in −9. However, you can simplify the expression by extending the properties of radicals.

$$x^2 = -9$$
$$x = \pm\sqrt{-9}$$
$$x = \pm\sqrt{9}\sqrt{-1}$$
$$x = \pm 3\sqrt{-1}$$

The solutions of the equation $x^2 = -9$ are not real numbers but are part of a number system called the complex numbers. The number $\sqrt{-1}$ is called the **imaginary unit** $i$. Replacing $\sqrt{-1}$ with $i$ allows you to write the solutions to the equation $x^2 = -9$ as $3i$ and $-3i$.

**Try It!** 1. Use square roots to solve each equation. Write your solutions using the imaginary unit, $i$.

**a.** $x^2 = -5$

**b.** $x^2 = -72$

## CONCEPT Complex Numbers

The imaginary unit, $i$, is the principal square root of $-1$. Then $i^2 = -1$.

An **imaginary number** is any number, $bi$, where $b$ is a non-zero real number and $i$ is the square root of $-1$.

**Complex numbers** are numbers that can be written in the form $a + bi$, where $a$ and $b$ are real numbers and $i$ is the square root of $-1$. They include all real and imaginary numbers, as well as the sums of real and imaginary numbers.

For example:

$-6 + 4i \qquad (a = -6,\ b = 4)$

$7 - i\sqrt{2} \qquad (a = 7,\ b = -\sqrt{2})$

$0.5i \qquad (a = 0,\ b = 0.5)$

## EXAMPLE 2 Add and Subtract Complex Numbers

**How can you add and subtract complex numbers?**

**A. What is the sum of $(4 - 7i)$ and $(-11 + 9i)$?**

When adding (or subtracting) two numbers in the form $a + bi$, combine the real parts and then combine the imaginary parts. The sum (or difference) may include both a real and imaginary part and can be written in the form $a + bi$.

**STUDY TIP**
Combine real parts and imaginary parts of complex numbers as you would combine like terms.

$$(4 - 7i) + (-11 + 9i) = (4 + -11) + (-7i + 9i)$$
$$= -7 + 2i$$

**B. What is the difference of $(6 + 8i)$ and $(2 - 5i)$?**

$$(6 + 8i) - (2 - 5i) = (6 + 8i) + (-2 + 5i)$$
$$= (6 + -2) + (8i + 5i)$$
$$= 4 + 13i$$

Remember to distribute the $-1$ over the complex number.

**Try It!** **2.** Find the sum or difference.

**a.** $(-4 + 6i) + (-2 - 9i)$

**b.** $(3 - 2i) - (-4 + i)$

## EXAMPLE 3 Multiply Complex Numbers

**How can you write each product in the form $a + bi$?**

**A.** $\mathbf{-2.5i(8 - 9i)}$

$-2.5i(8 - 9i) = -2.5i(8) - 2.5i(-9i)$ ........ Use the Distributive Property.

$= -20i + 22.5i^2$ ........ Multiply.

$= -20i + 22.5(-1)$ ........ Simplify using the definition of $i^2$.

$= -22.5 - 20i$ ........ Write in the form $a + bi$.

The product is $-22.5 - 20i$.

**COMMON ERROR**
Recall that $i^2 = -1$, so the product of 22.5 and $i^2$ is $-22.5$, not 22.5.

**B.** $\mathbf{(3 - 2i)(3 + 2i)}$

$(3 - 2i)(3 + 2i) = 3(3 + 2i) - 2i(3 + 2i)$ ........ Use the Distributive Property.

$= 9 + 6i - 6i - 4i^2$ ........ Use the Distributive Property.

$= 9 + 6i - 6i - 4(-1)$ ........ Simplify using the definition of $i^2$.

$= 13$ ........ Simplify.

The product is 13.

**Try It!** **3.** Write each product in the form $a + bi$.

**a.** $\frac{2}{5}i\left(10 - \frac{5}{2}i\right)$

**b.** $\left(\frac{1}{2} + 2i\right)\left(\frac{1}{2} - 2i\right)$

## CONCEPT Complex Conjugates

**Complex conjugates** are complex numbers with equivalent real parts and opposite imaginary parts. Their product is a real number.

For example:

$7 - 8i,\ 7 + 8i$ $\qquad$ $-2 + i,\ -2 - i$

$$(a + bi)(a - bi)$$
$$a^2 - abi + abi - b^2i^2$$
$$a^2 - b^2(-1)$$
$$a^2 + b^2$$

## EXAMPLE 4 Simplify a Quotient With Complex Numbers

**How can you write the quotient $\frac{10}{2 - i}$ in the form $a + bi$?**

When the denominator is a complex number with an imaginary component, you can create an equivalent fraction with a real denominator by multiplying by one using the complex conjugate of the denominator.

$\frac{10}{2 - i} = \frac{10}{2 - i} \times \frac{2 + i}{2 + i}$ ........ Use the complex conjugate of the denominator to multiply by 1.

$= \frac{10(2 + i)}{4 + 2i - 2i - i^2}$ ........ Use the Distributive Property.

$= \frac{10(2 + i)}{4 + 2i - 2i - (-1)}$ ........ Simplify using the definition of $i^2$.

$= \frac{10(2 + i)}{5}$ ........ Simplify.

$= 2(2 + i)$ ........ Simplify.

$= 4 + 2i$ ........ Write in the form $a + bi$.

**STUDY TIP**
Multiplying the denominator by its complex conjugate will result in a new denominator that is a real number.

CONTINUED ON THE NEXT PAGE

**Try It!** **4.** Write each quotient in the form $a + bi$.

**a.** $\dfrac{80}{2-6i}$ **b.** $\dfrac{4-3i}{-1+2i}$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 5 Factor a Sum of Squares

**How can you use complex numbers to factor the sum of two squares?**

**A. How can you factor the expression $x^2 + y^2$?**

Rewrite $x^2 + y^2$ as a difference of two squares: $x^2 - (-y^2)$.

How can $(-y^2)$ be a perfect square?

You can think of $(-y^2)$ as $(-1)(y^2)$.

Since $-1 = i^2$, $(-1)(y^2) = (i^2)(y^2) = (yi)^2$.

So $x^2 + y^2 = x^2 - (yi)^2$

$= (x + yi)(x - yi)$

Factor as the difference of two squares.

The factors of $x^2 + y^2$ are $(x + yi)$ and $(x - yi)$.

**STUDY TIP**
The product of complex conjugates $(a + bi)$ and $(a - bi)$ will always be equal to $a^2 + b^2$, which is the sum of two squares.

**B. How can you factor the expression $12x^2 + 3$?**

$12x^2 + 3 = 3(4x^2 + 1)$ Factor out the GCF.

$= 3(4x^2 - i^2)$ Rewrite as a difference of squares.

$= 3(2x + i)(2x - i)$ Factor the difference of squares.

The factors of $12x^2 + 3$ are 3, $(2x + i)$, and $(2x - i)$.

**Try It!** **5.** Factor each expression.

**a.** $4x^2 + 25$ **b.** $8y^2 + 18$

## EXAMPLE 6 Solve a Quadratic Equation With Complex Solutions

**How can you solve $x^2 + 4 = 0$ using factoring?**

$x^2 + 4 = 0$ Write the original equation.

$x^2 - (2i)^2 = 0$ Rewrite as a difference of squares.

$(x + 2i)(x - 2i) = 0$ Factor the difference of squares.

$x + 2i = 0 \quad x - 2i = 0$ Set each factor equal to 0.

$x = -2i \quad x = 2i$ Solve.

The solutions are $x = -2i$ and $x = 2i$.

**USE PATTERNS AND STRUCTURE**
In Example 1, you solved a similar problem by taking the square root of both sides. This example provides an alternative method that utilizes factoring.

**Try It!** **6.** Find the value(s) of $x$ that will solve each equation.

**a.** $x^2 + 49 = 0$ **b.** $9x^2 + 25 = 0$

## CONCEPT SUMMARY Complex Numbers and Operations

The imaginary unit $i$ is the number whose square is equal to $-1$: $\sqrt{-1} = i$, so $i^2 = -1$.

Complex numbers are written in the form $a + bi$.

real numbers imaginary unit

The four basic operations can be applied to complex numbers, such as $2 + 3i$ and $5 - i$.

### ADDITION

Add as you would with binomials with like terms.

$(2 + 3i) + (5 - i) = 7 + 2i$

### SUBTRACTION

Subtract as you would with binomials with like terms.

$(2 + 3i) - (5 - i) = -3 + 4i$

### MULTIPLICATION

Distribute as you would with binomials.

$(2 + 3i)(5 - i) = 10 - 2i + 15i - 3i^2 = 13 + 13i$

### DIVISION

Simplify so that the denominator is a real number. Multiply the numerator and denominator by the conjugate of the denominator.

$\frac{2 + 3i}{5 - i} = \frac{(2 + 3i)(5 + i)}{(5 - i)(5 + i)} = \frac{7 + 17i}{26} = \frac{7}{26} + \frac{17}{26}i$

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you represent and operate on numbers that are not real numbers?

2. **Vocabulary** How do you form the *complex conjugate* of a complex number $a + bi$?

3. **Error Analysis** Helena was asked to write the quotient $\frac{4}{3 - i}$ in the form $a + bi$. She began this way: $\frac{4}{3 - i} \times \frac{3 - i}{3 - i} = \frac{4(3 - i)}{3^2 + 1^2} = \frac{12 - 4i}{10}$. Explain the error Helena made.

4. **Generalize** The quadratic equation $x^2 + 9 = 0$ has solutions $x = 3i$ and $x = -3i$. How many times will the graph of $f(x) = x^2 + 9$ cross the $x$-axis? Explain.

## Do You KNOW HOW?

**Write each of the following in the form $a + bi$.**

5. $(2 + 5i) - (-6 + i)$

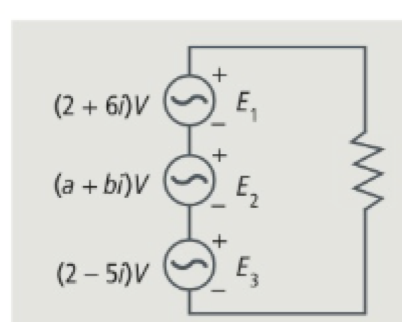
6. $(2i)(6 + 3i)$

**Solve each equation.**

7. $x^2 + 16 = 0$

8. $y^2 = -25$

9. **Apply Math Models** The total source voltage in the circuit is $6 - 3i$ V. What is the voltage at the middle source?

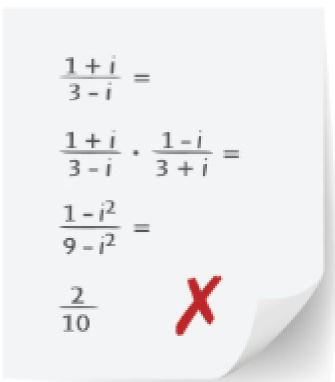


# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

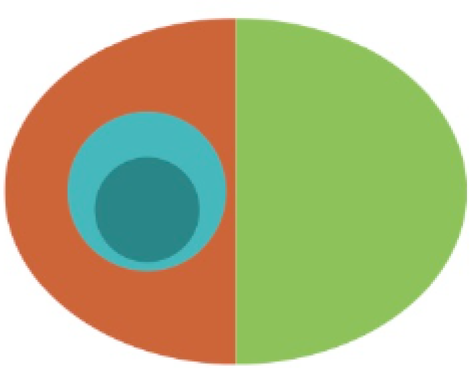
**10. Communicate and Justify** Tamara says that raising the number $i$ to any integer power results in either $-1$ or $1$ as the result, since $i^2 = -1$. Do you agree with Tamara? Explain.

**11. Error Analysis** Describe and correct the error a student made when dividing complex numbers.




**12. Higher Order Thinking** Label the diagram with the following sets of numbers:

1. complex numbers
2. real numbers
3. imaginary numbers
4. integers
5. rational numbers



Include an example of each type of number in the diagram.

**13. Generalize** Write an explicit formula, in standard form, to find the quotient of two complex numbers. Use the numbers $a + bi$ and $c + di$.

## PRACTICE

**Use square roots to solve each equation over the complex numbers.** SEE EXAMPLE 1

**14.** $x^2 = -5$

**15.** $x^2 = -0.01$

**16.** $x^2 = -18$

**17.** $x^2 = (-1)^2$

**Add or subtract. Write the answer in the form $a + bi$.** SEE EXAMPLE 2

**18.** $(3 - 2i) - (-9 + i)$

**19.** $(5 + 1.2i) + (-6 + 0.8i)$

**20.** $(2i) - (2i - 11)$

**21.** $13 + 2i - 4 - 8i$

**22.** $\frac{3-i}{4} - \frac{2+i}{3}$

**23.** $4.5i - 4.5 + 3.5i + 2.5$

**Write each product in the form $a + bi$.** SEE EXAMPLE 3

**24.** $(11i)(3i)$

**25.** $(3i)(5 - 4i)$

**26.** $(5 - 2i)(5 + 2i)$

**27.** $(8 + 3i)(8 + 3i)$

**28.** $\frac{1}{3}i(3 + 6i)$

**29.** $(-2i + 7)(7 + 2i)$

**Write each quotient in the form $a + bi$.**
SEE EXAMPLE 4

**30.** $\frac{12}{1-i}$

**31.** $\frac{5}{6+2i}$

**32.** $\frac{6+12i}{3i}$

**33.** $\frac{4-4i}{1+3i}$

**Factor the sums of two squares.** SEE EXAMPLE 5

**34.** $4x^2 + 49$

**35.** $x^2 + 1$

**36.** $36 + 100a^2$

**37.** $18y^2 + 8$

**38.** $\frac{1}{4}b^2 + 25$

**39.** $x^2 + y^2$

**Solve each equation.** SEE EXAMPLE 6

**40.** $x^2 + 81 = 0$

**41.** $25x^2 + 9 = 0$

**42.** $x^2 = -16$

**43.** $4 + 49y^2 = 0$

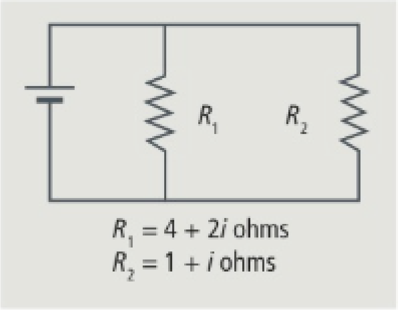
**44.** $y^2 + 1 = 0$

**45.** $x^2 + \frac{1}{4} = 0$

# PRACTICE & PROBLEM SOLVING

## APPLY

**46. Apply Math Models** The two resistors shown in the circuit are referred to as *in parallel*. The total resistance of the resistors is given by the formula $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$.

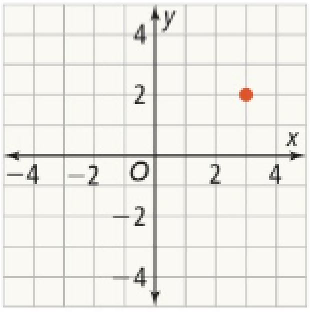



a. Find the total resistance. Write your answer in the form $a + bi$.

b. Show that the total resistance is equivalent to the expression $\frac{R_1 R_2}{R_1 + R_2}$.

c. Change the value of $R_2$ so that the total resistance is a real number. Explain how you chose the value.

**47. Use Patterns and Structure** The complex number $a + bi$ can be represented on a coordinate plane as the point $(a, b)$. You can use multiplication by $i$ to rotate a point about the origin in the coordinate plane.




a. Write the complex number, in the form $x + yi$, that corresponds to the given point.

b. Multiply the complex number by $i$. Interpret the new value as a new point in the plane.

c. Repeat the steps above for two other points. How does multiplication by $i$ rotate a point?

## ASSESSMENT PRACTICE

**48.** Select all powers of $i$ that are real numbers. NSO.2.1

☐ **A.** $i$

☐ **B.** $i^2$

☐ **C.** $i^3$

☐ **D.** $i^4$

☐ **E.** $i^5$

☐ **F.** $i^6$

**49. SAT/ACT** Which of the following is a solution to the equation $3x^2 = -12$?

Ⓐ $-4i$

Ⓑ $-2i$

Ⓒ $-2$

Ⓓ $2$

Ⓔ $4i$

**50. Performance Task** Abby wants to write the square root of $i$ in the form $a + bi$. She begins by writing the equation $\sqrt{i} = a + bi$.

**Part A** Square both sides of the equation. Then use the fact that the real part and imaginary part on each side of the equation are equal to write a system of equations involving the variables $a$ and $b$.

**Part B** Solve the system to find $b$. Then find $a$.

**Part C** List the possible solutions for $a$ and $b$.

**Part D** Square each of the possible solutions. What are the two square roots of $i$?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.3.4–Write a quadratic function to represent the relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. Also AR.3.8
MA.K12.MTR.7.1

## Swift Kick

Whether you call it soccer, football, or fùtbol, it's the most popular sport in the world by far. Even if you don't play soccer, you probably know several people who do.

There are many ways to kick a soccer ball: you can use any part of either foot. If you want the ball to end up in the goal, you also need to try different amounts of spin and power. You'll see one person's effort in the Mathematical Modeling in 3-Acts lesson.

### ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

### ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

### ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

# 2-5 Completing the Square

**I CAN…** solve quadratic equations by completing the square.

**VOCABULARY**

- completing the square

MA.912.AR.3.2–Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real and complex number systems. Also AR.3.4, AR.3.8

MA.K12.MTR.1.1, MTR.2.1, MTR.6.1

## CRITIQUE & EXPLAIN

**Hana and Enrique used different methods to solve the equation $x^2 - 6x + 9 = 16$.**

| Hana | Enrique |
|---|---|
| $x^2 - 6x + 9 = 16$ | $x^2 - 6x + 9 = 16$ |
| $x^2 - 6x - 7 = 0$ | $(x-3)^2 = 16$ |
| $(x-7)(x+1) = 0$ | I can square 4 or $-4$ to get 16. |
| $x - 7 = 0$ OR $x + 1 = 0$ | $x - 3 = 4$ OR $x - 3 = -4$ |
| $x = 7$ OR $x = -1$ | $x = 7$ OR $x = -1$ |
| The solutions are 7 and $-1$. | The solutions are 7 and $-1$. |

**A.** Does Hana's method work? If her method is valid, explain the reasoning she used. If her method is not valid, explain why not.

**B.** Does Enrique's method work? If his method is valid, explain the reasoning he used. If his method is not valid, explain why not.

**C.** Use Patterns and Structure Can you use either Hana's or Enrique's method to solve the equation $x^2 + 10x + 25 = 3$? Explain.

## ESSENTIAL QUESTION

**How can you solve a quadratic equation by completing the square?**

## EXAMPLE 1 Use Square Roots to Solve Quadratic Equations

**What are the solution(s) of $25 = x^2 + 14x + 49$?**

Previously, you solved a simple quadratic equation by finding the square root of both sides. You can use a similar method to solve more complicated quadratic equations.

$25 = x^2 + 14x + 49$ ········ Write the original equation.

$25 = x^2 + 2(7)x + 7^2$ ········ Recognize that the quadratic expression is a perfect square trinomial.

$25 = (x + 7)^2$ ········ Factor the perfect square trinomial.

$\sqrt{25} = \sqrt{(x+7)^2}$ ········ Take the square root of each side of the equation.

$5 = |x + 7|$ ········ Apply the definition of principal square root.

$\pm 5 = x + 7$ ········ Apply the definition of absolute value.

$5 = x + 7$ or $-5 = x + 7$

$-2 = x$ or $-12 = x$

The solutions of $25 = x^2 + 14x + 49$ are $x = -2$ and $x = -12$.

**ANALYZE AND PERSEVERE**
The principal square root returns only positive values, but you can square either 5 or $-5$ to get 25. How does the absolute value account for this?

**Try It!** 1. Find the solution(s) to the equations.

a. $81 = x^2 + 12x + 36$ b. $9 = x^2 - 16x + 64$

CONCEPTUAL UNDERSTANDING

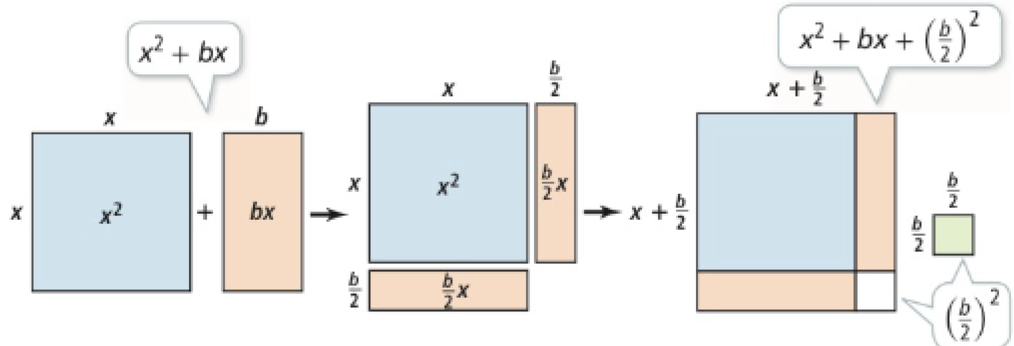
## EXAMPLE 2 Understand the Process of Completing the Square

**How can you complete the square to write an expression as a perfect square?**

**A. How can you rewrite the expression $x^2 + bx$ using the form $(x + p)^2$?**

Not every quadratic expression is a perfect square trinomial. **Completing the square** is the process of finding the constant to add to $x^2 + bx$ to create a perfect square trinomial.

The model below depicts the process of completing the square.




To create a perfect square trinomial, add $\left(\frac{b}{2}\right)^2$ to the variable expression.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

**B. Write $x^2 + 8x + 5 = 0$ in the form $(x + p)^2 = q$**

| | |
|---|---|
| $x^2 + 8x + 5 = 0$ | Write the original equation. |
| $x^2 + 8x = -5$ | Isolate the variable expression. |
| $x^2 + 8x + 16 = -5 + 16$ | Determine the constant needed to complete the square: $\left(\frac{b}{2}\right)^2 = \left(\frac{8}{2}\right)^2 = 16$. |
| $(x + 4)^2 = 11$ | Write the left side of the equation as a perfect square. |

The equation $x^2 + 8x + 5 = 0$ can be rewritten as $(x + 4)^2 = 11$.

**STUDY TIP**
Recall that you must keep the equation balanced. Any value that is added to one side of the equation must be added to the other side.

**Try It!** **2.** How can you write the equation $x^2 - 6x - 11 = 0$ in the form $(x - p)^2 = q$?

## EXAMPLE 3 Solve a Quadratic Equation by Completing the Square

**How can you solve $0 = x^2 - 2x + 3$ by completing the square?**

| | |
|---|---|
| $0 = x^2 - 2x + 3$ | Write the original equation. |
| $-3 = x^2 - 2x$ | Subtract 3 from each side. |
| $-3 + 1 = x^2 - 2x + 1$ | Add $\left(\frac{-2}{2}\right)^2$ to both sides of the equation. |
| $-2 = (x - 1)^2$ | Write the right side of the equation as a perfect square. |
| $\pm i\sqrt{2} = x - 1$ | Take the square root of each side of the equation. |
| $1 \pm i\sqrt{2} = x$ | Solve. |

The solutions of $0 = x^2 - 2x + 3$ are $x = 1 + i\sqrt{2}$ and $x = 1 - i\sqrt{2}$.

**HAVE A GROWTH MINDSET**
In what ways can you be inquisitive and open to learning new things?

CONTINUED ON THE NEXT PAGE

**Try It!** **3.** Solve the following equations by completing the square.

**a.** $0 = x^2 + 4x + 8$ **b.** $0 = x^2 - 8x + 17$

APPLICATION

## EXAMPLE 4 Complete the Square to Solve a Real-World Problem

**A rancher plans to create a rectangular pasturing enclosure. She has 340 m of fencing available for the enclosure's perimeter and wants it to have an area of 6,000 m$^2$. What dimensions should Libby use?**


**Formulate** Let $\ell$ and $w$ represent the length and width of the enclosure.

The perimeter is $2\ell + 2w = 340$, so:

$$2w = 340 - 2\ell$$
$$w = 170 - \ell$$

Libby wants the area to be 6,000 m$^2$. Write this as an equation:

$$A = \ell w$$
$$6{,}000 = \ell(170 - \ell)$$

Substitute for $A$ and $w$.

**Compute**

$$6{,}000 = 170\ell - \ell^2$$
$$\ell^2 - 170\ell = -6{,}000$$
$$\ell^2 - 170\ell + 7{,}225 = -6{,}000 + 7{,}225$$
$$(\ell - 85)^2 = 1{,}225$$
$$\ell - 85 = \pm 35$$
$$\ell = 85 \pm 35$$
$$\ell = 120 \text{ or } \ell = 50$$

Find the number to complete the square: $\frac{-170}{2} = -85$ and $(-85)^2 = 7{,}225$.

**Interpret** When $\ell = 120$, then $w = 170 - 120$, or 50.

When $\ell = 50$, then $w = 170 - 50$, or 120.

In each case, there is $2(120) + 2(50)$, or 340 m, of fencing used.

Likewise, the area is $(120)(50)$, or 6,000 m$^2$.

Libby should make two sides of the enclosure 120 m long and the other two sides 50 m long.

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**Try It!** 4. The relationship between the time since a ball was thrown and its height can be modeled by the equation $h = 32t - 16t^2 + 4$, where $h$ is the height of the ball after $t$ seconds. Complete the square to find how long it will take the ball to reach a height of 20 ft.

## EXAMPLE 5 Write a Quadratic Equation in Vertex Form

**Write the equation $y = -2x^2 + 10x + 1$ in vertex form and graph it. What is the maximum or minimum value of the graph of the equation?**

$y = -2x^2 + 10x + 1$ ······ Write the original equation.

$y - 1 = -2x^2 + 10x$ ······ Subtract 1 from each side.

$y - 1 = -2(x^2 - 5x)$ ······ Factor out the $x^2$ coefficient, $-2$.

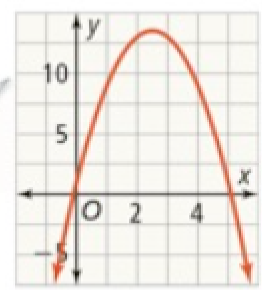
$y - 1 - (2)(6.25) = -2(x^2 - 5x + 6.25)$ ······ Complete the square: $\left(\frac{b}{2}\right)^2 = \left(\frac{-5}{2}\right)^2 = 6.25.$

$y - 13.5 = -2(x - 2.5)^2$ ······ Simplify and factor.

$y = -2(x - 2.5)^2 + 13.5$ ······ Write in vertex form.

**COMMON ERROR**
You may think that you have to add 6.25 to both sides; on the right side, 6.25 was added **with −2 already factored out**. So add −2(6.25), or −12.5, to the left side of the equation.

The vertex of the parabola is (2.5, 13.5).




The graph of this equation is a parabola that opens downward, so it has a maximum of $y = 13.5$, at $x = 2.5$.

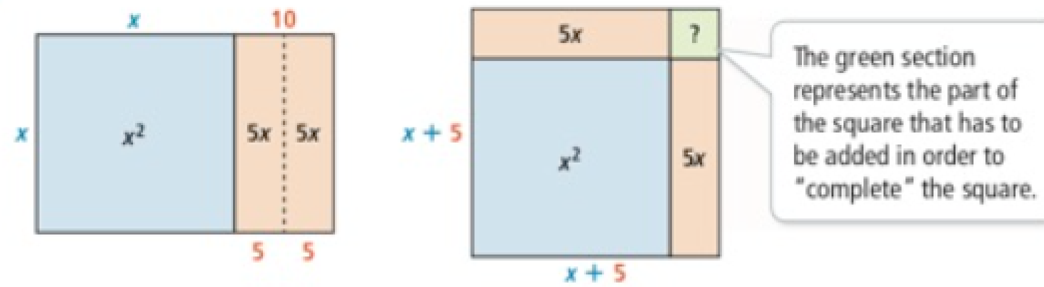
**Try It!** 5. Write each equation in vertex form. Identify the maximum or minimum value of the graph of each equation.

a. $y = -3x^2 - 9x + 7$

b. $y = 2x^2 + 12x + 9$

## CONCEPT SUMMARY Key Features of Completing the Square

**GEOMETRIC MODEL** The rectangles showing $x^2 + 10x$ are arranged into a square.




The square has side length $x + 5$, so the number needed to complete the square is 25.

**ALGEBRAIC MODEL** The number needed to complete the square is half the coefficient of the linear term, squared: the linear term coefficient is 10, half of 10 is 5, and $5^2 = 25$.

To solve $x^2 + 10x = 3$, add 25 to both sides of the equation, take the square root of both sides and solve for $x$:

$$\begin{aligned} x^2 + 10x + 25 &= 3 + 25 \\ (x+5)^2 &= 28 \\ x + 5 &= \pm 2\sqrt{7} \\ x &= -5 \pm 2\sqrt{7} \end{aligned}$$

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you solve a quadratic equation by completing the square?

2. **Error Analysis** Paula said that only quadratic equations with leading coefficients of 1 can be solved by completing the square. Is Paula correct? Explain.

3. **Generalize** Given the expression $x^2 + bx$, describe how to find $c$ so that $x^2 + bx + c$ is a perfect square trinomial.

4. **Analyze and Persevere** How can you complete the square to find the vertex of a parabola?

## Do You KNOW HOW?

**Solve each equation by completing the square.**

5. $0 = x^2 + 12x + 11$

6. $27 = 3x^2 + 12x$

7. $0 = 2x^2 + 6x - 14$

**Write the equation in vertex form, and identify the maximum or minimum point of the graph of the function.**

8. $y = x^2 + 6x - 6$

9. $y = -2x^2 + 20x - 42$

10. The daily profit, $P$, for a company is modeled by the function $P(x) = -0.5x^2 + 40x - 300$, where $x$ is the number of units sold. How many units does the company need to sell each day to maximize profits?

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

**11.** **Check for Reasonableness** How could you use a graphing calculator to determine whether you have correctly solved a quadratic equation by completing the square?

**12.** **Error Analysis** Describe and correct the error a student made in solving a quadratic equation by completing the square.

$$0 = x^2 + 16x - 5$$
$$5 = x^2 + 16x + 64$$
$$5 = (x + 8)^2$$
$$x = -8 \pm \sqrt{5}$$

✗

**13.** **Higher Order Thinking** What number do you need to add to $x^2 + \frac{7}{2}x$ in order to create a perfect square trinomial? Explain.

**14.** **Use Patterns and Structure** Does the geometric model hold for finding the number that completes the square of the expression $x^2 - 12x$? Explain.

**15.** **Error Analysis** When given the equation $-23 = x^2 + 8x$, a student says that you can add 64 to each side of the equation to complete the square. Is the student correct? If not, describe and correct the error.

**16.** **Communicate and Justify** Explain why you should not try to complete the square when solving $0 = x^2 - 4$.

**17.** **Represent and Connect** Jacob completed the square to rewrite the equation $f(x) = -2x^2 + 12x - 13$ as $f(x) = -2(x - 3)^2 + 5$. Which form of the equation is more helpful for identifying the key features of the graph? Explain.

### PRACTICE

**Use square roots to solve the quadratic equations.** SEE EXAMPLE 1

**18.** $9 = x^2 + 2x + 1$

**19.** $16 = x^2 - 10x + 25$

**20.** $50 = 2x^2 + 16x + 32$

**21.** $5 = 3x^2 - 36x + 108$

**22.** $7 = x^2 + 4x + 4$

**23.** $-4 = x^2 + 14x + 49$

**Rewrite the equations in the form $(x - p)^2 = q$.** SEE EXAMPLE 2

**24.** $0 = x^2 - 18x + 64$

**25.** $x^2 + 22x + 120.5 = 0$

**26.** $x^2 + 3x - \frac{27}{4} = 0$

**27.** $0 = 4x^2 + 4x - 14$

**28.** $0 = x^2 - \frac{3}{2}x - \frac{70}{8}$

**29.** $x^2 + 0.6x - 19.1 = 0$

**Solve the following quadratic equations by completing the square.** SEE EXAMPLES 3 AND 4

**30.** $x^2 + 8x + 60 = 0$

**31.** $x^2 + 14x = 51$

**32** $4x^2 + 16x - 65 = 0$

**33.** $7x^2 + 56x - 22 = 0$

**34.** $3x^2 - 6x + 13 = 0$

**35.** $x^2 - 0.4x - 1.2 = 0$

**36.** $x^2 + 6x = 59$

**37.** $8x^2 + 16x = 42$

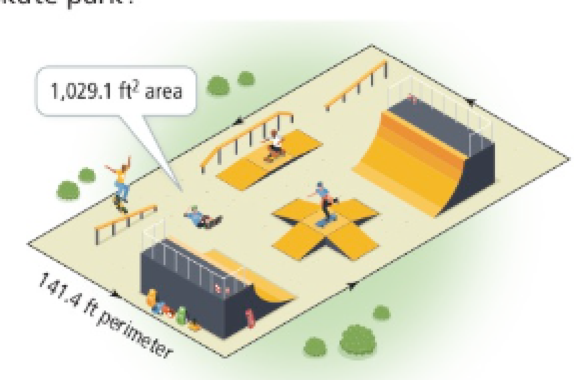
**38.** $5x^2 - 25 = 10x$

**39.** $-2x^2 - 12x + 18 = 0$

**40.** $-3x^2 - 24x - 19 = 0$

**41.** $17 - x^2 - 18x = 0$

**42.** What is the length and width of the skate park?




**Write the equation in vertex form. Identify the maximum or minimum value of the graph of the equation.** SEE EXAMPLE 5

**43.** $y = x^2 + 4x - 13$

**44.** $y = x^2 - 14x + 71$

**45.** $y = -2x^2 - 20x - 58$

**46.** $y = -3x^2 + 36x - 93$

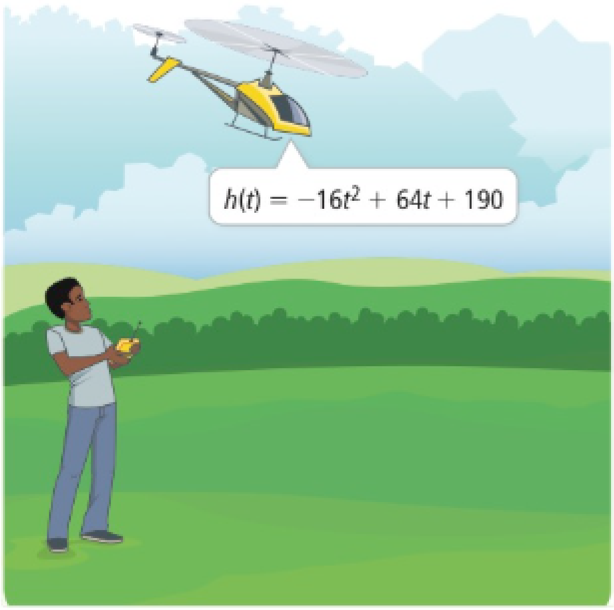
**47.** $y = 6x^2 - 42x + 74.5$

**48.** $y = 0.5x^2 + 0.5x + 2.125$

# PRACTICE & PROBLEM SOLVING

## APPLY

**49. Analyze and Persevere** Keenan launches a model helicopter. The height of the helicopter, in feet, is given by the equation $h = -16t^2 + 64t + 190$, where $t$ is the time in seconds. To the nearest hundredth, how many seconds will it take the helicopter to hit the ground? What is the maximum height of the helicopter?




**50. Represent and Connect** The decreasing population, $p$, of herons in a national park is being monitored by ecologists and is modeled by the equation $p = -0.4t^2 + 128t + 1{,}200$, where $t$ is the number of months since the ecologists started observing the herons.

**a.** If this model is accurate, when will the population reach its maximum?

**b.** What is the maximum population? Round to the nearest whole number.

**c.** Use the equation to determine in how many months the population of herons will disappear.

**51. Analyze and Persevere** Between 2000 and 2005, the number of skateboarders $s$ in the United States, in millions, can be approximated by the equation $s = 0.33t^2 + 2.27t + 3.96$, where $t$ represents the number of years since 2000. If this model is accurate, in what year did 9.8 million people skateboard?

## ASSESSMENT PRACTICE

**52.** Select all roots of the equation $3x^2 - 6x + 1 = 0$. AR.3.2

☐ **A.** $1 + \frac{\sqrt{6}}{3}$

☐ **B.** $-1 + \frac{\sqrt{6}}{3}$

☐ **C.** $1 - i\frac{\sqrt{3}}{3}$

☐ **D.** $-1 + i\frac{\sqrt{3}}{3}$

☐ **E.** $1 - \frac{\sqrt{6}}{3}$

☐ **F.** $1 + i\frac{\sqrt{3}}{3}$

☐ **G.** $-1 - i\frac{\sqrt{3}}{3}$

**53. SAT/ACT** Solve $x^2 + 2x - 5 = 0$.

Ⓐ $-5, 1$

Ⓑ $-1 \pm \sqrt{5}$

Ⓒ $-1 \pm \sqrt{6}$

Ⓓ $1 \pm \sqrt{5}$

Ⓔ $-3, 1$

**54. Performance Task** Yumiko has a rectangular-shaped patio. She wants to double the area of the patio by increasing the length and width by the same amount.


**Part A** Write a function to calculate the number of feet Yumiko would need to add to the length and width. Explain your reasoning.

**Part B** To the nearest hundredth, what are the new dimensions of the patio?

# 2-6 The Quadratic Formula

**I CAN…** solve quadratic equations using the Quadratic Formula.

**VOCABULARY**

- discriminant
- Quadratic Formula

MA.912.AR.3.2–Given a mathematical or real-world context, write and solve one-variable quadratic equations over the real and complex number systems.
Also AR.3.8
MA.K12.MTR.3.1, MTR.4.1

## EXPLORE & REASON

**You can complete the square to solve the general quadratic equation, $ax^2 + bx + c = 0$.**

**A. Communicate and Justify** Justify each step in this general solution.

**B.** What must be true of the value of $b^2 - 4ac$ if the equation $ax^2 + bx + c = 0$ has two non-real solutions? If it has just one solution?

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a}$$
$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## ESSENTIAL QUESTION

**How can you use the Quadratic Formula to solve quadratic equations or to predict the nature of their solutions?**

## EXAMPLE 1 Solve Quadratic Equations

**What are the solutions to the equation?**

**A.** $3x^2 - 4x - 9 = 0$

The **Quadratic Formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provides the solutions of the quadratic equation $ax^2 + bx + c = 0$, for $a \neq 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Write the Quadratic Formula.

$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-9)}}{2(3)}$ Substitute 3 for $a$, $-4$ for $b$, and $-9$ for $c$.

$= \frac{4 \pm \sqrt{124}}{6}$ Simplify.

$= \frac{4 \pm 2\sqrt{31}}{6}$ Simplify under the radical if possible.

$= \frac{2 \pm \sqrt{31}}{3}$ Simplify the fraction if possible.

The solutions are

$x = \frac{2 + \sqrt{31}}{3}$ and $x = \frac{2 - \sqrt{31}}{3}$.

CONTINUED ON THE NEXT PAGE

**CHOOSE EFFICIENT METHODS**
The Quadratic Formula is a useful method for finding solutions, particularly when an equation cannot be easily factored.

EXAMPLE 1 CONTINUED

**B.** How can you use the Quadratic Formula to solve $x^2 - 9x + 27 = 0$?

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Write the Quadratic Formula.

$= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(27)}}{2(1)}$ Substitute 1 for $a$, $-9$ for $b$, and 27 for $c$.

$= \frac{9 \pm \sqrt{-27}}{2}$ Simplify.

$= \frac{9 \pm i\sqrt{27}}{2}$ $\sqrt{-1} = i$

$= \frac{9 \pm 3i\sqrt{3}}{2}$ Simplify.

The solutions are $x = \frac{9 + 3i\sqrt{3}}{2}$ and $x = \frac{9 - 3i\sqrt{3}}{2}$.

**GENERALIZE**

Look for relationships between the coefficients of a quadratic equation and its solutions. If $a = 1$, then the sum of the solutions is the opposite of the $x$-coefficient, $b$, and their product is the constant coefficient, $c$.

**Try It!** **1.** Solve using the Quadratic Formula.

**a.** $2x^2 + 6x + 3 = 0$ **b.** $3x^2 - 2x + 7 = 0$

## EXAMPLE 2 Choose a Solution Method

**Solve the equation $6x^2 - 7x - 20 = 0$ using two different methods. Which do you prefer and why?**

**STUDY TIP**

When you substitute a negative number into a formula, such as the Quadratic Formula, use parentheses to help keep track of the effect of the sign.

**Method 1** OR

Using the Quadratic Formula:

Let $a = 6$, $b = -7$, and $c = -20$

$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-20)}}{2(6)}$

$= \frac{7 \pm \sqrt{49 + 480}}{12}$

$= \frac{7 \pm \sqrt{529}}{12}$

$= \frac{7 \pm 23}{12}$

$x = \frac{7 + 23}{12} = \frac{30}{12} = \frac{5}{2}$, and

$x = \frac{7 - 23}{12} = -\frac{16}{12} = -\frac{4}{3}$

**Method 2**

Factoring by Grouping:

$6x^2 - 7x - 20 = 0$

$6x^2 - 15x + 8x - 20 = 0$

$3x(2x - 5) + 4(2x - 5) = 0$

$(3x + 4)(2x - 5) = 0$

$x = -\frac{4}{3}$ and $x = \frac{5}{2}$

You may also find the factorization through trial and error.

Both solution methods give the same result. Factoring may be more efficient, but the Quadratic Formula *always works*, regardless of whether the function has real or imaginary roots.

**Try It!** **2.** Solve the equation $6x^2 + x - 15 = 0$ using the Quadratic Formula and another method.

CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Identify the Number of Real-Number Solutions

**How can you determine the number and type of roots or solutions for a quadratic equation?**

The radicand in the quadratic formula is what determines the nature of the roots.

The **discriminant** of a quadratic equation in the form $ax^2 + bx + c = 0$ is the value of the radicand, $b^2 - 4ac$.

If $b^2 - 4ac > 0$, then $ax^2 + bx + c = 0$ has two real roots.

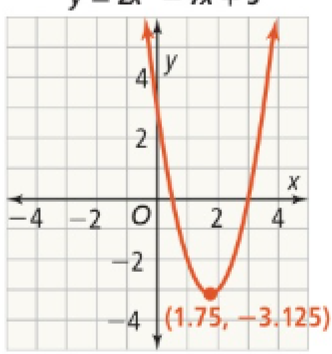
If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has one real root.

If $b^2 - 4ac < 0$, then $ax^2 + bx + c = 0$ has two non-real roots.

Graph each equation. Then use the quadratic formula to find the roots.

**Two Real Roots**

$y = 2x^2 - 7x + 3$




$$2x^2 - 7x + 3 = 0$$

$$x = \frac{7 \pm \sqrt{(-7)^2 - 4(2)(3)}}{2(2)}$$

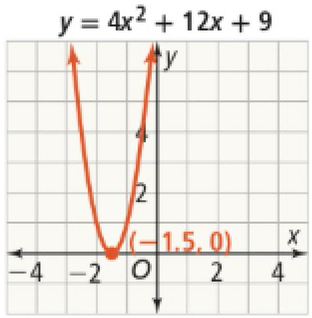
$$x = \frac{7 \pm \sqrt{25}}{4}$$

$$x = \frac{7 \pm 5}{4}$$

The graph shows two distinct $x$-intercepts and the quadratic formula shows two distinct real-number roots.

**One Real Root**

$y = 4x^2 + 12x + 9$




$$4x^2 + 12x + 9 = 0$$

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(4)(9)}}{2(4)}$$

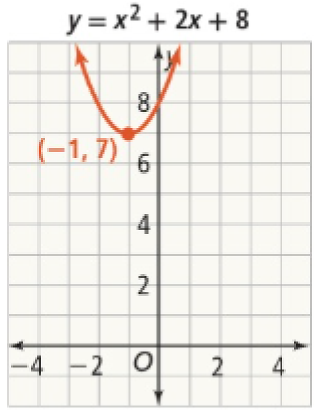
$$x = \frac{-12 \pm \sqrt{0}}{8}$$

$$x = \frac{-12}{8}$$

The graph shows one distinct $x$-intercept and the quadratic formula shows one real-number root.

**Two Non-Real Roots**

$y = x^2 + 2x + 8$




$$x^2 + 2x + 8 = 0$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(8)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{-28}}{2}$$

$$x = -1 \pm i\sqrt{7}$$

The graph shows no $x$-intercepts and the quadratic formula shows two distinct complex-number roots.

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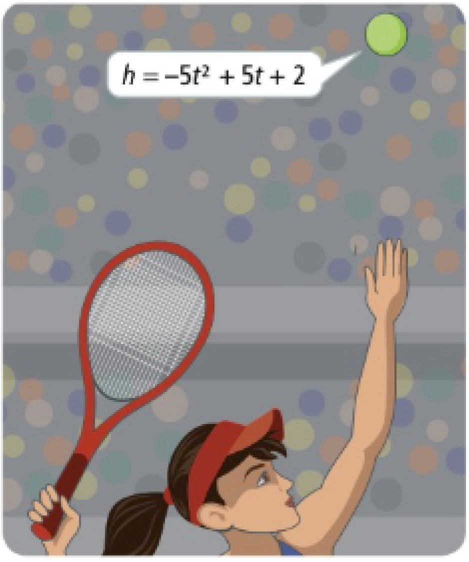
**Try It!** **3.** Describe the nature of the solutions for each equation.

**a.** $16x^2 + 8x + 1 = 0$ **b.** $2x^2 - 5x + 6 = 0$

APPLICATION

## EXAMPLE 4 Interpret the Discriminant

**Linh tosses a ball straight up into the air to serve. The height, *h*, in meters at time *t* in seconds is given by $h(t) = -5t^2 + 5t + 2$. Will the ball reach a height of 4 meters?**




To see if $h = 4$ for some value of $t$, set the equation for $h$ equal to 4, and solve.

$$-5t^2 + 5t + 2 = 4$$

Rewrite the equation in standard form:

$$-5t^2 + 5t - 2 = 0$$

$$a = -5,\ b = 5,\ c = -2$$

The discriminant is: $(5)^2 - 4(-5)(-2)$
$= 25 - 40$

$$25 - 40 = -15$$

$$-15 < 0$$

So the equation $h = 4$ does not have a real solution. Therefore, the ball does not reach 4 m.

**Try It!** **4.** According to the model of Linh's serve, will the ball reach a height of 3 meters?

## EXAMPLE 5 Use the Discriminant to Find a Particular Equation

**What value(s) of *b* will cause $2x^2 + bx + 18 = 0$ to have one real solution?**

For this equation, $a = 2$ and $c = 18$. The equation will have a single rational solution when the discriminant is equal to 0.

$$b^2 - 4ac = 0$$

$$b^2 - 4(2)(18) = 0$$

$$b^2 - 144 = 0$$

$$b^2 = 144$$

$$b = \pm 12$$

There are two possible equations: $2x^2 + 12x + 18 = 0$ and $2x^2 - 12x + 18 = 0$.

**STUDY TIP**
Note that the equation $2x^2 + bx + 18 = 0$ will have two real solutions if $b > 12$ or $b < -12$. It will have two non-real solutions if $-12 < b < 12$.

**Try It!** **5.** Determine the value(s) of $b$ that ensure $5x^2 + bx + 5 = 0$ has two non-real solutions.

## CONCEPT SUMMARY Key Features of the Quadratic Formula

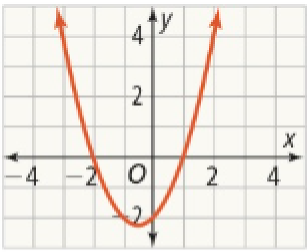
### QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is used to solve any quadratic equation: $ax^2 + bx + c = 0$, where $a \neq 0$.

### USING THE DISCRIMINANT

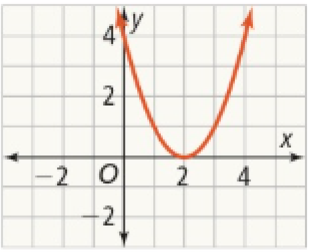
Predict the number and type of solutions using the discriminant, $b^2 - 4ac$.




$x^2 + x - 2 = 0$

$b^2 - 4ac > 0$

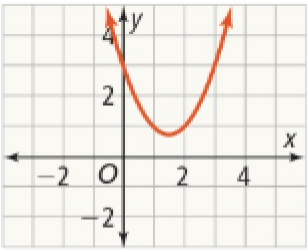
Two real solutions




$x^2 - 4x + 4 = 0$

$b^2 - 4ac = 0$

One real solution




$x^2 - 3x + 3 = 0$

$b^2 - 4ac < 0$

Two non-real solutions

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you use the Quadratic Formula to solve quadratic equations or to predict the nature of their solutions?

2. **Vocabulary** Why is the discriminant a useful tool to use when solving quadratic equations?

3. **Error Analysis** Rick claims that the equation $x^2 + 5x + 9 = 0$ has no solution. Jenny claims that there are two solutions. Explain how Rick could be correct, and explain how Jenny could be correct.

4. **Choose Efficient Methods** What methods can you use to solve quadratic equations?

## Do You KNOW HOW?

5. Describe the number and type of solutions of the equation $2x^2 + 7x + 11 = 0$.

6. Use the Quadratic Formula to solve the equation $x^2 + 6x - 10 = 0$.

7. At time $t$ seconds, the height, $h$, of a ball thrown vertically upward is modeled by the equation $h = -5t^2 + 33t + 4$. About how long will it take for the ball to hit the ground?

8. Use the Quadratic Formula to solve the equation $x^2 - 8x + 16 = 0$. Is this the only way to solve this equation? Explain.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

9. **Generalize** How can you use the Quadratic Formula to factor a quadratic equation?

10. **Error Analysis** Describe and correct the error a student made in solving an equation.

$$x^2 - 5x + 5 = 0$$
$$a = 1, b = -5, c = 5$$
$$x = \frac{-5 \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)}$$
$$= \frac{-5 \pm \sqrt{25 - 20}}{2}$$
$$= \frac{-5}{2} \pm \frac{\sqrt{5}}{2} \quad ✗$$

11. **Mathematical Connections** What does the Quadratic Formula tell you about the graph of a quadratic function?

12. **Communicate and Justify** Explain your process for choosing a method for solving quadratic equations.

13. **Higher Order Thinking** Kelsey wants to use the Quadratic Formula to solve the equation $x^4 + 5x^2 - 5 = 0$. Is this possible? If so, describe the steps she should follow.

14. **Use Patterns and Structure** Explain why the graph of the quadratic function $f(x) = x^2 + x + 5$ crosses the $y$-axis but does not cross the $x$-axis.

15. **Communicate and Justify** Sage said that the Quadratic Formula does not always work. Sage used it to solve the equation $x^2 - 3x - 2 = -4$, with $a = 1$, $b = -3$, and $c = -2$. The formula gave $x = \frac{3 \pm \sqrt{17}}{2}$ as the solutions to the equation. When Sage checked, neither one of them satisfied the equation. How could you convince Sage that the Quadratic Formula does always work?

### PRACTICE

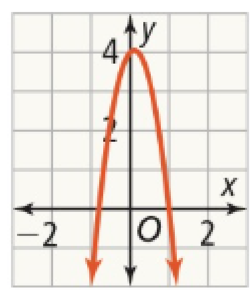
**Use the Quadratic Formula to solve each equation.** SEE EXAMPLE 1

16. $x^2 - 10x + 25 = 0$
17. $x^2 + 2x + 2 = 0$
18. $5x^2 - 8x + 4 = 0$
19. $x^2 + 9x - 1 = 3x - 10$
20. $3x^2 - 20x - 7 = 0$
21. $-x^2 + 3x - 8 = 0$

**Use the discriminant to identify the number and type of solutions for each equation.** SEE EXAMPLE 3

22. $25x^2 - 20x + 4 = 0$
23. $x^2 + 7x + 11 = 0$
24. $3x^2 - 8x - 10 = 0$
25. $2x^2 + 9x + 14 = 0$

**Deon throws a ball into the air. The height, $h$, of the ball, in meters, at time $t$ seconds is modeled by the function $h(t) = -5t^2 + t + 4$.** SEE EXAMPLE 4




26. When will the ball hit the ground?

27. Will the ball reach a height of 5 meters?

**Use any method to solve the equation.** SEE EXAMPLE 2

28. $4x^2 + 7x - 11 = 0$
29. $x^2 + 4x + 4 = 100$
30. $3x^2 + x + 7 = x^2 + 10$
31. $6x^2 + 2x + 3 = 0$

**Find the value(s) of $k$ that will cause the equation to have the given number and type of solutions.** SEE EXAMPLE 5

32. $5x^2 + kx + 5 = 0$, 1 real solution

33. $3x^2 + 12x + k = 0$, 2 real solutions

34. $kx^2 - 3x + 4 = 0$, 2 real solutions

## PRACTICE & PROBLEM SOLVING

### APPLY

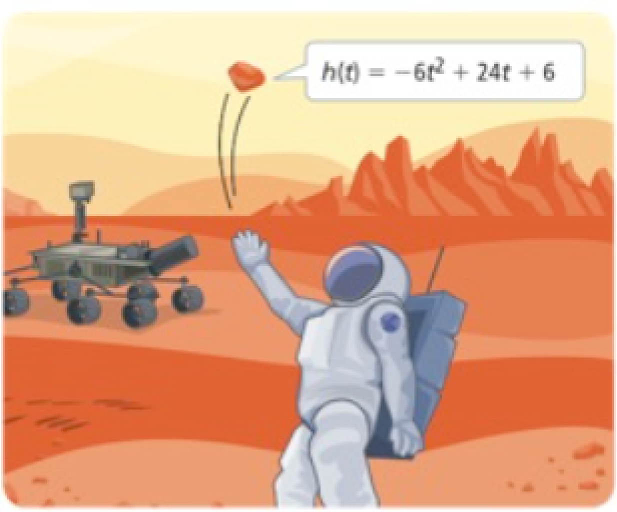
35. **Represent and Connect** The table shows the average cost of tuition and fees at a public four-year college for an in-state student in recent years.

| Academic Year | Tuition and Fees |
|---|---|
| 2012–13 | \$9,006 |
| 2013–14 | \$9,077 |
| 2014–15 | \$9,161 |
| 2015–16 | \$9,410 |

a. Write a quadratic equation that can be used to find the average cost, $C$, of tuition after $x$ years.

b. Use the model to predict when tuition will exceed \$10,000.

36. **Analyze and Persevere** The first astronaut on Mars tosses a rock straight up. The height, $h$, measured in feet after $t$ seconds, is given by the function $h(t) = -6t^2 + 24t + 6$.




a. After how many seconds will the rock be 30 feet above the surface?

b. After how many seconds will the rock be 10 feet above the surface?

c. How many seconds will it take for the rock to return to the surface?

d. The same action on Earth is modeled by the equation $g(t) = -16t^2 + 24t + 6$. On Earth, how many seconds would it take for the rock to hit the ground?

### ASSESSMENT PRACTICE

37. Select the equations that have two real solutions. AR.3.2

☐ A. $x^2 - 8x - 2 = 0$

☐ B. $2x^2 + 10x + 17 = 0$

☐ C. $4x^2 - 28x + 49 = 0$

☐ D. $x^2 + 10x - 25 = 4x + 2$

☐ E. $2x^2 + x + 10 = 5 - 4x - x^2$

38. **SAT/ACT** Which expression can be simplified to find the solution(s) of the equation $2x^2 - x - 15 = 0$?

Ⓐ $-1 \pm \frac{\sqrt{1 - 4(2)(-15)}}{2(2)}$

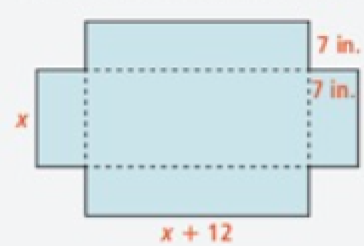
Ⓑ $\frac{1 \pm \sqrt{1 - 4(2)(-15)}}{2(2)}$

Ⓒ $\frac{1 \pm \sqrt{-1 - 4(2)(-15)}}{2(2)}$

Ⓓ $\frac{1 \pm \sqrt{1 - 4(2)(15)}}{2(2)}$

Ⓔ $\frac{1 \pm \sqrt{1 + 4(2)(-15)}}{2(2)}$

39. **Performance Task** Four congruent squares are cut from a rectangular piece of cardboard.




**Part A.** If the resulting flaps are folded up and taped together to make a box, write a function to represent the volume of the box in terms of the width of the original piece of cardboard.

**Part B.** What are the dimensions of the original cardboard, to the nearest tenth, if the volume of the box is 434 in.$^3$?

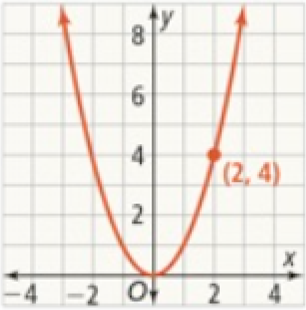
# 2-7 Quadratic Inequalities

**I CAN…** write and solve quadratic inequalities.

MA.912.AR.3.9–Given a mathematical or real-world context, write two-variable quadratic inequalities to represent relationships between quantities from a graph or a written description.
Also AR.3.10
MTR.2.1, MTR.6.1, MTR.7.1

## EXPLORE & REASON

**Given $y = x^2$, and the point (2, 4) on the parabola.**




A. Pick two points "above" and two points "below" the parabola that have an $x$-coordinate of 2. How do their $y$-coordinates compare with the $y$-value of (2, 4)?

B. The point $(a, a^2)$ is any point on the graph of $y = x^2$. Describe the location of the point $(a, b)$ in relation to the parabola if $b > a^2$. What if $b < a^2$?

C. **Generalize** How could you write an inequality to describe all points below the graph of $y = x^2$? How could you write an inequality to describe all points above the graph of $y = x^2$?

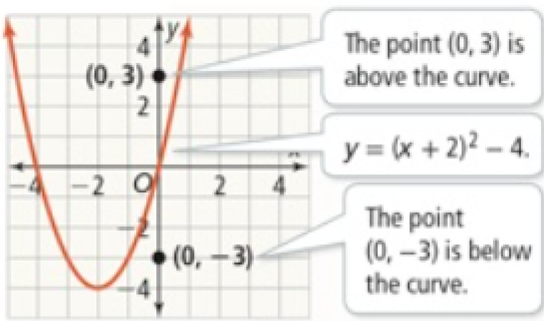
**ESSENTIAL QUESTION** **Why is the solution set for a two-variable inequality represented on a coordinate plane?**

CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Represent the Solution to a 2-Variable Quadratic Inequality

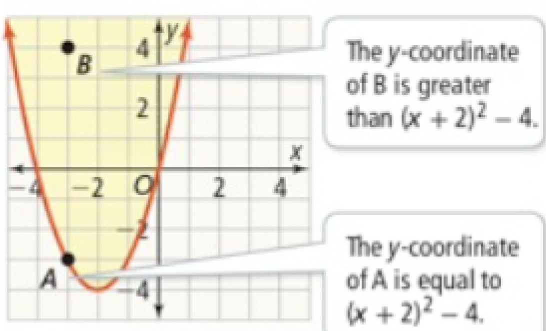
**Represent the solution set of the inequality $y \geq (x + 2)^2 - 4$.**

The solution is the set of pairs $\{(x, y) \mid y \geq (x + 2)^2 - 4\}$. The pair (0, 3) is a solution, but (0, −3) is not.




The parabola divides the coordinate plane into two regions: one above the parabola, and one below the parabola.

The points on the parabola, and the shaded region above the parabola, represent the solution to $y \geq (x + 2)^2 - 4$.




**USE PATTERNS AND STRUCTURE**
For strict inequalities, < or >, use a dashed boundary to indicate that the points on the parabola are not part of the solution. For ≤ or ≥, use a solid boundary to indicate that the points are part of the solution.

**Try It!** 1. Graph the solution for $y \geq 25(x + 6)(x - 2)$.

## EXAMPLE 2 Find and Graph the Solution Set

**Solve the inequality $y < (5x^2 + 24) - (2x^2 - 10x + 21)$. Graph the solution.**

**Step 1** Simplify the inequality.

$$y < (5x^2 + 24) - (2x^2 - 10x + 21)$$

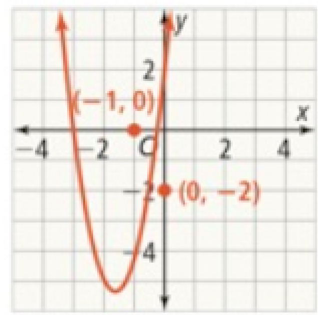
$$y < 5x^2 + 24 - 2x^2 + 10x - 21$$

$$y < 3x^2 + 10x + 3$$

**CHOOSE EFFICIENT METHODS**

Choose representative points for each region whose values are easy to calculate, such as the origin, (0, 0). But be careful not to choose a point on the boundary.

**Step 2** Graph $y = 3x^2 + 10x + 3$.



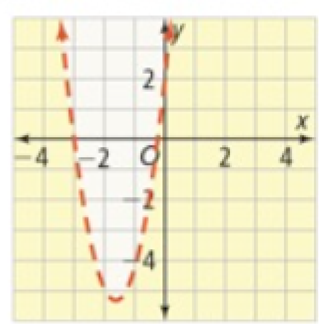

**Step 3** Pick representative points from each region, (0, −2) and (−1, 0), and determine if each pair makes the inequality true.

(0, −2) $3 \cdot 0^2 + 10 \cdot 0 + 3 > -2$ True

(−1, 0) $3 \cdot (-1)^2 + 10 \cdot (-1) + 3 > 0$ False

Shade the region containing (0, −2) to represent the solution set to the inequality.

**Step 4** Determine the boundary line. Any pair that makes $y = 3x^2 + 10x + 3$ true will make the inequality false. Use a dashed line to show these points are not included in the solution set.




**Try It!** 2. Graph the solution to each inequality.

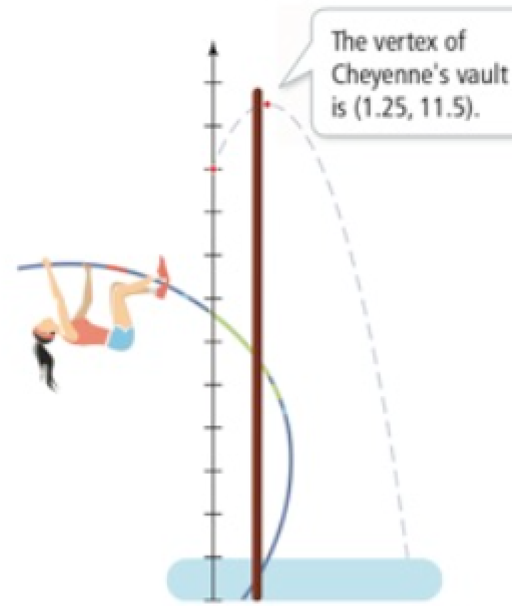
a. $y - 16 \geq -\frac{1}{3}x^2 + 4x + 1$

b. $(x + 11)^2 + 4x^2 - 6x \leq y + 11$

APPLICATION

## EXAMPLE 3 Write and Solve Quadractic Inequalities

**In a pole vault, a successful vault requires the crossbar to remain in place. From her release point, the trajectory of Cheyenne's vault can be modeled by a quadratic function. If the crossbar is 1 foot from the release point, at the height of 11 feet, does Cheyenne clear the crossbar on her vault?**



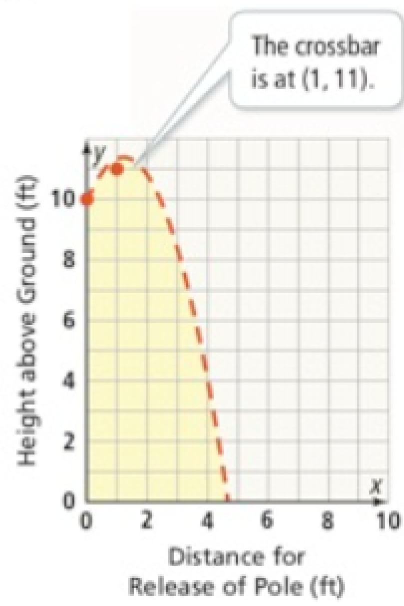

In order to vault over the crossbar, the crossbar must be below Cheyenne's trajectory. You can use a quadratic inequality to solve the problem.

Formulate ◀ First, find the function that describes Cheyenne's vault. The vertex is (1.25, 11.5), so write the function in vertex form: $y = a(x - 1.25)^2 + 11.5$.

Compute ◀ Use the point (0, 10) to find $a$:

$$y = a(x - 1.25)^2 + 11.5 \quad \text{Vertex form}$$
$$10 = a(0 - 1.25)^2 + 11.5 \quad \text{Substitute (0, 10)}$$
$$-1.5 = 1.5625a \quad \text{Solve for } a$$
$$a = -0.96$$

The function $y = -0.96(x - 1.25)^2 + 11.5$ models the trajectory of Cheyenne's vault. It is a successful vault if the position of the crossbar satisfies the inequality $y < -0.96(x - 1.25)^2 + 11.5$.



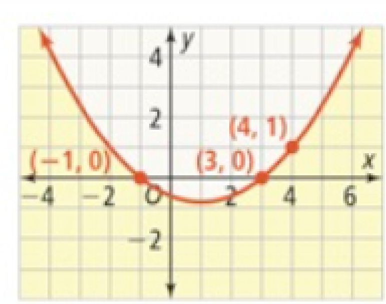

Interpret ◀ Substitute the coordinates (1, 11) in the inequality to determine if Cheyenne vaults over the crossbar.

$$11 < -0.96(1 - 1.25)^2 + 11.5$$
$$11 < 11.4$$

The inequality is true. Therefore, Cheyenne's vault should be successful.

**Try It!** 3. a. Write the quadratic inequality in factored form represented by the graph.

b. List two points that are part of the solution set and two points that are not part of the solution set.

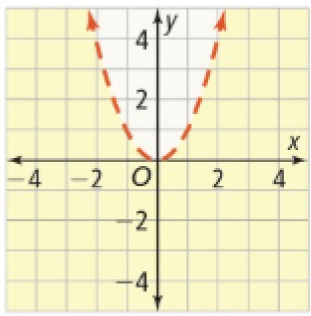


## CONCEPT SUMMARY Quadratic Inequalities

**INEQUALITY** $y < x^2$

**SOLUTION SET** $\{(x, y) | y < x^2\}$

**GRAPH**




1. $y < x^2$ is a strict inequality, so draw a dashed line.
2. (1, 0) is one solution. The region containing (1, 0) represents all of the solutions.
3. (0, 1) is not a solution. The region containing (0, 1) does not represent the solution.

## Do You UNDERSTAND?

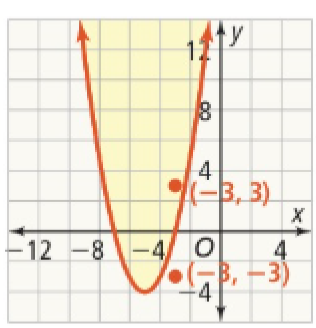
1. **ESSENTIAL QUESTION** Why is the solution set for a two-variable inequality represented on a coordinate plane?

2. **Vocabulary** Is the boundary on a coordinate plane always a line? Explain.

3. **Error Analysis** Taniqua claims that every quadratic boundary is included in the solution set for two-variable inequalities. Explain her error.

4. **Check for Reasonableness** Do all quadratic inequalities solutions require a coordinate plane to graph the solution?

5. **Represent and Connect** Isabel graphed $y > (x + 3)(x + 7)$, but she is unsure of her work. She used test points $(-3, 3)$ and $(-3, -3)$ to determine what area to shade. Explain to her what she did correct and how to correct her error.




## Do You KNOW HOW?

**Graph the solution set for each inequality.**

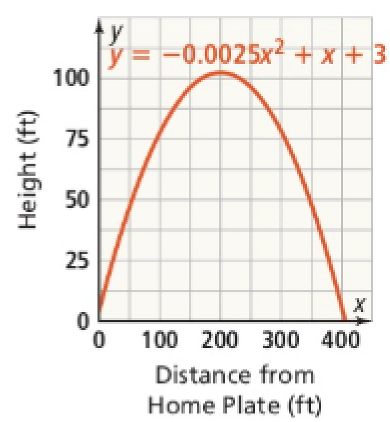
6. $y < -(x - 5)^2 + 3$

7. $y > \frac{1}{2}(x + 2)(x + 7)$

8. $x^2 + x + 1 \leq (2x^2 - 4x) - y$

9. $(x + 5)(x - 5) \geq y + 2x$

10. Johnson hit a baseball that follows the parabolic path shown. The point (400, 8) represents the top of the outfield wall on this path.




a. Write a quadratic inequality whose solutions represent the locations of the top of any wall that his ball would clear.

b. Describe the graph of the solution to the inequality in Part a.

c. Will Johnson's hit be a home run? Explain.

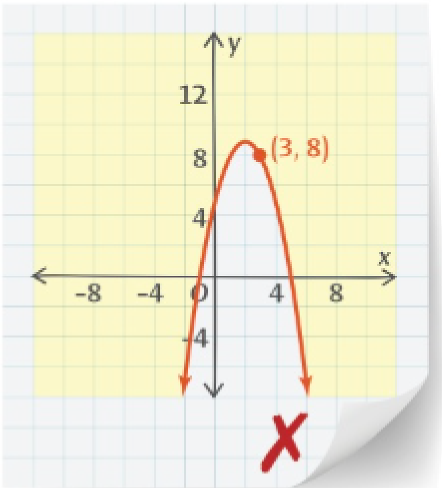
## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

11. **Generalize** Which inequality signs are needed when the parabola is part of the solution set? How do you indicate this in a graph?

12. Graph the solution that corresponds to $\{(x, y)|y \geq -\frac{1}{3}(x - 3)(x + 5)\}$.

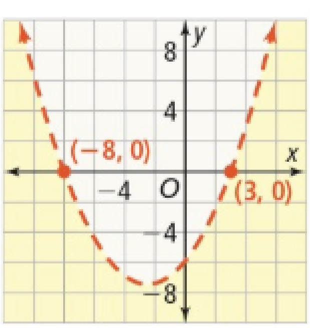
13. **Error Analysis** Li graphs the inequality $y > x^2 + 4x + 5$ and determines that (3, 8) is a solution to the inequality. Explain and correct Li's error.




14. **Higher Order Thinking** Explain why the solution set of a quadratic inequality in two-variables is a set of ordered pairs rather than a set of numbers.

15. **Check for Reasonableness** Your friend graphed the inequality $(x + 3)(x + 4) - 2x^2 - 3x \geq 3y + 5x - 5$. She is confident that she graphed the parabola correctly, but she wants you to check whether she shaded the correct part of the graph. How could you check her work efficiently?

16. **Represent and Connect** Choose the inequality represented by the graph.




Ⓐ $\frac{1}{4}x^2 + \frac{5}{4}x + 6 > y$ Ⓑ $\frac{1}{4}x^2 + \frac{5}{4}x - 6 < y$

Ⓒ $\frac{1}{4}x^2 + \frac{5}{4}x - 6 > y$ Ⓓ $\frac{1}{4}x^2 - \frac{5}{4}x - 6 < y$

### PRACTICE

**Sketch the solution set of each quadratic inequality.** SEE EXAMPLE 1

17. $y \geq 2(x - 3)^2 - 8$

18. $y < -\frac{1}{9}(x + 5)^2 + 4$

19. $y \leq \frac{1}{2}(x - 1)^2 + 1$

20. $y \geq -x^2 + 5$

21. $y < 2(x - 2)(x + 2)$

22. $y \leq -\frac{1}{3}(x - 5)(x + 1)$

23. $y > \frac{1}{12}(x + 7)(x - 5)$

24. $y > -\frac{5}{3}(x + 8)(x + 2)$

**Graph the solution set of each quadratic inequality.** SEE EXAMPLE 2

25. $y \leq \frac{1}{9}x^2 - \frac{1}{2}x + 6$

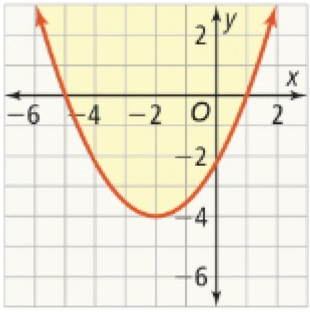
26. $y > 0.25x^2 - x - 4$

27. $y + 3x^2 - 4 > 2x^2 + 3x - 6$

28. $y \leq -(x - 5)^2 + 0.1(x + 2)^2$

29. Write a quadratic inequality to represent the area of a stained-glass mosaic that sits under a parabolic arch 10 feet tall and 8 feet wide at the base, if the left side of the base is the origin. SEE EXAMPLE 3

30. Write a quadratic inequality to describe the graph. SEE EXAMPLE 3



## PRACTICE & PROBLEM SOLVING

### APPLY

**31. Represent and Connect** A craftsman is designing a building where the roof is supported by outer poles and a single wooden column in the center. The maximum weight in pounds that the column can support is given by the equation $W = \pi r^2 C$, where $r$ is the radius of the column in inches, and $C$ is a constant representing the compressive strength of the wood. The craftsman plans to use pine wood for the column, which has a compressive strength of about 5,000 pounds per square inch. Write and graph an inequality representing the possible loads that can be placed on the column in terms of its radius.

**32. Apply Math Models** Profit on an item is sales minus cost and both of those are based on the price set, $P$. You want to manufacture a new mega water launcher. From research, you expect that per 1000 units, to make
Sales = $(500 - 1.2P)P$ and have
Costs = $650 + 160(500 - 1.2P)$.

**Part A** Find the equation for the expected profit.

**Part B** Sales fall flat. Graph the solution indicating you did not make the expected profit, but you also did not lose money.

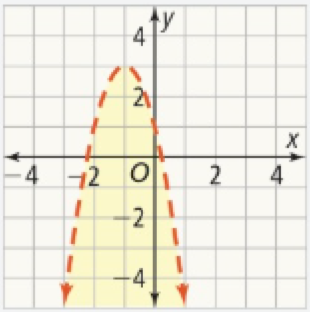
**33. Use Patterns and Structure** A rectangular parking lot must have a perimeter of 560 feet and an area of at least 10,000 square feet. Describe the possible lengths of the parking lot.

### ASSESSMENT PRACTICE

**34.** Choose *Yes* or *No* to tell whether each ordered pair is a solution of the inequality $y > 3x^2 - 2x + 1$. AR.3.9

| | Yes | No |
|---|---|---|
| (1, 5) | ❑ | ❑ |
| (0, 0) | ❑ | ❑ |
| $\left(\frac{1}{3}, \frac{2}{3}\right)$ | ❑ | ❑ |
| (–1, 5) | ❑ | ❑ |

**35. SAT/ACT** What inequality is shown by the graph?




Ⓐ $y < -2(x - 1)^2 - 3$
Ⓑ $y < -2(x + 1)^2 + 3$
Ⓒ $y \le -2(x - 1)^2 - 3$
Ⓓ $y \le -2(x + 1)^2 + 3$

**36. Performance Task** A rancher has 850 feet of fencing with which to make a rectangular pen for her animals.

**Part A** Write a function to represent the area of the pen in terms of the width of the fence, assuming she uses all the fencing. What is a reasonable domain for the function?

**Part B** The rancher decides not to use the entire length of fencing, to leave room for a garden outside the pen. Write a function to represent the possible areas of the pen.

**Part C** Graph the inequality. Write any other restrictions on the possible areas as equations or inequalities.

# 2-8 Systems Involving Quadratic Equations and Inequalities

**I CAN…** solve linear-quadratic systems.

MA.912.AR.9.2–Given a mathematical or real-world context, solve a system consisting of a two-variable linear equation and a non-linear equation algebraically or graphically. Also AR.9.3, AR.9.5, AR.9.7

MA.K12.MTR.1.1, MTR.2.1, MTR.4.1

CONCEPTUAL UNDERSTANDING

## EXPLORE & REASON

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**Draw a rough sketch of a parabola and a line on the coordinate plane.**

**A.** Count the number of points of intersection between the two graphs.

**B.** Sketch another parabola on a coordinate plane. Use a straightedge to investigate the different ways that a line and a parabola intersect. What conjectures can you make?

**C.** Communicate and Justify How many different numbers of intersection points are possible between a quadratic function and a linear function? Justify that you have found all of the possibilities.

## ESSENTIAL QUESTION

**How can you solve a system of two equations or inequalities in which one is linear and one is quadratic?**

## EXAMPLE 1 Determine the Number of Solutions

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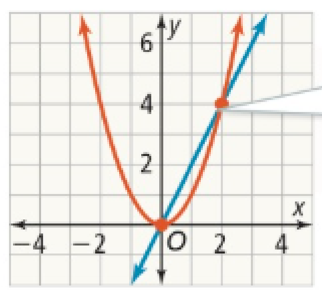
**How many solutions can there be for a linear-quadratic system?**

**A. How many real solutions does the system** $\begin{cases} y = x^2 \\ y = 2x \end{cases}$ **have?**

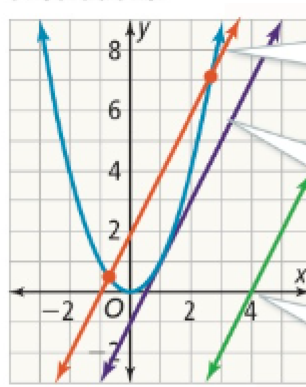



The graph seems to show that the quadratic function and linear function intersect at two points.

This system has two real solutions.

**B. How does modifying the linear function in the system** $\begin{cases} y = x^2 \\ y = 2x + b \end{cases}$ **affect the number of solutions?**

Test values for $b$ to determine when the system has different numbers of solutions.




When $b = 2$, there are two solutions. The graphs intersect at two points.

When $b = -1$, there is one solution. The graph of $y = 2x - 1$ is tangent to the graph of $y = x^2$.

When $b = -8$, there are no solutions. The graph of $y = 2x - 8$ does not intersect the graph of $y = x^2$.

Visually inspecting the graph suggests that there is no way for a line to cross a parabola more than twice. Thus, the system of a linear function and a quadratic function may have 0, 1, or 2 solutions.

**REPRESENT AND CONNECT**
A solution to a system of equations is an ordered pair that produces a true statement in all the equations of the system. In the graph, the solutions are the coordinates of the intersection points.

**Try It!** **1.** Determine the number of real solutions of the system $\begin{cases} y = 3x^2 \\ y = 3x - 2 \end{cases}$.

**USE PATTERNS AND STRUCTURE**
As with a system of linear equations, you can use substitution and elimination to find the values of $x$ and $y$ that make the system true. In this case, substitution yields a new quadratic equation to solve.

## EXAMPLE 2 Solve a Linear-Quadratic System Using Substitution

**How can you use substitution to solve this system?** $\begin{cases} y = 3x^2 + 3x - 5 \\ 2x - y = 3 \end{cases}$

The first equation provides an expression for $y$ in terms of $x$. Substitute this expression in the second equation.

$2x - (3x^2 + 3x - 5) = 3$ ........ Substitute $3x^2 + 3x - 5$ for $y$ in the second equation.

$2x - 3x^2 - 3x + 5 = 3$ ........ Distribute $-1$ to remove parentheses.

$3x^2 + x - 2 = 0$ ........ Simplify.

$(x + 1)(3x - 2) = 0$ ........ Factor.

So $x = -1$ and $x = \frac{2}{3}$ are solutions of this quadratic equation.

If the graphs of the equations have two solutions, there are two points of intersection for the graphs of the equations.

When $x = -1$, $y = 2(-1) - 3$, or $-5$. When $x = \frac{2}{3}$, $y = 2\left(\frac{2}{3}\right) - 3$, or $-\frac{5}{3}$.
The solutions of the system are $(-1, -5)$ and $\left(\frac{2}{3}, -\frac{5}{3}\right)$.

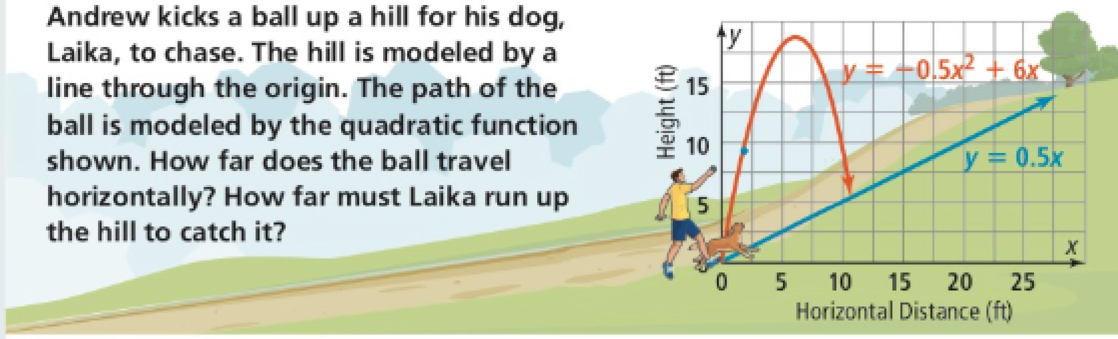
**Try It!** **2.** Solve each system by substitution.

**a.** $\begin{cases} y = 2x^2 - 6x - 8 \\ 2x - y = 16 \end{cases}$ **b.** $\begin{cases} y = -3x^2 + x + 4 \\ 4x - y = 2 \end{cases}$

**APPLICATION**

## EXAMPLE 3 Applying a Linear-Quadratic System

**Andrew kicks a ball up a hill for his dog, Laika, to chase. The hill is modeled by a line through the origin. The path of the ball is modeled by the quadratic function shown. How far does the ball travel horizontally? How far must Laika run up the hill to catch it?**




Create a system of equations and determine where the path of the ball intersects the hill.

$\begin{cases} y = -0.5x^2 + 6x \\ y = 0.5x \end{cases}$

$0.5x = -0.5x^2 + 6x$ ........ Substitute for $y$.

$0 = -0.5x^2 + 5.5x$ ........ Subtract $0.5x$ from both sides.

$0 = -0.5x(x - 11)$ ........ Factor.

$-0.5x = 0$ and $x - 11 = 0$ ........ Set each factor equal to 0 and solve.

$x = 0$ and $x = 11$

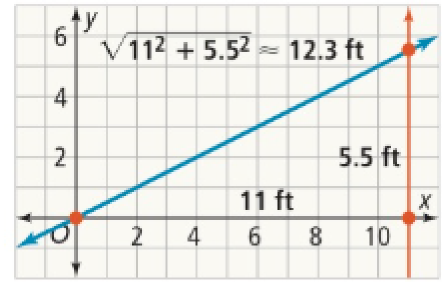
CONTINUED ON THE NEXT PAGE

**COMMON ERROR**

Make sure to answer every part of the question. After solving this equation, you still need to find the distance that Laika ran.

EXAMPLE 3 CONTINUED

The solution $x = 0$ represents the horizontal distance, in feet, when Andrew kicks the ball. The solution $x = 11$ represents the horizontal distance, in feet, when the ball lands on the hill. So the ball travels 11 ft horizontally and $y = 5.5$ feet vertically found by substitution.




The route Laika runs can be modeled as the hypotenuse of a right triangle.

So Laika runs approximately 12.3 ft to get the ball.

**Try It!** 3. Revenue for the high school band concert is given by the function $y = -30x^2 + 250x$, where $x$ is the ticket price, in dollars. The cost of the concert is given by the function $y = 490 - 30x$. At what ticket price will the band make enough revenue to cover their costs?

## EXAMPLE 4 Solve a Linear-Quadratic System of Inequalities

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**How can you solve this system of inequalities?** $\begin{cases} y < -2x^2 + 12x - 10 \\ 4x + y > 4 \end{cases}$

Graphing an inequality is similar to graphing an equation. You start in the same manner, but later you have to consider whether to sketch the graph as solid or dotted and how to shade the graph.

Graph the quadratic inequality:

Complete the square to write the inequality in vertex form.

$$y + 10 < -2(x^2 - 6x)$$
$$y + 10 - 18 < -2(x^2 - 6x + 9)$$
$$y - 8 < -2(x - 3)^2$$
$$y < -2(x - 3)^2 + 8$$

The parabola has vertex (3, 8).

Graph the linear inequality:

Solve the inequality for $y$ to write in slope-intercept form:

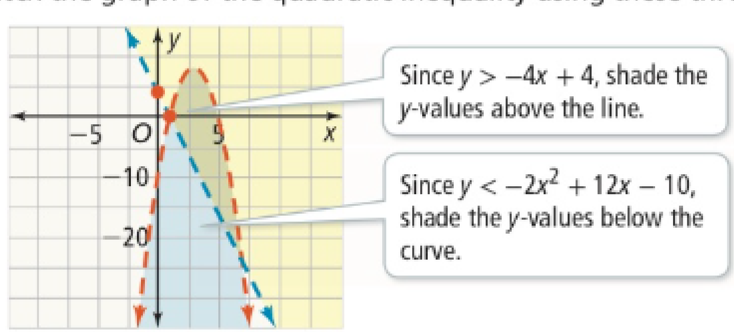
Sketch the graph of the linear inequality using the slope and $y$-intercept.

$$y > -4x + 4$$

Find two symmetric points on either side of the vertex:

$x = 2, y = 6 \rightarrow (2, 6)$ $\quad$ $x = 4, y = 6 \rightarrow (4, 6)$

Sketch the graph of the quadratic inequality using these three points.




The region where the two shaded areas overlap holds the solutions to the system.

CONTINUED ON THE NEXT PAGE

**Try It!** **4.** Solve the system of inequalities using shading. $\begin{cases} y > x^2 + 6x - 12 \\ 3x - y \geq -8 \end{cases}$

## EXAMPLE 5 Solve a Quadratic-Quadratic System

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**How can you solve this system of equations?** $\begin{cases} y = \frac{1}{4}(x+2)^2 - 2 \\ y = -2x^2 + 4x + 2 \end{cases}$

The coefficients are rational, so you can solve the system algebraically.

$\frac{1}{4}(x+2)^2 - 2 = -2x^2 + 4x + 2$ ······ Substitute $y$.

$\frac{1}{4}x^2 + x - 1 = -2x^2 + 4x + 2$ ······ Simplify.

$\frac{9}{4}x^2 - 3x - 3 = 0$ ······ Addition Property of Equality

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\left(\frac{9}{4}\right)(-3)}}{2\left(\frac{9}{4}\right)}$ ······ Quadratic Formula

$x = \frac{2 \pm 4}{3}$ ······ Simplify.

$x = 2$ or $x = -\frac{2}{3}$

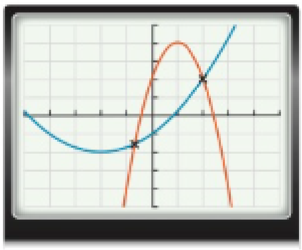
Now substitute each value of $x$ into one of the original equations to find the corresponding values for $y$:

$y = -2(2)^2 + 4(2) + 2$ or $y = -2\left(-\frac{2}{3}\right)^2 + 4\left(-\frac{2}{3}\right) + 2$

$y = 2$ or $y = -\frac{14}{9} \approx -1.56$

The solutions are (2, 2) and approximately (–0.67, –1.56).

**Check:** Graph both equations using a graphing calculator to find the points of intersection:



x scale: 1 y scale: 1

The curves intersect at the points (2, 2) and approximately (–0.67, –1.56).

**STUDY TIP**
When you graph both equations on your calculator, you can use the TRACE or INTERSECTION function to approximate the solution as a check.

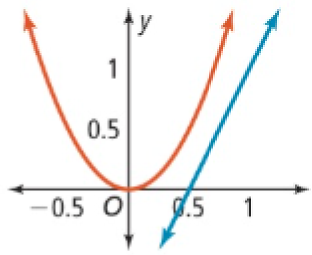
**Try It!** **5.** Solve the system of equations. $\begin{cases} y = 2(x+1)^2 - 3 \\ y = -\frac{3}{2}x^2 - 3x + \frac{3}{4} \end{cases}$

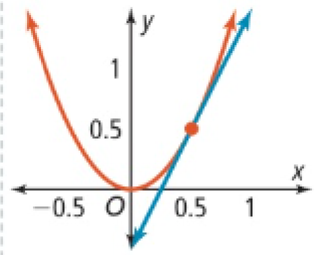
## CONCEPT SUMMARY Key Features of Linear-Quadratic Systems

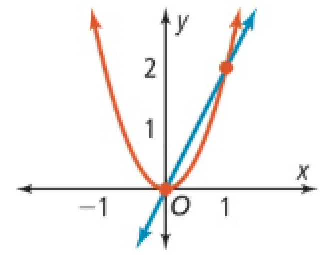
### Linear-Quadratic Systems of Equations

**WORDS** Use substitution or elimination to solve the system.

**GRAPHS**


no solution

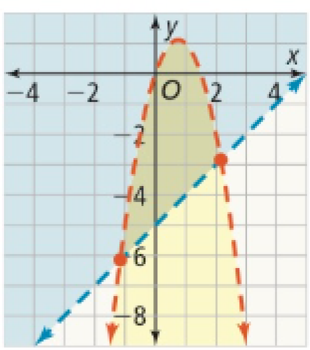

1 solution


2 solutions

### Linear-Quadratic Systems of Inequalities

**WORDS** Graph linear and quadratic inequalities, considering whether the graph is solid or dotted. Use shading to identify the solution region.

**GRAPH**



## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you solve a system of two equations or inequalities in which one is linear and one is quadratic?

2. **Error Analysis** Dyani was asked to use substitution to solve this system:

$$\begin{cases} y = 2x^2 - 6x + 4 \\ x - y = 7 \end{cases}$$

She began as follows, to find the $x$-coordinate(s) to the solution(s) of the system:

| | |
|---|---|
| $x + 2x^2 - 6x + 4 = 7$ | Substitute for $y$. |
| $2x^2 - 5x - 3 = 0$ | Simplify. |
| $(2x + 1)(x - 3) = 0$ | Factor. |
| $x = -\frac{1}{2}, x = 3$ | Set each factor equal to 0, solve for $x$. ✗ |

But Dyani has already made an error. What was her mistake?

## Do You KNOW HOW?

**Determine the number of solutions for the system of equations.**

3. $\begin{cases} y = \frac{2}{5}x^2 \\ y = x - 2 \end{cases}$

4. $\begin{cases} y = -x - 1 \\ 3x^2 + 2y = 4 \end{cases}$

**Use substitution to solve the system of equations.**

5. $\begin{cases} y = 3x^2 + 7x - 10 \\ y - 19x = 22 \end{cases}$

6. $\begin{cases} y = 3x^2 \\ y - 3x = -2 \end{cases}$

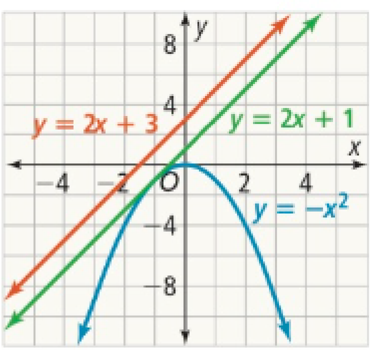
## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

7. **Communicate and Justify** Nora and William are asked to solve the system of equations $\begin{cases} y - 1 = 3x \\ y = 2x^2 - 4x + 9 \end{cases}$ without graphing.

Nora wants to use substitution, inserting $2x^2 - 4x + 9$ in place of $y$ in the upper equation and solving. William wants to rewrite $y - 1 = 3x$ as $y = 3x + 1$ and begin by setting $3x + 1$ equal to $2x^2 - 4x + 9$, and then solving. Which student is correct, and why?

8. **Error Analysis** Chris was given the system of equations $\begin{cases} y = -x^2 \\ y = 2x + b \end{cases}$ and asked to use graphing to test the number of solutions of the system for different values of $b$. He graphed the system as shown, and concluded that the system could have one solution or no solutions depending on the value of $b$. What was Chris's error?




9. **Analyze and Persevere** You are given the following system of equations: $\begin{cases} y = x^2 \\ y = -1 \end{cases}$. Without graphing or performing any substitutions, can you see how many solutions the system must have? Describe your reasoning.

10. **Communicate and Justify** Can a system of equations with one linear and one quadratic equation have more than two solutions? Give at least two arguments for your answer.

### PRACTICE

**Determine how many solutions each system of equations has by graphing them.** SEE EXAMPLE 1

11. $\begin{cases} y = 3 \\ y = x^2 - 4x + 7 \end{cases}$

12. $\begin{cases} y = 3x^2 - 2x + 7 \\ y + 5 = \frac{1}{2}x \end{cases}$

**Consider the system of equations** $\begin{cases} y = x^2 \\ y = mx + b \end{cases}$.
SEE EXAMPLE 1

13. Find values for $m$ and $b$ so that the system has two solutions.

14. Find values for $m$ and $b$ so that the system has no solutions.

15. Find values for $m$ and $b$ so that the system has one solution.

**Use substitution to solve the system of equations.**
SEE EXAMPLE 2

16. $\begin{cases} y = 5 \\ y = 2x^2 - 16x + 29 \end{cases}$

17. $\begin{cases} y = 3x^2 - 4x \\ 27 + y = 14x \end{cases}$

18. LaToya throws a ball from the top of a bridge. Her throw is modeled by the equation $y = -0.5x^2 + 3x + 10$, and the bridge is modeled by the equation $y = -0.2x + 7$. About how far does the ball travel horizontally before its first bounce? SEE EXAMPLE 3

**Solve each system of inequalities using shading.**
SEE EXAMPLE 4

19. $\begin{cases} y > x^2 \\ 5 > y \end{cases}$

20. $\begin{cases} -5 < y - x \\ y < -3x^2 + 6x + 1 \end{cases}$

**Solve each system of equations.** SEE EXAMPLE 5

21. $\begin{cases} y = \frac{3}{4}x^2 + \frac{3}{2}x - \frac{9}{4} \\ y = \frac{1}{2}x^2 + x - \frac{3}{2} \end{cases}$

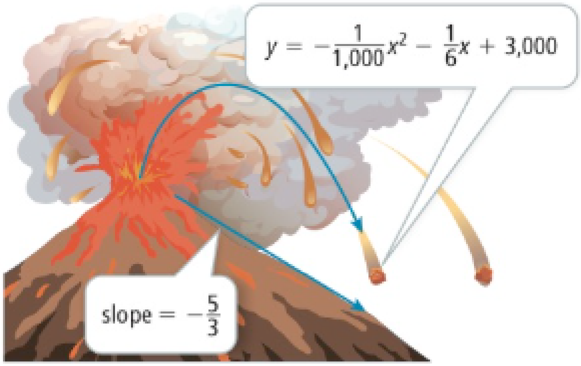
22. $\begin{cases} y = x^2 + x + 1 \\ y = -x^2 - x + 5 \end{cases}$

23. $\begin{cases} y = 2(x - 1)^2 - 1 \\ y = -2(x + 1)^2 + 1 \end{cases}$

# PRACTICE & PROBLEM SOLVING

## APPLY

**24. Apply Math Models** A boulder is flung out of the top of a 3,000 m tall volcano. The boulder's height, $y$, in meters, is a function of the horizontal distance it travels, $x$, in meters. The slope of the line representing the volcano's hillside is $-\frac{5}{3}$. At what height above the ground will the boulder strike the hillside? How far will it have traveled horizontally when it crashes?

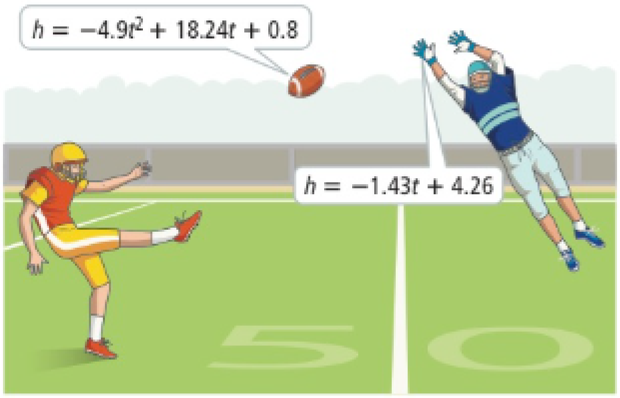



**25. Use Patterns and Structure** You are given the system of equations:

$$\begin{cases} y = x^2 + 7 \\ y + x^2 = 25 \end{cases}$$

Solve the system using any of the methods you have learned in this lesson. Explain why you selected the method you used.

**26. Analyze and Persevere** A football player punts the football, whose path is modeled by the equation $h = -4.9t^2 + 18.24t + 0.8$ for $h$, in meters, and $t$, in seconds. The height of a blocker's hands for the same time, $t$, is modeled as $h = -1.43t + 4.26$. Is it possible for the blocker to knock down the ball? What else would you have to know to be sure?




## ASSESSMENT PRACTICE

**27.** Select all functions that have exactly one point of intersection with the function $f(x) = x^2 + 8x + 11$. AR.9.2

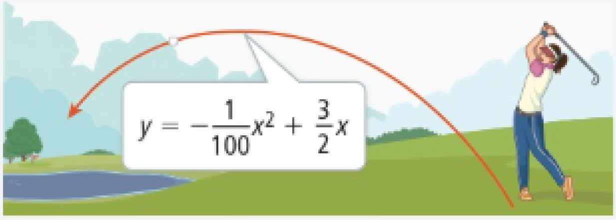
- ☐ **A.** $g(x) = 2x - 12$
- ☐ **B.** $g(x) = 12x + 7$
- ☐ **C.** $g(x) = -5$
- ☐ **D.** $g(x) = 11 + 8x$
- ☐ **E.** $g(x) = -6$

**28. SAT/ACT** How many solutions does the following system of equations have?

$$\begin{cases} y = 16x - 19 \\ y = 3x^2 + 4x - 7 \end{cases}$$

Ⓐ two solutions
Ⓑ no solutions
Ⓒ an infinite number of solutions
Ⓓ one solution
Ⓔ The number of solutions cannot be determined.

**29. Performance Task** A golfer accidentally hits a ball toward a water hazard that is downhill from her current position on the fairway. The hill can be modeled by a line through the origin with slope $-\frac{1}{8}$. The path of the ball can be modeled by the function $y = -\frac{1}{100}x^2 + \frac{3}{2}x$.




**Part A** If the golfer stands at the origin, and the water hazard is 180 yd away, will the golfer's ball bounce or splash?

**Part B** How far did the ball land from the edge of the water hazard?

**Part C** Does it matter whether you measure the 180 yd horizontally or along the hill? Explain.

TOPIC 2

# Topic Review

## TOPIC ESSENTIAL QUESTION

1. How do you use quadratic functions to model situations and solve problems?

## Vocabulary Review

**Choose the correct term to complete each sentence.**

2. According to the __________, a product is 0 only if one (or more) of its factors is 0.
3. The __________ of a quadratic function is $y = a(x - h)^2 + k$.
4. The __________ of a quadratic function is the value of the radicand, $b^2 - 4ac$.
5. A number with both real and imaginary parts is called a __________.
6. The __________ of a quadratic function is $y = ax^2 + bx + c$.
7. __________ is a method used to write a quadratic expression as a perfect square trinomial plus a constant.

- completing the square
- complex number
- discriminant
- imaginary number
- parabola
- quadratic function
- standard form
- vertex form
- Zero Product Property

## Concepts & Skills Review

**LESSON 2-1** Vertex Form of a Quadratic Function

### Quick Review

The parent **quadratic function** is $f(x) = x^2$. The graph of the function is represented by a **parabola**. All quadratic functions are transformations of $f(x) = x^2$.

The vertex form of a quadratic function is $y = a(x - h)^2 + k$, where $(h, k)$ is the vertex of a parabola.

### Example

**What is the equation of a parabola with vertex (3, 1) and *y*-intercept 10?**

$y = a(x - 3)^2 + 1$ ........ Substitute $(h, k) = (2, 3)$.

$10 = a(0 - 3)^2 + 1$ ........ Substitute *y*-intercept (0, 10).

$9 = a(-3)^2$ ........ Simplify.

$9 = 9a$

$a = 1$ ........ Solve for *a*.

$y = 1(x - 3)^2 + 1$ ........ Substitute *a*.

The equation of the parabola is $y = (x - 3)^2 + 1$.

### Practice & Problem Solving

**Describe the transformation of the parent function $f(x) = x^2$. Then graph the given function.**

8. $g(x) = (x + 2)^2 - 4$
9. $h(x) = -2(x - 1)^2 + 5$

**Identify the vertex, axis of symmetry, maximum or minimum, domain, and range of each function.**

10. $g(x) = -(x + 3)^2 + 2$
11. $h(x) = 3(x - 4)^2 - 3$

**Write the equation of each quadratic function in vertex form.**

12. Vertex: (2, 1); Point (0, 4)
13. Vertex: (1, 5); Point (3, −1)
14. **Use Patterns and Structure** The graph of the function $f(x) = x^2$ will be translated 4 units down and 2 units right. What is the resulting function $g(x)$?
15. **Analyze and Persevere** Find three additional points on the parabola that has vertex (5, 3) and passes through (2, 21).

## LESSON 2-2 Standard Form of a Quadratic Function

### Quick Review

The **standard form of a quadratic function** is $y = ax^2 + bx + c$ where $a$, $b$, and c are real numbers, and $a \neq 0$. Use the formula $h = -\frac{b}{2a}$ to find the $x$-coordinate of the vertex and the axis of symmetry. Substitue 0 for $x$ to find the $y$-intercept of the quadratic function.

### Example

**The function $y = -8x^2 + 880x - 5{,}000$ can be used to predict the profits for a company that sells eBook readers for a certain price, $x$. What is the maximum profit the company can expect to earn?**

The maximum value of a quadratic function occurs at the vertex of a parabola. Use the formula $h = -\frac{b}{2a}$ to find the $x$-coordinate of the vertex.

$h = -\frac{880}{2(-8)}$ ........ Substitute −8 for $a$ and 880 for $b$.

$h = 55$ ........ Simplify.

$x = 55$ ........ Substitute $h$ for $x$.

$y = -8(55)^2 + 880(55) - 5{,}000$ ........ Substitute 55 for $x$.

$y = 19{,}200$ ........ Simplify.

The vertex is (55, 19,200). The selling price of \$55 per item gives the maximum profit of \$19,200.

### Practice & Problem Solving

**Find the vertex and $y$-intercept of the quadratic function, and use them to graph the function.**

**16.** $y = x^2 - 6x + 15$

**17.** $y = 4x^2 - 15x + 9$

**Write an equation in standard form for the parabola that passes through the given points.**

**18.** (1, 5), (3, 7), (6, 25), (0, 7), (1.5, 4.75)

**19.** (−2, 48), (2, −16), (0, 0), (4, 0)

**20.** Higher Order Thinking A golfer is on a hill that is 60 meters above the hole. The path of the ball can be modeled by the equation $y = -5x^2 + 40x + 60$, where $x$ is the horizontal and $y$ the vertical distance traveled by the ball in meters. How would you use the function to find the horizontal distance traveled by the ball and its maximum height?

**21.** Analyze and Persevere The number of issues sold per month of a new magazine (in thousands) and its profit (in thousands of dollars) could be modeled by the function $y = -6x^2 + 36x + 50$. Determine the maximum profit.

## LESSON 2-3 Factored Form of a Quadratic Function

### Quick Review

Factor a quadratic equation by first setting the quadratic expression equal to 0. Then factor and use the **Zero Product Property** to solve. According to the Zero Product Property, if $ab = 0$, then $a = 0$ or $b = 0$ (or $a = 0$ and $b = 0$).

### Example

**Solve the equation $x^2 + x = 72$.**

$x^2 + x - 72 = 0$ ........ Set equation equal to 0.

$(x + 9)(x - 8)$ ........ Factor.

$x + 9 = 0$ or $x - 8 = 0$ ........ Zero Product Property.

$x = -9$ or $x = 8$ ........ Solve.

The solutions for equation $x^2 + x = 72$ are $x = -9$ or $x = 8$ .

### Practice & Problem Solving

**Solve each quadratic equation.**

**22.** $x^2 - 6x - 27 = 0$

**23.** $x^2 = 7x - 10$

**24.** $4x^2 + 4x = 3$

**25.** $5x^2 - 19x = -12$

**Identify the interval(s) on which each function is positive.**

**26.** $y = x^2 - x - 30$

**27.** $y = x^2 + 11x + 28$

**28.** Generalize For what values of $x$ is the expression $(x + 6)^2 > 0$?

**29.** Apply Math Models A prairie dog burrow has openings to the surface which, if they were graphed, correspond to points (2.5, 0) and (8, 0). At its deepest, it passes through point (5, −15). What quadratic function could model the burrow?

## LESSON 2-4 Complex Numbers and Operations

### Quick Review

The **imaginary unit *i*** is the number whose square is equal to −1. An **imaginary number** $bi$ is the product of any real number $b$ and the imaginary unit $i$. A **complex number** is a number that may be written in the form $a + bi$. **Complex conjugates** are complex numbers with equivalent real parts and opposite imaginary parts.

### Example

**Write the product of $3.5i(4 - 6i)$ in the form $a + bi$.**

$3.5i(4 - 6i)$

$= 3.5i(4) + 3.5i(-6i)$ ········ Distribute.

$= 14i - 21i^2$ ········ Simplify.

$= 14i - 21(-1)$ ········ Substitute −1 for $i$.

$= 14i + 21$ ········ Write in the form $a + bi$.

The product is $14i + 21$.

### Practice & Problem Solving

**Write each product in the form $a + bi$.**

**30.** $(5 - 3i)(2 + i)$ **31.** $(-3 + 2i)(2 - 3i)$

**Divide. Write the answer in the form $a + bi$.**

**32.** $\frac{5}{3 + i}$ **33.** $\frac{2 - 3i}{1 + 2i}$

**34. Error Analysis** Describe and correct the error a student makes when multiplying complex numbers.

$$\begin{aligned}(2 - 3i)(4 + i) &= 2(4) + 2(i) - 3i(4) - 3i(i)\\ &= 8 + 2i - 12i - 3i^2\\ &= 8 - 10i - 3i^2\end{aligned}$$

**35. Apply Math Models** The formula $E = IZ$ is used to calculate voltage, where $E$ is voltage, $I$ is current, and $Z$ is impedance. If the voltage in a circuit is $35 + 10i$ volts and the impedance is $4 + 4i$ ohms, what is the current (in amps)? Write your answer in the form $a + bi$.

## LESSON 2-5 Completing the Square

### Quick Review

**Completing the square** is a method used to rewrite a quadratic equation as a perfect square trinomial equal to a constant. A perfect square trinomial with the coefficient of $x^2$ equal to 1 has the form $(x - p)^2$ which is equivalent to $x^2 - 2px + p^2$.

### Example

**Solve the equation $0 = x^2 - 2x + 4$ by completing the square.**

$0 = x^2 - 2x + 4$ ···· Write the original equation.

$-4 = x^2 - 2x$ ········ Subtract 4 from both sides of the equation.

$1 - 4 = x^2 - 2x + 1$ ···· Complete the square

$-3 = (x - 1)^2$ ········ Write the right side of the equation as a perfect square.

$\pm\sqrt{-3} = x - 1$ ········ Take the square root of each side of the equation.

$1 \pm \sqrt{-3} = x$ ········ Add 1 to each side of the equation.

The solutions are $x = 1 \pm \sqrt{-3}$.

### Practice & Problem Solving

**Rewrite the equations in the form $(x - p)^2 = q$.**

**36.** $0 = x^2 - 16x + 36$ **37.** $0 = 4x^2 - 28x - 42$

**Solve the following quadratic equations by completing the square.**

**38.** $x^2 - 24x - 82 = 0$ **39.** $-3x^2 - 42x = 18$

**40.** $4x^2 = 16x + 25$ **41.** $12 + x^2 = 15x$

**42. Communicate and Justify** The height, in meters, of a punted football with respect to time is modeled using the function $f(x) = -4.9x^2 + 24.5x + 1$, where $x$ is time in seconds. You determine that the roots of the function $f(x) = -4.9x^2 + 24.5x + 1$ are approximately −0.04 and 5.04. When does the ball hit the ground? Explain.

**43. Analyze and Persevere** A bike manufacturer can predict profits, $P$, from a new sports bike using the quadratic function $P(x) = -100x^2 + 46{,}000x - 2{,}100{,}000$, where $x$ is the price of the bike. At what prices will the company make \$0 in profit?

## LESSON 2-6 The Quadratic Formula

### Quick Review

The **Quadratic Formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provides the solutions of the quadratic equation $ax^2 + bx + c = 0$ for $a \neq 0$. You can calculate the **discriminant** of a quadratic equation to determine the number of real roots.

$b^2 - 4ac > 0$: $ax^2 + bx + c = 0$ has 2 real roots.

$b^2 - 4ac = 0$: $ax^2 + bx + c = 0$ has 1 real root.

$b^2 - 4ac < 0$: $ax^2 + bx + c = 0$ has 2 non-real roots.

### Example

**How many real roots does $3x^2 - 8x + 1 = 0$ have?**

Find the discriminant.

$$\begin{aligned} b^2 - 4ac &= (-8)^2 - 4(3)(1) \\ &= 64 - 12 \\ &= 52 \end{aligned}$$

Since $52 > 0$, the equation has two real roots.

### Practice & Problem Solving

**Use the Quadratic Formula to solve the equation.**

**44.** $x^2 - 16x + 24 = 0$

**45.** $x^2 + 5x + 2 = 0$

**46.** $2x^2 - 18x + 5 = 0$

**47.** $3x^2 - 5x - 19 = 0$

**Use the discriminant to identify the number and type of solutions for each equation.**

**48.** $x^2 - 24x + 19 = 0$

**49.** $3x^2 - 8x + 12 = 0$

**50.** Find the value(s) of $k$ that will cause the equation $4x^2 - kx + 4 = 0$ to have one real solution.

**51.** **Communicate and Justify** Why does the graph of the quadratic function $f(x) = x^2 + 4x + 5$ cross the $y$-axis but not the $x$-axis?

**52.** **Apply Math Models** The function $C(x) = 0.0045x^2 - 0.47x + 139$ models the cost per hour of running a bus between two cities, where $x$ is the speed in kilometers per hour. At what speeds will the cost of running the bus exceed $130?

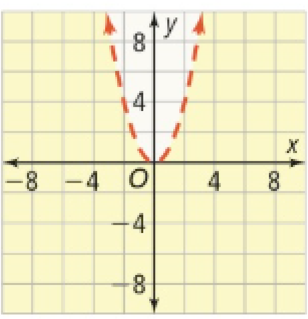
## LESSON 2-7 Quadratic Inequalities

### Quick Review

A quadratic inequality in two variables is an inequality that is in the same form as a quadratic equation in two variables but with an inequality symbol instead of an equal sign. Its solutions are all ordered pairs that satisfy the inequality, and the solution may be given in set builder notation. The inequality's solution may be shown as a shaded region of coordinate grid either above or below the parabola and perhaps the parabola itself.

### Example

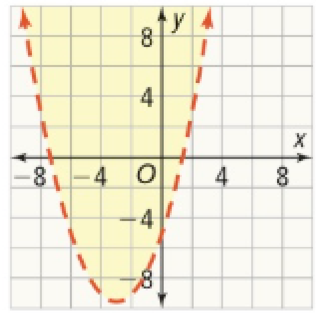
**What inequality is shown by the graph?**




The parabola represents the equation $y = x^2$. The boundary line is dashed, so the inequality is a strict inequality (< or >). The ordered pairs that make the inequality true are points below the curve, so the inequality shown by the graph is $y < x^2$.

### Practice & Problem Solving

**Use the graph to tell whether each ordered pair is a solution of the inequality $y > \frac{1}{2}x^2 + 3x - 5$.**




**53.** (2, 6)

**54.** (0, −5)

**55.** (−8, −4)

**Graph the inequality on the coordinate plane.**

**56.** $y \leq -(x-2)^2 + 5$

**57.** $y + x^2 - 3 > 3x^2 - 2x - 5$

**58.** **Use Patterns and Structure** Write a quadratic inequality in two variables for which (2, −1) and (5, −1) are solutions, and (3, −1) is not.

**59.** **Represent and Connect** A canal is 40 feet across and 20 feet deep at the middle. Its cross section resembles a parabola. Write and graph an inequality that represents the area filled by water in the cross section.

## LESSON 2-8 Systems Involving Quadratic Equations and Inequalities

### Quick Review

Solutions to a system of equations are points that produce a true statement for all the equations of the system. The solutions on a graph are the coordinates of the intersection points.

### Example

**Use substitution to solve the system of equations.**

$$\begin{cases} y = 2x^2 - 5x + 4 \\ 5x - y = 4 \end{cases}$$

Substitute $2x^2 - 5x + 4$ for $y$ in the second equation.

$$5x - (2x^2 - 5x + 4) = 4$$
$$-2x^2 + 10x - 8 = 0$$

Factor: $-2(x - 1)(x - 4) = 0$

So $x = 1$ and $x = 4$ are solutions.

When $x = 1$, $y = 2(1)^2 - 5(1) + 4 = 1$.

When $x = 4$, $y = 2(4)^2 - 5(4) + 4 = 16$.

The solutions of the system are (1, 1) and (4, 16).

### Practice & Problem Solving

**Determine the number of solutions of each system of equations.**

60. $\begin{cases} y = x^2 - 5x + 9 \\ y = 3 \end{cases}$

61. $\begin{cases} y = 3x^2 + 4x + 5 \\ y - 4 = 2x \end{cases}$

**Solve each system of equations.**

62. $\begin{cases} y = x^2 + 4x + 3 \\ y - 2x = 6 \end{cases}$

63. $\begin{cases} y = x^2 + 2x + 7 \\ y = 7 + x \end{cases}$

64. **Apply Math Models** An archer shoots an arrow to a height (meters) given by the equation $y = -5t^2 + 18t - 0.25$, where $t$ is the time in seconds. A target sits on a hill represented by the equation $y = 0.75x - 1$. At what height will the arrow strike the target, and how long will it take?

TOPIC

3

# Polynomial Functions

## TOPIC ESSENTIAL QUESTION

**What can the rule for a polynomial function reveal about its graph, and what can the graphs of polynomial functions reveal about the solutions of polynomial equations?**

## Topic Overview

**enVision® STEM Project**
Design a Stadium

**3-1** Graphing Polynomial Functions
AR.1.1, AR.6.5, MTR.1.1, MTR.2.1, MTR.5.1

**3-2** Adding, Subtracting, and Multiplying Polynomials
AR.1.3, AR.1.6, F.1.7, F.3.2, MTR.4.1, MTR.6.1, MTR.7.1

**3-3** Polynomial Identities
AR.1.3, AR1.8, AR.1.11, MTR.1.1, MTR.2.1, MTR.5.1

**3-4** Dividing Polynomials
AR.1.5, AR.1.6, AR.1.8, AR.6.2, MTR.4.1, MTR.5.1, MTR.6.1

**3-5** Zeros of Polynomial Functions
AR.1.8, AR.6.1, AR.6.5, F.1.1, MTR.1.1, MTR.5.1, MTR.7.1

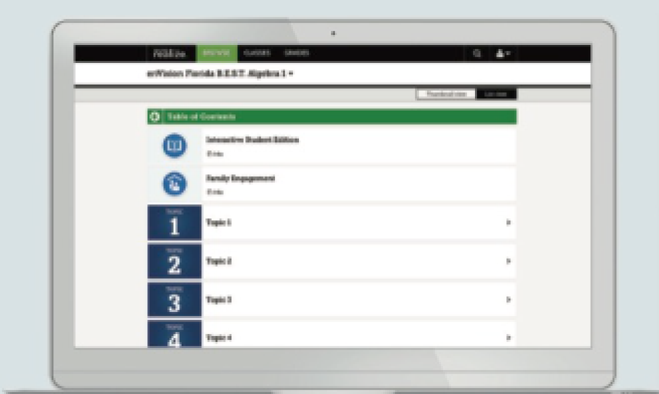
**Mathematical Modeling in 3 Acts:** What Are the Rules?
AR.1.3, AR.1.6, AR.6.1, MTR.2.1, MTR.6.1, MTR.7.1

**3-6** Roots of Polynomial Equations
AR.1.8, AR.6.1, MTR.1.1, MTR.4.1, MTR.5.1

**3-7** Transformations of Polynomial Functions
F.1.9, F.2.2, F.2.3, F.2.5, MTR.3.1, MTR.4.1, MTR.5.1

## Topic Vocabulary

- Binomial Theorem
- degree of a polynomial
- end behavior
- even function
- identity
- leading coefficient
- multiplicity of a zero
- odd function
- Pascal's triangle
- polynomial function
- relative maximum
- relative minimum
- Rational Root Theorem
- Remainder Theorem
- standard form of a polynomial
- synthetic division
- turning point



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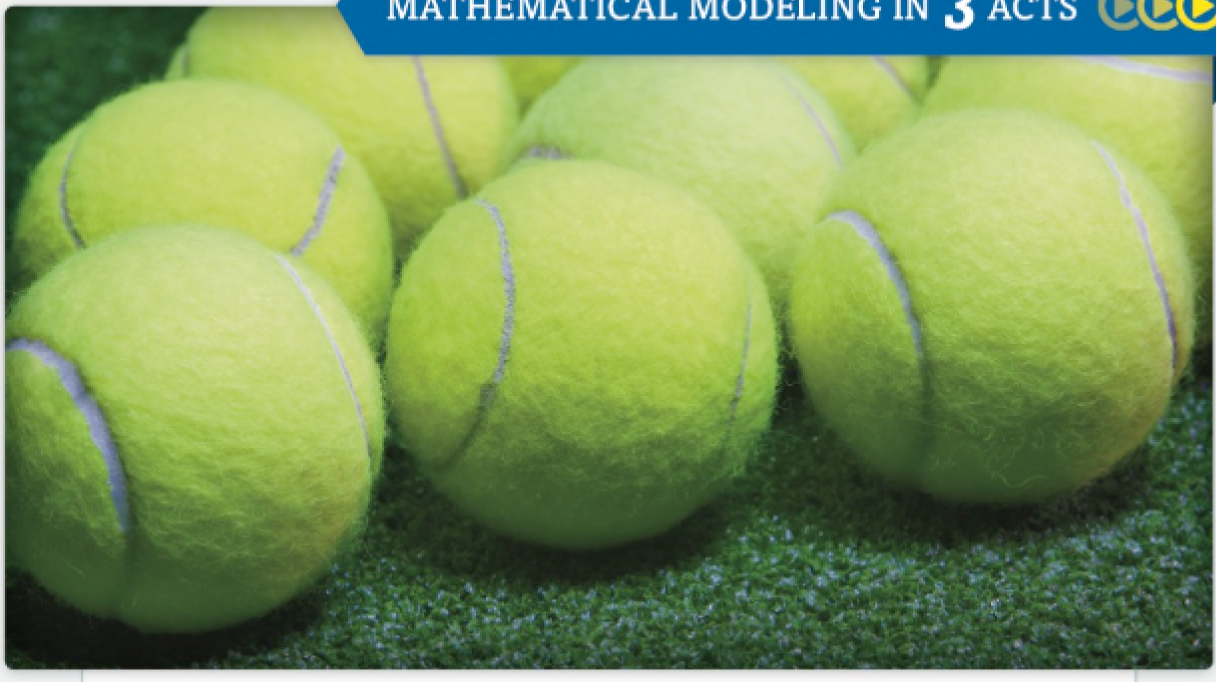
**ANIMATION** View and interact with real-world applications.

**PRACTICE** Practice what you've learned.



MATHEMATICAL MODELING IN 3 ACTS

## What Are the Rules?

All games have rules about how to play the game. The rules outline such things as when a ball is in or out, how a player scores points, and how many points a player gets for each winning shot.

If you didn't alreay know how to play tennis, or some other game, could you figure out what the rules were just by watching? What clues would help you understand the game? Think about this during the Mathematical Modeling in 3 Acts lesson.

**VIDEOS** Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

**ADAPTIVE PRACTICE** Practice that is *just right* and *just for you.*

**GLOSSARY** Read and listen to English and Spanish definitions.

**CONCEPT SUMMARY** Review key lesson content through multiple representations.

**ASSESSMENT** Show what you've learned.

**TUTORIALS** Get help from ***Virtual Nerd***, right when you need it.

**MATH TOOLS** Explore math with digital tools and manipulatives.

**DESMOS** Use Anytime and as embedded Interactives in Lesson content.

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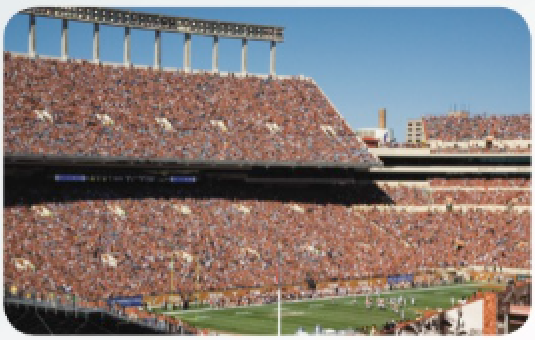
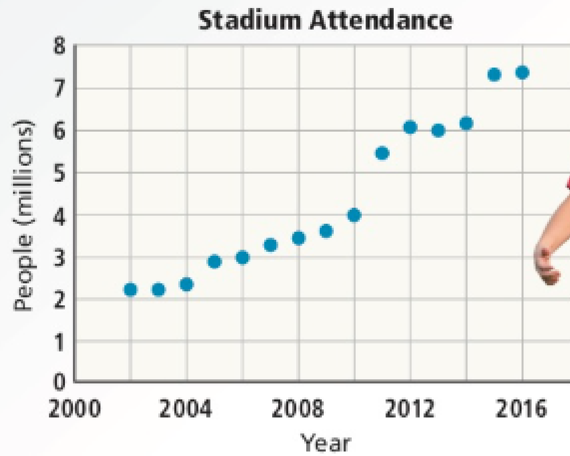
## Did You Know?

In a 2013–2015 overhaul, **Texas A&M's football stadium** increased its seating from **80,600 to 102,512**. The renovation cost $450 million and included lowering the field and adding overhangs to two sides.






Attendance at professional soccer games has **increased since 2002.**


Stadium Court at the **Crandon Park Tennis Center,** in Key Biscayne seats **13,800 spectators.** The Crandon Park Tennis Center features 26 tennis courts and hosts the annual Miami Open tennis tournament.

## Your Task: Design a Stadium

You and your classmates will plan the seating at a new stadium. You will explore attendance at the current stadium and use fitted curves to support your predictions of future attendance.

# 3-1 Graphing Polynomial Functions

**I CAN…** predict the behavior of polynomial functions.

**VOCABULARY**

- degree of a polynomial
- leading coefficient
- polynomial function
- relative maximum
- relative minimum
- standard form of a polynomial
- turning point

MA.912.AR.6.5–Sketch a rough graph of a polynomial function of degree 3 or higher using zeros, multiplicity and knowledge of end behavior. Also AR.1.1

MA.912.MTR.1.1, MTR.2.1, MTR.5.1

## EXPLORE & REASON

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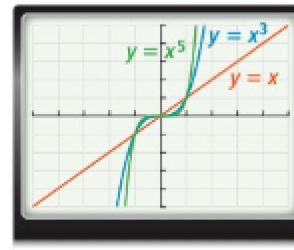
**Consider functions of the form $f(x) = x^n$, where $n$ is a positive integer.**




**A.** Graph $f(x) = x^n$ for $n = 1$, 3, and 5. Look at the graphs in Quadrant I. As the exponent increases, what is happening to the graphs? Which quadrants do the graphs pass through?

**B.** Use Patterns and Structure Now graph $f(x) = x^n$ for $n = 2$, 4, and 6. What happens to these graphs in Quadrant I as the exponent increases? Which quadrants do the graphs pass through?

**C.** Write two equations in the form $f(x) = x^n$ with graphs that you predict are in Quadrants I and II. Write two equations with graphs that you predict are in Quadrants I and III. Use graphing technology to test your predictions.

## ESSENTIAL QUESTION

**How do the key features of a polynomial function help you sketch its graph?**

## EXAMPLE 1 Classify Polynomials

**How can you write a polynomial in standard form and use it to identify the leading coefficient, the degree, and the number of terms?**

$$-4x + 9 + 2x^3$$

Recall that a polynomial is a monomial or the sum of one or more monomials, called terms. The degree of a term with one variable is the exponent of that variable.

Degree of $-4x$: 1     Degree of 9: 0     Degree of $2x^3$: 3

**Standard form of a polynomial** shows any like terms combined and the terms by degree in descending numerical order.

Standard form of this polynomial is:

$$2x^3 - 4x + 9$$

The **leading coefficient** refers to the non-zero factor that is multiplied by the greatest power of $x$. The leading coefficient of this polynomial is 2.

The polynomial has three terms, so it is called a *trinomial*.

The **degree of a polynomial** is the greatest degree of any of the terms. This is a polynomial of degree 3, also known as a *cubic polynomial*.

**USE PATTERNS AND STRUCTURE**
Note that there is no $x^2$-term in the polynomial $2x^3 - 4x + 9$. In some cases, it may be useful to write the polynomial as $2x^3 + 0x^2 - 4x + 9$.

**Try It!** 1. What is each polynomial in standard form and what are the leading coefficient, the degree, and the number of terms of each?

a. $2x - 3x^4 + 6 - 5x^3$

b. $x^5 + 2x^6 - 3x^4 - 8x + 4x^3$

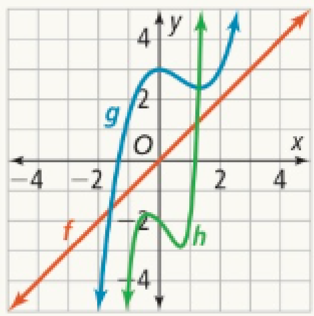
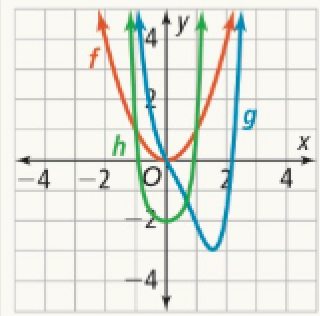
CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Understand End Behavior of Polynomial Functions

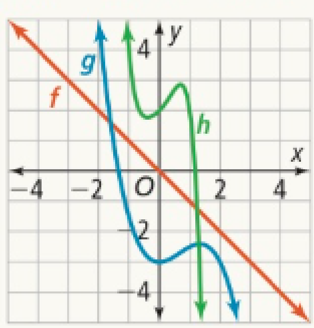
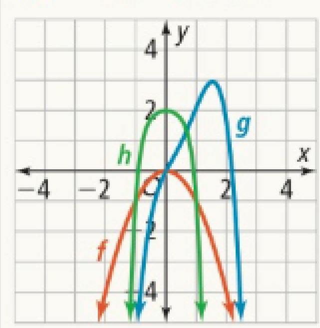
**How do the sign of the leading coefficient and the degree of a polynomial affect the end behavior of the graph of a polynomial function?**

A **polynomial function** is a function whose rule is a polynomial. The **end behavior** of a graph describes what happens to the function values as $x$ approaches positive and negative infinity.

**USE PATTERNS AND STRUCTURE**
Though a polynomial function may have many terms, the leading term determines the end behavior because it has the greatest exponent and therefore the greatest impact on function values when $x$ is very large or very small.

| Odd Degree<br>Positive Leading Coefficient | Even Degree<br>Positive Leading Coefficient |
|---|---|
| $f(x) = x$; degree 1<br>$g(x) = 0.5x^3 - x^2 + 3$; degree 3<br>$h(x) = 2x^5 - x^2 - x - 2$; degree 5 | $f(x) = x^2$; degree 2<br>$g(x) = 0.9x^4 - 2x^3 + x^2 - 2x$; degree 4<br>$h(x) = 2x^6 + x^2 - 2$; degree 6 |
|  |  |
| End behavior is similar to the linear parent function $f(x) = x$. As $x \to \infty$, $y \to \infty$. As $x \to -\infty$, $y \to -\infty$. | End is behavior similar to the quadratic parent function $f(x) = x^2$. As $x \to \pm\infty$, $y \to \infty$. |

Recall that a reflection of a function across the $x$-axis occurs when the function is negated: $f(x)$ becomes $-f(x)$. The end behavior of a function is similarly affected when the leading coefficient is negative.

| Odd Degree<br>Negative Leading Coefficient | Even Degree<br>Negative Leading Coefficient |
|---|---|
| $f(x) = -x$; degree 1<br>$g(x) = -0.5x^3 - x^2 - 3$; degree 3<br>$h(x) = -2x^5 + x^2 + x + 2$; degree 5 | $f(x) = -x^2$; degree 2<br>$g(x) = -0.9x^4 + 2x^3 - x^2 + 2x$; degree 4<br>$h(x) = -2x^6 - x^2 + 2$; degree 6 |
|  |  |
| End behavior is similar to $f(x) = -x$. As $x \to \infty$, $y \to -\infty$. As $x \to -\infty$, $y \to \infty$. | End behavior is similar to $f(x) = -x^2$. As $x \to \pm\infty$, $y \to -\infty$. |

**Try It!** 2. Use the leading coefficient and degree of the polynomial function to determine the end behavior of each graph.

a. $f(x) = 2x^6 - 5x^5 + 6x^4 - x^3 + 4x^2 - x + 1$

b. $g(x) = -5x^3 + 8x + 4$

## EXAMPLE 3 Graph a Polynomial Function

**Consider the polynomial function $f(x) = -0.5x^4 + 3x^2 + 2$.**

**How can you use a table of values to identify key features and sketch a graph of the function?**

Make a table of values and identify intervals where the function is increasing and decreasing.

| $x$ | $f(x)$ |
|---|---|
| −3 | −11.5 |
| −2 | 6 |
| −1 | 4.5 |
| 0 | 2 |
| 1 | 4.5 |
| 2 | 6 |
| 3 | −11.5 |

increasing

decreasing

increasing

decreasing

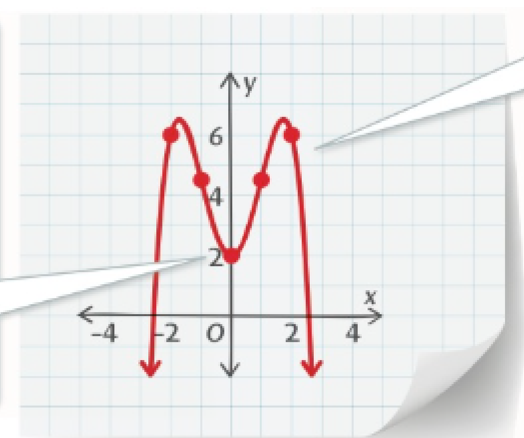
Points where the function values change from increasing to decreasing, or vice-versa, are **turning points**. This function has approximate turning points when the value of $x$ is between −2 and −1, −1 and 0, and 1 and 2.

**CHOOSE EFFICIENT METHODS**
It can be very difficult to locate the precise turning points and zeros of polynomial functions. Graphing technology can help identify these points.

This is a polynomial function with an even degree and a negative leading coefficient, so both ends of the graph will trend toward $-\infty$.

Plot the points and sketch the graph with a smooth curve.

A point where the function has the least value over an interval is a **relative minimum**. This function has a relative minimum near (0, 2).




A point where the function has the greatest value over an interval is a **relative maximum**. This function has two relative maximums near (−2, 6) and (2, 6).

**Try It!** 3. Consider the polynomial function $f(x) = x^5 + 18x^2 + 10x + 1$.

Make a table of values to identify key features and sketch a graph of the function.

## EXAMPLE 4 Sketch the Graph from a Verbal Description

**COMMON ERROR**
Positive/negative behavior only tells where the function's graph lies above or below the $x$-axis. It does not indicate whether the function is increasing or decreasing.

**How can you sketch a graph of the polynomial function $f$ from a verbal description?**

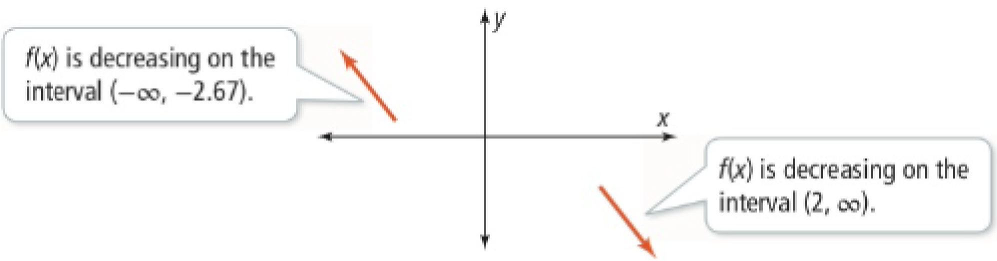
- **$f(x)$ is positive on the intervals $(-\infty, -4)$ and $(-1, 4)$.**
- **$f(x)$ is negative on the intervals $(-4, -1)$ and $(4, \infty)$.**
- **$f(x)$ is decreasing on the intervals $(-\infty, -2.67)$ and $(2, \infty)$.**
- **$f(x)$ is increasing on the interval $(-2.67, 2)$.**

**Step 1:** Identify or estimate $x$-intercepts. The function values change signs at $x = -4$, $x = -1$, and $x = 4$.

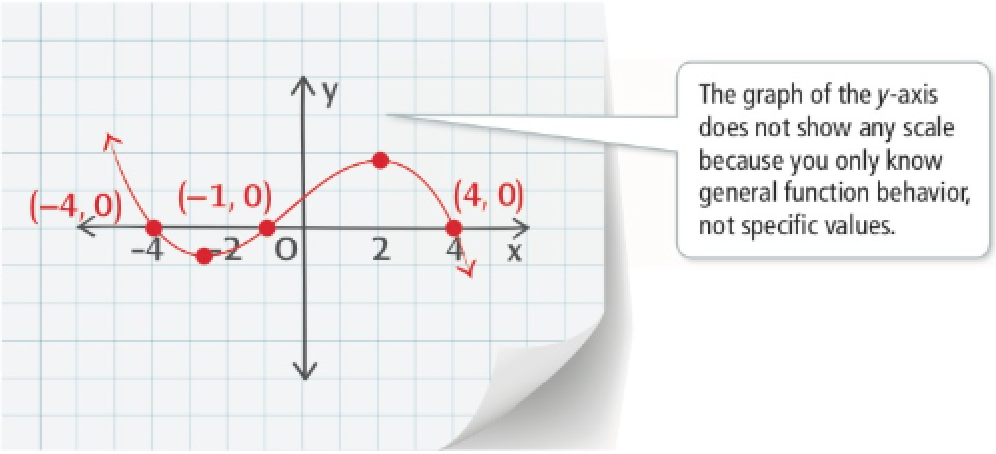
**Step 2:** Identify or estimate turning points. The function changes direction at $x = -2.67$ and $x = 2$.

- There is a relative minimum at $x = -2.67$.
- There is a relative maximum at $x = 2$.

**Step 3:** Evaluate end behavior.




**Step 4:** Sketch the graph.




**Try It!** **4.** Use the information below to sketch a graph of the polynomial function $y = f(x)$.

- $f(x)$ is positive on the intervals $(-2, -1)$ and $(1, 2)$.
- $f(x)$ is negative on the intervals $(-\infty, -2)$, $(-1, 1)$, and $(2, \infty)$.
- $f(x)$ is increasing on the interval $(-\infty, -1.5)$ and $(0, 1.5)$.
- $f(x)$ is decreasing on the intervals $(-1.5, 0)$ and $(1.5, \infty)$.

APPLICATION

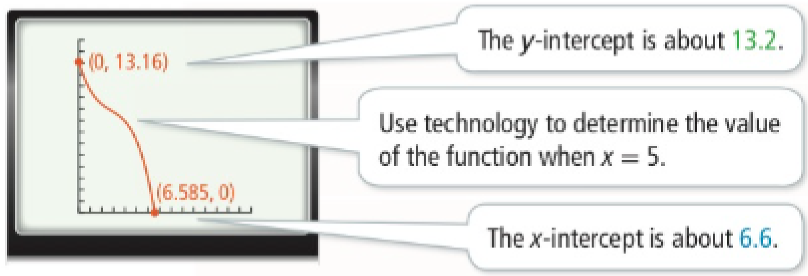
## EXAMPLE 5 Interpret a Polynomial Model

**In science class, Abby mixes a fixed amount of baking soda with different amounts of vinegar in a bottle capped by a balloon. She records the amount of time it takes the gases produced by the reaction to inflate the balloon.**


**From her data, Abby created a function to model the situation. For $x$ quarter-cups of vinegar, it takes $t(x) = -0.12x^3 + x^2 - 3.38x + 13.16$ seconds to inflate the balloon.**

**A. How long would it take to inflate the balloon with 5 quarter-cups of vinegar?**

Use technology to sketch the graph.




When $x = 5$, the value of the function is about 6.3. This means that if Abby uses 5 quarter-cups of vinegar, the balloon will inflate in approximately 6.3 seconds.

**STUDY TIP**
Recall that when you are using a graph in a real-world context, you need to consider the context when thinking about domain and range. Does it make sense for $x$ to be negative? Does it make sense for $y$ to be negative?

**B. What do the $x$- and $y$-intercepts of the graph mean in this context? Do those values make sense?**

The $x$-intercept is approximately 6.6 which means that if 6.6 cups of vinegar are used, the balloon would inflate in 0 seconds.

The $y$-intercept is approximately 13.2, which means that if no vinegar is used, the balloon will inflate in 13.2 seconds.

Neither the $x$- nor the $y$-intercept make sense in this context. Therefore, we must limit the domain and range when considering this model.

**Try It!** 5. Danielle is engineering a new brand of shoes. For $x$ shoes sold, in thousands, a profit of $p(x) = -3x^4 + 4x^3 - 2x^2 + 5x + 10$ dollars, in ten thousands, will be earned.

a. How much will be earned in profit for selling 1,000 shoes?

b. What do the $x$- and $y$-intercepts of the graph mean in this context? Do those values make sense?

# CONCEPT SUMMARY Graphing Polynomial Functions

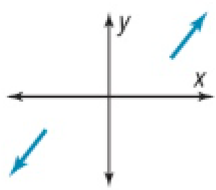
**WORDS** A **polynomial function** is a function whose rule is either a monomial or a sum of monomials.

**KEY FEATURES** **Turning points** – function values change from increasing to decreasing, or vice-versa

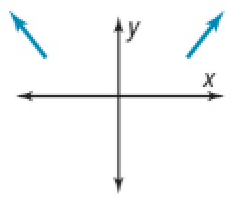
**Relative minimum** – changes from decreasing to increasing

**Relative maximum** – changes from increasing to decreasing

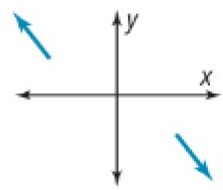
**GRAPHS** End behavior depends on the degree of the polynomial and the sign of its leading coefficient.



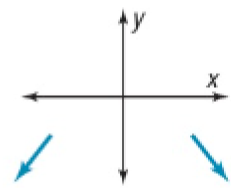
Degree: odd
Leading Coefficient: +



Degree: even
Leading Coefficient: +



Degree: odd
Leading Coefficient: −



Degree: even
Leading Coefficient: −

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do the key features of a polynomial function help you sketch its graph?

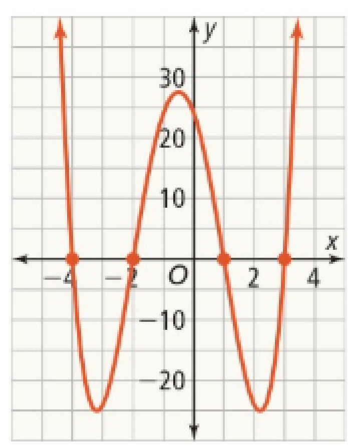
2. **Error Analysis** Allie said the degree of the polynomial function $f(x) = x^5 + 2x^4 + 3x^3 - 2x^6 - 9x^2 - 6x + 4$ is 5. Explain and correct Allie's error.

3. **Vocabulary** Explain how to determine the leading coefficient of a polynomial function.

4. **Use Patterns and Structure** What is the relationship between the degree and leading coefficient of a polynomial function and the end behavior of the polynomial?

## Do You KNOW HOW?

**The graph shows the function $f(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$. Find the following.**




5. number of terms
6. degree
7. leading coefficient
8. end behavior
9. turning point(s)
10. $x$-intercept(s)
11. relative minimum(s)
12. relative maximum(s)

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

13. **Analyze and Persevere** The table shows some values of a polynomial function. Deshawn says there are turning points between the $x$-values $-3$ and $-2$ and between 0 and 1. He also says there is a relative minimum between the $x$-values $-3$ and $-2$, and a relative maximum between 0 and 1. Sketch a graph that shows how Deshawn could be correct and another graph that shows how Deshawn could be incorrect.

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-1004$ | 129 | 220 | 85 | 12 | 1 | 4 | 165 |

14. **Higher Order Thinking** Use the information below about a polynomial function in standard form to write a possible polynomial function. Explain how you determined your function and graph it to verify that it satisfies the criteria.

- 6 terms
- $y$-intercept at 1
- end behavior: As $x \to -\infty$, $y \to +\infty$.
  As $x \to +\infty$, $y \to -\infty$.

15. **Represent and Connect** An analyst for a new company used the first three years of revenue data to project future revenue for the company. The analyst predicts the function $f(x) = -2x^5 + 6x^4 - x^3 + 5x^2 + 6x + 50$ will give the revenue after $x$ years. Should the CEO expect the company to be successful? Explain.

16. **Use Patterns and Structure** Sketch a graph of each of the functions described below.

- a cubic function with one $x$-intercept
- a cubic function with 2 $x$-intercepts
- a cubic function with 3 $x$-intercepts

17. **Analyze and Persevere** For any 4th degree polynomial function, answer the following questions.

a. What is the maximum number of real zeros?

b. What is the minimum number of real zeros?

c. What is the maximum number of turning points?

d. What is the minimum of turning points?

## PRACTICE

**Write each polynomial function in standard form. For each function, find the degree, number of terms, and leading coefficient.**
SEE EXAMPLE 1

18. $f(x) = -3x^3 + 2x^5 + x + 8x^3 - 6 + x^4 - 3x^2$

19. $f(x) = 8x^2 + 10x^7 - 7x^3 - x^4$

20. $f(x) = -x^3 + 9x + 12 - x^4 + 5x^2$

**Use the leading coefficient and degree of the polynomial function to determine the end behavior of the graph.** SEE EXAMPLE 2

21. $f(x) = -x^5 + 2x^4 + 3x^3 + 2x^2 - 8x + 9$

22. $f(x) = 7x^4 - 4x^3 + 7x^2 + 10x - 15$

23. $f(x) = -x^6 + 7x^5 - x^4 + 2x^3 + 9x^2 - 8x - 2$

**Use a table of values to estimate the intercepts and turning points of the function. Then graph the function.** SEE EXAMPLE 3

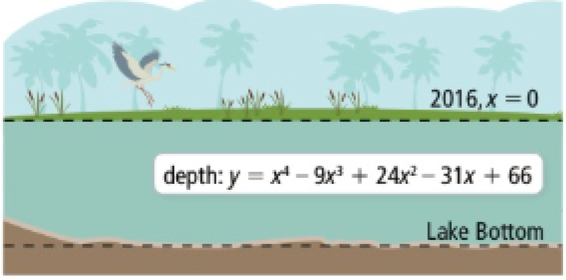
24. $f(x) = x^3 + 2x^2 - 5x - 6$

25. $f(x) = x^4 - x^3 - 21x^2 + x + 20$

26. Use the information below to sketch a graph of the polynomial function $y = f(x)$. SEE EXAMPLE 4

- $f(x)$ is positive on the intervals $(-\infty, -3)$, $(-2, 0)$, and $(2, 3)$.
- $f(x)$ is negative on the intervals $(-3, -2)$, $(0, 2)$, and $(3, \infty)$.
- $f(x)$ is increasing on the interval $(-2.67, -1)$ and $(1, 2.5)$.
- $f(x)$ is decreasing on the intervals $(-\infty, -2.67)$, $(-1, 1)$, and $(2.5, \infty)$.

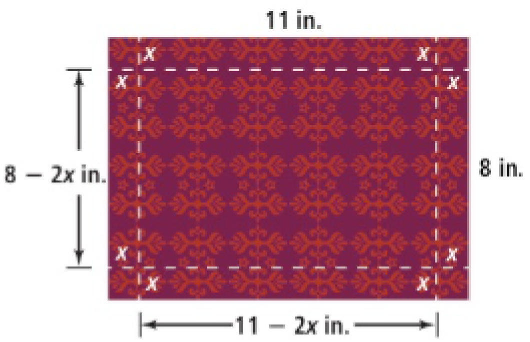
27. The equation shown models the average depth $y$, in feet, of a lake, $x$ years after 2016, where $0 < x < 6$. Use technology to graph the function. In what year does this model predict a relative minimum value for the depth? SEE EXAMPLE 5



## PRACTICE & PROBLEM SOLVING

### APPLY

**28. Represent and Connect** Allie has a piece of construction paper that she wants to use to make an open rectangular prism. She will cut a square with side length $x$ from each corner of the paper, so the length and width is decreased by $2x$ as shown in the diagram.



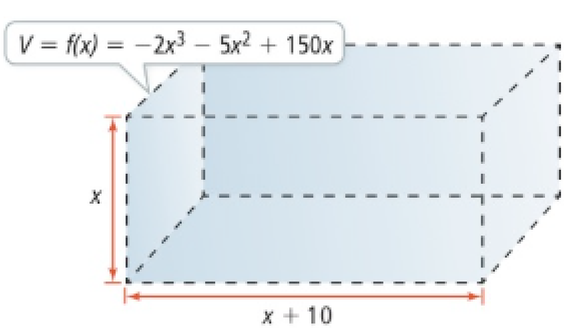

a. Write a function that models the volume of the rectangular prism.

b. Graph the function and identify a reasonable domain.

c. What do the $x$-intercepts of the graph mean in this context?

d. If Allie wants to maximize the volume of the box, what is the side length of the squares that should be cut from each corner of the piece of construction paper? Explain.

**29. Analyze and Persevere** Alberto is designing a container in the shape of a rectangular prism to ship electronic devices. The length of the container is 10 inches longer than the height. The sum of the length, width, and height is 25 inches. The volume of the container, in terms of height $x$, is shown. Use a graphing calculator to graph the function. What do the $x$-intercepts of the graph mean in this context? What dimensions of the container will maximize the volume?



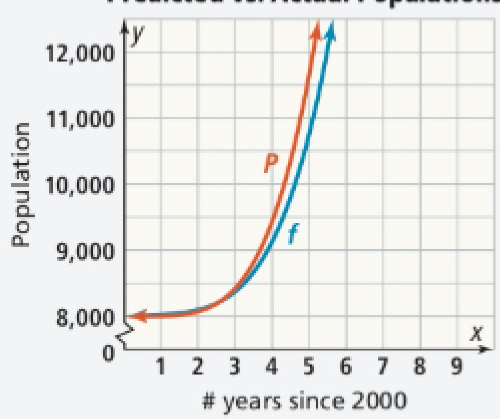

### ASSESSMENT PRACTICE

**30.** Graph the function $f(x) = -(x-2)(x+1)(x+3)(x+7)$. The zeros of this function divide its domain into five separate intervals. In which of these intervals does the function have relative maximums? AR.6.5

**31. SAT/ACT** What is the maximum number of terms a fourth-degree polynomial function in standard form can have?

Ⓐ 1 Ⓑ 2 Ⓒ 3 Ⓓ 4 Ⓔ 5

**32. Performance Task** In the year 2000, a demographer predicted the estimated population of a city, which can be modeled by the function $f(x) = 5x^4 - 4x^3 + 25x + 8{,}000$. Several years later, a statistician, using data from the U.S. Census Bureau, modeled the actual population with the function $P(x) = 7x^4 - 6x^3 + 5x + 8{,}000$. The graphs of the functions are shown.




**Part A** What is the $y$-intercept of each function, and what does it represent?

**Part B** Identify the end behaviors of $f$ and $P$. What do the end behaviors of these functions tell about the quality of these functions as models?

# 3-2 Adding, Subtracting, and Multiplying Polynomials

**I CAN…** add, subtract, and multiply polynomials.

MA.912.AR.1.6–Solve mathematical and real-world problems involving addition, subtraction, multiplication or division of polynomials.
Also AR.1.3, F.1.7, F.3.2
MA.912. MTR.4.1, MTR.6.1, MTR.7.1

## EXPLORE & REASON

**Let $S$ be the set of expressions that can be written as $ax + b$ where $a$ and $b$ are real numbers.**

**A.** Describe the Associative Property, Commutative Property, and the Distributive Property. Then, explain the role of each in simplifying the sum $(3x + 2) + (7x - 4)$. Identify the leading coefficient and the constant term in the result.

**B.** Is the sum you found in part A a member of $S$? Explain.

**C.** **Communicate and Justify** Is the product of two expressions in $S$ also a member of $S$? Explain why or produce a counterexample.

## ESSENTIAL QUESTION

**How do you add, subtract, and multiply polynomials?**

## EXAMPLE 1 Add and Subtract Polynomials

**How do you add or subtract the polynomials?**

To add and subtract polynomials, use the Commutative and Associative Properties to group like terms. Then combine like terms.

**A.** $(6x^3 + 4x + x^2 - 7) + (2x^3 - 8x^2 + 3)$

$= (6x^3 + 2x^3) + 4x + (x^2 - 8x^2) + (-7 + 3)$ Apply the Commutative and Associative Properties.

$= 8x^3 + 4x - 7x^2 - 4$ Combine like terms.

$= 8x^3 - 7x^2 + 4x - 4$ Write in standard form.

**GENERALIZE**
In order for terms to be *like terms*, the variables and their corresponding exponents must be identical.

**B.** $\left(\frac{1}{3}x^2y^2 + 2xy^2 + \frac{1}{4}x^2\right) - \left(\frac{4}{9}x^2y^2 + \frac{1}{2}xy^2 - \frac{3}{8}x^2\right)$

$= \frac{1}{3}x^2y^2 + 2xy^2 + \frac{1}{4}x^2 - \frac{4}{9}x^2y^2 - \frac{1}{2}xy^2 + \frac{3}{8}x^2$ Distribute the factor of –1.

$= \left(\frac{1}{3}x^2y^2 - \frac{4}{9}x^2y^2\right) + \left(2xy^2 - \frac{1}{2}xy^2\right) + \left(\frac{1}{4}x^2 + \frac{3}{8}x^2\right)$

$= -\frac{1}{9}x^2y^2 + \frac{3}{2}xy^2 + \frac{5}{8}x^2$

The degree of a multi-variable polynomial is the greatest sum of powers in any term.

**Try It!** **1.** Add or subtract the polynomials.

**a.** $\left(\frac{2}{5}a^4 - 6a^3 - \frac{5}{6}a^2 + \frac{a}{2} + 1\right) + \left(\frac{9}{4}a^3 + \frac{2a^2}{3} + \frac{5}{3}a - \frac{8}{5}\right)$

**b.** $(2a^2b^2 + 3ab^2 - 5a^2b) - (3a^2b^2 - 9a^2b + 7ab^2)$

## EXAMPLE 2 Multiply Polynomials

**How do you multiply the polynomials?**

To multiply polynomials, use the Distributive Property, then group like terms and combine.

**A.** $(2m + 5)(3m^2 - 4m + 2)$

$= 2m(3m^2 - 4m + 2) + 5(3m^2 - 4m + 2)$ Use the Distributive Property.

$= 6m^3 - 8m^2 + 4m + 15m^2 - 20m + 10$ Use the Distributive Property.

$= 6m^3 + (-8m^2 + 15m^2) + (4m - 20m) + 10$ Group like terms.

$= 6m^3 + 7m^2 - 16m + 10$ Combine like terms.

**LEARN TOGETHER**
How do you value other perspectives and points of view respectfully?

**B.** $(mn + 1)(mn - 2)(mn + 4)$

$= [(mn + 1)(mn - 2)](mn + 4)$ Multiply two binomials first.

$= (m^2n^2 - 2mn + mn - 2)(mn + 4)$ Use the Distributive Property.

$= (m^2n^2 - mn - 2)(mn + 4)$ Combine like terms.

$= m^2n^2(mn + 4) + (-mn)(mn + 4) + (-2)(mn + 4)$ Use the Distributive Property.

$= m^3n^3 + 4m^2n^2 - m^2n^2 - 4mn - 2mn - 8$ Use the Distributive Property.

$= m^3n^3 + (4m^2n^2 - m^2n^2) + (-4mn - 2mn) - 8$ Group like terms.

$= m^3n^3 + 3m^2n^2 - 6mn - 8$ Combine like terms.

**Try It!** **2.** Multiply the polynomials.

**a.** $\left(6n^2 - \frac{1}{2}\right)\left(n^2 + n + \frac{3}{5}\right)$ **b.** $(mn + 1)(m^2n - 1)(mn^2 + 2)$

**CONCEPTUAL UNDERSTANDING**

## EXAMPLE 3 Understand Closure

**Is the set of polynomials closed under addition and subtraction? Explain.**

**COMMUNICATE AND JUSTIFY**
Can you think of two real numbers such that when you add them, the result is NOT a real number?

The set of real numbers is closed under addition: if $a$ and $b$ are real and $a + b = c$, then $c$ is also real.

Add two polynomials:

Adding like terms does not change the variable factor(s) of the terms, only the coefficient: $-3x^2y^2 + 9x^2y^2 = 6x^2y^2$.

$$\begin{array}{r} (a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0) \\ +\ (b_nx^n + b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0) \\ \hline (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots + (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) \end{array}$$

Since $a$ and $b$ are real, $(a + b)$ is also real. The exponents are unchanged. The sum is still a polynomial, so the set of polynomials is closed under addition.

Using the same logic, you can determine that the set of polynomials is closed under subtraction.

**Try It!** **3.** Is the set of monomials closed under multiplication? Explain.

APPLICATION

## EXAMPLE 4 Write a Polynomial Function

**Hachi makes Seminole Indian dolls to sell at the local street market.**

**As Hachi produces a greater number of dolls, she can lower the price per unit. The function $v(x) = 49 - 2x$ relates the price $v$ to the number produced $x$. The cost $c$ of making $x$ dolls can be represented with the function $c(x) = 12x + 64$.**


**How many Seminole Indian dolls should Hachi sell each week to maximize her profit $P$?**

Formulate ◀ Write a function for revenue $R$ by multiplying the price $v(x) = 49 - 2x$ of each item by the number sold $x$.

$$R(x) = (49 - 2x)x$$

Then write the function for profit $P$.

$$P(x) = R(x) - c(x) \quad \text{Profit} = \text{Revenue} - \text{Cost}$$

$$= (49 - 2x)x - (12x + 64) \quad \text{Substitute for } R(x) \text{ and } c(x).$$

Compute ◀ Simplify the function.

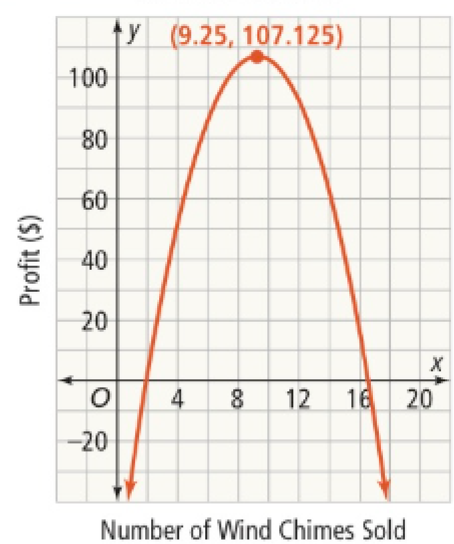
$$P(x) = (49 - 2x)x - (12x + 64) \quad \text{Write the profit function.}$$

$$= (49x - 2x^2) - (12x + 64) \quad \text{Use the Distributive Property.}$$

$$= 49x - 2x^2 - 12x - 64 \quad \text{Distribute the factor of } -1.$$

$$= -2x^2 + 37x - 64 \quad \text{Combine like terms.}$$

Hachi's profit function is $P(x) = -2x^2 + 37x - 64$.




Interpret ◀ Hachi's profit is modeled by a quadratic function. The domain of the function is the set of whole numbers. Her maximum profit cannot correspond to the vertex of the graph because she cannot sell a part of a doll. Consider $P(9) = 107$ and $P(10) = 106$ and choose the maximum profit of the two.

Hachi's best business plan is to produce and sell 9 Seminole Indian dolls per week, for a weekly profit of $107.

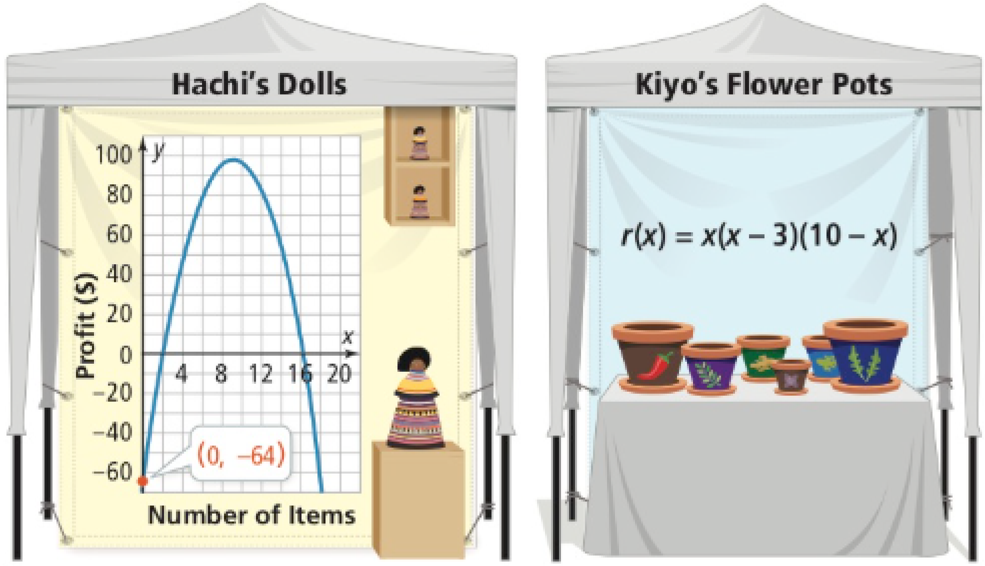
**Try It!** 4. The cost of Hachi's materials changes so that her new cost function is $c(x) = 4x + 42$.

Find the new profit function. Then find the quantity that maximizes profit and calculate the profit.

APPLICATION

## EXAMPLE 5 Compare Two Polynomial Functions

**Hachi's profit function, $y = P(x)$ is represented by the graph. Kiyo's profit from selling $x$ flowerpots can be modeled by the function shown.**




**A. Find the $y$-intercept of each function. Who would lose more money if neither person sold any items?**

The $y$-intercept of $P(x)$ is $-64$. If Hachi does not sell any wind chimes this week, she will lose \$64.

Hachi has startup costs, which she pays whether she sells any dolls or not.

Substitute 0 for $x$ in $r$ to find the $y$-intercept.

$r(0) = 0(0 - 3)(10 - 0)$ Substitute 0 for $x$.

$= 0$ Simplify.

Kiyo has no startup costs.

The $y$-intercept of $r$ is 0. If Kiyo does not sell any flowerpots, he will not lose any money. So, Hachi loses more money by not making any sales.

**B. Interpret the end behavior of the functions.**

The domain of each function includes only non-negative values.

The graph of $P(x)$ shows that the end behavior, as $x \to \infty$, is $y \to -\infty$.

Rewrite the function $r$ in standard form to identify end behavior.

$r(x) = x(x - 3)(10 - x)$

$= (x^2 - 3x)(10 - x)$ Use the Distributive Property.

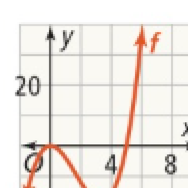
$= -x^3 + 13x^2 - 30x$ Combine like terms.

Because the leading coefficient is negative, we know that as $x \to \infty$, $y \to -\infty$.

GENERALIZE
As $x$ approaches positive infinity or negative infinity, the leading term of the polynomial determines the end behavior of the graph of the function.

Kiyo and Hachi each have a finite number of items they should sell to maximize profits. They will both lose money if they sell too many items.

**Try It!** 5. Compare the profit functions of two additional market sellers modeled by the graph of $f$ and the equation $g(x) = (x + 1)(5 - x)$. Compare and interpret the $y$-intercepts of these functions and their end behavior.



## CONCEPT SUMMARY Adding, Subtracting, and Multiplying Polynomials

**ADD** To add polynomials, use the Associative and Commutative Properties to group like terms. Then use the Distributive Property to combine like terms.

$$(2x^2 + 5x - 7) + (3x^2 - 9x + 12)$$
$$= (2x^2 + 3x^2) + (5x - 9x) + (-7 + 12)$$
$$= 5x^2 - 4x + 5$$

**SUBTRACT** To subtract polynomials, distribute the factor of −1. Then, group and combine like terms.

$$(6x^3 + 2x^2 + 14) - (4x^3 + 4x^2 - 8)$$
$$= 6x^3 + 2x^2 + 14 - 4x^3 - 4x^2 + 8$$

Distribute the factor of −1 to each term.

$$= (6x^3 - 4x^3) + (2x^2 - 4x^2) + (14 + 8)$$
$$= 2x^3 - 2x^2 + 22$$

**MULTIPLY** To multiply polynomials, use the Distributive Property. Then, group and combine like terms.

$$(x + 5)(3x^2 - 2x + 4)$$
$$= x(3x^2 - 2x + 4) + 5(3x^2 - 2x + 4)$$
$$= 3x^3 - 2x^2 + 4x + 15x^2 - 10x + 20$$
$$= 3x^3 + (-2x^2 + 15x^2) + (4x - 10x) + 20$$
$$= 3x^3 + 13x^2 - 6x + 20$$

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do you add, subtract, and multiply polynomials?

2. **Error Analysis** Chen subtracted two polynomials as shown. Explain Chen's error.

$$p^2 + 7mp + 4 - (-2p^2 - mp + 1)$$
$$p^2 + 2p^2 + 7mp - mp + 4 + 1$$
$$3p^2 + 6mp + 5 \quad ✗$$

3. **Communicate and Justify** Why do we often write the results of polynomial calculations in standard form?

4. **Represent and Connect** Is the set of whole numbers closed under subtraction? Explain why you think so, or provide a counterexample.

## Do You KNOW HOW?

**Add or subtract the polynomials.**

5. $(-3a^3 + 2a^2 - 4) + (a^3 - 3a^2 - 5a + 7)$

6. $\left(\frac{7}{12}x^2y^2 - 6x^3 + \frac{19}{4}xy\right) - \left(\frac{1}{3}x^2y^2 - x^3 + \frac{3}{2}xy + x\right)$

**Multiply the polynomials.**

7. $(7a + 2)(2a^2 - 5a + 3)$

8. $(xy - 1)(xy + 6)(xy - 8)$

9. The length of a rectangular speaker is three times its width, and the height is four more than the width. Write an expression for the volume $V$ of the rectangular prism in terms of its width $w$.

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

10. **Generalize** Explain two methods by which $(2m^3 + 4n^2)^2$ can be simplified. Which method do you prefer and why?

11. **Use Patterns and Structure** Polynomial function $P$ is the sum of two polynomial functions, one with degree 2 and a positive leading coefficient and one with degree 3 and a negative leading coefficient. Describe the end behavior of $P$. Write an example of two polynomial functions and their sum, $P$, to justify your description.

12. **Generalize** Multiply the polynomials $(a + b)(a + b)(a + b)$ to develop a general formula for cubing a binomial, $(a + b)^3$.

13. **Represent and Connect** Polynomial function $R$ is the difference of two degree-two polynomial functions. What are the possible degrees for $R$? Explain.

14. **Error Analysis** Describe and correct the error a student made in multiplying the polynomials.

$$(y - 2)(3y^2 - y - 7)$$
$$= y(3y^2 - y - 7) - 2(3y^2 - y - 7)$$
$$= 3y^3 - y^2 - 7y + (-6y^2) + (-2y) - 14$$
$$= 3y^3 - 7y^2 - 9y - 14 \quad ✗$$

15. **Higher Order Thinking** Do you think polynomials are closed under division? Explain why you think so, or provide a counterexample.

16. **Communicate and Justify** Explain why the expression $9x^3 + \frac{1}{2}x^2 + 3x^{-1}$ is not a polynomial.

17. **Check for Reasonableness** Explain the difference between the graphs of polynomial functions with a degree of 3 that have a positive leading coefficient and the graphs of those with a negative leading coefficient.

## PRACTICE

**Add or subtract the polynomials.** SEE EXAMPLE 1

18. $\left(\frac{1}{2}x^3 + 3x^2 + \frac{27}{4}\right) + \left(\frac{9}{7}x^3 - \frac{4}{5}x^2 - 5x\right)$

19. $(5y^4 + 3y^3 - 6y^2 + 14) - (-y^4 + y^2 - 7y - 1)$

20. $(4p^2q^2 + 2p^2q - 7pq) - (9p^2q^2 + 5pq^2 - 11pq)$

**Multiply the polynomials.** SEE EXAMPLE 2

21. $-\frac{2}{3}xy(5x^2 - \frac{12}{5}xy - \frac{2}{3}y^2)$

22. $(3c - 4)(2c^2 - 5c + 7)$

23. $(z + 5)(z - 9)(1 - z)$

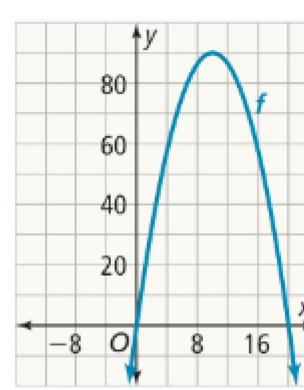
24. Is the set of monomials closed under addition? Explain why you think so, or provide a counterexample. SEE EXAMPLE 3

25. An online shopping club has 13,500 members when it charges \$8 per month for membership. For each \$1 monthly increase in membership fee, the club loses approximately 500 of its existing members.


Write and simplify a function $R$ to represent the monthly revenue received by the club when $x$ represents the price increase.

**Hint** Monthly revenue = # members • monthly fee SEE EXAMPLE 4

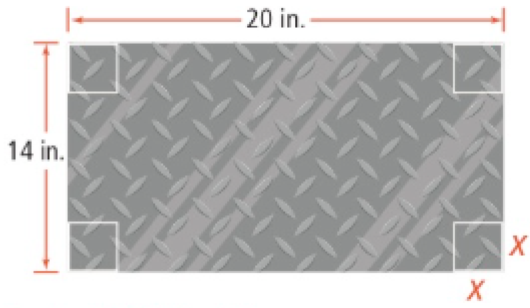
26. The graph shows a polynomial function $f$. Polynomial function $g$ is defined by $g(x) = x^2(6 - x)$. Compare the maximum values and the end behavior of the functions $f$ and $g$ when $x > 0$.
SEE EXAMPLE 5



## PRACTICE & PROBLEM SOLVING

### APPLY

**Use this information for 27 and 28. A foundry manufactures aluminum trays from pieces of sheet metal as shown.**

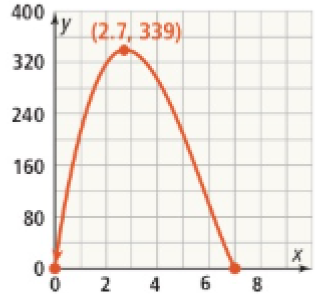



**27. Apply Math Models** Let $x$ represent the side length of the squares cut from the corners.

a. Write expressions for the length, width, and height of the metal tray.

b. Write and simplify a polynomial function $V$ to represent the volume of the tray. What is the domain of the function? Explain.

c. Using the graph of the function $V$, explain what the marked relative maximum represents.




**28. Analyze and Persevere** Suppose the foundry manufacturer has a new design where the squares cut from the corners have sides that are half the length of the squares in the previous design.

a. Write expressions for the length, width, and height of this tray.

b. Write and simplify the polynomial function $v(x)$, to represent the volume of the new tray.

c. Write and interpret the function $D(x)$ that represents the difference, $V(x) - v(x)$.

**29. Represent and Connect** Jacy has \$1,000 to invest in a fund that pays approximately 4.6% per year or in a savings account with an annual interest rate of 1.8%. Write a polynomial function $S(x)$ to represent the interest Jacy will earn in 1 year by investing $x$ dollars in the fund and the remainder in the savings account.

### ASSESSMENT PRACTICE

**30.** Which statement implies that polynomials are closed under multiplication? AR.1.3

Ⓐ The product of two polynomials is sometimes a polynomial.

Ⓑ The product of two polynomials is usually a polynomial.

Ⓒ The product of two polynomials is never a polynomial.

Ⓓ The product of two polynomials is always a polynomial.

**31. SAT/ACT** Which of the following functions is NOT a polynomial function?

Ⓐ $2y^2 + 9y - 8$

Ⓑ $-\frac{1}{2}x^3 + 8$

Ⓒ $(x - 1)(5 - x)(x + 4)$

Ⓓ $9z^4 + 2z + \frac{1}{z}$

**32. Performance Task** Consider the polynomial functions $P(x) = x^2 - 4$ and $R(x) = -x^2 - 2x$.

**Part A** Write and simplify a polynomial function $T(x)$ that is the product of $P$ and $R$.

**Part B** Copy and complete the table of values for all three functions.

| $x$ | $P(x)$ | $R(x)$ | $T(x)$ |
|---|---|---|---|
| –3 | | | |
| –2 | | | |
| –1 | | | |
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |

**Part C** Graph the functions on the same coordinate grid.

**Part D** How do the zeros of $T$ relate to the zeros of $P$ and $R$?

**Part E** Explain how you can identify the intervals in which $T$ is positive by analyzing the $R$ and $P$.

**Part F** Explain what happens to the domain of $T(x)$ if the domain of $R(x)$ is only valid for non-negative numbers.

# 3-3 Polynomial Identities

**I CAN…** prove and use polynomial identities.

**VOCABULARY**

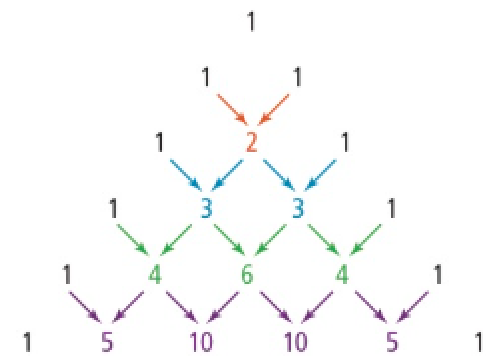
- Binomial Theorem
- identity
- Pascal's Triangle

MA.912.AR.1.8–Rewrite a polynomial expression as a product of polynomials over the real or complex number system.
Also AR.1.3, AR.1.11
MA.912. MTR.1.1, MTR.2.1, MTR.5.1

## EXPLORE & REASON

**Look at the following triangle.**

Each number is the sum of the two numbers diagonally above. If there is not a second number, think of it as 0.




A. Write the numbers in the next three rows.

B. **Communicate and Justify** What other patterns do you see?

C. Find the sum of the numbers in each row of the triangle. Write a formula for the sum of the numbers in the $n^{\text{th}}$ row.

## ESSENTIAL QUESTION

**How can you use polynomial identities to rewrite expressions efficiently?**

## CONCEPT Polynomial Identities

A mathematical statement that equates two polynomial expressions is an **identity** if one side can be transformed into the other side using mathematical operations. These polynomial identities are helpful tools used to multiply and factor polynomials. Identities are equations that are true for all values.

**Difference of Squares**

$a^2 - b^2 = (a + b)(a - b)$

**Example:** $25x^2 - 36y^2$

Substitute $5x$ for $a$ and $6y$ for $b$.

$25x^2 - 36y^2 = (5x + 6y)(5x - 6y)$

**Square of a Sum**

$(a + b)^2 = a^2 + 2ab + b^2$

**Example:** $(3x + 4y)^2$

Substitute $3x$ for $a$ and $4y$ for $b$.

$$(3x + 4y)^2 = (3x)^2 + 2(3x)(4y) + (4y)^2$$
$$= 9x^2 + 24xy + 16y^2$$

**Difference of Cubes**

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

**Example:** $8m^3 - 27$

Substitute $2m$ for $a$ and $3$ for $b$.

$$8m^3 - 27 = (2m - 3)[(2m)^2 + (2m)(3) + 3^2]$$
$$= (2m - 3)(4m^2 + 6m + 9)$$

**Sum of Cubes**

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

**Example:** $g^3 + 64h^3$

Substitute $g$ for $a$ and $4h$ for $b$.

$$g^3 + 64h^3 = (g + 4h)[g^2 - (g)(4h) + (4h)^2]$$
$$= (g + 4h)(g^2 - 4gh + 16h^2)$$

## EXAMPLE 1 Prove a Polynomial Identity

**How can you prove the Sum of Cubes Identity, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$?**

To prove an identity, start with the expression on one side of the equation and use properties of operations on polynomials to transform it into the expression on the other side.

**REPRESENT AND CONNECT**
Another way to establish the identity is to multiply each term of the second factor by $(a + b)$, and then combine like terms.

$(a + b)(a^2 - ab + b^2)$

$= a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$ Use the Distributive Property.

$= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$ Use the Distributive Property.

$= a^3 + (-a^2b + a^2b) + (ab^2 - ab^2) + b^3$ Group like terms.

$= a^3 + b^3$ Combine like terms.

So, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

**Try It!** **1.** Prove the Difference of Cubes Identity.

## EXAMPLE 2 Use Polynomial Identities to Multiply

**How can you use polynomial identities to multiply expressions?**

**A.** $(2x^2 + y^3)^2$

The sum is a binomial, and the entire sum is being raised to the second power.

**COMMON ERROR**
When finding $(a + b)^2$, recall that it is not sufficient to square the first term and square the second term. You must distribute the two binomials.

Use the Square of a Sum Identity to find the product:

$(a + b)^2 = a^2 + 2ab + b^2$

$(2x^2 + y^3)^2 = (2x^2)^2 + 2(2x^2)(y^3) + (y^3)^2$ Substitute $2x^2$ for $a$ and $y^3$ for $b$.

$= 4x^4 + 4x^2y^3 + y^6$ Simplify.

So, $(2x^2 + y^3)^2 = 4x^4 + 4x^2y^3 + y^6$.

**B.** $41 \cdot 39$

$41 \cdot 39 = (40 + 1)(40 - 1)$ Rewrite values using 40 and 1.

Use the Difference of Squares Identity:

$(40 + 1)(40 - 1) = 40^2 - 1^2$

$= 1{,}600 - 1$

$= 1{,}599$

So $41 \cdot 39 = 1{,}599$.

**Try It!** **2.** Use polynomial identities to multiply the expressions.

**a.** $(3x^2 + 5y^3)(3x^2 - 5y^3)$ **b.** $(12 + 15)^2$

## EXAMPLE 3 Use Polynomial Identities to Factor and Simplify

**How can you use polynomial identities to factor polynomials and simplify numerical expressions?**

**A.** $9m^4 - 25n^6$

$9m^4$ and $25n^6$ are both perfect squares.

A square term includes an even exponent, not necessarily an exponent that is a perfect square.

$$9m^4 = (3m^2)^2$$

$$25n^6 = (5n^3)^2$$

Use the Difference of Squares Identity: $a^2 - b^2 = (a + b)(a - b)$.

$$9m^4 - 25n^6 = (3m^2)^2 - (5n^3)^2 \quad \text{Express each term as a square.}$$

$$= (3m^2 + 5n^3)(3m^2 - 5n^3) \quad \text{Write the factors.}$$

So, $9m^4 - 25n^6 = (3m^2 + 5n^3)(3m^2 - 5n^3)$.

**B.** $16g^8 + 49$

Use the Difference of Squares Identity to factor over the complex numbers.

$$16g^8 + 49 = (4g^4)^2 + (7)^2 \quad \text{Express each term as a square.}$$

$$= (4g^4)^2 - i^2 \cdot (7)^2 \quad \text{Substitute } 1 = -i^2.$$

$$= (4g^4)^2 - (7i)^2 \quad \text{Rewrite the expression as a difference of squares.}$$

$$= (4g^4 + 7i)(4g^4 - 7i) \quad \text{Write the factors.}$$

So, $16g^8 + 49 = (4g^4 + 7i)(4g^4 - 7i)$.

**COMMON ERROR**

The second factor is *almost* a Square of a Sum. Remember that the middle term of the Difference of Cubes Identity is the product *ab*, not 2*ab*.

**C.** $x^3 - 216$

$x^3$ and 216 are both perfect cubes.

$$x^3 = (x)^3$$

$$216 = 6^3$$

Use the Difference of Cubes Identity: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$$x^3 - 216 = (x)^3 - (6)^3 \quad \text{Express each term as a cube.}$$

$$= (x - 6)(x^2 + 6x + 36) \quad \text{Write the factors.}$$

So, $x^3 - 216 = (x - 6)(x^2 + 6x + 36)$.

**D.** $11^3 + 5^3$

Use the Sum of Cubes Identity: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$$11^3 + 5^3 = (11 + 5)(11^2 - 11(5) + 5^2)$$

$$= (16)(121 - 55 + 25)$$

$$= 16(91)$$

$$= 1{,}456$$

So, $11^3 + 5^3 = 1{,}456$.

**Try It!** **3.** Use polynomial identities to factor each polynomial.

**a.** $m^8 - 9n^{10}$ **b.** $25 + 4b^4$

**c.** $27x^9 - 343y^6$ **d.** $12^3 + 2^3$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 4 Expand a Power of a Binomial

**How is $(x + y)^n$ obtained from $(x + y)^{n-1}$?**

**A. What are $(x + y)^3$ and $(x + y)^4$?**

$$(x + y)^3 = (x + y)(x + y)^2$$
$$= (x + y)(x^2 + 2xy + y^2)$$

$$1x^2 \quad + \quad 2xy \quad + \quad 1y^2$$
$$(x \;+\; y) \qquad (x \;+\; y) \qquad (x \;+\; y)$$
$$x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3$$
$$1x^3 + 3x^2y + 3xy^2 + 1y^3 = (x + y)^3$$

Multiply each term by $(x + y)$ to get $(x + y)^3$.

$$(x \quad y) \quad (x \quad y) \quad (x \quad y) \quad (x \quad y)$$
$$x^4 + x^3y + 3x^3y + 3x^2y^2 + 3x^2y^2 + 3xy^3 + xy^3 + y^4$$
$$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 = (x + y)^4$$

Multiply each term in $(1x^3 + 3x^2y + 3xy^2 + y^3)$ by $(x + y)$ to get $(x + y)^4$.

The coefficients of $(x + y)^n$ are produced by adding the coefficients of $(x + y)^{n-1}$, producing an array known as Pascal's Triangle. **Pascal's Triangle** is the triangular pattern of numbers where each number is the sum of the two numbers diagonally above it. If there is not a second number diagonally above in the triangle, think of the missing number as 0.

| | | | |
|---|---|---|---|
| Row 0 | 1 | $1$ | $(x + y)^0$ |
| Row 1 | 1 1 | $1x + 1y$ | $(x + y)^1$ |
| Row 2 | 1 2 1 | $1x^2 + 2xy + 1y^2$ | $(x + y)^2$ |
| Row 3 | 1 3 3 1 | $1x^3 + 3x^2y + 3xy^2 + 1y^3$ | $(x + y)^3$ |
| Row 4 | 1 4 6 4 1 | $1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$ | $(x + y)^4$ |

**STUDY TIP**
Notice the patterns of the powers. The powers of $x$ decrease from $n$ to 0 and the powers of $y$ increase from 0 to $n$ when reading the terms from left to right.

You can obtain $(x + y)^n$ by adding adjacent pairs of coefficients from $(x + y)^{n-1}$.

**B. Use Pascal's Triangle to expand $(x + y)^5$.**

Add pairs of coefficients from Row 4 to complete Row 5.

| | | | | | | |
|---|---|---|---|---|---|---|
| Row 4 | 1 | 4 | 6 | 4 | 1 | |
| Row 5 | 1 | 5 | 10 | 10 | 5 | 1 |

Write the expansion. Use the coefficients from Row 5 with powers of $x$ starting at 5 and decreasing to 0 and with powers of $y$ starting at 0 and increasing to 5.

The sum of the exponents in each term is equal to the exponent on the original binomial.

$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

**Try It!** **4.** Use Pascal's Triangle to expand $(x + y)^6$.

## CONCEPT Binomial Theorem

The **Binomial Theorem** states that, for every positive integer $n$,

$$(a+b)^n = C_0a^n + C_1a^{n-1}b + C_2a^{n-2}b^2 + \ldots + C_{n-1}ab^{n-1} + C_nb^n.$$

The coefficients $C_0, C_1, C_2, \ldots, C_{n-1}, C_n$ are the numbers in Row $n$ of Pascal's Triangle.

Notice that the powers of $a$ are decreasing while the powers of $b$ are increasing, and that the sum of the powers of $a$ and $b$ in each term is always $n$.

## EXAMPLE 5 Apply the Binomial Theorem

**Use the Binomial Theorem to expand the expressions.**

**A. Find $(x-3)^4$.**

**Step 1** Use the Binomial Theorem to write the expansion when $n = 4$.

$$C_0a^4 + C_1a^3b + C_2a^2b^2 + C_3ab^3 + C_4b^4$$

**Step 2** Use Row 4 in Pascal's Triangle to write the coefficients.

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

**Step 3** Identify $a$ and $b$.

$$a = x \text{ and } b = -3$$

**COMMON ERROR**
Remember that the base of $(a+b)^n$ in the Binomial Theorem is $(a+b)$. If the terms are being subtracted, use the opposite of $b$ in the expansion.

**Step 4** Substitute $x$ for $a$ and $-3$ for $b$ in the pattern. Then simplify.

$$x^4 + 4x^3(-3) + 6x^2(-3)^2 + 4x(-3)^3 + (-3)^4$$

$$x^4 - 12x^3 + 54x^2 - 108x + 81$$

So $(x-3)^4 = x^4 - 12x^3 + 54x^2 - 108x + 81$.

**B. Find $(s^2+3)^5$.**

The expansion of $(a+b)^5$ is $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

Since $a = s^2$ and $b = 3$, the expansion is:

$$(s^2+3)^5 = (s^2)^5 + 5(s^2)^4(3) + 10(s^2)^3(3)^2 + 10(s^2)^2(3)^3 + 5(s^2)(3)^4 + (3)^5$$
$$= s^{10} + 15s^8 + 90s^6 + 270s^4 + 405s^2 + 243$$

So $(s^2+3)^5 = s^{10} + 15s^8 + 90s^6 + 270s^4 + 405s^2 + 243$.

**Try It!** **5.** Use the Binomial Theorem to expand each expression.

**a.** $(x-1)^7$ **b.** $(2c+d)^6$

## CONCEPT SUMMARY Polynomial Identities

**POLYNOMIAL IDENTITIES**

Special polynomial identities can be used to multiply and factor polynomials.

**Difference of Squares**

$a^2 - b^2 = (a + b)(a - b)$

**Square of a Sum**

$(a + b)^2 = a^2 + 2ab + b^2$

**Difference of Cubes**

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

**Sum of Cubes**

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

**BINOMIAL EXPANSION**

The binomial expansion of $(a + b)^n$ has the following properties:

1) The expansion contains $n + 1$ terms.

2) The coefficients of each term are numbers from the $n$th row of Pascal's Triangle.

3) The exponent of $a$ is $n$ in the first term and decreases by 1 in each successive term.

4) The exponent of $b$ is 0 in the first term and increases by 1 in each successive term.

5) The sum of the exponents in any term is $n$.

| | | | |
|---|---|---|---|
| Row 0 | 1 | $1$ | $(x + y)^0$ |
| Row 1 | 1 1 | $1x + 1y$ | $(x + y)^1$ |
| Row 2 | 1 2 1 | $1x^2 + 2xy + 1y^2$ | $(x + y)^2$ |
| Row 3 | 1 3 3 1 | $1x^3 + 3x^2y + 3xy^2 + 1y^3$ | $(x + y)^3$ |
| Row 4 | 1 4 6 4 1 | $1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$ | $(x + y)^4$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you use polynomial identities to rewrite expressions efficiently?

2. **Analyze and Persevere** Explain why the middle term of $(x + 5)^2$ is $10x$.

3. **Use Patterns and Structure** Explain how to use a polynomial identity to factor $8x^6 - 27y^3$.

4. **Communicate and Justify** How are Pascal's Triangle and a binomial expansion, such as $(a + b)^5$, related?

5. **Analyze and Persevere** What number does $C_3$ represent in the expansion $C_0a^5 + C_1a^4b + C_2a^3b^2 + C_3a^2b^3 + C_4ab^4 + C_5b^5$? Explain.

6. **Error Analysis** Dakota said the third term of the expansion of $(2g + 3h)^4$ is $36g^2h^2$. Explain Dakota's error. Then correct the error.

## Do You KNOW HOW?

**Use polynomial identities to multiply each expression.**

7. $(2x + 8y)(2x - 8y)$

8. $(5c - 6i)(5c + 6i)$

9. $(x + 3y^3)^3$

**Use polynomial identities to factor each polynomial.**

10. $36a^6 - 4b^2$

11. $121y^{12} + 49z^4$

12. $8x^6 - y^3$

13. $m^9 + 27n^6$

**Find the term of the binomial expansion.**

14. fifth term of $(x + y)^5$

15. third term of $(a - 3)^6$

**Use Pascal's Triangle or the Binomial Theorem to expand each expression.**

16. $(x + 1)^5$

17. $(a - b)^6$

18. $(d - 3)^4$

19. $(2x + 4y)^7$

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

20. **Error Analysis** Emma factored $625g^{16} - 25h^4$. Describe and correct the error Emma made in factoring the polynomial.

$$625g^{16} - 25h^4$$
$$= (25g^4)^2 - (5h^2)^2$$
$$= (25g^4 + 5h^2)(25g^4 - 5h^2)$$

21. **Use Patterns and Structure** Factor $x^3 - 125y^6$ in the form $(x - A)(x^2 + Bx + C)$. What are the values of $A$, $B$, and $C$?

22. **Analyze and Persevere** How could you use polynomial identities to factor the expression $x^6 - y^6$?

23. **Use Patterns and Structure** Expand $(3x + 4y)^3$ using Pascal's Triangle and the Binomial Theorem.

24. **Higher Order Thinking** Use Pascal's Triangle and the Binomial Theorem to expand and simplify the complex number $(x + i)^4$. Justify your work.

25. **Use Patterns and Structure** Expand the expression $(2x - 1)^4$. What is the sum of the coefficients?

26. **Error Analysis** A student says that the expansion of the expression $(-4y + z)^7$ has seven terms. Describe and correct the error the student may have made.

27. **Use Patterns and Structure** The sum of the coefficients in the expansion of the expression $(a + b)^n$ is 64. Use Pascal's Triangle to find the value of $n$.

28. **Generalize** How many terms will there be in the expansion of the expression $(x + 3)^n$? Explain how you know.

### PRACTICE

29. Prove the polynomial identity.
$x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$
SEE EXAMPLE 1

**Use polynomial identities to multiply the expressions.** SEE EXAMPLE 2

30. $(x + 9)(x - 9)$
31. $(x + 6)^2$
32. $(3x - 7)^2$
33. $(2x - 5)(2x + 5)$
34. $(4x^2 + 6y^2)(4x^2 - 6y^2)$
35. $(x^2 + y^6)^2$
36. $(8 - x^2)(8 + x^2)$
37. $(6 - y^3)^2$
38. $18 \bullet 22$
39. $103 \bullet 97$
40. $(7 + 9)^2$
41. $(10 + 5)^2$

**Use polynomial identities to factor the polynomials or simplify the expressions.** SEE EXAMPLE 3

42. $x^8 - 9$
43. $x^9 - 8$
44. $8x^3 + y^9$
45. $x^6 - 27y^3$
46. $4x^2 + 144$
47. $216 + 27y^{12}$
48. $64x^3 - 125y^6$
49. $\frac{1}{16}x^6 - 25y^4$
50. $9^3 + 6^3$
51. $10^3 + 5^3$
52. $10^3 - 3^3$
53. $8^3 - 2^3$

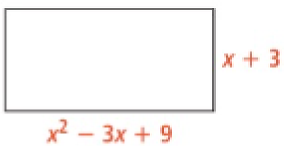
**Use the Binomial Theorem to expand the expressions.** SEE EXAMPLES 4 and 5

54. $(x + 3)^3$
55. $(2a - b)^5$
56. $\left(b - \frac{1}{2}\right)^4$
57. $(x^2 + 1)^4$
58. $\left(2x + \frac{1}{3}\right)^3$
59. $(x^3 + y^2)^6$
60. $(d - 3)^4$
61. $(2m + 2n)^6$
62. $(n + 5)^5$
63. $(3x - 0.2)^3$
64. $(4g + 2h)^4$
65. $\left(m^2 + \frac{1}{2}n\right)^3$

## PRACTICE & PROBLEM SOLVING

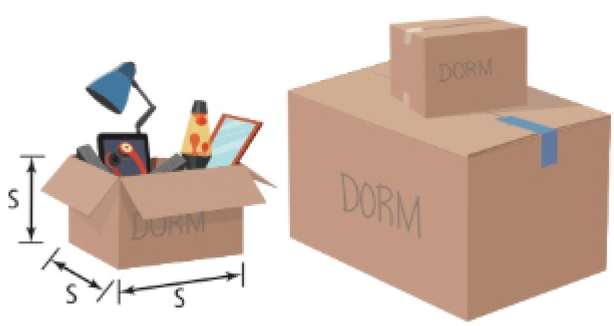
### APPLY

**66. Use Structure** The dimensions of a rectangle are shown. Write the area of the rectangle as a sum of cubes.




**67.** A Pythagorean triple is a set of three positive integers $a$, $b$, and $c$ that satisfy $a^2 + b^2 = c^2$. The identity $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$ can be used to generate Pythagorean triples. Use the identity to generate a Pythagorean triple when $x = 5$ and $y = 4$.

**68. Reason** A medium-sized shipping box with side length $s$ units has a volume of $s^3$ cubic units.




**a.** A large shipping box has side lengths that are 3 units longer than the medium shipping box. Write a binomial expression for the volume of the large shipping box.

**b.** Expand the polynomial in part a to simplify the volume of the large shipping box.

**c.** A small shipping box has side lengths that are 2 units shorter than the medium shipping box. Write a binomial expression for the volume of the small shipping box.

**d.** Expand the polynomial in part c to simplify the volume of the small shipping box.

### ASSESSMENT PRACTICE

**69.** Select all the perfect-square trinomials.
AR.1.8

☐ **A.** $x^2 + 16x + 64$

☐ **B.** $4x^2 - 44x + 121$

☐ **C.** $9x^2 - 15x + 25$

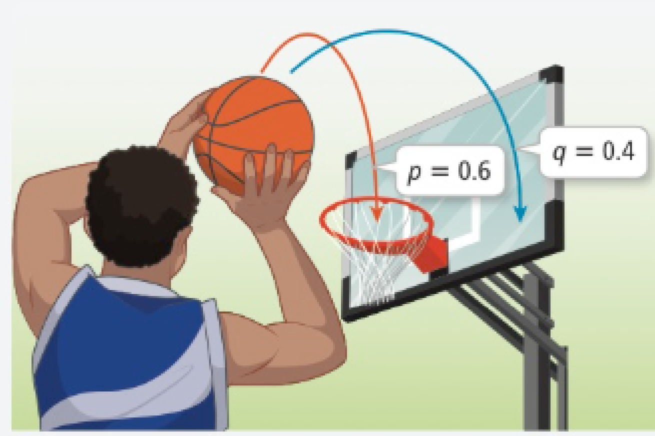
☐ **D.** $4x^2 + 64x + 16$

☐ **E.** $9x^2 - 42x + 49$

**70. SAT/ACT** How many terms are in the expansion of $(2x + 7y)^9$?

Ⓐ 2 Ⓑ 7 Ⓒ 8 Ⓓ 9 Ⓔ 10

**71. Performance Task** If an event has a probability of success $p$ and a probability of failure $q$, then each term in the expansion of $(p + q)^n$ represents a probability. To find the probability the basketball player will make exactly $h$ out of $k$ free throws, find $C_{k-h}p^h q^{k-h}$, where $C_{k-h}$ is a coefficient of row $k$ of Pascal's Triangle, $p$ is the probability of success, and $q$ is the probability of failure.




**Part A** If a player makes 60% of his free throw attempts, $p = 0.6$ and $q = 0.4$. What is the probability the basketball player will make exactly 6 out of 10 free throws? Round to the nearest percent.

**Part B** Another basketball player makes 80% of her free throw attempts. Write an expression to find the probability of this basketball player making exactly 7 out of 10 free throws. Describe what each variable in the expression represents.

**Part C** Evaluate the expression found in Part B.

# 3-4 Dividing Polynomials

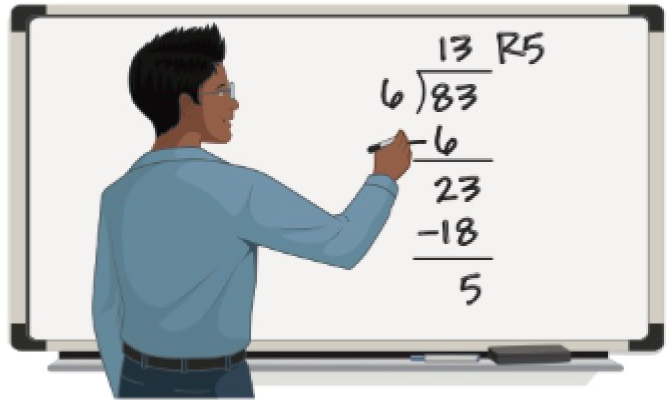
**I CAN…** divide polynomials.

**VOCABULARY**
- Remainder Theorem
- synthetic division

MA.912.AR.1.5–Divide polynomial expressions using long division, synthetic division or algebraic manipulation.
Also AR.1.6, AR.1.8, AR.6.2
MA.912. MTR.4.1, MTR.5.1, MTR.6.1

## EXPLORE & REASON

**Benson recalls how to divide whole numbers by solving a problem with 6 as the divisor and 83 as the dividend. He determines that the quotient is 13 with remainder 5.**




**A.** Explain the process of long division using Benson's example.

**B.** Use the result of Benson's long division to write $\frac{83}{6}$ as a mixed fraction.

**C.** **Generalize** Use the results of the division problem to write two expressions for 83 that include the divisor, quotient, and remainder.

## ESSENTIAL QUESTION

**How can you divide polynomials?**

## EXAMPLE 1 Use Long Division to Divide Polynomials

**How can you use long division to divide $P(x)$ by $D(x)$? Write the polynomial $P(x)$ in terms of the quotient and remainder.**

**USE PATTERNS AND STRUCTURE**
Compare the long division of these two polynomials to this numerical long division problem.

$$\begin{array}{r} 120 \\ 13\overline{)1{,}569} \\ -13 \\ \hline 26 \\ -26 \\ \hline 09 \end{array}$$

**A.** **Let $P(x) = x^3 + 5x^2 + 6x + 9$ and $D(x) = x + 3$.**

Long division of polynomials is similar to long division of numbers.

$$\begin{array}{r} x^2 + 2x \\ x+3\overline{)x^3 + 5x^2 + 6x + 9} \\ -(x^3 + 3x^2) \\ \hline 2x^2 + 6x + 9 \\ -(2x^2 + 6x) \\ \hline 9 \end{array}$$

Divide the leading terms: $x^3 \div x = x^2$.

Multiply: $x^2(x + 3) = x^3 + 3x$. Then subtract.

Divide the leading terms again: $2x^2 \div x = 2x$.

Multiply: $2x(x + 3) = 2x + 6x$.

Subtract. The remainder is 9.

When you divide polynomials, you can express the relationship of the quotient and remainder to the dividend and divisor in two ways.

$$\frac{P(x)}{D(x)} = \frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$\frac{x^3 + 5x^2 + 6x + 9}{x + 3} = x^2 + 2x + \frac{9}{x + 3}$$

dividend = quotient × divisor + remainder

$$x^3 + 5x^2 + 6x + 9 = (x^2 + 2x)(x + 3) + 9$$

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

**B. Let $P(x) = 8x^3 + 27$ and $D(x) = 2x + 3$.**

The dividend is a cubic polynomial with no first- or second-degree term.

$$\begin{array}{r}
4x^2 - 6x + 9 \\
2x + 3\overline{)8x^3 + 0x^2 + 0x + 27} \\
\underline{-(8x^3 + 12x^2)} \phantom{+ 0x + 27} \\
-12x^2 + 0x + 27 \\
\underline{-(-12x^2 - 18x)} \phantom{+ 27} \\
18x + 27 \\
\underline{-(18x + 27)} \\
0
\end{array}$$

Use 0 as the coefficient for the missing 1st- and 2nd-degree terms.

The remainder is 0. This means the divisor is a factor of the dividend.

**CHECK FOR REASONABLENESS**
How do you know when you have completed the long division?

So, $\frac{8x^3 + 27}{2x + 3} = 4x^2 - 6x + 9$ and $8x^3 + 27 = (2x + 3)(4x^2 - 6x + 9)$.

**Try It!** **1.** Use long division to divide the polynomials. Then write the dividend in terms of the quotient and remainder.

**a.** $x^3 - 6x^2 + 11x - 6$ divided by $x^2 - 4x + 3$

**b.** $16x^4 - 85$ divided by $4x^2 + 9$

## EXAMPLE 2 Use Synthetic Division to Divide by $x - a$

**What is $2x^3 - 7x^2 - 4$ divided by $x - 3$? Use synthetic division.**

**Synthetic division** is a method used to divide a polynomial by a linear expression in the form $x - a$. Note that the leading coefficient of the divisor is 1, and that $a$ is the zero of the divisor.

**Step 1** To change from long division format to synthetic division format, write only the zero of the divisor and the coefficients of the dividend.

$$\begin{array}{rrrrr}
x - 3 \,\big) & 2x^3 & -\,7x^2 & +\,0x & -\,4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 \,\big| & 2 & -7 & 0 & -4
\end{array}$$

zero of the divisor, $a$

coefficients of the dividend

Separate $a$ from the dividend.

**Step 2** Bring down the first coefficient. Multiply the zero of the divisor by the first coefficient. Add the result to the second coefficient.

$$\begin{array}{r|rrrr}
3 & 2 & -7 & 0 & -4 \\
 & & 6 & & \\
\hline
 & 2 & -1 & &
\end{array}$$

$3 \cdot 2 = 6$

$-7 + 6 = -1$

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

**Step 3** Repeat the process until all the columns are complete.

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & -4 \\ & & 6 & -3 & \\ \hline & 2 & -1 & -3 & \end{array}$$

$3 \cdot -1 = -3$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & -4 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -13 \end{array}$$

$3 \cdot -3 = -9$

**Step 4** Use the numbers in the last row to write the quotient and remainder.

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & -4 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -13 \end{array}$$

coefficients of the quotient

remainder

$$2x^2 - 1x - 3 - \frac{13}{x-3}$$

Since the dividend is cubic and the divisor is linear, the quotient is quadratic.

So, $\frac{2x^3 - 7x^2 - 4}{(x-3)} = 2x^2 - x - 3 - \frac{13}{x-3}$.

**COMMON ERROR**

Remember to keep track of all positive and negative signs when multiplying.

**USE PATTERNS AND STRUCTURE**

You can use the result to write the dividend $2x^3 - 7x - 4$ in the form $(2x^2 - x - 3)(x - 3) - 13$.

**Try It!** **2.** Use synthetic division to divide $3x^3 - 5x + 10$ by $x - 1$.

CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Relate $P(a)$ to the Remainder of $P(x) \div (x - a)$

**How is the value of $P(a)$ related to the remainder of $P(x) \div (x - a)$?**

To explore this question, let $P(x) = x^3 + 10x^2 + 29x + 24$. Use synthetic division to divide $P(x)$ by $x + 5$.

To identify the value of $a$, write the divisor $x + 5$ in the form $x - a$.

$x + 5 = x - (-5)$ $a = -5$

$$\begin{array}{r|rrrr} -5 & 1 & 10 & 29 & 24 \\ & & -5 & -25 & -20 \\ \hline & 1 & 5 & 4 & 4 \end{array}$$

The remainder is 4.

The quotient is $x^2 + 5x + 4$.

So, $P(x) = (x^2 + 5x + 4)(x + 5) + 4$. Use this form to evaluate $P(-5)$.

$$\begin{aligned} P(-5) &= [((-5)^2 + 5(-5) + 4)(-5 + 5)] + 4 \\ &= [(25 - 25 + 4)(0)] + 4 \\ &= 0 + 4 \\ &= 4 \end{aligned}$$

$-5$ is the zero of the divisor, so the product of the quotient and the divisor is 0.

So, $P(-5)$ is the remainder, 4, of $P(x)$ divided by $x -(-5)$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

In general, dividing $P(x)$ by $x - a$ results in a quotient $Q(x)$ and a remainder $r$.

$$\begin{aligned} P(x) &= Q(x)(x-a) + r \\ P(a) &= Q(x)(a-a) + r \\ &= Q(x)(0) + r \\ &= r \end{aligned}$$

Evaluating $P(x)$ at $a$, the zero of the divisor, shows that $P(a) = r$.

So, for a polynomial $P(x)$ the value of $P(a)$ is equal to the remainder of the division $P(x) \div (x - a)$.

**Try It!** **3.** Use synthetic division to show that the remainder of $f(x) = x^3 + 8x^2 + 12x + 5$ divided by $x + 2$ is equal to $f(-2)$.

## CONCEPT Remainder Theorem

The **Remainder Theorem** states that if a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

When $x - a$ is a factor of $P(x)$, we can show that $P(a) = 0$.

$$\begin{aligned} P(x) &= Q(x)(x-a) \\ P(a) &= Q(x)(a-a) \\ P(a) &= 0 \end{aligned}$$

Conversely, when $P(a) = 0$, we can show that $x - a$ is a factor.

$$\begin{aligned} P(x) &= Q(x)(x-a) + r \\ P(x) &= Q(x)(x-a) + P(a) \\ P(x) &= Q(x)(x-a) + 0 \\ P(x) &= Q(x)(x-a) \end{aligned}$$

Use the Remainder Theorem.

So, $x - a$ is a factor of a polynomial $P(x)$ if and only if $P(a) = 0$.

APPLICATION

## EXAMPLE 4 Use the Remainder Theorem to Evaluate Polynomials

**The population of tortoises on an island is modeled by the function $P(x) = -x^3 + 6x^2 + 12x + 325$ where $x$ is the number of years since 2015. Use the Remainder Theorem to estimate the population in 2023.**

Use synthetic division to find $P(a)$ when $a = 8$.

$$\begin{array}{r|rrrr} 8 & -1 & 6 & 12 & 325 \\ & & -8 & -16 & -32 \\ \hline & -1 & -2 & -4 & 293 \end{array}$$

The estimated population in 2023 is 293 tortoises.

**Try It!** **4.** A technology company uses the function $R(x) = -x^3 + 12x^2 + 6x + 80$ to model expected annual revenue, in thousands of dollars, for a new product, where $x$ is the number of years after the product is released. Use the Remainder Theorem to estimate the revenue in year 5.

## EXAMPLE 5 Check Whether $x - a$ is a Factor of $P(x)$

**How can you use the Remainder Theorem to determine whether the given binomial is a factor of $P(x)$? If it is a factor, write the polynomial in factored form.**

**A.** $P(x) = x^4 - 8x^3 + 16x^2 - 23x - 6$**; binomial:** $x - 6$

The binomial $x - 6$ is a factor of $P(x)$ if 6 is a zero of $P(x)$.

**Method 1** Use synthetic substitution.

$$\begin{array}{r|rrrrr} 6 & 1 & -8 & 16 & -23 & -6 \\ & & 6 & -12 & 24 & 6 \\ \hline & 1 & -2 & 4 & 1 & 0 \end{array}$$

**Method 2** Use direct substitution.

$$\begin{aligned} P(6) &= 6^4 - 8(6^3) + 16(6^2) - 23(6) - 6 \\ &= 1{,}296 - 1{,}728 + 576 - 138 - 6 \\ &= 0 \end{aligned}$$

Because $P(6) = 0$, you can conclude that $x - 6$ is a factor of $P(x)$: $P(x) = (x^3 - 2x^2 + 4x + 1)(x - 6)$.

**B.** $P(x) = x^5 - 5x^3 + 9x^2 - x + 3$**; binomial:** $x + 3$

**COMMON ERROR**
When using synthetic division, remember to include 0 coefficients for any missing terms.

**Method 1** Use synthetic substitution.

$$\begin{array}{r|rrrrrr} -3 & 1 & 0 & -5 & 9 & -1 & 3 \\ & & -3 & 9 & -12 & 9 & -24 \\ \hline & 1 & -3 & 4 & -3 & 8 & -21 \end{array}$$

**Method 2** Use direct substitution.

$$P(-3) = (-3)^5 - 5((-3)^3) + 9((-3)^2) - (-3) + 3$$

$$P(-3) = -243 + 135 + 81 + 3 + 3$$

$$P(-3) = -21$$

Because $-3$ is a not a zero of $P(x)$, you can conclude that $x + 3$ is not a factor of $P(x)$.

**Try It!** **5.** Use the Remainder Theorem to determine whether the given binomial is a factor of $P(x)$.

**a.** $P(x) = x^3 - 10x^2 + 28x - 16$; binomial: $x - 4$

**b.** $P(x) = 2x^4 + 9x^3 - 2x^2 + 6x - 40$; binomial: $x + 5$

## CONCEPT SUMMARY Dividing Polynomials

**Example:** Divide $x^3 - 8x^2 - 5x - 30$ by $x - 9$.

### LONG DIVISION

Can be used for any polynomial division.

$$\begin{array}{r} x^2 + \phantom{0}x + \phantom{0}4 \\ x - 9\overline{)x^3 - 8x^2 - 5x - 30} \\ \underline{-(x^3 - 9x^2)\phantom{- 5x - 30}} \\ x^2 - 5x - 30 \\ \underline{-(x^2 - 9x)\phantom{- 30}} \\ 4x - 30 \\ \underline{-(4x - 36)} \\ 6 \end{array}$$

### SYNTHETIC DIVISION

Most readily used when the divisor is linear and its leading coefficient is 1.

$$\begin{array}{r|rrrr} 9 & 1 & -8 & -5 & -30 \\ & & 9 & 9 & 36 \\ \hline & 1 & 1 & 4 & 6 \end{array}$$

Either method shows that $x^3 - 8x^2 - 5x - 30 = (x^2 + x + 4)(x - 9) + 6$

### REMAINDER THEOREM

If a polynomial $P(x)$ is divided by a linear divisor $x - a$, the remainder is $P(a)$.

$P(x) = x^3 - 2x + 1$ divided by $x - 2$ has remainder 5.

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -2 & 1 \\ & & 2 & 4 & 4 \\ \hline & 1 & 2 & 2 & 5 \end{array}$$

$P(2) = 5$

The binomial $x - a$ is a factor of $P(x)$ if and only if $P(a) = 0$.

$P(x) = 2x^4 - 5x^3 - 12x^2 + x - 4$ divided by $x - 4$ has remainder 0.

$$\begin{array}{r|rrrrr} 4 & 2 & -5 & -12 & 1 & -4 \\ & & 8 & 12 & 0 & 4 \\ \hline & 2 & 3 & 0 & 1 & 0 \end{array}$$

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you divide polynomials?

2. **Check for Reasonableness** You divide a polynomial $P(x)$ by a linear expression $D(x)$. You find a quotient $Q(x)$ and a remainder $R(x)$. How can you check your work?

3. **Error Analysis** Ella said the remainder of $x^3 + 2x^2 - 4x + 6$ divided by $x + 5$ is 149. Is Ella correct? Explain.

## Do You KNOW HOW?

4. Use long division to divide $x^4 - 4x^3 + 12x^2 - 3x + 6$ by $x^2 + 8$.

5. Use synthetic division to divide $x^3 - 8x^2 + 9x - 5$ by $x - 3$.

6. Use the Remainder Theorem to find the remainder of $2x^4 + x^2 - 10x - 1$ divided by $x + 2$.

7. Is $x + 9$ a factor of the polynomial $P(x) = x^3 + 11x^2 + 15x - 27$? If so, write the polynomial as a product of two factors. If not, explain how you know.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

8. **Error Analysis** Alicia divided the polynomial $2x^3 - 4x^2 + 6x + 10$ by $x^2 + x$. Describe and correct the error Alicia made in dividing the polynomials.

$$2x - 6 + \frac{10}{x^2 + x}$$

$$x^2 + x \overline{)\, 2x^3 - 4x^2 + 6x + 10}$$

$$\underline{-(2x^3 + 2x^2)}$$

$$-6x^2 + 6x$$

$$\underline{-(-6x^2 - 6x)}$$

$$10$$

9. **Higher Order Thinking** When dividing polynomial $P(x)$ by polynomial $d(x)$, the remainder is $R(x)$. How can you use the degrees of $R(x)$ and $d(x)$ to determine whether you are finished dividing?

10. **Mathematical Connections** Use polynomial long division to divide $8x^3 + 27$ by $2x + 3$. How can you use multiplication to check your answer? Show your work.

11. **Check for Reasonableness** Write a polynomial division problem with a quotient of $x^2 - 5x + 7$ and a remainder of 2. Explain your reasoning. How can you verify your answer?

12. **Represent and Connect** Show that $x - 3$ and $x + 5$ are factors of $x^4 + 2x^3 - 16x^2 - 2x + 15$. Explain your reasoning.

13. **Generalize** When dividing polynomial $P(x)$ by polynomial $x - n$, the remainder is 0. When graphing $P(x)$, what is an $x$-intercept of the graph?

14. **Communicate and Justify** When dividing $x^3 + nx^2 + 4nx - 6$ by $x + 3$, the remainder is $-48$. What is the value of $n$?

### PRACTICE

**Use long division to divide.** SEE EXAMPLE 1

15. $x^3 + 5x^2 - x - 5$ divided by $x - 1$

16. $2x^3 + 9x^2 + 10x + 3$ divided by $2x + 1$

17. $3x^3 - 2x^2 + 7x + 9$ divided by $x^2 - 3x$

18. $2x^4 - 6x^2 + 3$ divided by $2x - 6$

**Use synthetic division to divide.** SEE EXAMPLE 2

19. $x^4 - 25x^2 + 144$ divided by $x - 4$

20. $x^3 + 6x^2 + 3x - 10$ divided by $x + 5$

21. $x^5 + 2x^4 - 3x^3 + x - 1$ divided by $x + 2$

22. $-x^4 + 7x^3 + x^2 - 2x - 12$ divided by $x - 3$

23. Use synthetic division to show that the remainder of $f(x) = x^4 - 6x^3 - 33x^2 + 46x + 75$ divided by $x - 9$ is equivalent to $f(9)$. SEE EXAMPLE 3

**Use the Remainder Theorem to evaluate each polynomial for the given value of $x$.** SEE EXAMPLE 4

24. $f(x) = x^3 + 9x^2 + 3x - 7;\ x = -5$

25. $f(x) = 2x^3 - 3x^2 + 4x + 13;\ x = 3$

26. $f(x) = -x^4 + 2x^3 - x^2 + 4x + 8;\ x = -2$

27. $f(x) = x^5 - 3x^4 - 2x^3 + x^2 - 2x - 1;\ x = 4$

**Is each given binomial a factor of the given polynomial? If so, write the polynomial as a product of two factors.** SEE EXAMPLE 5

28. polynomial: $P(x) = 8x^3 - 10x^2 + 28x - 16$; binomial: $x - 3$

29. polynomial: $P(x) = 4x^4 - 9x^3 - 7x^2 - 2x + 25$; binomial: $x + 4$

30. polynomial: $P(x) = -x^5 + 12x^3 + 6x^2 - 23x + 1$; binomial: $x - 2$

31. polynomial: $P(x) = 2x^3 + 3x^2 - 8x - 12$; binomial: $2x + 3$

## PRACTICE & PROBLEM SOLVING

### APPLY

**32. Apply Math Models** Darren is placing shipping boxes in a storage unit with a floor area of $x^4 + 5x^3 + x^2 - 20x - 14$ square units. Each box has a volume of $x^3 + 10x^2 + 29x + 20$ cubic units and can hold a stack of items with a height of $x + 5$ units.

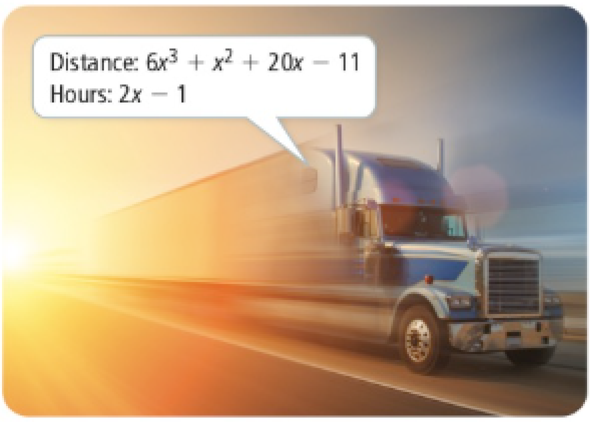

a. How much floor space will each box cover?

b. What is the maximum number of boxes Darren can place on the floor of the storage unit?

c. Assume Darren places the maximum number of boxes on the floor of the storage unit, with no overlap. How much of the floor space is not covered by a box?

**33. Analyze and Persevere** Lauren wants to determine the length and height of her DVD stand. The function $f(x) = x^3 + 14x^2 + 57x + 72$ represents the volume of the DVD stand, where the width is $x + 3$ units. What are possible dimensions for the length and height of the DVD stand? Explain.

**34. Analyze and Persevere** A truck traveled $6x^3 + x^2 + 20x - 11$ miles in $2x - 1$ hours. At what rate did the semi-truck travel? (*Hint:* Use the formula $d = rt$, where $d$ is the distance, $r$ is the rate, and $t$ is the time.)




### ASSESSMENT PRACTICE

**35.** $(3x^5 + 9x^2 + 4x - 2) \div (x^2 - 1)$ AR.1.5

Ⓐ $3x^3 + 12 + \frac{4x + 10}{x^2 - 1}$

Ⓑ $3x^3 + 3x + 9 + \frac{7x + 7}{x^2 - 1}$

Ⓒ $3x^4 + 3x^3 + 3x^2 + 12x + 16 + \frac{14}{x^2 - 1}$

Ⓓ $3x^4 - 3x^3 + 3x^2 + 6x - 2$

**36. SAT/ACT** $x + 3$ is a factor of the polynomial $x^3 + 2x^2 - 5x + n$. What is the value of $n$?

Ⓐ $-6$

Ⓑ $-3$

Ⓒ $-2$

Ⓓ $3$

Ⓔ $6$

**37. Performance Task** The table shows some quotients of the polynomial $x^n - 1$ divided by the linear factor $x - 1$.

| Dividend | Divisor | Quotient |
|---|---|---|
| $x^2 - 1$ | $x - 1$ | $x + 1$ |
| $x^3 - 1$ | $x - 1$ | $x^2 + x + 1$ |
| $x^4 - 1$ | $x - 1$ | |
| $x^5 - 1$ | $x - 1$ | |
| $x^6 - 1$ | $x - 1$ | |

**Part A** Use long division or synthetic division to find the missing quotients to complete the table.

**Part B** Look for a pattern. Then describe the pattern when $x^n - 1$ is divided by $x - 1$.

**Part C** Use the pattern to find the quotient when $x^{10} - 1$ is divided by $x - 1$.

# 3-5 Zeros of Polynomial Functions

**I CAN…** model and solve problems using the zeros of a polynomial function.

**VOCABULARY**

- multiplicity of a zero

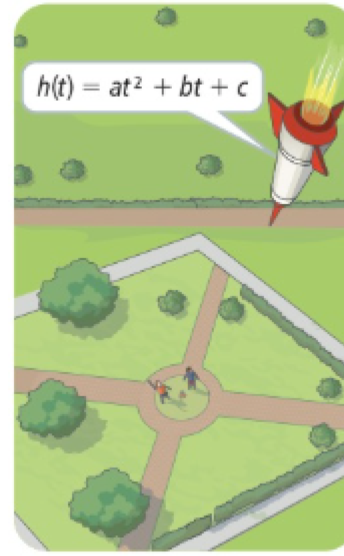
**MA.912.AR.6.1**–Given a mathematical or real-world context, when suitable factorization is possible, solve one-variable polynomial equations of degree 3 or higher over the real and complex number systems. **Also AR.1.8, AR.6.5, F.1.1**

**MA.912. MTR.1.1, MTR.5.1, MTR.7.1**

## MODEL & DISCUSS

Charlie and Aisha built a small rocket and launched it from their backyard. The rocket fell to the ground 10 s after it launched.

The height $h$, in feet, of the rocket relative to the ground at time $t$ seconds can be modeled by the function shown.




**A.** How are the launch and landing times related to the modeling function?

**B.** What additional information about the rocket launch could you use to construct an accurate model for the rocket's height relative to the ground?

**C.** **Communicate and Justify** Charlie believes that the function $h(t) = -16t^2 + 160t$ models the height of the rocket with respect to time. Do you agree? Explain your reasoning and indicate the domain of this function.

## ESSENTIAL QUESTION

**How are the zeros of a polynomial function related to an equation and graph of the function?**

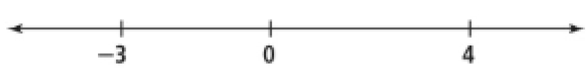
**CONCEPTUAL UNDERSTANDING**

## EXAMPLE 1 Use Zeros to Graph a Polynomial Function

**What are the zeros of $f(x) = x(x - 4)(x + 3)$? Graph the function.**

A *zero* of a polynomial function is a value for which the function is equal to 0. By the Zero-Product Property, the zeros of the function are –3, 0, and 4.

The zeros divide a number line into four intervals.




**GENERALIZE**
Recall applying the Zero-Product Property to quadratic polynomials. The zeros of a quadratic function relate to the factors: when $x = a$ is a zero of the function, then $(x - a)$ is a factor of the polynomial. The Zero-Product Property applies to polynomials of any degree.

To see how the graph of the function behaves on each interval, look at the sign of each factor on each interval, and the sign of the product.

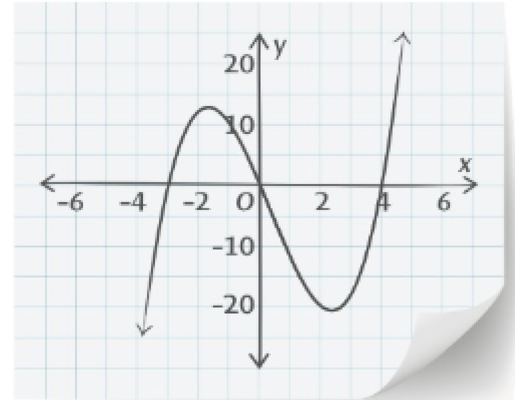
| Interval | Sign | | | |
|---|---|---|---|---|
| | $x$ | $x - 4$ | $x + 3$ | Product |
| $x < -3$ | – | – | – | – |
| $-3 < x < 0$ | – | – | + | + |
| $0 < x < 4$ | + | – | + | – |
| $x > 4$ | + | + | + | + |

The last column shows the sign of the product of the three factors.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Sketch the graph. Draw a continuous curve that passes through each zero on the $x$-axis, and is below the $x$-axis when the function is negative, and above the $x$-axis when the function is positive.




**Try It!** **1.** Factor each function. Then use the zeros to sketch its graph.

**a.** $f(x) = 4x^3 + 4x^2 - 24x$ **b.** $g(x) = x^4 - 13x^2 + 36$

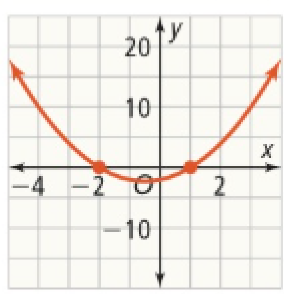
## EXAMPLE 2 Understand How a Multiple Zero Can Affect a Graph

Powered By desmos

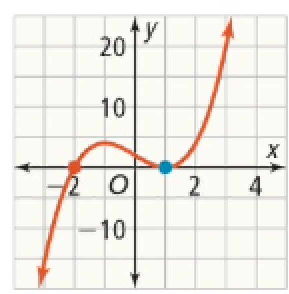
**How does a multiple zero affect the graph of a polynomial function?**

The **multiplicity of a zero** of a polynomial function is the number of times its related factor appears in the factored form of the polynomial. Notice the behavior of each graph as it approaches the $x$-axis. What can you conclude about the multiplicity of a zero and its effect on the graph of the function?

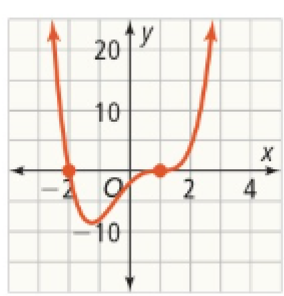
STUDY TIP
Close to a zero, the graph looks like a polynomial function with degree equal to the multiplicity of the zero.



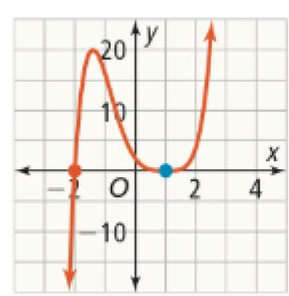

$f(x) = (x - 1)(x + 2)$




$f(x) = (x - 1)^2(x + 2)$




$f(x) = (x - 1)^3(x + 2)$




$f(x) = (x - 1)^4(x + 2)$

When the multiplicity of a zero is odd, the function crosses the $x$-axis.

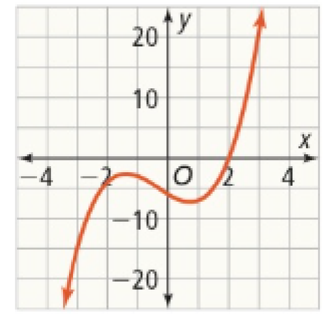
When the multiplicity of a zero is even, the graph has a turning point at the $x$-axis.

**Try It!** **2.** Describe the behavior of the graph of the function at each of its zeros.

**a.** $f(x) = x(x + 4)(x - 1)^4$ **b.** $f(x) = (x^2 + 9)(x - 1)^5(x + 2)^2$

## EXAMPLE 3 Find Real and Complex Zeros

**What are all the real and complex zeros of the polynomial function shown in the graph?**



$f(x) = x^3 + x^2 - 3x - 6$

**GENERALIZE**

These statements are equivalent:

- The function's graph crosses the $x$-axis at 2.
- The $x$-intercept of the graph is 2.
- $f(2) = 0$
- 2 is a zero of the function.
- $x - 2$ is a factor of the polynomial.

**Step 1** Use the graph to determine one of the zeros of the polynomial. The function appears to cross the $x$-axis at $x = 2$.

Confirm that 2 is a zero of the function.

$$f(2) = (2)^3 + (2)^2 - 3(2) - 6$$
$$= 0$$

So 2 is a zero of the function, and by algebra, $x - 2$ is a factor of the related polynomial.

**Step 2** Use synthetic division to factor the polynomial.

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -3 & -6 \\ & & 2 & 6 & 6 \\ \hline & 1 & 3 & 3 & 0 \end{array}$$

So $f(x) = (x - 2)(x^2 + 3x + 3)$.

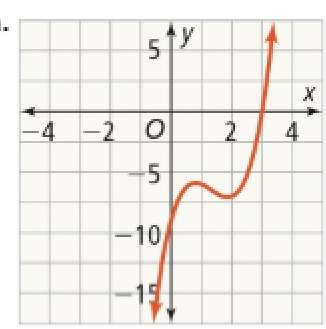
**Step 3** Use the Quadratic Formula to find the remaining zeros.

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(3)}}{2(1)}$$
$$= -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$$

The zeros of the polynomial function $f$ are 2, $-\frac{3}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{3}{2} - \frac{\sqrt{3}}{2}i$.

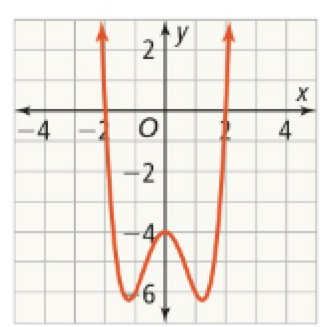
**Try It!** 3. What are all the real and complex zeros of the polynomial function shown in the graph?

a.



$f(x) = 2x^3 - 8x^2 + 9x - 9$

b.



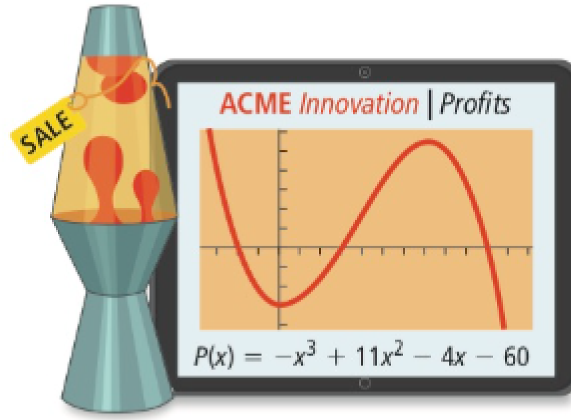
$f(x) = x^4 - 3x^2 - 4$

APPLICATION

## EXAMPLE 4 Interpret the Zeros of a Function

**Acme Innovations makes and sells lamps. Their profit *P*, in hundreds of dollars earned, is a function of the number of lamps sold *x*, in thousands.**

**From historical data, they know that their company's profit is modeled by the function shown.**




**What do the zeros of the function tell you about the number of lamps that Acme Innovations should produce?**

Formulate ◀ A profit for the company corresponds to the portions of the graph that lie above the $x$-axis. Find the zeros of the function to determine where the graph crosses the $x$-axis.

Based on the graph, the zeros of the function appear to be –2, 3, and 10. If these are the zeros of the function, then you can determine the factors of the related polynomial.

| Zero of $P$ | Factor of $P(x)$ |
|---|---|
| $-2$ | $x + 2$ |
| 3 | $x - 3$ |
| 10 | $x - 10$ |

Compute ◀ Multiply these factors to verify that the product is equal to the polynomial given.

$$\begin{aligned}(x+2)(x-3)(x-10) &= (x+2)(x^2-13x+30)\\ &= x^3-13x^2+30x+2x^2-26x+60\\ &= x^3-11x^2+4x+60\end{aligned}$$

The result is equal to $-P(x)$, not $P(x)$. But both polynomials have the same factors since $P(x) = -(x+2)(x-3)(x-10)$.

So the zeros of the function are –2, 3, and 10.

Interpret ◀ When $P(x)$ is positive, Acme Innovations earns a profit. The profit is positive when $x < -2$ or $3 < x < 10$. Acme Innovations cannot produce a negative number of lamps, so disregard the interval $x < -2$. Since $x$ represents the number of lamps in thousands, the company should make between 3,000 and 10,000 lamps.

**Try It!** 4. Due to a decrease in the cost of materials, the profit function for Acme Innovations has changed to $Q(x) = -x^3 + 10x^2 + 13x - 22$. How many lamps should they make in order to make a profit?

## EXAMPLE 5 Solve Polynomial Equations

**What are the solutions of $2x^3 + 5x^2 - 3x = 3x^3 + 8x^2 + 1$?**

Rewrite the equation in the form $P(x) = 0$.

$$2x^3 + 5x^2 - 3x = 3x^3 + 8x^2 + 1$$

$$x^3 + 3x^2 + 3x + 1 = 0$$

Combine like terms on one side of the equation.

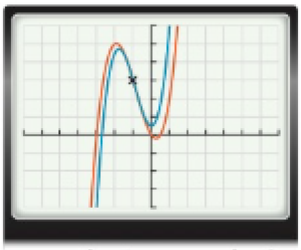
**VOCABULARY**
A *polynomial equation* is an equation that can be written in the form $P(x) = 0$, where $P(x)$ is a polynomial.

The roots are the zeros of the function $P(x) = x^3 + 3x^2 + 3x + 1$.

$(x + 1)^3 = 0$ ········ Cube of a binomial.

$x + 1 = 0$ ········ Zero-Product Property.

$x = -1$ ········ Subtract 1 from each side.



*x scale: 1* *y scale: 2*

To check, write each side of the equation as a separate polynomial and graph. Use the INTERSECT feature to confirm that the graphs intersect at $x = -1$.

**Try It!** **5.** What are the solutions of the equations?

**a.** $x^3 - 7x + 6 = x^3 + 5x^2 - 2x - 24$ **b.** $x^4 + 2x^2 = -x^3 - 2x$

## EXAMPLE 6 Solve a Polynomial Inequality by Graphing

**What are the solutions of $x^3 - 16x < 0$?**

The solutions are all values of x that make the inequality true. The polynomial $x^3 - 16x$ defines a polynomial function $P(x) = x^3 - 16x$.

**HAVE A GROWTH MINDSET**
How can you take on challenges with positivity?

Factor to find the zeros of the function.

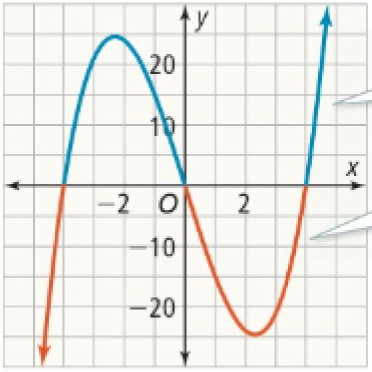
$x^3 - 16x = 0$

$x(x^2 - 16) = 0$ ········ Factor out the greatest common factor.

$x(x - 4)(x + 4) = 0$ ········ Difference of squares.

By the Zero-Product Property, the zeros of $P$ are –4, 0, and 4.

Sketch the function, and use the graph to determine where $P(x) < 0$.




The blue portions show where $P(x) > 0$.

The red portions show where $P(x) < 0$.

The solutions of the inequality $x^3 - 16x < 0$ are all real numbers such that $x < -4$ or $0 < x < 4$.

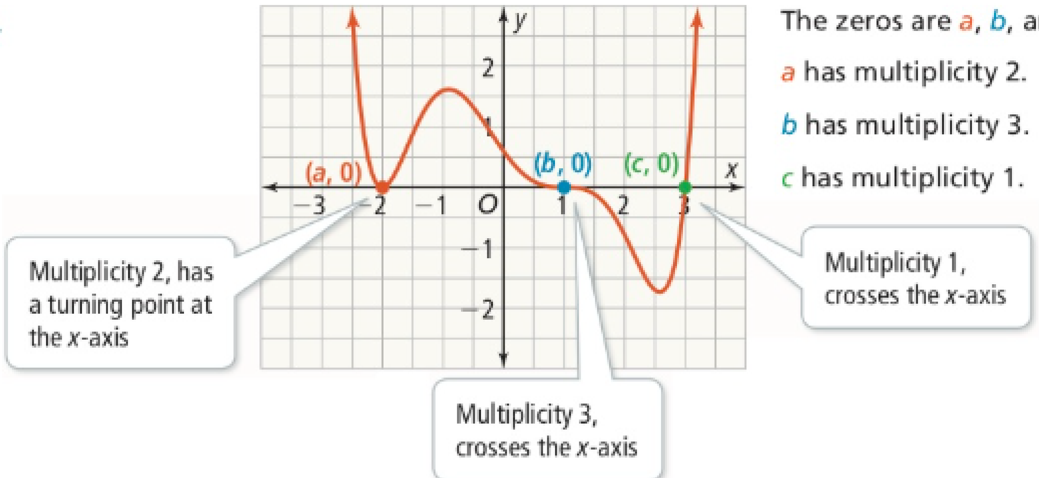
**Try It!** **6.** What are the solutions of the inequality?

**a.** $2x^3 + 12x^2 + 12x < 0$ **b.** $(x^2 - 1)(x^2 - x - 6) > 0$

## CONCEPT SUMMARY Zeros of Polynomials

**FUNCTION** $f(x) = (x - a)^2(x - b)^3(x - c)$

**GRAPH**




The zeros are $a$, $b$, and $c$.

$a$ has multiplicity 2.

$b$ has multiplicity 3.

$c$ has multiplicity 1.

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the zeros of a polynomial function related to the equation and graph of a function?

2. **Error Analysis** In order to identify the zeros of the function, a student factored the cubic function $f(x) = x^3 - 3x^2 - 10x$ as follows:

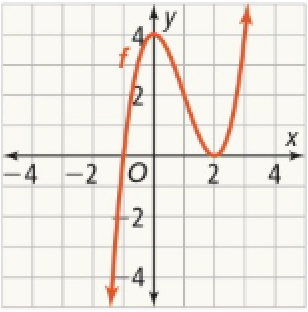
$$f(x) = x^3 - 3x^2 - 10x$$
$$= x(x^2 - 3x - 10)$$
$$= x(x - 5)(x + 2)$$
$$x = 0, x = -5, x = 2$$

Describe and correct the error the student made.

3. **Analyze and Persevere** Explain how you can determine that the function $f(x) = x^3 + 3x^2 + 4x + 2$ has both real and complex zeros.

## Do You KNOW HOW?

4. If the graph of the function $f$ has a multiple zero at $x = 2$, what is a possible exponent of the factor $x - 2$? Justify your reasoning.




5. Energy Solutions manufactures LED light bulbs. The profit $p$, in thousands of dollars earned, is a function of the number of bulbs sold, $x$, in ten thousands. Profit is modeled by the function $-x^3 + 9x^2 - 11x - 21$.

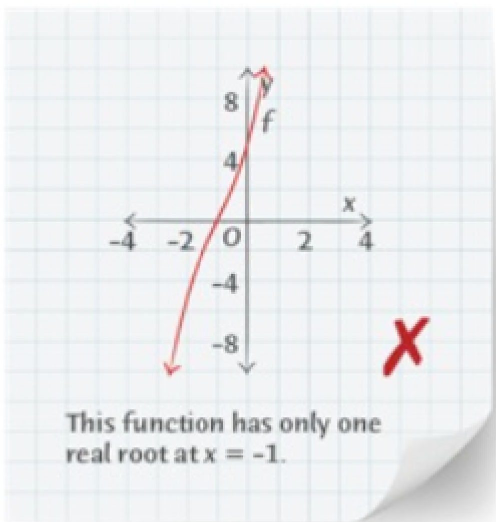
For what number of bulbs manufactured does the company make a profit?

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

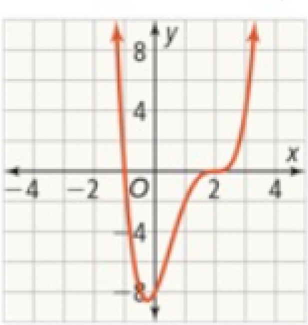
6. **Represent and Connect** If you use zeros to sketch the graph of a polynomial function, how can you verify that your graph is correct?

7. **Error Analysis** Describe and resolve two errors that Tonya may have made in finding all the roots of the polynomial function, $f(x) = x^3 + 3x^2 + 7x + 5$.




8. **Higher Order Thinking** How could you use your graphing calculator to determine that $f(x) = (x + 2)(x + 6)(x - 1)$ is not the correct factorization of $f(x) = x^3 + 7x^2 + 16x + 12$? Explain.

9. **Generalize** How can you determine that the polynomial function shown does not have any zeros with even multiplicity? Explain.




10. **Use Patterns and Structure** Factor the polynomial $x^4 - 16$. How many real zeros does the function $g(x) = x^4 - 16$ have?

11. At what points do the graphs of $f(x) = x^3 - 2x^2 - 16x + 20$ and $g(x) = -12$ intersect?

## PRACTICE

**Sketch the graph of the function by finding the zeros.** SEE EXAMPLE 1

12. $f(x) = 3x^3 - 9x^2 - 12x$

13. $g(x) = (x + 3)(x - 1)(x - 4)$

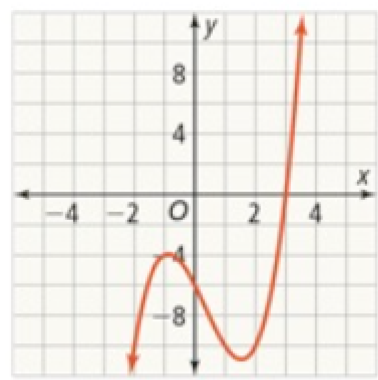
**Find the zeros of the function, and describe the behavior of the graph at each zero.** SEE EXAMPLE 2

14. $f(x) = x^3 - 8x^2 + 16x$

15. $g(x) = x^3 - x^2 - 25x + 25$

16. $f(x) = 9x^4 - 40x^2 + 16$

17. What are all the real and complex zeros of the polynomial function shown in the graph? SEE EXAMPLE 3




$f(x) = x^3 - x^2 - 4x - 6$

18. Waterworks is a company that manufactures and sells paddleboards. Their profit $P$, in hundreds of dollars earned, is a function of the number of paddleboards sold $x$, measured in thousands. Profit is modeled by the function $P(x) = -3x^3 + 48x^2 - 144x$. What do the zeros of the function tell you about the number of paddleboards that Waterworks should produce? SEE EXAMPLE 4

**What are the solution(s) of the equation?** SEE EXAMPLE 5

19. $-3x^3 - x^2 + 54x - 40 = 2x^2 + 6x + 20$

20. $2x^3 + 3x^2 - 36 = x^3 - x^2 + 9x$

21. $-5x^4 + 4x^2 - 12x = -6x^4 + 3x^3$

**What are the solutions of the inequality?** SEE EXAMPLE 6

22. $x^3 - 9x > 0$

23. $0 > 4x^3 + 8x^2 - x - 2$

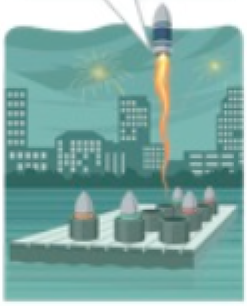
24. $64x^2 > -4x^3 - x - 16$

# PRACTICE & PROBLEM SOLVING

## APPLY

25. **Analyze and Persevere** A firework is launched vertically into the air. Its height in meters is given by the function shown, where $t$ is measured in seconds.

$h = -4.9t^2 + 49t$

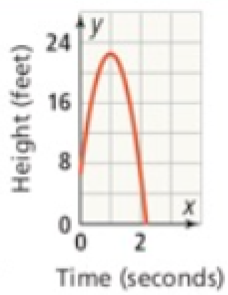


a. What is a reasonable domain of the function?

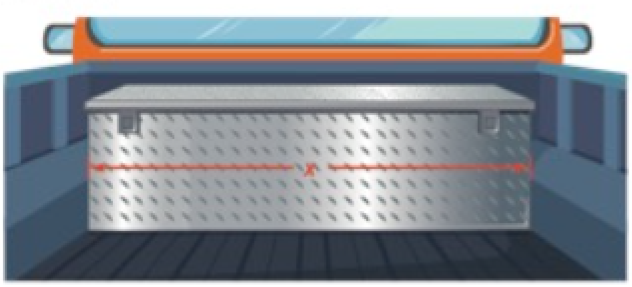
b. What are the zeros of the function? Explain what they represent in this situation.

c. Use technology to find the vertex. What does it represent in this situation?

26. The height of a baseball thrown in the air can be modeled by the function $h(t) = -16t^2 + 32t + 6.5$, where $h(t)$ represents the height in feet of the baseball after $t$ seconds. Explain why the graph of this function only shows one zero.




27. **Apply Math Models** The height of a rectangular storage box is less than both its length and width. The function $f(x) = x^3 + 2x^2 - 3x$ represents the volume of the rectangular box, where $x$ represents the width of the box, in feet.




a. Find the factored form of $f(x)$.

b. Find the zeros of the function.

c. You know $x$ represents the width of the box. What do the other two factors represent?

d. Find the dimensions of the box when the volume is 10 ft$^3$.

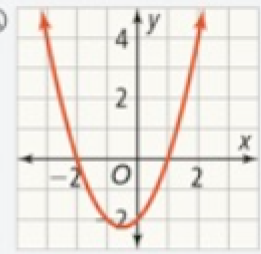
## ASSESSMENT PRACTICE

28. Complete each statement so it means the same as 4 is *a zero of the function.* AR.6.5

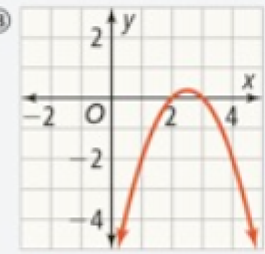
The graph of the function crosses the _________ at 4. _____________ is a factor of the polynomial.

29. **SAT/ACT** Without the use of a graphing calculator, determine which of the following functions is the graph of $f(x) = x^3 + x^2 - 4x$.

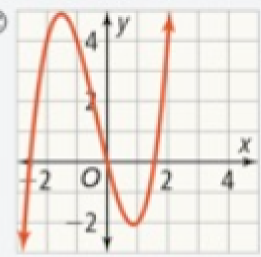
Ⓐ



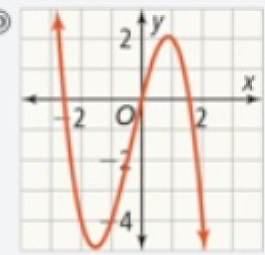
Ⓑ



Ⓒ



Ⓓ



30. **Performance Task** Venetta opened several deli sandwich franchises in 2000. The profit $P$ (in hundreds of dollars) of the franchises in $t$ years (since the franchises opened) can be modeled by the function $P(t) = t^3 + t^2 - 6t$.

**Part A** Sketch a graph of the function.

**Part B** Based on the model, during what years did Venetta not make a profit?

**Part C** If the model is appropriate, predict the amount of profit Venetta will receive from her franchises in 2020.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.6.1–Given a mathematical or real-world context, when suitable factorization is possible, solve one-variable polynomial equations of degree 3 or higher over the real and complex number systems. Also AR.1.3, AR.1.6

MA.912. MTR.2.1, MTR.6.1, MTR.7.1

## What Are the Rules?

All games have rules about how to play the game. The rules outline such things as when a ball is in or out, how a player scores points, and how many points a player gets for each winning shot.

If you didn't already know how to play tennis, or some other game, could you figure out what the rules were just by watching? What clues would help you understand the game? Think about this during the Mathematical Modeling in 3-Acts lesson.

### ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

### ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

### ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

# 3-6 Roots of Polynomial Equations

**I CAN…** use roots of a polynomial equation to find other roots.

MA.912.AR.6.1–Given a mathematical or real-world context, when suitable factorization is possible, solve one-variable polynomial equations of degree 3 or higher over the real and complex number systems. Also AR.1.8
MA.912. MTR.1.1, MTR.4.1, MTR.5.1

## CRITIQUE & EXPLAIN

Look at the polynomial functions shown.

$$g(x) = x^2 - 7x - 18$$
$$h(x) = 5x^2 + 24x + 16$$

A. Avery has a conjecture that the zeros of a polynomial function have to be positive or negative factors of its constant term. Factor $g(x)$ completely. Are the zeros of $g$ factors of $-18$?

B. **Use Patterns and Structure** Now test Avery's conjecture by factoring $h(x)$. Does Avery's conjecture hold? If so, explain why. If not, make a new conjecture.

## ESSENTIAL QUESTION

**How are the roots of a polynomial equation related to the coefficients and degree of the polynomial?**

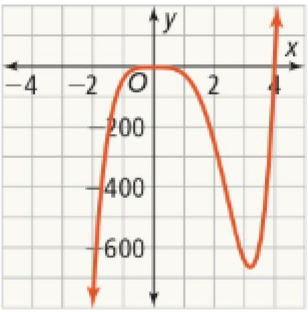
## CONCEPT The Rational Root Theorem

Let $P(x) = a_nx^n + a_{n-1}x^n + \ldots + a_1x + a_0$ be a polynomial with integer coefficients.

If the polynomial equation $P(x) = 0$ has any rational roots, then each rational root is of the form $\frac{p}{q}$, where $p$ is a factor of the constant term, $a_0$, and $q$ is a factor of the leading coefficient, $a_n$.

## EXAMPLE 1 Identify Possible Rational Solutions

**From the graph it appears that 4 is a zero of the function $P(x) = 8x^5 - 32x^4 + x^2 - 4$. Without substituting, how can you determine if 4 is a possible solution to $P(x) = 0$?**




List all the factors of the leading coefficient and the constant term of $P(x)$.

constant term $= -4$ factors: $\pm1, \pm2, \pm4$

leading coefficient $= 8$ factors: $\pm1, \pm2, \pm4, \pm8$

The Rational Root Theorem states that the possible rational roots of $P(x) = 0$ are

$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{2}{4}, \pm\frac{2}{8}, \pm\frac{4}{1}, \pm\frac{4}{2}, \pm\frac{4}{4}, \pm\frac{4}{8}$

Values in the numerator are factors of the constant term.

Values in the denominator are factors of the leading coefficient.

$\frac{4}{1}$ is equal to 4, so it is a possible solution to $8x^5 - 32x^4 + x^2 - 4 = 0$ according to the Rational Root Theorem.

**COMMON ERROR**
These are *possible* roots of the equation. You still need to test them to determine whether they are *actual* roots.

**Try It!** 1. List all the possible rational solutions for each equation.

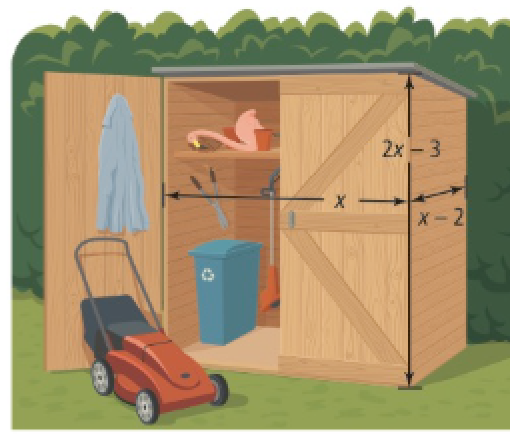
a. $4x^4 + 13x^3 - 124x^2 + 212x - 8 = 0$

b. $7x^4 + 13x^3 - 124x^2 + 212x - 45 = 0$

APPLICATION

## EXAMPLE 2 Use the Rational Root Theorem

**A storage company is designing a new storage unit. Based on the dimensions shown, the volume of a container is modeled by the polynomial $v(x) = 2x^3 - 7x^2 + 6x$, where $x$ is the width in feet. What are the dimensions of the container in feet if the volume of the unit is 154 ft$^3$?**




Formulate

The volume is 154 ft$^3$, so find solutions to the equation $2x^3 - 7x^2 + 6x = 154$.

The zeros of the polynomial will be rational roots of the equation in standard form: $2x^3 - 7x^2 + 6x - 154 = 0$.

Compute

List the factors of the constant term and the leading coefficient.

constant term = $-154$

factors: $\pm1, \pm2, \pm7, \pm11, \pm14, \pm22, \pm77, \pm154$

leading coefficient = $2$

factors: $\pm1, \pm2$

Use a spreadsheet or a programmable calculator to test the possible roots.

List all possible rational roots, eliminating repeated values.

$$\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{7}{1}, \pm\frac{11}{1}, \pm\frac{14}{1}, \pm\frac{22}{1}, \pm\frac{77}{1}, \pm\frac{154}{1}, \pm\frac{1}{2}, \pm\frac{7}{2}, \pm\frac{11}{2}, \pm\frac{77}{2}$$

Look for an $x$-value where $2x^3 - 7x^2 + 6x - 154 = 0$.

Testing shows that $\frac{11}{2}$ is a solution to the equation. Once you find one root, you can use synthetic division to find the other factor.

$$\begin{array}{r|rrrr} \frac{11}{2} & 2 & -7 & 6 & -154 \\ & & 11 & 22 & 154 \\ \hline & 2 & 4 & 28 & 0 \end{array}$$

The factored form of the equation is $\left(x - \frac{11}{2}\right)(2x^2 + 4x + 28) = 0$.

The discriminant of the quadratic factor is –208, so there are no real zeros for this factor. Therefore $\frac{11}{2}$ is the only real solution to the original equation.

Interpret

The width of the container is $\frac{11}{2}$, or 5.5 ft; its length is $\frac{11}{2} - 2$, or 3.5 ft; and its height is $2\left(\frac{11}{2}\right) - 3$, or 8 ft.

**Try It!** 2. A jewelry box measures $2x + 1$ in. long, $2x - 6$ in. wide, and $x$ in. tall. The volume of the box is given by the function $v(x) = 4x^3 - 10x^2 - 6x$. What is the height of the box, in inches, if its volume is 28 in.$^3$?

## CONCEPT Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly $n$ solutions in the set of complex numbers.

If $P(x)$ has any factor of multiplicity $m$, count the solution associated with that factor $m$ times. For example, the equation $(x - 3)^4 = 0$ has four solutions, each equal to 3.

## EXAMPLE 3 Find All Complex Roots

**What are all the complex roots of the polynomial equation?**

$$3x^4 + 4x^3 + 2x^2 - x - 2 = 0$$

**Step 1** List the factors of the constant term and leading coefficient

constant term $= -2$ factors: $\pm 1, \pm 2$

leading coefficient $= 3$ factors: $\pm 1, \pm 3$

**Step 2** List the possible rational roots.

$\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2$

**STUDY TIP**
Remember that if you use synthetic division to test for factors, the result also tells you the quotient after division by the factor.

**Step 3** Testing with synthetic division reveals that $\frac{2}{3}$ and $-1$ are roots.

Divide $3x^4 + 4x^3 + 2x^2 - x - 2$ by $x - \frac{2}{3}$.

$$\begin{array}{r|rrrrr} \frac{2}{3} & 3 & 4 & 2 & -1 & -2 \\ & & 2 & 4 & 4 & 2 \\ \hline & 3 & 6 & 6 & 3 & 0 \end{array}$$

So $3x^4 + 4x^3 + 2x^2 - x - 2 = \left(x - \frac{2}{3}\right)(3x^3 + 6x^2 + 6x + 3)$.

Now divide the cubic factor by $x - (-1)$.

$$\begin{array}{r|rrrr} -1 & 3 & 6 & 6 & 3 \\ & & -3 & -3 & -3 \\ \hline & 3 & 3 & 3 & 0 \end{array}$$

After factoring out 3 from the final quotient, the polynomial equation can be written as $3(x - \frac{2}{3})(x + 1)(x^2 + x + 1) = 0$.

**Step 4** Use the Quadratic Formula to find the last two roots.

If $x^2 + x + 1 = 0$, then $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

The complex numbers $\frac{2}{3}$, $-1$, $\frac{-1 - i\sqrt{3}}{2}$, and $\frac{-1 + i\sqrt{3}}{2}$ are all roots of the equation. Since the polynomial has degree 4, the Fundamental Theorem of Algebra states that these are the only four roots of the equation.

**Try It!** 3. What are all the complex roots of the equation $x^3 - 2x^2 + 5x - 10 = 0$?

## CONCEPT SUMMARY Roots of Polynomial Equations

| | Words | Example |
|---|---|---|
| **RATIONAL ROOT THEOREM** | For the polynomial equation with integer coefficients $0 = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0$, there are a limited number of possible rational roots.<br><br>Rational roots must have reduced form $\frac{p}{q}$ where $p$ is an integer factor of $a_0$ and $q$ is an integer factor of $a_n$.<br><br>Use substitution or synthetic division to check roots. | $2x^3 + 3x^2 - 10x - 15 = 0$<br>$p = -15$; Factors of $p$: $\pm1, \pm3, \pm5, \pm15$<br>$q = 2$; Factors of $q$: $\pm1, \pm2$<br>Possible rational roots:<br>$\pm1, \pm3, \pm5, \pm15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}$<br>$-\frac{3}{2}$ is a root of the equation. |
| **FUNDAMENTAL THEOREM OF ALGEBRA** | If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly $n$ solutions counting multiplicity in the set of complex numbers. | |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the roots of a polynomial equation related to the coefficients and degree of the polynomial?

2. **Error Analysis** Renaldo says the polynomial equation $0 = 4x^2 - 2x + 1$ has only integer roots. Is Renaldo correct? Explain.

3. **Use Patterns and Structure** A fifth degree polynomial $P(x)$ with rational coefficients has zeros only at $-5$ and 2, and it has no complex or irrational roots. What do you know about the multipllicities of these zeros? Explain.

4. **Communicate and Justify** If one root of a polynomial equation with real coefficients is $4 + 2i$, is it certain that $4 - 2i$ is also a root of the equation? Explain.

## Do You KNOW HOW?

**List all the possible rational solutions for each equation according to the Rational Roots Theorem. Then find all of the rational roots.**

5. $0 = x^3 + 4x^2 - 9x - 36$

6. $0 = x^4 - 2x^3 - 7x^2 + 8x + 12$

7. $0 = 4x^3 + 8x^2 - x - 2$

8. $0 = 9x^4 - 40x^2 + 16$

**Find all rational and complex roots of the polynomial equations. Indicate any multiplicity greater than one.**

9. $0 = x^4 - 9x^2$

10. $0 = 5x^3 - 30x + 100$

11. $0 = 15x^3 - 89x^2 - 12x + 36$

12. $0 = x^4 - 6x^3 + 10x^2 - 6x + 9$

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

13. **Communicate and Justify** Consider the polynomial $P(x) = 5x^3 + ms^2 + nx + 6$, where $m$ and $n$ are rational coefficients. Is 3 *sometimes, always,* or *never* a root? Explain.

14. **Choose Efficient Methods** What are the possible rational roots for any polynomial with a leading coefficient of −6 and a constant of 1?

15. **Error Analysis** A student says that a fifth-degree polynomial equation with rational coefficients has roots −5, −3, 1, and 2. Describe possible errors the student may have made.

16. **Generalize** Write the leading coefficient and the constant for a polynomial with rational coefficients that has the following possible roots. Explain your reasoning.

$$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{5}{1}, \pm\frac{5}{2}, \pm\frac{10}{1}, \pm\frac{10}{2}$$

17. **Error Analysis** Describe and correct the error a student made in finding the roots of the polynomial equation $2x^3 - x^2 - 10x + 5 = 0$.

> List all possible rational roots.
>
> $\pm 1, \pm\frac{1}{2}, \pm 5, \pm\frac{5}{2}$
>
> Testing reveals that $\frac{1}{2}$ is a root.
> Dividing the polynomial by the binomial $x - \frac{1}{2}$ results in the factored form
>
> $f(x) = (x - \frac{1}{2})(2x^2 - 10)$
>
> The equation $2x^2 - 10 = 0$ has two irrational roots, $\sqrt{10}$ and $-\sqrt{10}$.
>
> The complete set of roots is $\{\frac{1}{2}, \sqrt{10}, -\sqrt{10}\}$.

18. **Higher Order Thinking** What must be true about any polynomial if a correct list of its possible rational zeros contains only integers? Explain.

19. **Use Patterns and Structure** Show that the Fundamental Theorem of Algebra is true for all quadratic equations with real coefficients. (*Hint*: Use the Quadratic Formula and examine the possibilities for the value of the discriminant.)

### PRACTICE

**List all the possible rational solutions for each equation.** SEE EXAMPLE 1

20. $0 = x^3 - 3x^2 + 4x - 12$

21. $0 = 2x^4 + 13x^3 - 47x^2 - 13x + 45$

22. $0 = 4x^3 + 64x^2 - x - 16$

23. $0 = 8x^3 + 11x^2 - 13x - 6$

24. A closet in the shape of a rectangular prism has the measurements shown. What is the height of the closet, in feet, if its volume is 220 ft$^3$?
SEE EXAMPLE 2


**What are all real and complex roots of the following functions?** SEE EXAMPLE 3

25. $0 = x^3 - 3x - 52$

26. $0 = x^3 + 9x^2 - 7x - 63$

27. $0 = x^4 + 34x^2 - 72$

28. $0 = x^6 + 4x^4 - 41x^2 + 36$

29. $0 = x^3 + 4x^2 + x$

30. $0 = x^2 - 2x + 37$

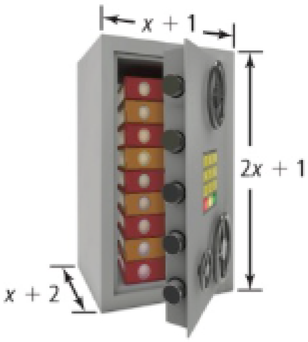
31. $0 = x^3 + 3x^2 - 56x - 18$

32. $0 = x^3 - 3x^2 + 40x + 400$

# PRACTICE & PROBLEM SOLVING

## APPLY

33. **Analyze and Persevere** A fireproof safe has the measurements shown.



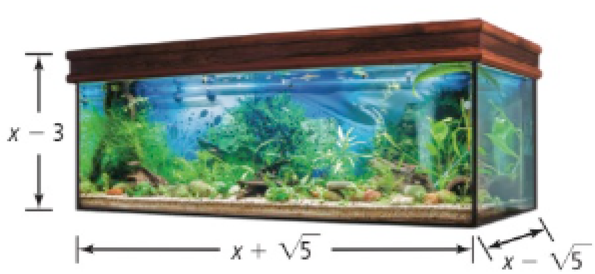

a. Write an equation to represent the situation when the volume of the fireproof safe is 270 in.$^3$. Rewrite the equation in the form $P(x) = 0$.

b. List all of the possible factors of the polynomial expression.

c. What are the real roots of the equation? Explain how you know these are the only real roots.

d. What are the length, width, and height of the fireproof safe?

34. **Apply Math Models** What are the dimensions of the fish tank, in feet, if its volume is 176 ft$^3$?




35. **Apply Math Models** The cost of producing $x$ video game consoles is modeled by the function $C(x) = x^4 - 5x^3 - 12x^2 - 22x - 40$. If a company spent $1,706 to produce video game consoles, how many consoles were made?

## ASSESSMENT PRACTICE

36. Select the list that has all the possible rational zeros for $P(x) = 3x^5 + 2x^2 - 48 + 9$. AR.6.1

Ⓐ $\pm 1, \pm 3, \pm 9, \pm\frac{1}{3}, \pm\frac{1}{9}$

Ⓑ $\pm 1, \pm 3, \pm\frac{1}{3}, \pm\frac{1}{9}$

Ⓒ $\pm 1, \pm 3, \pm 9, \pm\frac{1}{3}$

Ⓓ $\pm 9, \pm\frac{1}{3}$

37. **SAT/ACT** Which is a third-degree polynomial equation with rational coefficients that has roots $-2$ and $6i$?

Ⓐ $x^3 + 2x^2 + 36x + 72$

Ⓑ $x^3 - 2x^2 + 36x - 72$

Ⓒ $x^3 + 2x^2 - 36x - 72$

Ⓓ $x^2 + (6i - 2)x - 12$

Ⓔ $x^2 - (6i - 2)x - 12$

38. **Performance Task** The table shows the number of possible real and imaginary roots, counting multiplicity, for an $n$th degree polynomial equation with rational coefficients.

| Degree | Real Roots | Imaginary Roots |
|---|---|---|
| 3 | 3 | 0 |
| 3 | 1 | 2 |
| 5 | 5 | 0 |
| 5 | 3 | 2 |
| 5 | 1 | 4 |

**Part A** List all of the possible combinations of real and imaginary roots for a seventh-degree polynomial equation.

**Part B** What do you notice about the number of real roots of a polynomial equation with an odd degree?

# 3-7 Transformations of Polynomial Functions

**I CAN…** identify symmetry in and transform polynomial functions.

**VOCABULARY**

- even function
- odd function

MA.912.F.1.9–Determine whether a function is even, odd or neither when represented algebraically, graphically or in a table. Also F.2.2, F.2.3, F2.5

MA.912. MTR.3.1, MTR.4.1, MTR.5.1

## EXPLORE & REASON

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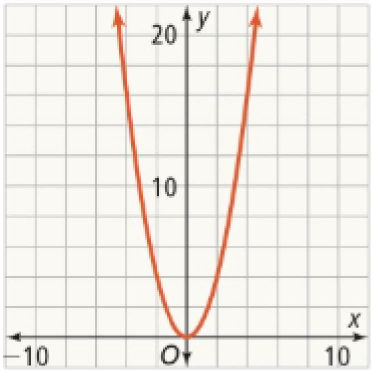
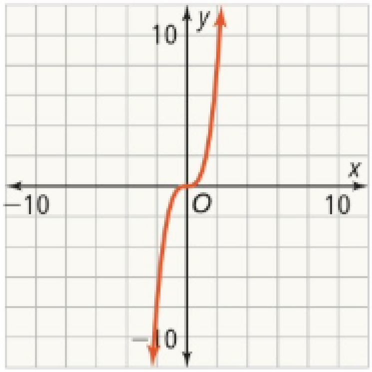
**Look at the polynomial graphs below.**

$f(x) = x^2$




$g(x) = x^3$




**A.** Is the graph of $f$ or $g$ symmetric about the $y$-axis? Is the graph of $f$ or $g$ symmetric about the origin? Explain.

**B. Use Patterns and Structure** Graph more functions of the form $y = x^n$ where $n$ is a natural number. Which of these functions are symmetric about the origin? Which are symmetric about the $y$-axis? What conjectures can you make?

## ESSENTIAL QUESTION

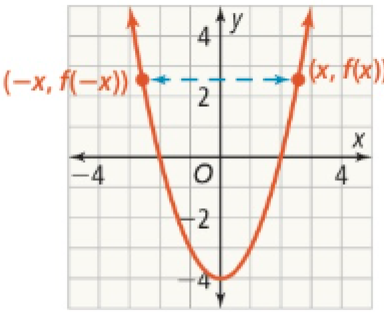
**How are symmetry and transformations represented in the graph and equation of a polynomial function?**

## CONCEPT Odd and Even Functions

A polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$ is an **even function** if it is symmetric about the $y$-axis and an **odd function** if it is symmetric about the origin.

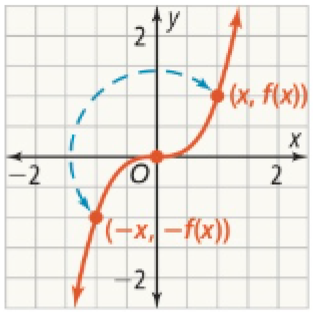
Other types of functions can also be classified as odd or even. For example, the function $y = |x|$ is an even function.

**Even Function**



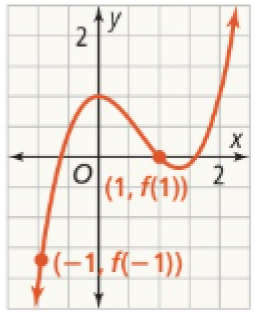

For all $x$ in the domain, $f(x) = f(-x)$.

**Odd Function**




For all $x$ in the domain, $f(-x) = -f(x)$.

**Neither**




$f(1) \neq f(-1)$ (not even)

$f(-1) \neq -f(1)$ (not odd)

## EXAMPLE 1 Identify Even and Odd Functions From Their Graphs

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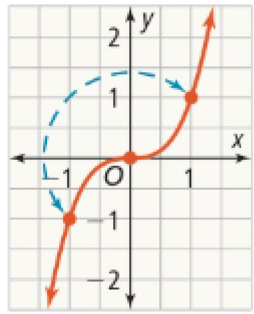
**Use the graph to classify the polynomial function. Is it even, odd, or neither?**

**A.**




What happens when you reflect the graph across the $y$-axis?

New graph – not even.

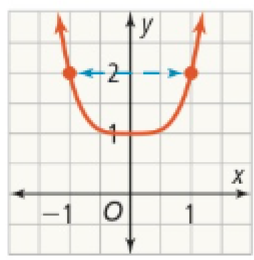
What happens when you rotate the graph 180° about the origin?

Same graph – odd

Test points to confirm: (1, 1) and (−1, −1) are both on the graph.

This function is odd.

**B.**




What happens when you reflect the graph across the $y$-axis?

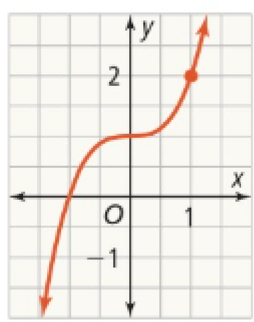
Same graph – even

Test points to confirm: (1, 2) and (−1, 2) are both on the graph.

This function is even.

**COMMON ERROR**
An odd degree polynomial function may have rotational symmetry around a point other than (0, 0). It is only an odd function if the symmetry is around the origin.

**C.**




What happens when you reflect the graph across the $y$-axis?

New graph – not even

What happens when you rotate the graph 180° about the origin?

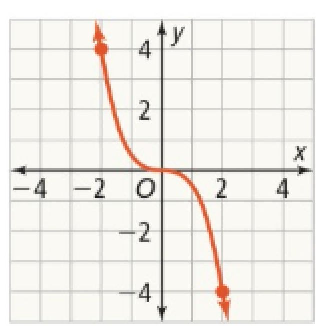
New graph – not odd

Test points to confirm: (1, 2) is on the graph, but (−1, 2) and (−1, −2) are not.

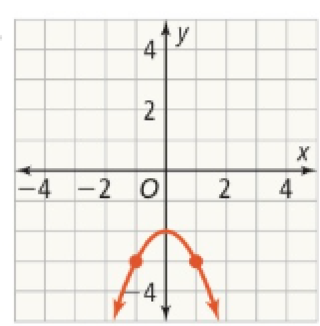
This function is neither even nor odd.

**Try It!** **1.** Classify the polynomial functions as even or odd based on the graphs.

**a.**




**b.**



## EXAMPLE 2 Identify Even and Odd Functions From Their Equations

**COMMUNICATE AND JUSTIFY**
Why use a variable rather than a specific value? You must show that $f(x) = f(-x)$ or that $-f(x) = f(-x)$ for all $x$ in the domain, not just one value.

**Is the function odd, even, or neither?**

**A.** $f(x) = 4x^4 + 5$

$f(-x) = 4(-x)^4 + 5$ ······ Replace $x$ with $-x$.

$f(-x) = 4x^4 + 5$ ······ Simplify.

Since $f(x) = f(-x)$, $f(x) = 4x^4 + 5$ is an even function.

**B.** $g(x) = 2x^3 + 3x$

$g(-x) = 2(-x)^3 + 3(-x)$ ······ Replace $x$ with $-x$.

$g(-x) = -2x^3 - 3x = -(2x^3 + 3x)$ ······ Simplify.

Since $g(-x) = -g(x)$, $g(x) = 2x^3 + 3x$ is an odd function.

**Try It!** 2. Is the function odd, even, or neither?

a. $f(x) = 7x^5 - 2x^2 + 4$

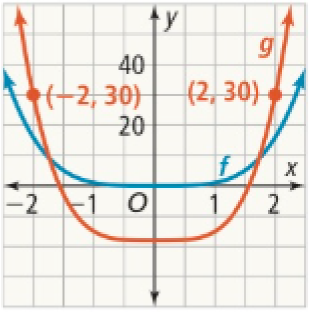
b. $f(x) = x^6 - 2$

## EXAMPLE 3 Graph Transformations of Cubic and Quartic Parent Functions

**CHOOSE EFFICIENT METHODS**
The graph of the parent function is a valuable tool, because it provides the foundation for sketching the graphs of related functions using transformations.

**How do transformed graphs compare to the graph of the parent function?**

**A.** $g(x) = 3x^4 - 18$



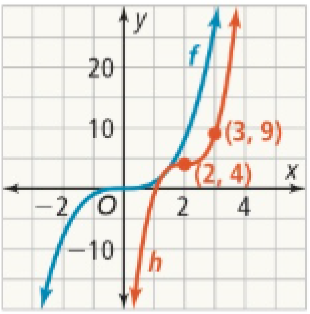

Identify the transformations: $g(x) = 3x^4 - 18$.

Parent function: $f(x) = x^4$

Leading coefficient, 3, stretches the graph vertically, making it narrower than the graph of the parent function.

Subtracting 18 translates the graph down 18 units.

**B.** $h(x) = 5(x - 2)^3 + 4$




Identify the transformations: $h(x) = 5(x - 2)^3 + 4$.

Parent function: $f(x) = x^3$

Subtracting 2 (before calculating the cube) shifts the parent graph to the right 2 units.

Multiplying by 5 stretches the translated graph vertically, making it narrower than the graph of the parent function.

Adding 4 translates the stretched graph up 4 units.

**Try It!** 3. How does the graph of the function $g(x) = 2x^3 - 5$ differ from the graph of its parent function?

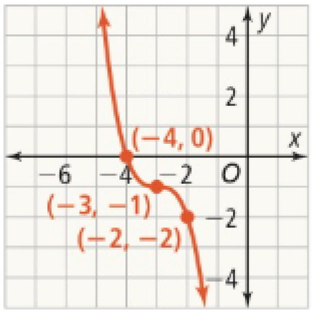
CONCEPTUAL UNDERSTANDING

**VOCABULARY**
A cubic function is 3rd degree. A quartic function is 4th degree.

## EXAMPLE 4 Identify a Transformation

**Each of the given graphs is a transformation of the parent cubic function or parent quartic function. How can you determine the equation of the graph?**

**A.**




Since the ends extend in opposite directions, the end behavior shows that the parent function has odd degree: $y = x^3$.

The end behavior indicates a negative leading coefficient so this graph is a reflection across the $x$-axis, such as $y = -x^3$.

The point (0, 0) has shifted to (−3, −1), which shows that the graph has been translated:

Left 3 units

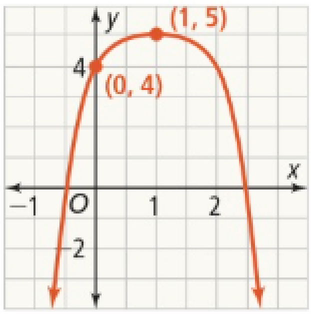
$y = -(x + 3)^3$

Down 1 unit

$y = -(x + 3)^3 - 1$

The function is
$f(x) = -(x + 3)^3 - 1$.

**B.**




Since the ends extend in the same direction, the end behavior shows that the parent function has even degree: $y = x^4$.

The end behavior indicates a negative leading coefficient so this graph is a reflection across the $x$-axis, such as $y = -x^4$.

The point (0, 0) has shifted to (1, 5), which shows that the graph has been translated:

Right 1 unit

$y = -(x - 1)^4$

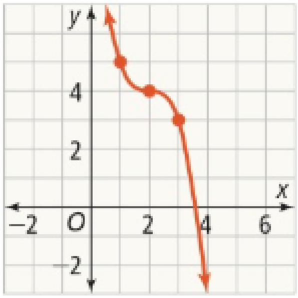
Up 5 units

$y = -(x - 1)^4 + 5$

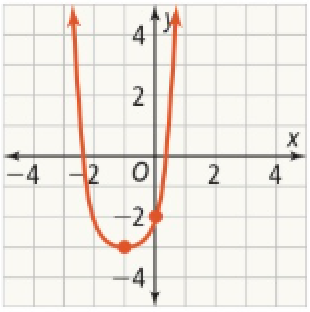
The function is
$f(x) = -(x - 1)^4 + 5$.

**Try It!** 4. Determine the equation of each graph as it relates to its parent cubic function or quartic function.

a.




b.



APPLICATION

## EXAMPLE 5 Apply a Transformation of a Cubic Function

**A. The volume of a box, in cubic yards, is given by the function $V(x) = x^3$. The post office lists permissible shipping volumes in cubic feet. Write a function for the volume in cubic feet if $x$ is the edge length in yards.**

Replace $x$ with $3x$ ······ Convert yards to feet.

$V(3x) = (3x)^3$ ······ Evaluate $V(x)$ for the value $3x$.

$V(x) = 27x^3$ ······ Simplify to write the function in units of cubic feet.

**COMMON ERROR**
Remember to apply exponents correctly: $(3x)^3 \neq 3x^3$.

The function that represents the volume of the box, in cubic feet, is $V(x) = 27x^3$.

**B. A terrarium is in the shape of a rectangular prism. The volume of the tank is given by $V(x) = (x)(2x)(x + 5) = 2x^3 + 10x^2$, where $x$ is measured in inches. The manufacturer wants to compare the volume of this tank with one that has a width 2 inches shorter but maintains the relationships between the width and the other dimensions. Write a new function for the volume of this smaller tank.**

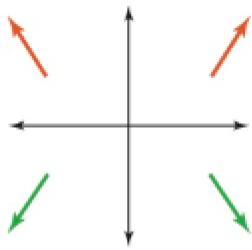
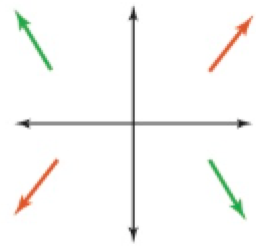
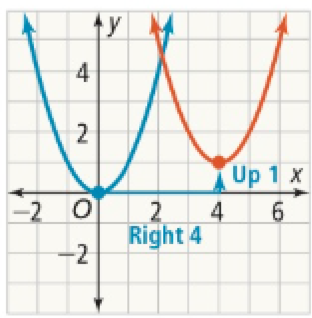
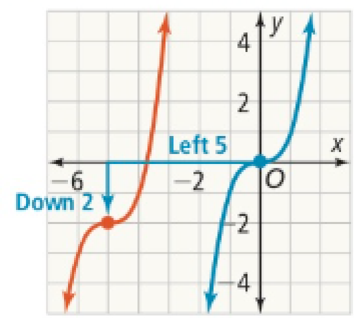

$$
\begin{aligned}
V(x-2) &= (x-2)[2(x-2)][(x-2)+5] \quad \text{Replace } x \text{ with } x-2.\\
&= (x-2)(2x-4)(x+3)\\
&= (2x^2-8x+8)(x+3)\\
&= 2x^3-2x^2-16x+24
\end{aligned}
$$

The function that represents the volume of the smaller tank is $V(x - 2) = 2x^3 - 2x^2 - 16x + 24$.

**Try It!** **5a.** The volume of a cube, in cubic feet, is given by the function $V(x) = x^3$. Write a function for the volume of the cube in cubic inches if $x$ is the edge length in feet.

**b.** A storage unit is in the shape of a rectangular prism. The volume of the storage unit is given by $V(x) = (x)(x)(x - 1) = x^3 - x^2$, where $x$ is measured in feet. A potential customer wants to compare the volume of this storage unit with that of another storage unit that is 1 foot longer in every dimension. Write a function for the volume of this larger unit.

## CONCEPT SUMMARY Understand Polynomial Functions

| | Even Function | Odd Function |
|---|---|---|
| **DEFINITION** | Line of symmetry: $y$-axis<br>For all $x$, $f(x) = f(-x)$. | Point of symmetry: origin<br>For all $x$, $f(-x) = -f(x)$. |
| **PARENT FUNCTION** | Has even degree:<br>$y = x^2, y = x^4, y = x^6, \ldots$ | Has odd degree:<br>$y = x, y = x^3, y = x^5, \ldots$ |
| **END BEHAVIOR**<br>(positive leading coefficient)<br>(negative leading coefficient) |  |  |
| **TRANSLATION** | <br>The vertex moves to the right 4 units and up 1 unit.<br>$y = x^2 \rightarrow y = (x - 4)^2 + 1$ | <br>The graph of the function moves to the left 5 units and down 2 units.<br>$y = x^3 \rightarrow y = (x + 5)^3 - 2$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are symmetry and transformations represented in the graph and equation of a polynomial function?

2. **Vocabulary** What is the difference between the graph of an even function and the graph of an odd function?

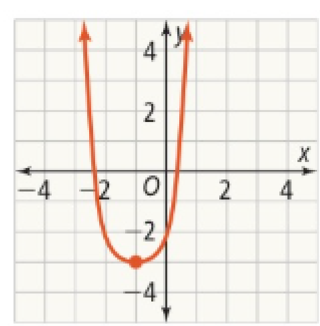
3. **Error Analysis** A student identified the transformations of the polynomial function $f(x) = 3(x - 1)^3 - 6$ as follows:

   *The function shifted to the left 1 unit, stretched vertically, and shifted downward 6 units.*

   Describe and correct the error the student made.

## Do You KNOW HOW?

4. Classify the function on the graph as odd, even, or neither.




5. Use the equation to classify the function as odd, even, or neither.

$$g(x) = 4x^3 - x$$

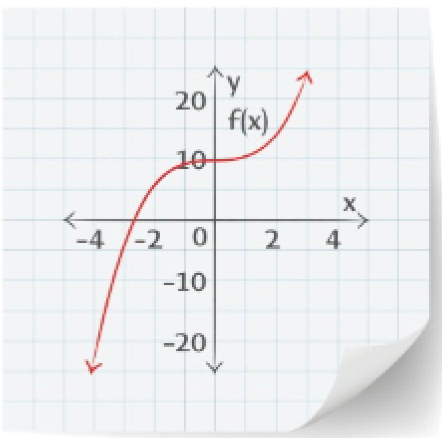
6. The volume of a cardboard box is given by the function $V(x) = x(x - 2)(x) = x^3 - 2x^2$. Write a new function for the volume of a cardboard box that is 2 units longer in every dimension.

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

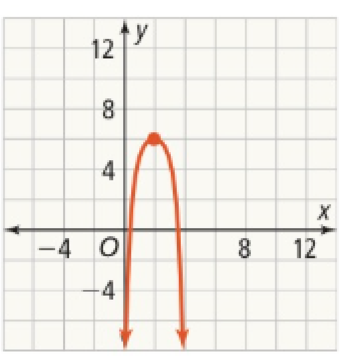
7. **Analyze and Persevere** If you use a graph to determine the equation of a function, explain how to check that your equation is correct.

8. **Error Analysis** Describe the error Terrence made in graphing the transformation of the cubic function $g(x) = x^3$ to $f(x) = -\frac{1}{2}x^3 + 10$.




9. **Higher Order Thinking** Explain how to identify a transformation of the function $y = x^3$ by looking at a graph. What do you look for to determine a translation? A reflection? A stretch or compression?

10. **Use Patterns and Structure** Describe the steps used to determine the equation of the graph of the transformed parent quartic function.



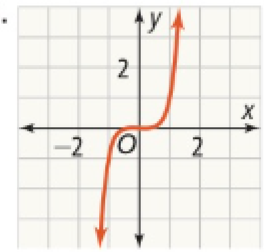

11. **Communicate and Justify** Explain why the function $g(x) = 2x^5 + 3x^4 + 1$ is neither even nor odd.

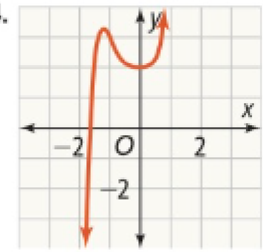
12. **Communicate and Justify** Provide an example that demonstrates the following statement is not true.

    *If the degree of a function is an even number, then the function is an even function.*

## PRACTICE

**Use the graph to classify the polynomial function. Is it even, odd, or neither?** SEE EXAMPLE 1

13. 


14. 

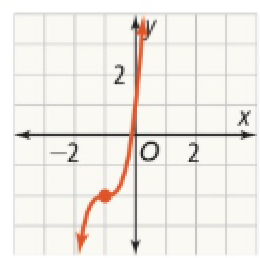

**Use the equation to classify the polynomial function. Is it even, odd, or neither?** SEE EXAMPLE 2

15. $f(x) = 2x^5 + 4x^2$

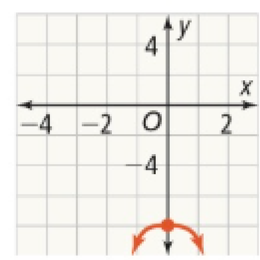
16. $g(x) = 6x^4 + 2x^2$

**How do the graphs of transformations compare to the graph of the parent function?** SEE EXAMPLE 3

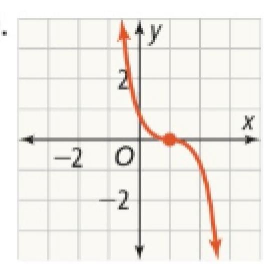
17. $f(x) = 3(x + 1)^3 - 2$

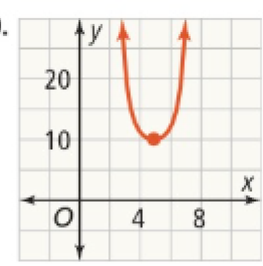



18. $g(x) = -x^4 - 8$

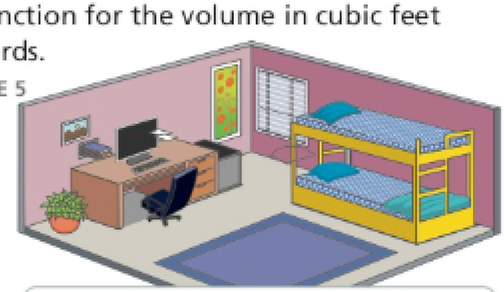



**Each graph is a transformation of the parent cubic function or quartic function. Determine the equation of the graph.** SEE EXAMPLE 4

19. 


20. 


21. The volume of a rectangular room, in cubic yards, is given by the function shown. Write a function for the volume in cubic feet if $x$ is in yards.
SEE EXAMPLE 5



Volume in cubic yards $= x(3x)(x + 4) = 3x^3 + 12x^2$

## PRACTICE & PROBLEM SOLVING

### APPLY

22. **Analyze and Persevere** Last season the approximate number of guests in week $x$ at an amusement park could be modeled by the function, $f$, where $x$ represents the number of weeks since the park opened for the season. This year, since the park opened its new water slide, the approximate number of guests in week $x$ at the park can be modeled by $g$.

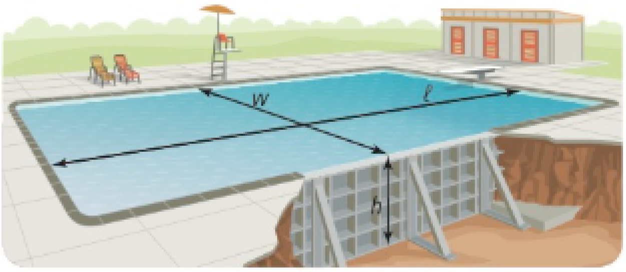

a. Write the function $g$ in terms of $x$.

b. Describe the transformation of the graph of $g$ compared to $f$.

c. Compare the number of weekly visitors from last year to this year.

23. **Generalize** The volume of a storage box, in cubic feet, is given by the function $V(x) = (x)(x + 1)^2$. A freight company lists the shipping rates of items in cubic inches. Write a function for the volume of the box in cubic inches if $x$ is its width in feet.

24. **Apply Math Models** A pool is in the shape of a rectangular prism. The width is one more than five times the height, and the length is one less than eleven times the height.




a. Using $x$ for the height, write a function $V(x)$ to represent the volume of the pool.

b. Compare the volume of this pool with a larger one that is the same height, but twice the length and twice the width of this pool. Write a function $Z(x)$ for the volume of this larger pool.

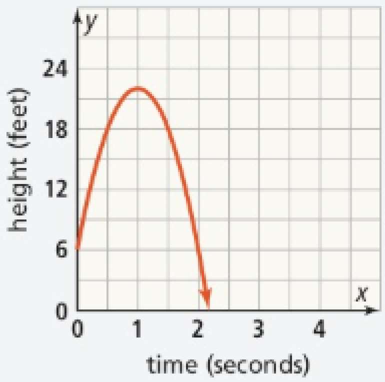
### ASSESSMENT PRACTICE

25. Identify the effects of the values 2, 1, and 5 in graph of the function $g(x) = 2(x - 1)^4 + 5$ as a transformation of its parent function, $f(x) = x^4$.
F.2.2

26. **SAT/ACT** Which of the following functions is neither even nor odd?

Ⓐ $f(x) = x^4 + 3x^2$
Ⓑ $g(x) = 5x^3 - x$
Ⓒ $h(x) = x^5 + 4x^3 + x^2$
Ⓓ $k(x) = 9 - 8x^2$
Ⓔ $p(x) = 5$

27. **Performance Task** The height of a ball thrown in the air can be modeled by the function $h(x) = -16t^2 + 32t + 6$, where $h(x)$ represents the height in feet of the ball after $t$ seconds. The graph of this function is shown below.




**Part A** What do the vertex, $y$-intercept, and $x$-intercept represent?

**Part B** If the ball is thrown from a height of 10 ft, how will this transform the graph?

**Part C** About how much longer will the ball be in the air when it is thrown from 10 ft compared to when it was thrown from 6 ft? (Hint: You may want to use your graphing calculator to compare the two graphs.)

TOPIC 3

# Topic Review

## TOPIC ESSENTIAL QUESTION

1. What can the rule for a polynomial function reveal about its graph, and what can the graphs of polynomial functions reveal about the solutions of polynomial equations?

## Vocabulary Review

**Choose the correct term to complete each sentence.**

2. The __________ is the greatest power of the variable in a polynomial expression.
3. The __________ is the non-zero constant multiplied by the greatest power of the variable in a polynomial expression.
4. The __________ of a function describes what happens to its graph as $x$ approaches positive and negative infinity.
5. __________ is a triangular pattern of numbers where each number is the sum of two numbers above it.
6. The __________ determines whether the graph of the function will cross the $x$-axis at the point or merely touch it.
7. The __________ is a formula that can be used to expand powers of binomial expressions.
8. __________ is a method to divide a polynomial by a linear factor whose leading coefficient is 1.

- Binomial Theorem
- degree of a polynomial
- end behavior
- even function
- identity
- leading coefficient
- multiplicity of a zero
- Pascal's Triangle
- synthetic division

## Concepts & Skills Review

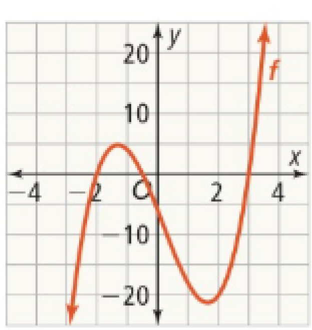
### LESSON 3-1 Graphing Polynomial Functions

#### Quick Review

A **polynomial** can be either a monomial or a sum of monomials. When a polynomial has more than one monomial, the monomials are also referred to as **terms.**

#### Example

**Graph the function $f(x) = 2x^3 - x^2 - 13x - 6$.**




There are zeros at $x = -2$, $x = -0.5$, and $x = 3$.

There are turning points between –2 and –0.5 and between –0.5 and 3.

As $x \to -\infty$, $y \to -\infty$.

As $x \to +\infty$, $y \to +\infty$.

#### Practice & Problem Solving

**Graph the polynomial function. Estimate the zeros and the turning points of the graph.**

9. $f(x) = x^5 + 2x^4 - 10x^3 - 20x^2 + 9x + 18$

10. $f(x) = x^4 + x^3 - 16x^2 - 4x + 48$

11. **Generalize** A polynomial function has the following end behavior: As $x \to -\infty$, $y \to +\infty$. As $x \to +\infty$, $y \to -\infty$. Describe the degree and leading coefficient of the polynomial function.

12. **Analyze and Persevere** After $x$ hours of hiking, Sadie's elevation is $p(x) = -x^3 + 11x^2 - 34x + 24$, in meters. After how many hours will Sadie's elevation be 18 m below sea level? What do the $x$- and $y$-intercepts of the graph mean in this context?

## LESSONS 3-2 & 3-3 Adding, Subtracting, and Multiplying Polynomials and Polynomial Identities

### Quick Review

To add or subtract polynomials, add or subtract like terms. To multiply polynomials, use the Distributive Property.

Polynomial identities can be used to factor or multiply polynomials.

### Example

**Add $(-2x^3 + 5x^2 + 2x - 3) + (x^3 - 6x^2 + x + 12)$.**

Use the Commutative and Associative Properties. Then combine like terms.

$(-2x^3 + 5x^2 + 2x - 3) + (x^3 - 6x^2 + x - 12)$

$= (-2x^3 + x^3) + (5x^2 - 6x^2) + (2x + x) + (-3 + 12)$

$= -x^3 - x^2 + 3x + 9$

### Example

**Use polynomial identities to factor $8x^3 + 27y^3$.**

Use the Sum of Cubes Identity. Express each term as a square. Then write the factors.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$8x^3 + 27y^3 = (2x)^3 + (3y)^3$

$= (2x + 3y)(4x^2 - 6xy + 9y^2)$

### Practice & Problem Solving

**Add or subtract the polynomials.**

13. $(-8x^3 + 7x^2 + x - 9) + (5x^3 + 3x^2 - 2x - 1)$

14. $(9y^4 - y^3 + 4y^2 + y - 2) - (2y^4 - 3y^3 + 6y - 7)$

**Multiply the polynomials.**

15. $(9x - 1)(x + 5)(7x + 2)$

**Use polynomial identities to multiply each polynomial.**

16. $(5x + 8)^2$

17. $(7x - 4)(7x + 4)$

**Factor the polynomial.**

18. $x^6 - 64$

19. $27x^3 + y^6$

**Use Pascal's Triangle or the Binomial Theorem to expand the expressions.**

20. $(x - 2)^4$

21. $(x + 5y)^5$

22. **Represent and Connect** Explain why the set of polynomials is closed under subtraction.

23. **Apply Math Models** The length of a rectangle is represented by $3x^3 - 2x^2 + 10x - 4$, and the width is represented by $-x^3 + 6x^2 - x + 8$. What is the perimeter of the rectangle?

## LESSON 3-4 Dividing Polynomials

### Quick Review

Polynomials can be divided using long division or synthetic division. **Synthetic division** is a method to divide a polynomial by a linear factor whose leading coefficient is 1.

### Example

**Use synthetic division to divide $x^4 - 5x^3 - 6x^2 + 2x - 8$ by $x + 3$.**

$$\begin{array}{r|rrrrr} -3 & 1 & -5 & -6 & 2 & -8 \\ & & -3 & 24 & -54 & 156 \\ \hline & 1 & -8 & 18 & -52 & 148 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & x^3 & -8x^2 & +18x & -52 & +\frac{148}{x+3} \end{array}$$

The quotient is $x^3 - 8x^2 + 18x - 52$, and the remainder is 148.

### Practice & Problem Solving

**Use long division to divide.**

24. $x^4 + 2x^3 - 8x^2 - 3x + 1$ divided by $x + 2$

**Use synthetic division to divide.**

25. $x^4 + 5x^3 + 7x^2 - 2x + 17$ divided by $x - 3$

26. **Analyze and Persevere** A student divided $f(x) = x^3 + 8x^2 - 9x - 3$ by $x - 2$ and got a remainder of 19. Explain how the student could verify the remainder is correct.

27. **Apply Math Models** The area of a rectangle is $4x^3 + 14x^2 - 18$ in.$^2$. The length of the rectangle is $x + 3$ in. What is the width of the rectangle?

## LESSONS 3-5 & 3-6 Zeros of Polynomial Functions and Roots of Polynomial Equations

### Quick Review

You can factor and use synthetic division to find zeros of polynomial functions. Then you can use the zeros to sketch a graph of the function.

The **Rational Root Theorem** states that the possible rational roots, or zeros, of a polynomial equation with integer coefficients come from the list of numbers of the form: $\pm \frac{\textit{factor of } a_0}{\textit{factor of } a_n}$.

### Example

List all the possible rational solutions for the equation $0 = 2x^3 + x^2 - 7x - 6$. Then find all of the rational roots.

$\pm1, \pm2, \pm3, \pm6$ Factors of the constant term

$\pm1, \pm2$ Factors of the leading coefficient

List the possible roots, eliminating duplicates.

$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{6}{1}$

Use synthetic division to find that the roots are $-\frac{3}{2}$, $-1$, and 2.

### Practice & Problem Solving

**Sketch the graph of the function.**

**28.** $f(x) = 2x^4 - x^3 - 32x^2 + 31x + 60$

**29.** $g(x) = x^3 - x^2 - 20x$

**30.** What $x$-values are solutions to the equation $x^3 + 2x^2 - 4x + 8 = x^2 - x + 4$?

**31.** What values of $x$ are solutions to the inequality $x^3 + 3x^2 - 4x - 12 > 0$?

**32.** What are all of the real and complex roots of the function $f(x) = x^4 - 4x^3 + 4x^2 - 36x - 45$?

**33.** A graph of a polynomial has 3 sign changes. What is the minimum number of roots it could have?

**34.** **Generalize** What does the graph of a function tell you about the multiplicity of a zero?

**35.** **Analyze and Persevere** A storage unit in the shape of a rectangular prism measures $2x$ ft long, $x + 8$ ft wide, and $x + 9$ ft tall. What are the dimensions of the storage unit, in feet, if its volume is 792 ft$^3$?

## LESSON 3-7 Transformations of Polynomial Functions

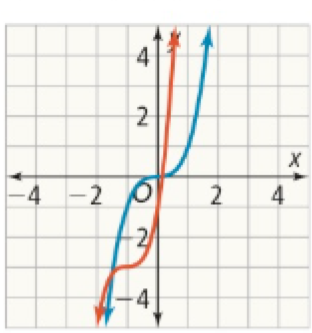
### Quick Review

Polynomial functions can be translated, reflected, and stretched in similar ways to other functions you have studied.

### Example

**How does the graph of $f(x) = 2(x + 1)^3 - 3$ compare to the graph of the parent function?**

Parent function: $y = x^3$




Adding 1 shifts the graph to the left 1 unit.

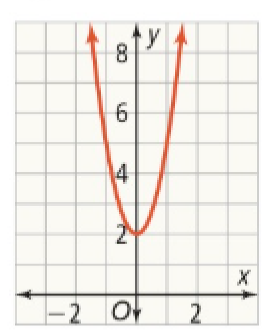
Multiplying by 2 stretches the graph vertically.

Subtracting 3 shifts the graph down 3 units.

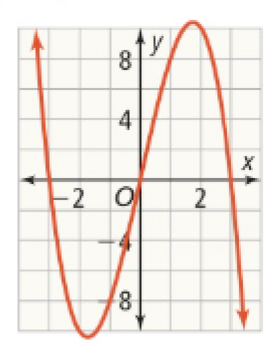
### Practice & Problem Solving

Classify each function as even, odd, or neither.

**36.**




**37.**




**38.** **Error Analysis** A student says the graph of $f(x) = 0.5x^4 + 1$ is a vertical stretch and a translation up 1 unit of the parent function. Explain the student's error.

**39.** **Analyze and Persevere** The volume of a refrigerator, in cubic centimeters, is given by the function $V(x) = (x)(x + 1)(x - 2)$. Write a new function for the volume of the refrigerator in cubic millimeters if $x$ is in centimeters.

TOPIC 4

# Rational Functions

## TOPIC ESSENTIAL QUESTION

**How do you calculate with functions defined as quotients of polynomials, and what are the key features of their graphs?**

## Topic Overview


**enVision® STEM Project:**
Manufacturing Costs

**4-1** Inverse Variation and the Reciprocal Function
AR.8.2, AR.8.3, F.1.1, F.2.2, MTR.1.1, MTR.5.1, MTR.7.1

**4-2** Graphing Rational Functions
AR.8.2, AR.8.3, MTR.2.1, MTR.4.1, MTR.6.1

**4-3** Multiplying and Dividing Rational Expressions
AR.1.9, MTR.2.1, MTR.4.1, MTR.6.1

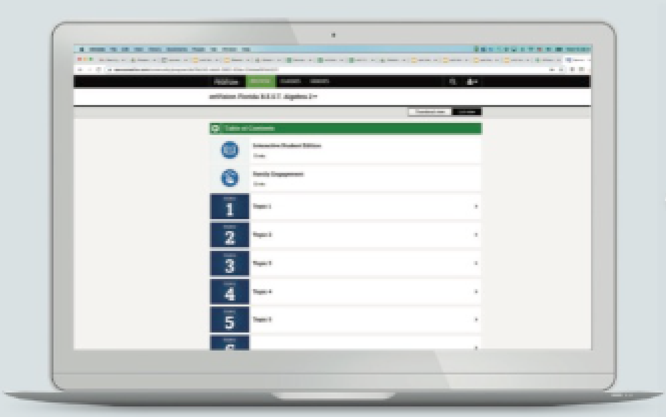
**4-4** Adding and Subtracting Rational Expressions
AR.1.9, MTR.3.1, MTR.5.1, MTR.7.1

**4-5** Solving Rational Equations
AR.8.1, AR.8.3, MTR.1.1, MTR.3.1, MTR.5.1

**Mathematical Modeling in 3 Acts:**
Real Cool Waters
AR.1.9, AR.8.3, MTR.7.1


## Topic Vocabulary

- asymptote
- compound fraction
- constant of variation
- extraneous solution
- inverse variation
- rational equation
- rational expression
- rational function
- reciprocal function
- simplified form of a rational expression



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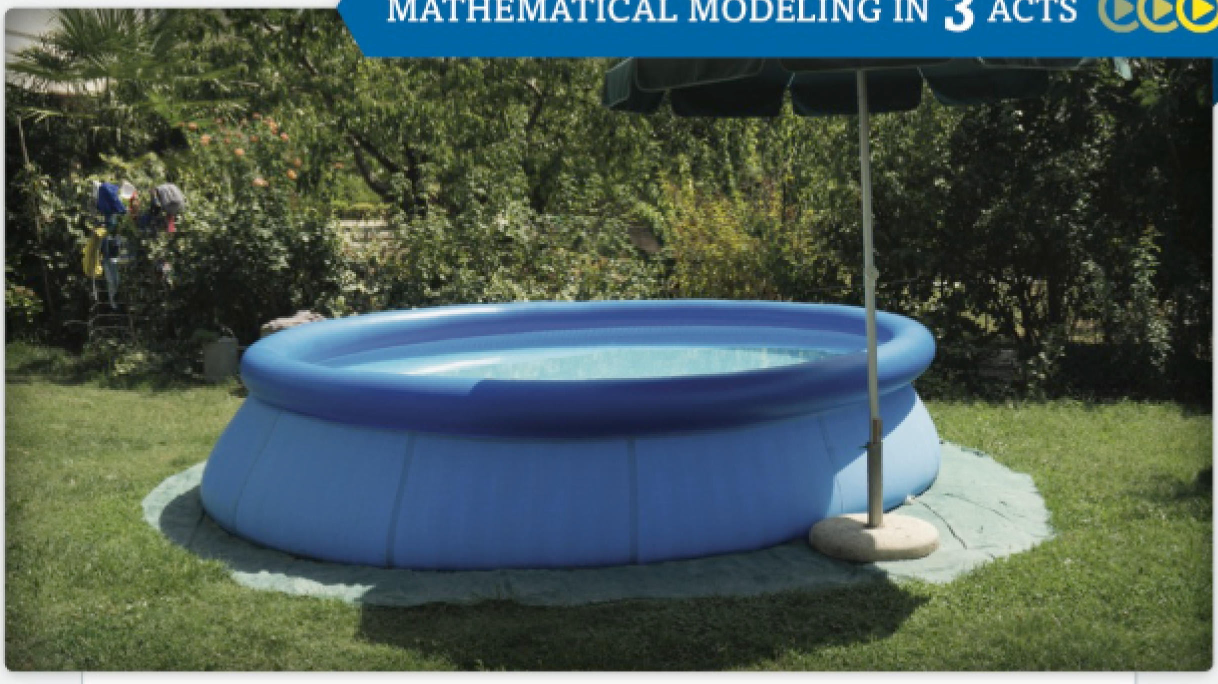
**ANIMATION** View and interact with real-world applications.

**PRACTICE** Practice what you've learned.

## MATHEMATICAL MODELING IN 3 ACTS



### Real Cool Waters

Nothing feels better on a hot day than jumping into a pool! Many cities have swimming pools that people can go to for a small fee. Some people have swimming pools in their backyards that they can enjoy any time. If neither of these options are available, you can always create your own beach paradise! Get a kiddie pool, a lawn chair, and a beach umbrella. Think about your beach paradise during the Mathematical Modeling in 3 Acts lesson.

**VIDEOS** Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

**ADAPTIVE PRACTICE** Practice that is *just right* and *just for you*.

**GLOSSARY** Read and listen to English and Spanish definitions.

**CONCEPT SUMMARY** Review key lesson content through multiple representations.

**ASSESSMENT** Show what you've learned.

**TUTORIALS** Get help from *Virtual Nerd*, right when you need it.

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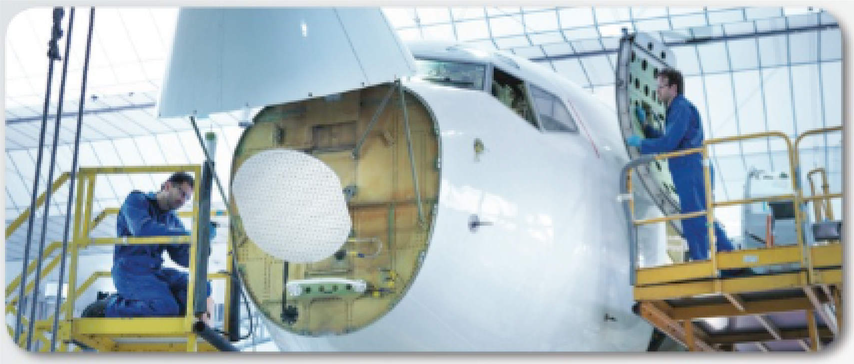
**DESMOS** Use Anytime and as embedded Interactives in Lesson content.

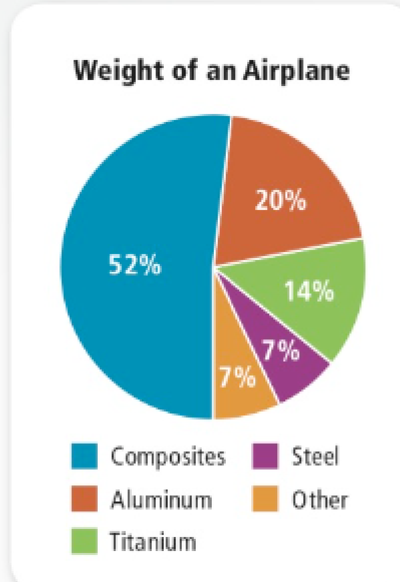
**QR CODES** Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

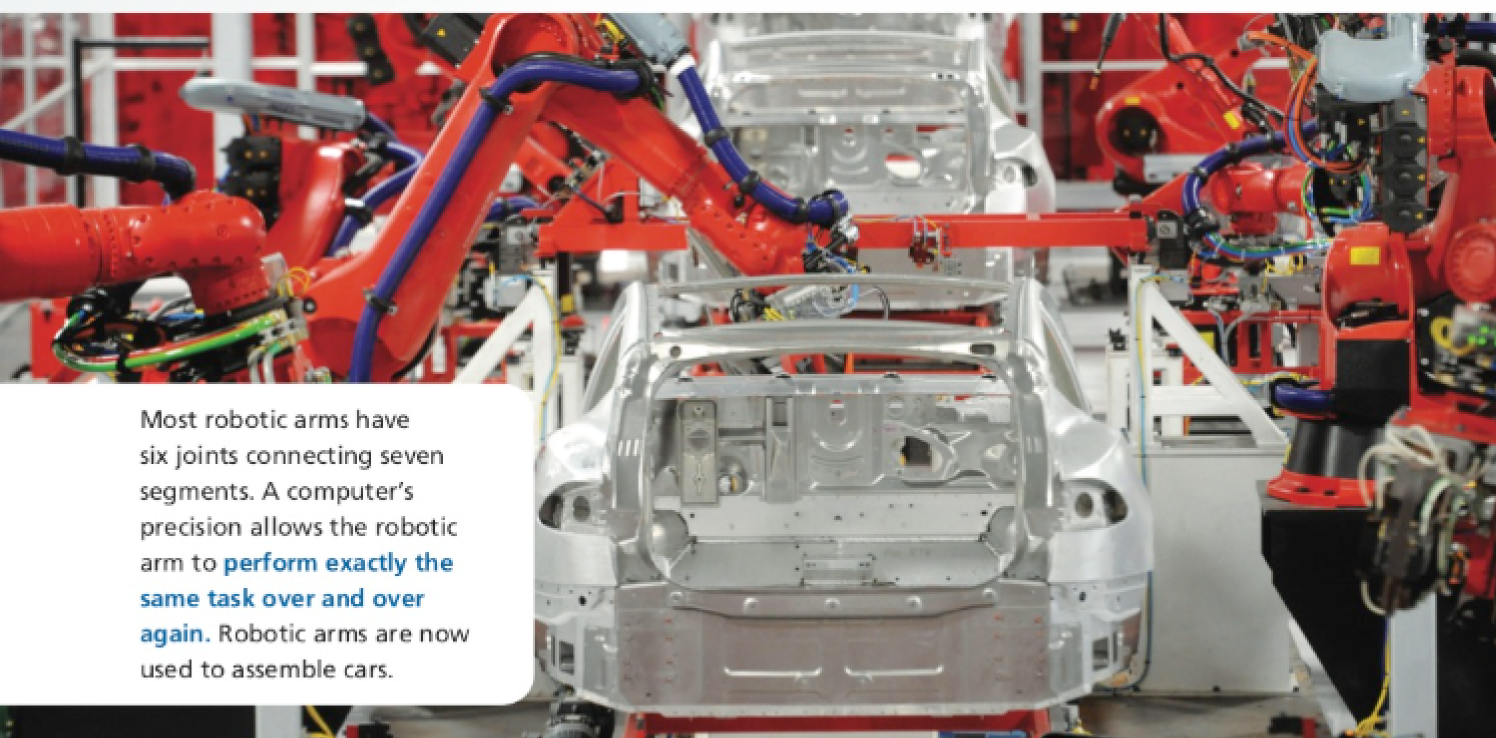
## Did You Know?

All business-related costs are either *fixed* or *variable*. **Fixed costs** for a business include rent and machinery to make items to be sold by the business. **Variable costs** include the materials needed to make the items.



In 2013, **Delta Airlines** began purchasing used aircraft instead of new aircraft, lowering its fixed costs (buying aircraft) and raising its variable costs (maintaining its aircraft).



Most robotic arms have six joints connecting seven segments. A computer's precision allows the robotic arm to **perform exactly the same task over and over again.** Robotic arms are now used to assemble cars.

## Your Task: Manufacturing Costs

You and your classmates will collect data about a potential business venture, and determine the number of items that must be built and sold in order for the business to be profitable. Based on this and other information, you will give advice about whether or not the business venture is viable and how to improve it.

# 4-1 Inverse Variation and the Reciprocal Function

**I CAN…** use inverse variation and graph translations of the reciprocal function.

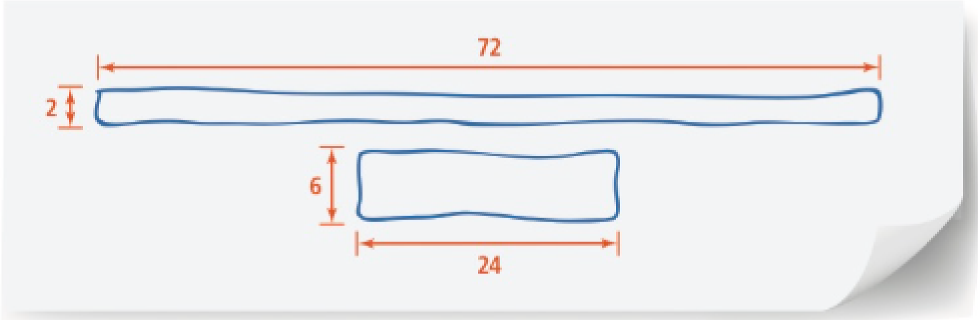
**VOCABULARY**

- asymptote
- constant of variation
- inverse variation
- reciprocal function

MA.912.AR.8.2–Given a table, equation or written description of a rational function, graph that function and determine its key features. Also AR.8.3, F.1.1, F.2.2
MA.912.MTR.1.1, MTR.5.1, MTR.7.1

## MODEL & DISCUSS

**The two rectangles shown both have an area of 144 square units.**




**A.** Sketch as many other rectangles as you can that have the same area. Organize and record your data for the lengths and widths of the rectangles.

**B.** **Use Patterns and Structure** Considering rectangles with an area of 144 square units, what happens to the width of the rectangle as the length increases?

**C.** Examine at least five other pairs of rectangles, each pair sharing the same area. How would you describe the relationship between the lengths and widths?

**ESSENTIAL QUESTION** How are inverse variations related to the reciprocal function?

CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Identify Inverse Variation

**How do you determine if a relationship represents an inverse variation?**

**A. Does the table of values represent an inverse variation?**

| $x$ | 1 | 2 | 3 | 4 | 6 | 12 |
|---|---|---|---|---|---|---|
| $y$ | 12 | 6 | 4 | 3 | 2 | 1 |

An **inverse variation** is a relation between two variables such that as one variable increases, the other decreases proportionally. For the table to represent an inverse variation, the product of $x$ and $y$ must be constant. Find the product, $xy$, for each column in the table.

| $x$ | 1 | 2 | 3 | 4 | 6 | 12 |
|---|---|---|---|---|---|---|
| $y$ | 12 | 6 | 4 | 3 | 2 | 1 |
| $xy$ | 12 | 12 | 12 | 12 | 12 | 12 |

Since the product of the values is constant, *the table of values represents an inverse variation.*

**STUDY TIP**
Be sure to check the products for every pair of values before drawing a conclusion.

**B. Does the table of values represent an inverse variation?**

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 20 | 17 | 14 | 11 | 8 | 5 |

Find the products.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 20 | 17 | 14 | 11 | 8 | 5 |
| $xy$ | 20 | 34 | 42 | 44 | 40 | 30 |

Since the products are not constant, *the table does not represent an inverse variation.*

CONTINUED ON THE NEXT PAGE

**Try It!** **1.** Determine if each table of values represents an inverse variation.

**a.**

| $x$ | 1 | 2 | 3 | 5 | 6 | 15 |
|---|---|---|---|---|---|---|
| $y$ | 25.5 | 12.75 | 8.50 | 5.10 | 4.25 | 1.70 |

**b.**

| $x$ | 6.6 | 5.5 | 4.4 | 3.3 | 2.2 | 1.1 |
|---|---|---|---|---|---|---|
| $y$ | 3 | 5 | 7 | 9 | 11 | 13 |

## CONCEPT Inverse Variation

When a relation between $x$ and $y$ is an inverse variation, we say that $x$ varies inversely as $y$. Inverse variation is modeled by the equation $y = \frac{k}{x}$, or with an equivalent form $x = \frac{k}{y}$ or $xy = k$, where $k \neq 0$. The variable $k$ represents the **constant of variation**, the number that relates the two variables in an inverse variation.

In this table, the constant of variation is 24.

| $x$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 24 | 12 | 8 | 6 | 4 | 3 | 2 | 1 |

Notice how as $x$ doubles in value from 1 to 2 to 4 to 8, . . .

. . . the value of $y$ is halved from 24 to 12 to 6 to 3.

## EXAMPLE 2 Use Inverse Variation

**In an inverse variation, $x = 10$ when $y = 3$. Write an equation to represent the inverse variation. Then find the value of $y$ when $x = -6$.**

$y = \frac{k}{x}$ Write the equation for an inverse variation.

$3 = \frac{k}{10}$ Substitute 10 and 3 for $x$ and $y$.

$30 = k$ Multiply both sides by 10 to solve for $k$.

After solving for $k$, write an equation for the inverse variation.

$y = \frac{30}{x}$ Write the equation to represent the inverse variation.

$y = \frac{30}{-6}$ Substitute −6 for $x$ in the equation.

$y = -5$ Divide.

The equation that represents the inverse relation is $y = \frac{30}{x}$. When $x = -6$, $y = -5$.

**COMMON ERROR**
Remember to keep track of any negative signs when substituting into equations and performing calculations.

**Try It!** **2.** In an inverse variation, $x = 6$ and $y = \frac{1}{2}$.

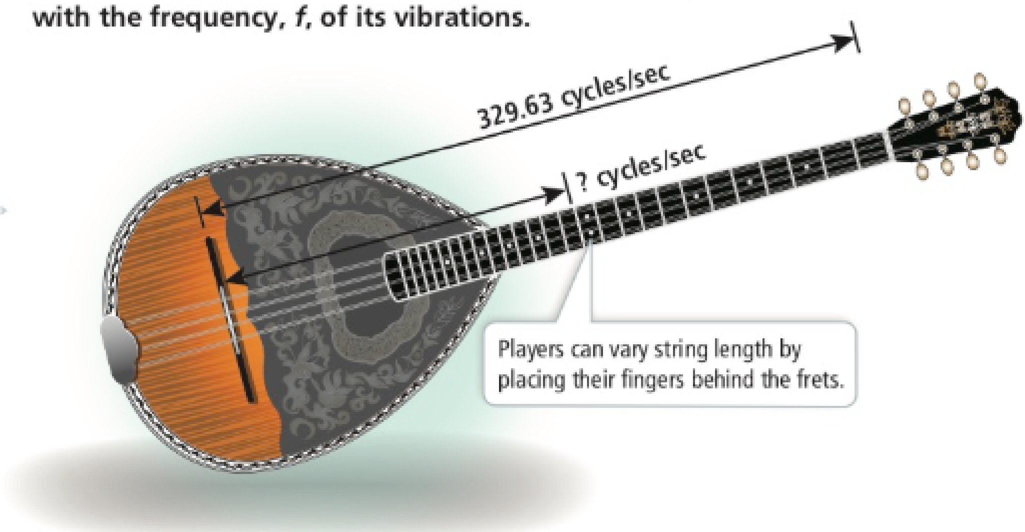
**a.** What is the equation that represents the inverse variation?

**b.** What is the value of $y$ when $x = 15$?

APPLICATION

## EXAMPLE 3 Use an Inverse Variation Model

**On a bouzouki, the string length, $s$, varies inversely with the frequency, $f$, of its vibrations.**




**ANALYZE AND PERSEVERE**
Use what you know about inverse variation to mentally compute an approximate value of your answer.

**The frequency of a 26-inch E-string is 329.63 cycles per second. What is the frequency when the string length is 13 inches?**

$s = \frac{k}{f}$ ······ Write the equation for an inverse variation.

$26 = \frac{k}{329.63}$ ······ Substitute 26 for $s$ and 320.63 for $f$.

$8{,}570.38 = k$ ······ Multiply by 329.63 to solve for $k$.

After solving for $k$, write an equation for the inverse variation.

$s = \frac{8{,}570.38}{f}$ ······ Substitute 8,570.38 for $k$ in the equation.

$13 = \frac{8{,}570.38}{f}$ ······ Substitute 13 for $s$ in the equation.

$f = 659.26$ ······ Solve for $f$.

So the frequency of the 13-inch string is 659.26 cycles per second.

**Try It!** **3.** The amount of time it takes for an ice cube to melt varies inversely to the air temperature, in degrees. At 20° Celsius, the ice will melt in 20 minutes. How long will it take the ice to melt if the temperature is 30° Celsius?

CONCEPTUAL UNDERSTANDING

## EXAMPLE 4 Graph the Reciprocal Function

**How do you graph the reciprocal function, $y = \frac{1}{x}$?**

The reciprocal function maps every non-zero real number to its reciprocal.

**Step 1:** Consider the domain and range of the function.

Domain: $\{x \mid x \neq 0\}$

Range: $\{y \mid y \neq 0\}$

If $x = 0$ that will result in an undefined expression, so $x \neq 0$.

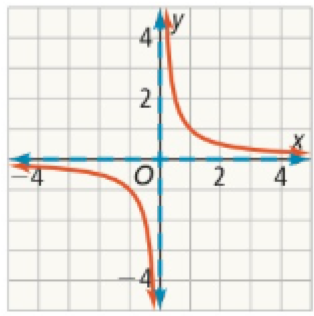
CONTINUED ON THE NEXT PAGE

EXAMPLE 4 CONTINUED

**Step 2:** Graph the function.

| $x$ | $-3$ | $-2$ | $-1$ | $-\frac{1}{2}$ | $-\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $\frac{1}{2}$ | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-\frac{1}{3}$ | $-\frac{1}{2}$ | $-1$ | $-2$ | $-3$ | Undefined | 3 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ |

**CHOOSE EFFICIENT METHODS**
For an equation such as this one, that does not involve a lot of parameters, graphing by hand makes sense. As you encounter more complex equations, it may be appropriate to graph with technology.




Use a table of values or technology to graph the function.

**Step 3:** Observe the graph of $y = \frac{1}{x}$ as it approaches positive infinity and negative infinity.

| $x$ | 1 | 10 | 100 | 1,000 | 10,000 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | $\frac{1}{10}$ | $\frac{1}{100}$ | $\frac{1}{1,000}$ | $\frac{1}{10,000}$ |

As $x$ gets larger, the denominator gets larger and the value of the function approaches zero.

An **asymptote** is a line that a graph approaches. Vertical asymptotes may never be crossed. Horizontal asymptotes guide the end behavior of a function.

As $x$ approaches infinity, $f(x)$ approaches 0. The same is true as $x$-values approach negative infinity, so the line $y = 0$ is a *horizontal asymptote*.

**Step 4:** Observe the graph of $y = \frac{1}{x}$ as $x$ approaches 0 for positive and negative $x$-values.

| $x$ | 1 | $\frac{1}{10}$ | $\frac{1}{100}$ | $\frac{1}{1,000}$ | $\frac{1}{10,000}$ |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 10 | 100 | 1,000 | 10,000 |

As $x$ gets closer to 0, the value of the function gets larger and larger.

For positive values of $x$, as $x$ approaches 0, $f(x)$ approaches positive infinity.

| $x$ | $-1$ | $-\frac{1}{10}$ | $-\frac{1}{100}$ | $-\frac{1}{1,000}$ | $-\frac{1}{10,000}$ |
|---|---|---|---|---|---|
| $f(x)$ | $-1$ | $-10$ | $-100$ | $-1,000$ | $-10,000$ |

As $x$ gets closer to 0, the value of the function approaches negative infinity.

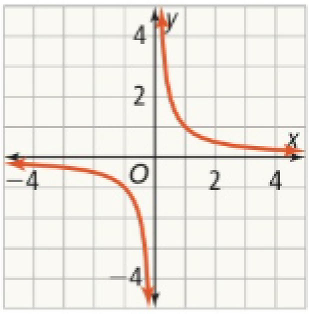
For negative values of $x$, as $x$ approaches 0, $f(x)$ approaches negative infinity. The domain of the function excludes 0, so the graph will never touch the line $x = 0$. The line $x = 0$ is a *vertical asymptote*.

**Try It!** 4. Graph the function $y = \frac{10}{x}$. What are the domain, range, and asymptotes of the function?

## EXAMPLE 5 Graph Translations of the Reciprocal Function

**Graph $g(x) = \frac{1}{x-3} + 2$. What are the equations of the asymptotes? What are the domain and range?**

Start with the graph of the parent function, $f(x) = \frac{1}{x}$.

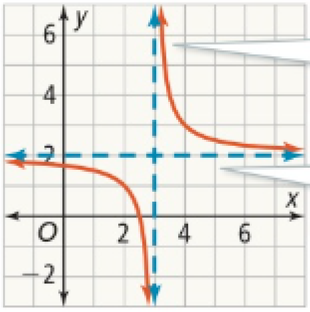


Recall that subtracting $h$ from $x$ in the definition of $f$ translates the graph of $f$ horizontally. Subtracting $k$ from $f(x)$ translates the graph of $f$ vertically.

The function $q(x) = \frac{1}{x-h} + k$ is a transformation of the parent function $f$ that shifts the graph of $f$ horizontally by $h$ units and then shifts the graph of $f$ vertically by $k$ units.

The graph of $g(x) = \frac{1}{x-3} + 2$ is a translation of the graph of the parent function 3 units right and 2 units up.

**ANALYZE AND PERSEVERE**
Not only are the points of the graph translated, but the asymptotes are translated as well.




The line $x = 3$ is a vertical asymptote. The line $y = 2$ is a horizontal asymptote.

The domain is $\{x \mid x \neq 3\}$.

The range is $\{y \mid y \neq 2\}$.

**Try It!** 5. Graph $g(x) = \frac{1}{x+2} - 4$. What are the equations of the asymptotes? What are the domain and range?

## CONCEPT SUMMARY Inverse Variation and the Reciprocal Function

### Inverse Variation

**WORDS** An inverse variation is a relation between two variables such that as one variable increases, the other decreases proportionally.

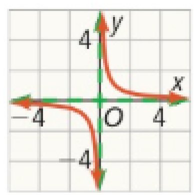
**ALGEBRA** $y = \frac{k}{x}$, where $k \neq 0$

**EXAMPLES** $y = \frac{1}{x}$

asymptotes:

$x = 0$

$y = 0$




### Transformations of the Reciprocal Function

**WORDS** The reciprocal function models the inverse variation, $y = \frac{1}{x}$. Like other functions, it can be transformed.

**ALGEBRA** $y = \frac{a}{x - h} + k$

**EXAMPLES** $y = \frac{1}{x - 4} - 2$

$h = 4$

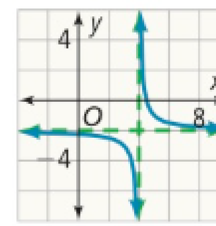
$k = -2$

Parent is transformed down 2 and right 4.

asymptotes:

$x = 4$

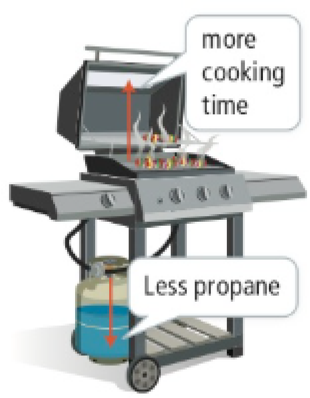
$y = -2$




## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are inverse variations related to the reciprocal function?

2. **Communicate and Justify** Explain why the amount of propane in a grill's tank and the time spent grilling could represent an inverse variation.




3. **Vocabulary** Why is it impossible for the graph of the function $y = \frac{1}{x}$ to intersect the horizontal asymptote at the $x$-axis?

4. **Error Analysis** Carmen said the table of values shown represents an inverse variation. Explain why Carmen is mistaken.

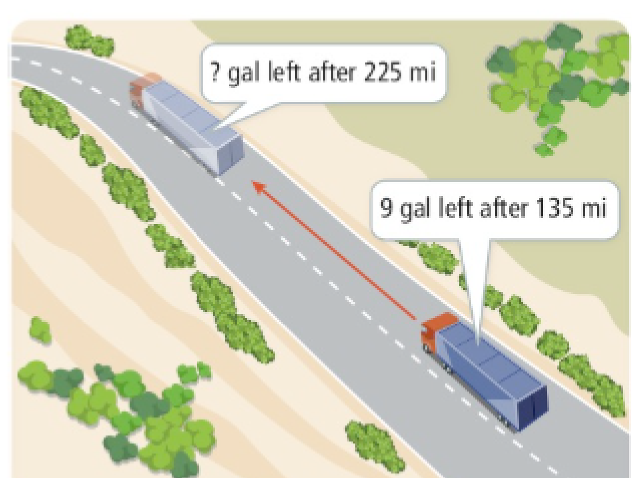
| $x$ | 1 | 2 | 3 | 4 | 8 | 16 |
|---|---|---|---|---|---|---|
| $y$ | 24 | 12 | 8 | 6 | 3 | 2 |

## Do You KNOW HOW?

5. In an inverse variation, $x = -8$ when $y = -\frac{1}{4}$. What is the value of $y$ when $x = 4$?

6. What are the equations of the asymptotes of the function $f(x) = \frac{1}{x - 5} + 3$? What are the domain and range?

7. Until the truck runs out of gas, the amount of gas in its fuel tank varies inversely with the number of miles traveled. Model a relationship between the amount of gas in a fuel tank of a truck and the number of miles traveled by the truck as an inverse variation.



## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

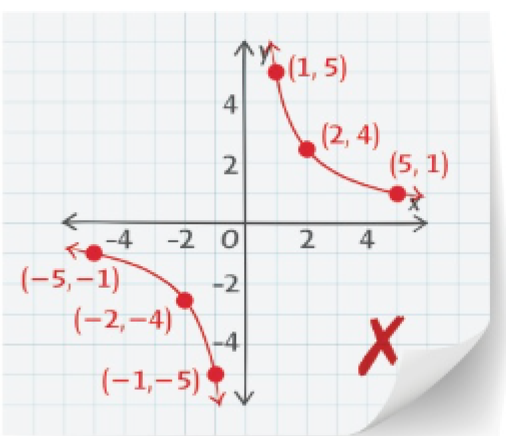
8. **Represent and Connect** Explain the difference between the graphs of inverse variation functions when $k > 0$ and when $k < 0$.

9. **Generalize** Just from looking at the table of values, how can you determine that the data do *not* represent an inverse variation?

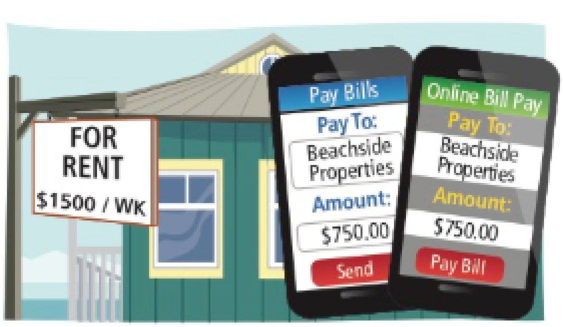
| $x$ | −2 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| $y$ | −6 | 6 | 12 | 18 | 24 | 30 |

10. **Communicate and Justify** Explain why zero cannot be in the domain of an inverse variation.

11. **Error Analysis** Describe and correct the error a student made in graphing the function $y = \frac{5}{x}$.




12. **Higher Order Thinking** The cost to rent a condominium at the beach is \$1,500 per week. If two people share the cost, they each have to pay \$750. Explain why the cost per person varies inversely with the number of persons sharing the cost. Then write an inverse variation function that can be used to calculate the cost per person, $c$, of $p$ persons sharing the rental fee.




13. **Generalize** For an inverse variation, write an equation that gives the value of $k$ in terms of $x$ and $y$.

### PRACTICE

**Do the tables of values represent inverse variations? Explain.** SEE EXAMPLE 1

14.

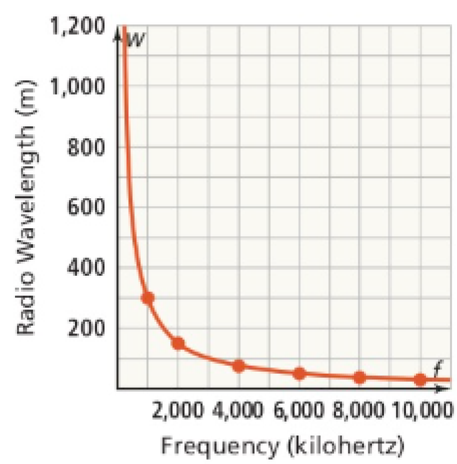
| $x$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $\frac{1}{3}$ | 2 | 5 | 11 |
|---|---|---|---|---|---|---|
| $y$ | $-\frac{9}{2}$ | −9 | 6 | 36 | 90 | 198 |

15.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 60 | 30 | 20 | 15 | 12 | 10 |

16. If $x$ and $y$ vary inversely and $x = 3$ when $y = \frac{2}{3}$, what is the value of $y$ when $x = -1$? SEE EXAMPLE 2

17. The wavelength, $w$, of a radio wave varies inversely to its frequency, $f$, as shown in the graph.




A radio wave with a frequency of 1,000 kilohertz has a length of 300 m. What is the frequency when the wave-length is 375 m? SEE EXAMPLE 3

18. Graph the function $y = \frac{-2}{x}$. What are the domain, range, and asymptotes of the function? SEE EXAMPLE 4

19. Graph $g(x) = \frac{1}{x-2} + 6$. What are the equations of the asymptotes? What are the domain and range? SEE EXAMPLE 5

## PRACTICE & PROBLEM SOLVING

### APPLY

**20. Apply Math Models** The time $t$ required to empty a water tank varies inversely as the rate of pumping $p$. A pump can empty a water tank in 40 min at the rate of 120 gal/min. Write the equation of the inverse variation. How long will it take the pump to empty the water tank at the rate of 200 gal/min?

**21. Use Patterns and Structure** The number of downloaded games that can be stored on a video game system varies inversely with the average size of a video game. A certain video game system can store 160 games when the average size of a game is 2.0 gigabytes (GB).

a. Write an equation that relates the number of games $n$ that will fit on the video game system as a function of the average game size $s$ in GB.

b. Use the relationship to complete the table of values.

| Game Size (GB), $s$ | 1.0 | 2.5 | 3.0 | 4.0 |
|---|---|---|---|---|
| Number of Games, $n$ | ■ | ■ | ■ | ■ |

c. Sketch a graph of this relationship on a coordinate plane.

**22. Apply Math Models** The voltage $V$, in volts, in an electrical circuit varies inversely as the resistance $R$ in ohms. The voltage in the circuit is 15 volts when the resistance is 192 ohms.

a. Write the equation of the inverse variation.

b. Find the voltage in the circuit when the resistance is 144 ohms.

**23.** Boyle's Law states that the pressure exerted by fixed quantity of a gas, $P$, varies inversely with the volume the gas occupies, $V$, assuming constant temperature.

The volume and air pressure of a volleyball are 300 in.$^3$ and 4.5 psi. The volume and air pressure of a basketball are 415 in.$^3$ and 8 psi. How much smaller would the volleyball have to be to equal the air pressure of the basketball?

### ASSESSMENT PRACTICE

**24.** Katie ordered a wedding cake with a volume of 1,200 in.$^3$. She knows that the amount of cake each guest will get is inversely proportional to the number of people she invites. Write an equation to model this situation. AR.8.2

**25. SAT/ACT** Suppose $y$ varies inversely as the square of $x$. If $x$ is multiplied by 4, which of the following is true for the value of $y$?

Ⓐ It is multiplied by 4.

Ⓑ It is multiplied by 16.

Ⓒ It is multiplied by $\frac{1}{4}$.

Ⓓ It is multiplied by $\frac{1}{16}$.

**26. Performance Task** Suppose Cameron takes a road trip. He starts from his home in Tampa and travels to Daytona Beach to visit his aunt and uncle.


**Part A** The distance Cameron drives from Tampa to Daytona Beach is 139 miles. The trip takes him 2 hours. The distance $d$ in miles that Cameron drives varies directly with the amount of time $t$ in hours, he spends driving. Write the equation of the direct variation. Use the given relationship and the equation to find the number of miles Cameron would travel if he continues on for 5 more hours.

**Part B** The amount of gas in Cameron's car is 9 gal after he drives for 2 h. The amount of gas $g$ in gallons in his tank varies inversely with the amount of time $t$, in hours, he spends driving. Write the equation of the inverse variation. Use the given relationship and the equation to find the number of gallons in Cameron's tank after 5 more hours of driving.

# 4-2 Graphing Rational Functions

**I CAN…** graph rational functions.

**VOCABULARY**
- rational expression
- rational function

MA.912.AR.8.2–Given a table, equation or written description of a rational function, graph that function and determine its key features. Also AR.8.3
MA.912.MTR.2.1, MTR.4.1, MTR.6.1

## EXPLORE & REASON

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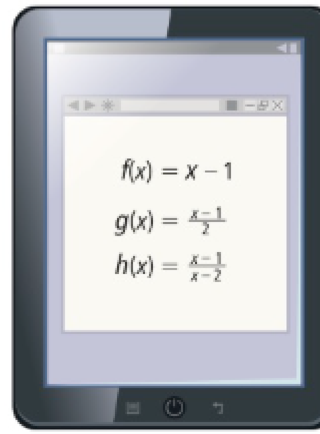
**Look at the three functions shown.**




**A.** **Use Patterns and Structure** Graph each function. Determine which of the functions are linear. Find the $y$-intercept of each function and the slope, if appropriate.

**B.** What is the effect on the graph of $f$ when dividing $x - 1$ by 2?

**C.** What happens to the graph of $h$ as $x$ approaches 2?

**D.** **Communicate and Justify** What is the effect on the graph of $f(x)$ when dividing $x - 1$ by $x - 2$? (Hint: Compare it to what you found in part (B).)

## ESSENTIAL QUESTION

How can you graph a rational function?

## EXAMPLE 1 Rewrite a Rational Function to Identify Asymptotes

**Rewrite $g(x) = \frac{4x}{x-3}$ using long division. How is the quotient related to the reciprocal function, $f(x) = \frac{1}{x}$? Sketch the graph.**

$g(x) = \frac{4x}{x-3}$ ........ Write the equation.

$= x - 3\overline{)\,4x}$ with quotient $4$ ........ Divide $4x$ by $x$.

$\quad -(4x - 12)$
$\quad\quad 12$ ........ Multiply $x - 3$ by 4 and subtract from $4x$.

$g(x) = 4 + \frac{12}{x-3}$ ........ Write the remainder as a fraction in the quotient.

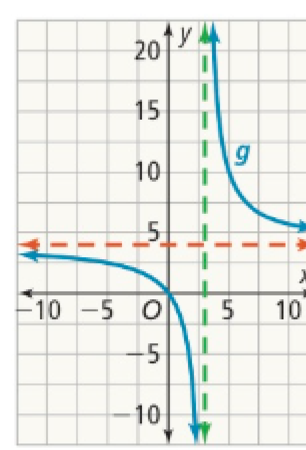
**USE PATTERNS AND STRUCTURE**
Rewriting $g$ in this way is similar to rewriting an improper fraction as a mixed number.

Rewrite $g$ in the form $g(x) = \frac{a}{x-h} + k$ to identify the transformation of the parent function, $f(x) = \frac{1}{x}$.

$g(x) = \frac{12}{x-3} + 4$

In the graph, the parent function $f$ has been shifted up 4 units and then right 3 units. The resulting graph has been stretched vertically by a factor of 12.

There is a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 4$.




## Try It!

1. Use long division to rewrite each rational function. Find the asymptotes of $f$ and sketch the graph.

   a. $f(x) = \frac{6x}{2x+1}$

   b. $f(x) = \frac{x}{x-6}$

**CONCEPT** Rational Functions

Just as a rational number is a number that can be expressed as the ratio of two integers, a **rational expression** is an expression that can be expressed as the ratio of two polynomials, such as $\frac{P(x)}{Q(x)}$.

A **rational function** is any function defined by a rational expression, such as $R(x) = \frac{P(X)}{Q(X)}$. The domain of $R(x)$ is all values of $x$ that are in the domains of both $P(x)$ and $Q(x)$ and for which $Q(x) \neq 0$.

The function $g(x) = \frac{4x}{x-3}$ is a rational function.

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Find Asymptotes of a Rational Function

**How do you find vertical and horizontal asymptotes of a rational function?**

**A. What are the vertical asymptotes for the graph of $f(x) = \frac{3x-2}{x^2+7x+12}$?**

*Vertical asymptotes* can occur at the $x$-values where the function is undefined. Determine where the denominator of the rational function is equal to 0.

$x^2 + 7x + 12 = 0$ ........ Set the denominator equal to 0.

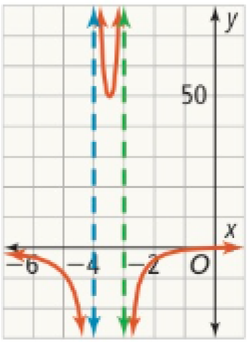
$(x+3)(x+4) = 0$ ........ Factor.

$x + 3 = 0$ or $x + 4 = 0$ ........ Use the Zero Product Property.

$x = -3$ or $x = -4$ ........ Solve using the Addition Property of Equality.

The possible vertical asymptotes are $x = -3$ and $x = -4$.

Graph the function to determine if there are asymptotes at $x = -3$ or $x = -4$.




Use the TRACE feature on the graphing calculator to confirm that the graph is not defined at $x = -3$ or $x = -4$.

**REPRESENT AND CONNECT**
When creating a graph with asymptotes, it is important to identify the lines that are vertical or horizontal asymptotes.

The graph is not defined at $x = -3$ or $x = -4$. These lines are vertical asymptotes. The domain of $f(x)$ is $\{x \mid x \neq -3 \text{ or } x \neq -4\}$ whose restrictions correspond to the vertical asymptotes or when the denominator equals zero.

CONTINUED ON THE NEXT PAGE

**STUDY TIP**

The vertical asymptote(s) are found by factoring the denominator of the function.

The horizontal asymptote(s) are found using the relationship between the degree of the numerator and the degree of the denominator.

EXAMPLE 2 CONTINUED

**B. What are the horizontal asymptotes for the graph $f(x) = \frac{3x-2}{x^2+7x+12}$?**

To identify *horizontal asymptotes*, we have to consider three cases.

**Case 1:** The degree of the numerator is less than the degree of the denominator.

Consider $g(x) = \frac{x+4}{x^2+1}$

As the value of $x$ increases, the value of the denominator gets very large in relation to the numerator. The value of the function gets closer and closer to 0.

When the degree of the numerator is less than the degree of the denominator, there exists a horizontal asymptote at $y = 0$.

**Case 2:** The degree of the numerator is greater than the degree of the denominator.

Consider $h(x) = \frac{x^2+1}{x+2}$

As the value of $x$ increases, the value of the numerator gets very large in relation to the denominator. The value of the function continues to increase.

When the degree of the numerator is greater than the degree of the denominator, there are no horizontal asymptotes.

**Case 3:** The degree of the numerator and the denominator are the same.

Consider $k(x) = \frac{2x^2+x+1}{x^2-1}$

Using long division, then we can rewrite this as $k(x) = 2 + \frac{x+3}{x^2-1}$.

As the value of $x$ increases, the value of the rational part of the quotient approaches 0, so the value of the function approaches 2.

$k(x) = \frac{2x^2+x+1}{x^2-1}$ has a horizontal asymptote at $y = 2$.

Another way to think about this is when the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is the ratio of the leading coefficients.

**ANALYZE AND PERSEVERE**

To show that the horizontal asymptote is accurate, try substituting different values for $x$ and see if the values for $y$ approach the asymptote(s).

**Which condition applies** to the function $f(x) = \frac{3x-2}{x^2+7x+12}$? The degree of the numerator is less than the degree of the denominator.

It has a horizontal asymptote at $y = 0$. To determine the range of $f(x)$, use technology to find local maximums and minimums. The range is $\{y \mid y \leq 0.181 \text{ or } y \geq 49.819\}$. Notice the horizontal asymptote only affects end behaviors and may be crossed in the graph.

**Try It!** 2. What are the vertical and horizontal asymptotes of the graph of each function?

a. $g(x) = \frac{2x^2+x-9}{x^2-2x-8}$

b. $R(x) = \frac{x^2+5x+4}{3x^2-12}$

## EXAMPLE 3 Graph a Function of the Form $\frac{ax + b}{cx + d}$

**What is the graph of the function $f(x) = \frac{2x + 1}{3x - 4}$?**

**Step 1:** Determine if there is a vertical asymptote.

$3x - 4 = 0$ Set the denominator equal to 0.

$3x = 4$ Solve.

$x = \frac{4}{3}$ Divide to isolate the variable.

At $x = \frac{4}{3}$, the value of the denominator is 0.

There is a vertical asymptote at $x = \frac{4}{3}$.

**COMMON ERROR**
The ratio of the leading terms cannot be used as an approximation unless $x$ is approaching positive or negative infinity.

**Step 2:** Determine if there is a horizontal asymptote.

$y \approx \frac{2x}{3x}$ Approximate $f$ with ratio of leading terms.

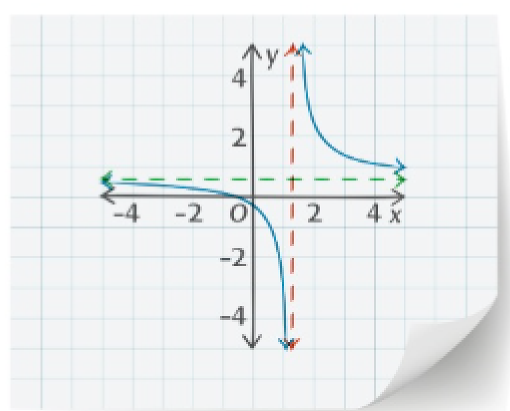
$y \approx \frac{2}{3}$ Simplify.

As $x \to \pm\infty$, $y \to \frac{2}{3}$.

There is a horizontal asymptote at $y = \frac{2}{3}$.

**Step 3:** Graph the function.

- Indicate the asymptotes.
- Choose $x$-values on either side of the vertical asymptote, and evaluate the function for those $x$-values to create coordinate points.
- Plot the points.




**Try It!** **3.** Graph each function.

**a.** $f(x) = \frac{4x - 3}{x + 8}$

**b.** $g(x) = \frac{3x + 2}{x - 1}$

APPLICATION

## EXAMPLE 4 Use a Rational Function Model

**The cost of removing an algae bloom is modeled by the given function where $f(p)$ is the cost, in millions of dollars, of removing $p$ percent of the algae. What percent of the algae can be removed for \$78.3 million?**

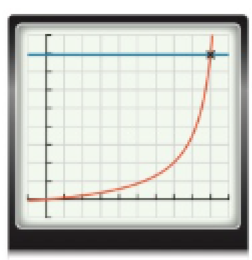

**Formulate** ◀ Since $p$ is a percent, you know that $0 \leq p \leq 100$.

**Compute** ◀ Find the vertical asymptote by solving $100 - p = 0$:

$p = 100$.

Refine the domain: $0 \leq p < 100$.

Graph $y = 78.3$ and $y = \frac{8.7p}{100 - p}$.



Use graphing technology to find the point of intersection.

The point (90, 78.3) lies on both graphs.

**Interpret** ◀ 90% of the algae can be removed for \$78.3 million.

**Try It!** **4.** New techniques have changed the cost function. For the new function $g(p) = \frac{3.2p + 1}{100 - p}$, what percent of the algae can be removed for \$50 million?

## EXAMPLE 5 Graph a Rational Function

**What is the graph of $f(x) = \dfrac{4x^2 - 9}{x^2 + 2x - 15}$?**

**Step 1** Determine if there are any vertical asymptotes.

$x^2 + 2x - 15 = 0$ Set the denominator equal to 0.

$(x + 5)(x - 3) = 0$ Factor.

$x + 5 = 0$ or $x - 3 = 0$ Use the Zero Product Property.

$x = -5$ or $x = 3$

Neither of these values makes the numerator equal to 0, but they each make the denominator equal to 0.

Vertical asymptotes: $x = -5$ and $x = 3$

**Step 2** Determine if there is a horizontal asymptote.

$y \approx \dfrac{4x^2}{x^2}$ Approximate $f$ with ratio of leading terms.

$\approx 4$ Simplify.

As $x \to \pm\infty$, $y \to 4$.

There is a horizontal asymptote at $y = 4$.

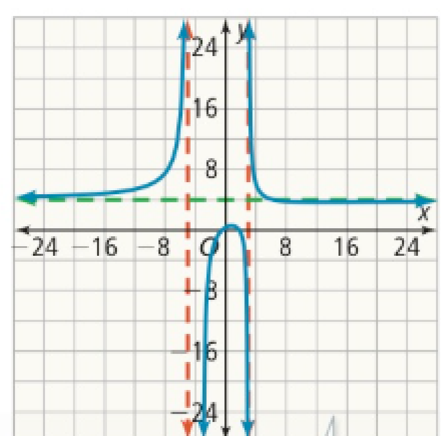
**REPRESENT AND CONNECT**
When graphing, it is important to indicate and clearly label both horizontal and vertical asymptotes.

**Step 3** Sketch the graph.

Indicate the asymptotes.

Plot points, choosing some $x$-values from each area of the graph.

| $x$ | $f(x)$ |
|---|---|
| −20 | 4.61 |
| −15 | 4.95 |
| −10 | 6.02 |
| −4 | −7.86 |
| 0 | 0.60 |
| 4 | 6.11 |
| 7 | 3.90 |
| 13 | 3.71 |
| 20 | 3.74 |




For $x > 3$ as the $x$ values get larger, this graph drops below the horizontal asymptote and then approaches it from below.

**Try It!** 5. Identify the asymptotes and sketch the graph of $g(x) = \dfrac{x^2 - 5x + 6}{2x^2 - 10}$.

## CONCEPT SUMMARY Graphing Rational Functions

**RATIONAL FUNCTION**

A function that is expressible as a fraction with polynomials in the numerator and the denominator

**ASYMPTOTES**

**Vertical**

Vertical asymptotes are guides for the behavior of a graph as it approaches a vertical line.

- The line $x = a$ is a vertical asymptote of $\frac{P(x)}{Q(x)}$, if $Q(a) = 0$ and $P(a) \neq 0$.
- The up or down behavior of the function as it approaches the asymptote can be determined by substituting values close to $a$ on either side of the asymptote.

**Horizontal**

Horizontal asymptotes are guides for the end behavior of a graph as it approaches a horizontal line.

If the degree of the numerator is

- less than the degree of the denominator, the horizontal asymptote is at $y = 0$.
- greater than the denominator, there is no horizontal asymptote.
- equal to the degree of the denominator, set $y$ equal to the ratio of the leading coefficients. The graph of this line is the horizontal asymptote.

**ALGEBRA**

$f(x) = \frac{8x-3}{4x+1}$

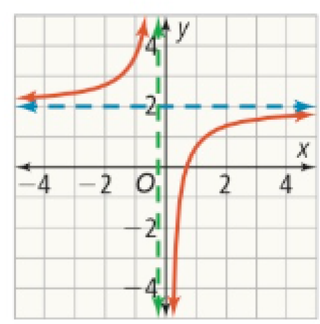
Vertical Asymptote: Let $4x + 1 = 0$ and solve.

$x = -\frac{1}{4}$

Horizontal Asymptote: Find the ratio of the leading coefficients $\left(\frac{8}{4}\right)$.

$y = 2$

**GRAPH**




## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you graph a rational function?

2. **Vocabulary** Why does it make sense to call the expressions in this lesson *rational* functions?

3. **Error Analysis** Ashton said the graph of $f(x) = \frac{x+2}{2x^2+4x-6}$ has a horizontal asymptote at $y = \frac{1}{2}$. Describe and correct Ashton's error.

4. **Communicate and Justify** When will the graph of a rational function have no vertical asymptotes? Give an example of such a function.

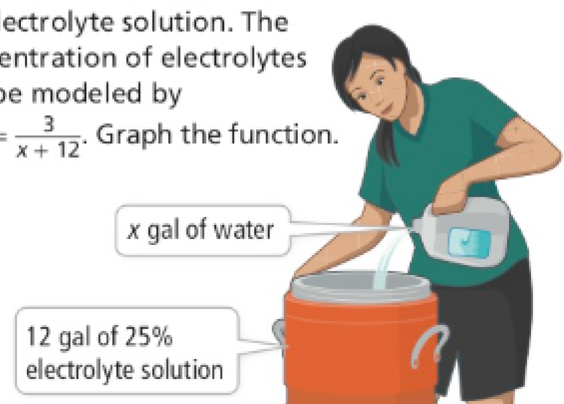
## Do You KNOW HOW?

**Find the vertical asymptote(s) and horizontal asymptote(s) of the rational function. Then graph the function.**

5. $f(x) = \frac{x+2}{x-3}$

6. $f(x) = \frac{x-1}{2x+1}$

7. A trainer mixed water with an electrolyte solution. The concentration of electrolytes can be modeled by $f(x) = \frac{3}{x+12}$. Graph the function.



## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

8. **Communicate and Justify** What is the horizontal asymptote of the rational function $f(x) = \frac{ax^2 + bx + c}{dx^2 + ex + f}$? Explain.

9. **Error Analysis** Juanita is trying to determine the vertical and horizontal asymptotes for the graph of the function $f(x) = \frac{x^2 + 3x - 4}{x^2 - x - 12}$. Describe and correct the error Juanita made in determining the vertical and horizontal asymptotes.

$$f(x) = \frac{x^2 + 3x - 4}{x^2 - x - 12}$$
$$= \frac{(x+4)(x-1)}{(x+3)(x-4)}$$

vertical asymptote: $x = -3$, $x = 4$
horizontal asymptote: $y = -4$, $y = 1$ ✗

10. **Higher Order Thinking** Suppose the numerator and denominator of a rational function are factored, and the numerator and denominator have a common factor of $x + a$. What happens on the graph of the function at $x = -a$? Explain your reasoning.

11. **Check for Reasonableness** The graph of a rational function has vertical asymptotes at $x = -3$ and $x = 1$ and a horizontal asymptote at $y = 3$.
   a. Write a function that has these attributes.
   b. Graph your function to verify it is correct.
   c. Is it possible to have a different graph with the same attributes? Explain.

12. **Communicate and Justify** Explain how to use the end behavior of the function $f(x) = \frac{x^2 + 6}{4x^2 - 3x - 1}$ to determine the horizontal asymptote of the graph. Then explain why using end behavior for finding the horizontal asymptote works the same as using the ratio of the leading terms.

### PRACTICE

**Use long division to rewrite each rational function. What are the asymptotes of *f*? Sketch the graph.**
SEE EXAMPLE 1

13. $f(x) = \frac{2x}{x + 4}$
14. $f(x) = \frac{5x}{x - 2}$
15. $f(x) = \frac{6x^2}{3x^2 + 1}$
16. $f(x) = \frac{x^2}{2x^2 - 2}$

**Identify the vertical and horizontal asymptotes of each rational function.** SEE EXAMPLE 2

17. $f(x) = \frac{3x^2}{4x^2 - 1}$
18. $f(x) = \frac{5x + 6}{x^2 - 9x + 18}$
19. $f(x) = \frac{4x + 3}{x^2 - 4}$
20. $f(x) = \frac{5x^2 - 19x - 4}{2x^2 - 2}$

**Graph each function.** SEE EXAMPLE 3

21. $f(x) = \frac{-1}{x + 3}$
22. $f(x) = \frac{3x}{x - 1}$
23. $f(x) = \frac{x + 2}{-x + 1}$
24. $f(x) = \frac{2x - 3}{3x + 4}$

25. An owner tracks her sales each day since opening her marketing company. The daily sales, in dollars, after day $x$ is given by the function $f(x) = \frac{200{,}000x}{x^2 + 150}$. On approximately which day(s) will the daily sales be \$3,000?
SEE EXAMPLE 4

AC ADVERTISING COMPANY

**DAILY SALES TRACKER**

| DAYS | SALES |
|---|---|
| 1 | \$1,324.50 |
| 2 | \$2,597.40 |
| 3 | \$3,773.58 |
| 4 | \$4,819.28 |

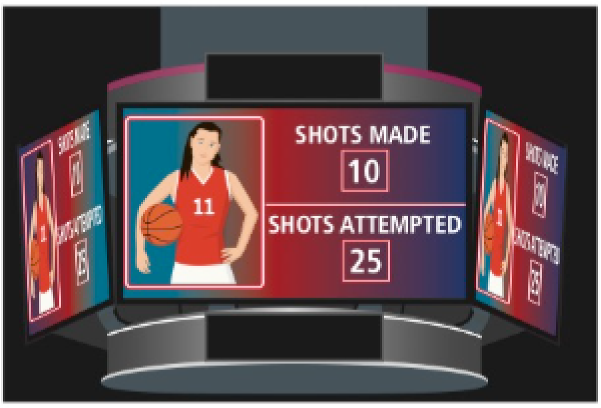
**Graph each function, labeling all horizontal or vertical asymptotes of the form $x = a$ or $y = b$.**
SEE EXAMPLE 5

26. $f(x) = \frac{x + 4}{2x^2 - 13x - 7}$
27. $f(x) = \frac{2x - 1}{x^2 - 3x - 10}$
28. $f(x) = \frac{x^2 + x - 2}{2x^2 - 9x - 18}$
29. $f(x) = \frac{6x^2 - 12x}{x^2 + 5x - 24}$

# PRACTICE & PROBLEM SOLVING

## APPLY

30. **Analyze and Persevere** Amaya made 10 three-point shots out of 25 attempts. If she then goes on to make $x$ consecutive three-point shots, her success would be given by the function $f(x) = \frac{x + 10}{x + 25}$.




a. Identify the vertical asymptote(s) and horizontal asymptote(s).

b. Graph the function.

31. **Apply Math Models** A software CD can be manufactured for \$0.10 each. The development cost to produce the software is \$500,000. The first 200 CDs were used by testers to test the functionality of the software and were not sold.

a. Write a function $f$ for the average cost, in dollars, of a salable software CD where $x$ is the number of salable software CDs.

b. What are the vertical asymptotes of the graph?

c. What are the horizontal asymptotes of the graph?

d. Graph the function.

e. What do the asymptotes mean?

32. **Check for Reasonableness** After diluting salt water, the concentration of salt in the water is given by the function $f(x) = \frac{0.5x}{x^2 - 1}$, where $x$ is the time in hours since the dilution.

a. What is the concentration of salt in the water after 4 hours?

b. After how many hours will the concentration of salt in the water be 0.2? Round to the nearest hundredth.

## ASSESSMENT PRACTICE

33. Which function has a graph with a vertical asymptote at $x = 3$? Select all that apply. AR.8.2

☐ **A.** $f(x) = \frac{x - 2}{x^2 + 2x - 15}$

☐ **B.** $f(x) = \frac{x - 3}{x^2 + 7x + 12}$

☐ **C.** $f(x) = \frac{x^2 - 9}{x + 9}$

☐ **D.** $f(x) = \frac{x^2 + 6x + 5}{x^2 - 9}$

☐ **E.** $f(x) = \frac{x + 3}{x - 3}$

34. **SAT/ACT** Which function has a graph with a horizontal asymptote at $y = -1$?

Ⓐ $f(x) = \frac{x + 5}{x - 3}$

Ⓑ $f(x) = \frac{-x + 9}{x - 8}$

Ⓒ $f(x) = \frac{x^2 + 4}{x^2 - 1}$

Ⓓ $f(x) = \frac{2x^2}{x^2 - x - 2}$

35. **Performance Task** There is a relationship between the degree of the numerator and denominator of a rational function and the function's horizontal asymptote.

| Function | Horizontal Asymptote |
|---|---|
| $f(x) = \frac{2x}{x^2}$ | |
| $f(x) = \frac{5x^2}{2x^3}$ | |
| $f(x) = \frac{9x^6}{7x}$ | |
| $f(x) = \frac{-3x^7}{4x^4}$ | |

**Part A** Complete the right column of the table.

**Part B** What is the relationship between the degree of the numerator and denominator when the horizontal asymptote is $y = 0$?

**Part C** What is the relationship between the degree of the numerator and denominator when there is no a horizontal asymptote?

# 4-3 Multiplying and Dividing Rational Expressions

**I CAN…** find the product and the quotient of rational expressions.

**VOCABULARY**

- simplified form of a rational expression

MA.912.AR.1.9–Apply previous understanding of rational number operations to add, subtract, multiply and divide rational algebraic expressions.
MA.912.MTR.2.1, MTR.4.1, MTR.6.1

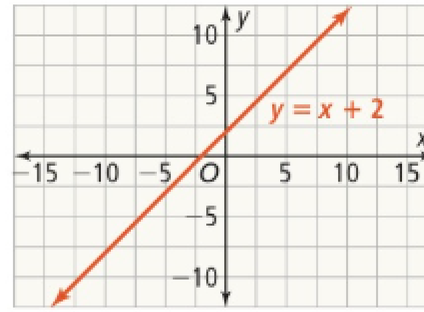
## EXPLORE & REASON

**Consider the following graph of the function $y = x + 2$.**

**A.** What is the domain of this function?

**B.** Sketch a function that resembles the graph, but restrict its domain to exclude 2.

**C.** **Use Patterns and Structure** Consider the function you have sketched. What kind of function might have a graph like this? Explain.




## ESSENTIAL QUESTION

**How does understanding operations with fractions help you multiply and divide rational expressions?**

## CONCEPT Rational Expression

A *rational expression* is the quotient of two polynomials. The domain is all real numbers except those for which the denominator is equal to 0.

$\frac{x^2}{x^2 - 9}$ is an example of a rational expression.

Since the denominator cannot equal 0, $x^2 - 9 \neq 0$.

$x^2 \neq 9 \rightarrow x \neq 3$ or $-3$.

So the domain of $\frac{x^2}{x^2 - 9}$ is all real numbers except 3 and –3 which is the same as $\{x | x \neq \pm 3\}$.

CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Write Equivalent Rational Expressions

**When are two rational expressions equivalent?**

Rational expressions can be simplified in a process that is similar to the process for simplifying rational numbers.

$$\frac{12}{16} = \frac{3 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2} = \frac{3}{2 \cdot 2} \cdot \frac{2}{2} \cdot \frac{2}{2} = \frac{3}{2 \cdot 2} \cdot 1 \cdot 1 = \frac{3}{4}$$

By replacing quotients of common factors between the numerator and denominator with 1, you learn that $\frac{12}{16}$ is equivalent to $\frac{3}{4}$.

**LEARN TOGETHER**
How can you share your ideas and communicate your thinking with others?

Write an expression that is equivalent to $\frac{x^3 - 5x^2 - 24x}{x^3 + x^2 - 72x}$.

**Step 1** Factor the numerator and the denominator.

$$\frac{x^3 - 5x^2 - 24x}{x^3 + x^2 - 72x} = \frac{x(x^2 - 5x - 24)}{x(x^2 + x - 72)} = \frac{x(x - 8)(x + 3)}{x(x - 8)(x + 9)}$$

**Step 2** Find the domain of the rational expression.

The domain is all real numbers except 0, 8, and –9.

Both $\frac{x^3 - 5x^2 - 24x}{x^3 + x^2 - 72x}$ and $\frac{x(x - 8)(x + 3)}{x(x - 8)(x + 9)}$ have the same domain.

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**CHECK FOR REASONABLENESS**
A statement of equivalence between two expressions is an identity. The identity is only valid where both expressions are defined.

EXAMPLE 1 CONTINUED

**Step 3** Recognize that the ratio of the common factors in the numerator and denominator are equal to 1.

$$\frac{x(x-8)(x+3)}{x(x-8)(x+9)} = \frac{x}{x} \cdot \frac{(x-8)}{(x-8)} \cdot \frac{(x+3)}{(x+9)} = 1 \cdot 1 \cdot \frac{(x+3)}{(x+9)} = \frac{x+3}{x+9}$$

So $\frac{x^3-5x^2-24x}{x^3+x^2-72x}$ is equivalent to $\frac{x+3}{x+9}$ for all $x$ except −9, 0, and 8.

Like the original expression, the domain of $\frac{x+3}{x+9}$ excludes −9. But the domain of the original expression also excludes 0 and 8.

**Try It!** **1.** Write an expression equivalent to $\frac{3x^5-18x^4-21x^3}{2x^6-98x^4}$. Remember to give the domain for your expression.

## EXAMPLE 2 Simplify a Rational Expression

**What is the simplified form of the rational expression? What is the domain for which the identity between the two expressions is valid?**

$$\frac{4-x^2}{x^2+3x-10}$$

The **simplified form of a rational expression** has no common factors, other than 1, in the numerator and the denominator.

$$\frac{4-x^2}{x^2+3x-10} = \frac{(2-x)(2+x)}{(x-2)(x+5)}$$ Factor the polynomials.

The domain is all real numbers except 2 and −5. Identify the domain from the original expression.

$$= \frac{-\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x+5)}$$ Divide out common factors.

**COMMON ERROR**
Be sure to factor out −1 from $2-x$ before dividing out common factors: $2-x = -(x-2)$

The simplified form of $\frac{4-x^2}{x^2+3x-10}$ is $-\frac{x+2}{x+5}$ for all real numbers except 2 and −5.

**Try It!** **2.** Simplify each expression and state the domain.

a. $\frac{x^2+2x+1}{x^3-2x^2-3x}$

b. $\frac{x^3+4x^2-x-4}{x^2+3x-4}$

## EXAMPLE 3 Multiply Rational Expressions

**A. What is the product of $\frac{2xy}{z}$ and $\frac{3x^2}{4yz}$?**

To multiply rational expressions, follow a similar method to that for multiplying two numerical fractions.

The domain is $z \neq 0$ and $y \neq 0$.

$$\frac{2xy}{z} \cdot \frac{3x^2}{4yz} = \frac{(2xy)(3x^2)}{z(4yz)}$$ Multiply the expressions.

$$= \frac{\cancel{2} \cdot 3 \cdot x^3 \cdot \cancel{y}}{\cancel{2} \cdot 2 \cdot \cancel{y} \cdot z^2}$$ Divide out common factors.

$$= \frac{3x^3}{2z^2}$$

The product of $\frac{2xy}{z}$ and $\frac{3x^2}{4yz}$ is $\frac{3x^3}{2z^2}$ for $y \neq 0$ and $z \neq 0$.

**USE PATTERNS AND STRUCTURE**
Recall that when multiplying $\frac{a}{b} \times \frac{c}{d}$ you can often simplify by dividing both $a$ and $d$ (or both $b$ and $c$) by the greatest common factor.

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EXAMPLE 3 CONTINUED

**STUDY TIP**
It is easier to find the domain after factoring the denominator. Use the Zero Product Property to find values that will make the expression undefined. In this example, $x$ cannot be $-3$, $-1$, or 2.

**B. What is the simplified form of $\frac{5x}{x+3} \cdot \frac{x^2+x-6}{x^2+2x+1} \cdot \frac{x^2+x}{5x-10}$?**

$$\frac{5x}{x+3} \cdot \frac{x^2+x-6}{x^2+2x+1} \cdot \frac{x^2+x}{5x-10} = \frac{5x(x+3)(x-2)x(x+1)}{(x+3)(x+1)^2 5(x-2)}$$ Multiply and factor the expressions.

$$= \frac{\cancel{5}\, x\, \cancel{(x+3)}\, \cancel{(x-2)}\, x\, \cancel{(x+1)}}{\cancel{(x+3)}\, \cancel{(x+1)}\, (x+1)\, \cancel{5}\, \cancel{(x-2)}}$$ Divide out common factors.

$$= \frac{x^2}{x+1}$$ Simplify.

So $\frac{5x}{x+3} \cdot \frac{x^2+x-6}{x^2+2x+1} \cdot \frac{x^2+x}{5x-10} = \frac{x^2}{x+1}$ for $x \neq -3, -1$, or 2.

**Try It!** 3. Find the simplified form of each product, and give the domain.

a. $\frac{x^2-16}{9-x} \cdot \frac{x^2+x-90}{x^2+14x+40}$

b. $\frac{x+3}{4x} \cdot \frac{3x-18}{6x+18} \cdot \frac{x^2}{4x+12}$

## EXAMPLE 4 Multiply a Rational Expression by a Polynomial

**STUDY TIP**
This process is similar to writing a whole number with a denominator of 1 when multiplying a fraction and a whole number.

**What is the product of $\frac{x+2}{x^4-16}$ and $x^3+4x^2-12x$?**

$$\frac{x+2}{x^4-16} \cdot (x^3+4x^2-12x) = \frac{x+2}{x^4-16} \cdot \frac{x^3+4x^2-12x}{1}$$

$$= \frac{(x+2)x(x^2+4x-12)}{1(x^2+4)(x^2-4)}$$

$$= \frac{\cancel{(x+2)}x(x+6)\cancel{(x-2)}}{1(x^2+4)\cancel{(x+2)}\cancel{(x-2)}}$$

The domain is $x \neq -2$ or 2. Remember $x^2+4$ is never zero so this provides no domain restriction.

So $\frac{x+2}{x^4-16} \cdot (x^3+4x^2-12x) = \frac{x(x+6)}{x^2+4}$ for $x \neq -2$ or 2.

**Try It!** 4. Find the simplified form of each product and the domain.

a. $\frac{x^3-4x}{6x^2-13x-5} \cdot (2x^3-3x^2-5x)$

b. $\frac{3x^2+6x}{x^2-49} \cdot (x^2+9x+14)$

## EXAMPLE 5 Divide Rational Expressions

**What is the quotient of $\frac{x^3+3x^2+3x+1}{1-x^2}$ and $\frac{x^2+5x+4}{x^2+3x-4}$?**

$$\frac{x^3+3x^2+3x+1}{1-x^2} \div \frac{x^2+5x+4}{x^2+3x-4} = \frac{x^3+3x^2+3x+1}{1-x^2} \cdot \frac{x^2+3x-4}{x^2+5x+4}$$

Multiply by the reciprocal of the divisor.

$$= \frac{(x+1)(x+1)(x+1)(x+4)(x-1)}{-(x-1)(x+1)(x+1)(x+4)}$$

The domain is $x \neq -4, -1$, or 1.

$$= \frac{(x+1)}{-1} \cdot \frac{(x+1)}{(x+1)} \cdot \frac{(x+1)}{(x+1)} \cdot \frac{(x-1)}{(x-1)} \cdot \frac{(x+4)}{(x+4)}$$

$$= -(x+1)$$

**COMMON ERROR**
Remember to include the factor of $-1$!

The quotient is $-(x+1)$, $x \neq -4, -1$, or 1.

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**Try It!** 5. Find the simplified quotient and the domain of each expression.

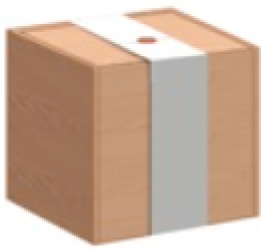
a. $\frac{1}{x^2+9x} \div \left(\frac{6-x}{3x^2-18x}\right)$

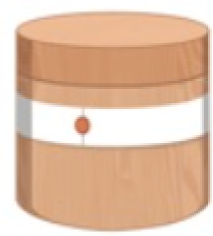
b. $\frac{2x^2-12x}{x+5} \div \left(\frac{x-6}{x+5}\right)$

APPLICATION

## EXAMPLE 6 Use Division of Rational Expressions

**A company is evaluating two packaging options for its product line. The more efficient design will have the lesser ratio of surface area to volume. Should the company use packages that are cylinders or rectangular prisms?**



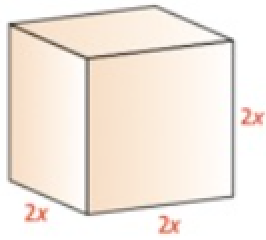


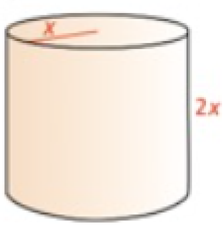
**Option 1:** A rectangular prism with a square base

**Option 2:** A cylinder with the same height as the prism, and diameter equal to the side length of the prism's base

STUDY TIP

Recall that surface area tells how much packaging material is needed and volume tells how much product the package can hold.




Surface Area: $2(2x)^2 + 4(2x)^2$
Volume: $(2x)^3$

Surface Area: $2\pi x^2 + 2\pi x(2x)$
Volume: $\pi x^2(2x)$

The efficiency ratio is $\frac{SA}{V}$, where $SA$ represents surface area and $V$ represents volume.

**Option 1:**

$$\frac{SA}{V} = \frac{2(4x^2)+4(4x^2)}{8x^3} = \frac{24x^2}{8x^3} = \frac{3}{x}$$

**Option 2:**

$$\frac{SA}{V} = \frac{2\pi x^2 + 4\pi x^2}{2\pi x^3} = \frac{6\pi x^2}{2\pi x^3} = \frac{3}{x}$$

The company can now compare the efficiency ratio of the package designs.

Prism: $\frac{3}{x}$ Cylinder: $\frac{3}{x}$

Regardless of what positive value is selected for $x$, the efficiency ratios for these two package designs will be the same.

In this example, the efficiency ratio of the cylinder is equal to that of the prism. So the company should choose their package design based on other criteria.

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**Try It!** 6. The company compares the ratios of surface area to volume for two more containers. One is a rectangular prism with a square base. The other is a rectangular prism with a rectangular base. One side of the base is equal to the side-length of the first container, and the other side is twice as long. The surface area of this second container is $4x^2 + 6xh$. The heights of the two containers are equal. Which has the smaller surface area-to-volume ratio?

## CONCEPT SUMMARY Products and Quotients of Rational Expressions

| | Multiply | Multiply an Integer or a Polynomial | Divide |
|---|---|---|---|
| **RATIONAL EXPRESSIONS** | $\frac{3x}{x+1} \cdot \frac{x^2+x}{3x-6}$ The domain is $x \neq -1$ or 2. | $\frac{x+2}{x^2-4} \cdot (x^2 - 2x)$ $= \frac{x+2}{x^2-4} \cdot \frac{x^2-2x}{1}$ The domain is $x \neq -2$ or 2. | $\frac{1-x^2}{x^2+3x-4} \div \frac{x+1}{x+4}$ $= \frac{1-x^2}{x^2+3x-4} \cdot \frac{x+4}{x+1}$ The domain is $x \neq -4$, $-1$, or 1. |
| **WORDS** | Identify common factors and simplify. | Write the polynomial as a rational expression with 1 in the denominator. Then multiply. | Multiply by the reciprocal of the divisor. |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does understanding operations with fractions help you multiply and divide rational expressions?

2. **Vocabulary** In your own words, define *rational expression* and provide an example of a rational expression.

3. **Error Analysis** A student divided the rational expressions as follows:

$$\frac{4x}{5y} \div \frac{20x^2}{25y^2} = \frac{4x}{\cancel{5}y} \div \frac{\overset{4}{\cancel{20}}x^2}{25y^2} = \frac{16x^3}{25y^3}.$$

Describe and correct the errors the student made.

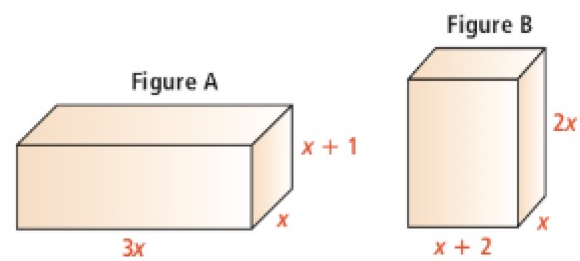
4. **Communicate and Justify** Why do you have to state the domain when simplifying rational expressions?

## Do You KNOW HOW?

5. What is the simplified form of the rational expression $\frac{x^2-36}{x^2+3x-18}$? What is the domain?

6. Find the product and give the domain of $\frac{y+3}{y+2} \cdot \frac{y^2+4y+4}{y^2-9}$.

7. Find and simplify the ratio of the volume of Figure A to the volume of Figure B.



## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

8. **Communicate and Justify** Explain why $\frac{4x^2-7}{4x^2-7} = 1$ is a valid identity under the domain of all real numbers except $\pm\frac{\sqrt{7}}{2}$.

9. **Error Analysis** Describe the error a student made in multiplying and simplifying $\frac{x+2}{x-2} \cdot \frac{x^2-4}{x^2+x-2}$.

$$\frac{x+2}{x-2} \cdot \frac{x^2-4}{x^2+x-2}$$
$$= \frac{\cancel{x}+2}{\cancel{x-2}} \cdot \frac{\cancel{(x+2)}\cancel{(x-2)}}{\cancel{(x+2)}(\cancel{x}-1)}$$
$$= \frac{2}{-1}$$

10. **Higher Order Thinking** Explain why the process of dividing by a rational number is the same as multiplying by its reciprocal.

11. **Represent and Connect** Explain how you can use your graphing calculator to show that the rational expressions $\frac{-6x^2+21x}{3x}$ and $-2x+7$ are equivalent under a given domain. What is true about the graph at $x = 0$ and why?

12. **Generalize** Explain the similarities between rational numbers and rational expressions.

13. **Choose Efficient Methods** Determine whether $\frac{5x+11}{6x+11} = \frac{5}{6}$ is *sometimes*, *always*, or *never* true. Justify your reasoning.

14. **Communicate and Justify** Explain how you can tell whether a rational expression is in simplest form.

15. **Analyze and Persevere** When multiplying $\frac{15}{x} \cdot \frac{x}{3} = 5$, is it necessary to make the restriction $x \neq 0$? Why or why not?

16. **Communicate and Justify** If the denominator of a rational expression is $x^3 + 3x^2 - 10x$, what value(s) must be restricted from the domain for $x$?

### PRACTICE

**Write an equivalent expression. State the domain.** SEE EXAMPLE 1

17. $\frac{x^3+4x^2-12x}{x^2+x-30}$

18. $\frac{3x^2+15x}{x^2+3x-10}$

**What is the simplified form of each rational expression? What is the domain?** SEE EXAMPLE 2

19. $\frac{y^2-5y-24}{y^2+3y}$

20. $\frac{ab^3-9ab}{12ab^2+12ab-144a}$

21. $\frac{x^2+8x+15}{x^2-x-12}$

22. $\frac{x^3+9x^2-10x}{x^3-9x^2-10x}$

**Find the product and the domain.** SEE EXAMPLE 3

23. $\frac{x^2+6x+8}{x^2+4x+3} \cdot \frac{x+3}{x+2}$

24. $\frac{(x-y)^2}{x+y} \cdot \frac{3x+3y}{x^2-y^2}$

**Find the product and the domain.** SEE EXAMPLE 4

25. $\frac{(x+5)}{(x^3-25x)} \cdot (2x^3-11x^2+5x)$

26. $\frac{(2x^2-10x)}{(x-5)(x^2-1)} \cdot (3x^2+4x+1)$

**Find the quotient and the domain.** SEE EXAMPLE 5

27. $\frac{y^2-16}{y^2-10y+25} \div \frac{3y-12}{y^2-3y-10}$

28. $\frac{(x-y)^2}{x+y} \div \frac{3x+3y}{x^2-y^2}$

29. $\frac{25x^2-4}{x^2-9} \div \frac{5x-2}{x+3}$

30. $\frac{x^4+x^3-30x^2}{x^2-3x-18} \div \frac{x^3+x^2-30x}{x^2-36}$

31. An orange crate is in the shape of a rectangular prism. It has a volume of $3x^3+7x^2+2x$ cubic units and a base area of $x^2+2x$ square units. Find the height of the orange crate. SEE EXAMPLE 6

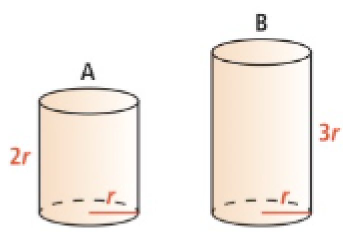
## PRACTICE & PROBLEM SOLVING

### APPLY

**32. Analyze and Persevere** An engineering firm wants to construct a cylindrical structure that will maximize the volume for a given surface area. Compare the ratios of the volume to surface area of each of the cylindrical structures shown, using the following formulas for volume and surface area of cylinders.

Volume $(V) = \pi r^2 h$

Surface Area $(SA) = 2\pi rh + 2\pi r^2$

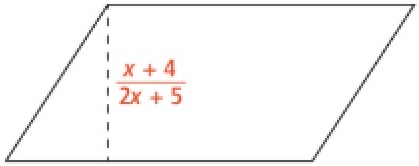



a. Calculate the ratio of volume to surface area for cylinder A.

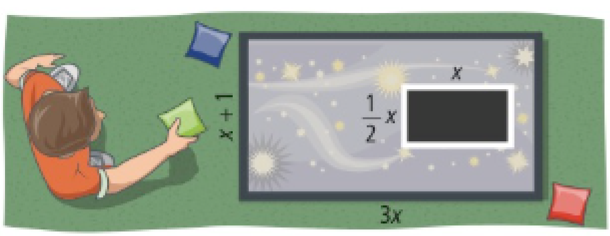
b. Calculate the ratio of volume to surface area for cylinder B.

c. Which of these cylinders has a greater ratio of volume to surface area?

**33. Use Patterns and Structure** A parallelogram with an area of $\frac{3x+12}{10x+25}$ square units has a height shown. Find the length of the base of the parallelogram.




**34. Apply Math Models** Brie designed a carnival game that involves tossing a beanbag into the box shown. In order to win a prize, the beanbag must fall inside the black rectangle. The probability of winning is equal to the ratio of the area of the black rectangle to the total area of the face of the box shown. Find this probability in simplified form.




### ASSESSMENT PRACTICE

**35.** Which of the following rational expressions simplify to $\frac{y}{y+3}$, ignoring domain restrictions? Select all that apply. AR.1.9

☐ A. $\frac{(2y^2+y)(y+3)}{(4y+2)(y+3)^2}$

☐ B. $\frac{3y^2+y}{3y^2+10y+3}$

☐ C. $\frac{2y^3+3y^2+y}{(2y+1)(y^2+4y+3)}$

☐ D. $\frac{y^2+2y}{y^2+4y+3}$

☐ E. $\frac{\frac{1}{y}}{y+3}$

**36. SAT/ACT** For which of the following values of $x$ is $\frac{2x^2+8x}{(x+4)(x^2-9)}$ undefined?

Ⓐ $-8$

Ⓑ $-3$

Ⓒ $0$

Ⓓ $4$

Ⓔ $9$

**37. Performance Task** The approximate annual interest rate $r$ of a monthly installment loan is given by the formula:

$$r = \frac{\left[\frac{24(nm-p)}{n}\right]}{\left(p + \frac{nm}{12}\right)},$$

where $n$ is the total number of payments, $m$ is the monthly payment, and $p$ is the amount financed.

**Part A** Find the approximate annual interest rate (to the nearest percent) for a four-year signature loan of \$20,000 that has monthly payments of \$500.

**Part B** Find the approximate annual interest rate (to the nearest tenth percent) for a five-year auto loan of \$40,000 that has monthly payments of \$750.

# 4-4 Adding and Subtracting Rational Expressions

**I CAN…** find the sum or difference of rational expressions.

**VOCABULARY**

- compound fraction

MA.912.AR.1.9–Apply previous understanding of rational number operations to add, subtract, multiply and divide rational algebraic expressions.
MA.912.MTR.3.1, MTR.5.1, MTR.7.1

## CRITIQUE & EXPLAIN

**Teo and Shannon find the following exercise in their homework:**

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{9}$$


**A.** Teo claims that a common denominator of the sum is $2 + 3 + 9 = 14$. Shannon claims that it is $2 \cdot 3 \cdot 9 = 54$. Is either student correct? Explain why or why not.

**B.** Find the sum, explaining the method you use.

**C.** **Communicate and Justify** Timothy states that the quickest way to find the sum of any two fractions with unlike denominators is to multiply their denominators to find a common denominator, and then rewrite each fraction with that denominator. Do you agree?

## ESSENTIAL QUESTION

How do you rewrite rational expressions to find sums and differences?

## EXAMPLE 1 Add Rational Expressions With Like Denominators

**What is the sum?**

**USE PATTERNS AND STRUCTURE**
Compare addition of numerical and algebraic fractions:
$\frac{1}{5} + \frac{3}{5} = \frac{1+3}{5} = \frac{4}{5}$
In the same way,
$\frac{x}{x+4} + \frac{5}{x+4} = \frac{x+5}{x+4}$.

**A.** $\frac{x}{x+4} + \frac{5}{x+4}$

$= \frac{x+5}{x+4}$ ........ When denominators are the same, add the numerators.

So $\frac{x}{x+4} + \frac{5}{x+4} = \frac{x+5}{x+4}$

**B.** $\frac{2x+1}{x^2+3x} + \frac{3x-8}{x(x+3)}$

$= \frac{(2x+1)+(3x-8)}{x^2+3x}$ ........ Add the numerators.

$= \frac{(2x+3x)+(1-8)}{x^2+3x}$ ........ Use the Commutative and Associative Properties.

$= \frac{5x-7}{x^2+3x}$ ........ Combine like terms.

So $\frac{2x+1}{x^2+3x} + \frac{3x-8}{x(x+3)} = \frac{5x-7}{x^2+3x}$

## Try It!

**1.** Find the sum.

**a.** $\frac{10x-5}{2x+3} + \frac{8-4x}{2x+3}$

**b.** $\frac{x-5}{x+5} + \frac{3x-21}{x+5}$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Identify the Least Common Multiple of Polynomials

**How can you find the least common multiple (LCM) of polynomials?**

**STUDY TIP**
Using the LCM of the denominators can mean less work simplifying later.

**A.** $(x+2)^2$, $x^2+5x+6$

Factor each polynomial.

$(x+2)^2 = (x+2)(x+2)$

$x^2+5x+6 = (x+2)(x+3)$

The LCM is the product of the factors. Duplicate factors are raised to the greatest power represented.

LCM: $(x+2)(x+2)(x+3)$ or $(x+2)^2(x+3)$

**B.** $x^3-9x$, $x^2-2x-15$, $x^2-5x$

Factor each polynomial.

$x^3-9x = x(x^2-9) = x(x+3)(x-3)$

$x^2-2x-15 = (x+3)(x-5)$

$x^2-5x = x(x-5)$

LCM: $x(x+3)(x-3)(x-5)$

Each original polynomial is a factor of the LCM.

**Try It!** **2.** Find the LCM for each set of expressions.

**a.** $x^3+9x^2+27x+27$, $x^2-4x-21$

**b.** $10x^2-10y^2$, $15x^2-30xy+15y^2$, $x^2+3xy+2y^2$

## EXAMPLE 3 Add Rational Expressions With Unlike Denominators

**What is the sum of $\frac{x+3}{x^2-1}$ and $\frac{2}{x^2-3x+2}$?**

Follow a similar procedure to the one you use to add numerical fractions with unlike denominators.

**USE PATTERNS AND STRUCTURE**
The LCM of 6 and 15 is 30, not 90.

$\frac{1}{6}+\frac{1}{15}=\frac{1}{2\cdot 3}+\frac{1}{3\cdot 5}=\frac{1\cdot 5+1\cdot 2}{2\cdot 3\cdot 5}$

The LCM does not contain the common factor 3 twice. In the example problem, the common factor $(x-1)$ is not used twice.

$\frac{x+3}{x^2-1}+\frac{2}{x^2-3x+2} = \frac{x+3}{(x+1)(x-1)}+\frac{2}{(x-1)(x-2)}$ Factor each denominator.

$= \frac{(x+3)(x-2)}{(x+1)(x-1)(x-2)}+\frac{2(x+1)}{(x+1)(x-1)(x-2)}$ Use the LCM as the least common denominator (LCD). Multiply fractions by a form of 1.

$= \frac{(x+3)(x-2)+2(x+1)}{(x+1)(x-1)(x-2)}$ Add the numerators.

$= \frac{(x^2+x-6)+(2x+2)}{(x+1)(x-1)(x-2)}$ Distribute.

$= \frac{x^2+3x-4}{(x+1)(x-1)(x-2)}$ Combine like terms.

$= \frac{(x+4)(x-1)}{(x+1)(x-1)(x-2)}$ Factor.

$= \frac{(x+4)}{(x+1)(x-2)}\cdot\frac{(x-1)}{(x-1)}$ Rewrite to identify unit factors.

$= \frac{x+4}{(x+1)(x-2)}$ for $x \neq -1, 1,$ and $2$ Simplify and state the domain.

The sum of $\frac{x+3}{x^2-1}$ and $\frac{2}{x^2-3x+2}$ is $\frac{x+4}{(x+1)(x-2)}$ for $x \neq -1, 1,$ and 2.

**Try It!** **3.** Find the sum.

**a.** $\frac{x+6}{x^2-4}+\frac{2}{x^2-5x+6}$

**b.** $\frac{2x}{3x+4}+\frac{4x^2-11x-12}{6x^2+5x-4}$

## EXAMPLE 4 Subtract Rational Expressions

**What is the difference between $\frac{x+1}{x^2-6x-16}$ and $\frac{x+1}{x^2+6x+8}$?**

$$\frac{x+1}{x^2-6x-16} - \frac{x+1}{x^2+6x+8} = \frac{x+1}{(x-8)(x+2)} - \frac{x+1}{(x+2)(x+4)}$$

The LCD is $(x-8)(x+2)(x+4)$.

$$= \frac{(x+1)(x+4)}{(x-8)(x+2)(x+4)} - \frac{(x-8)(x+1)}{(x-8)(x+2)(x+4)}$$

$$= \frac{(x^2+5x+4)-(x^2-7x-8)}{(x-8)(x+2)(x+4)}$$

$$= \frac{x^2+5x+4-x^2+7x+8}{(x-8)(x+2)(x+4)}$$

$$= \frac{12x+12}{(x-8)(x+2)(x+4)}$$

Check for common factors in the numerator and denominator and simplify, if possible.

$$= \frac{12(x+1)}{(x-8)(x+2)(x+4)}$$

The difference between $\frac{x+1}{x^2-6x-16}$ and $\frac{x+1}{x^2+6x+8}$ is $\frac{12(x+1)}{(x-8)(x+2)(x+4)}$ for $x \neq -4, -2,$ and $8$.

**COMMON ERROR**
When subtracting polynomials, remember to distribute $-1$ when removing the parentheses.

**Try It!** **4.** Simplify.

**a.** $\frac{1}{3x} + \frac{1}{6x} - \frac{1}{x^2}$

**b.** $\frac{3x-5}{x^2-25} - \frac{2}{x+5}$

**APPLICATION**

## EXAMPLE 5 Find a Rate

**Leah drives her car to the mechanic, then she takes the commuter rail train back to her neighborhood. The average speed for the 10-mile trip is 15 miles per hour faster on the train. Find an expression for Leah's total travel time. If she drove 30 mph, how long did this take?**


**CHOOSE EFFICIENT METHODS**
Use a table to organize information and help create an accurate model of the situation.

| | Distance | Rate | Time |
|---|---|---|---|
| Car | 10 | $r$ | $\frac{10}{r}$ |
| Commuter Rail | 10 | $r+15$ | $\frac{10}{r+15}$ |

Remember: distance = rate • time, so time = $\frac{\text{distance}}{\text{rate}}$.

Total time for the trip:

$$\frac{10}{r} + \frac{10}{r+15} = \frac{10(r+15)}{r(r+15)} + \frac{10r}{r(r+15)}$$

Add the times for each part of the trip.

$$= \frac{10r+150+10r}{r(r+15)}$$

$$= \frac{20r+150}{r(r+15)}$$

CONTINUED ON THE NEXT PAGE

EXAMPLE 5 CONTINUED

At a driving rate of 30 mph, you can find the total time.

$$\frac{20r+150}{r(r+15)} = \frac{20(30)+150}{30(30+15)}$$

$$= \frac{750}{1{,}350}$$

$$= \frac{5}{9}$$

Substitute 30 mph for the rate, and simplify.

The expression for Leah's total travel time is $\frac{20r+150}{r(r+15)}$. The total time is $\frac{5}{9}$ h, or about 33 min.

**Try It!** **5.** On the way to work Juan carpools with a fellow co-worker, then takes the city bus back home in the evening. The average speed of the 20-mile trip is 5 miles per hour faster in the carpool. Write an expression that represents Juan's total travel time.

## EXAMPLE 6 Simplify a Compound Fraction

**A compound fraction is in the form of a fraction and has one or more fractions in the numerator and/or the denominator. How can you write a simpler form of a compound fraction?**

$$\frac{\frac{1}{x}+\frac{2}{x+1}}{\frac{1}{y}}$$

**Method 1** Find the Least Common Multiple (LCM) of the fractions in the numerator and denominator. Multiply the numerator and the denominator by the LCM.

$$\frac{\frac{1}{x}+\frac{2}{x+1}}{\frac{1}{y}} = \frac{\left[\frac{1}{x}+\frac{2}{x+1}\right]\cdot[x(x+1)y]}{\frac{1}{y}\cdot[x(x+1)y]}$$ Multiply the numerator and the denominator by the LCD.

$$= \frac{(x+1)y+2xy}{x(x+1)}$$ Use the Distributive Property to eliminate the fractions.

$$= \frac{xy+y+2xy}{x(x+1)}$$ Simplify.

$$= \frac{(3x+1)y}{x(x+1)}$$ Factor.

**Method 2** Express the numerator and denominator as single fractions. Then multiply the numerator by the reciprocal of the denominator.

$$\frac{\frac{1}{x}+\frac{2}{x+1}}{\frac{1}{y}} = \frac{\frac{1}{x}\cdot\frac{x+1}{x+1}+\frac{2}{x+1}\cdot\frac{x}{x}}{\frac{1}{y}}$$ Multiply the numerator by the LCD.

$$= \frac{\frac{(x+1)+2x}{x(x+1)}}{\frac{1}{y}}$$ Simplify.

$$= \frac{3x+1}{x(x+1)}\cdot\frac{y}{1}$$ Multiply the numerator by the reciprocal of the denominator.

$$= \frac{(3x+1)y}{x(x+1)}$$ Simplify.

Using either method, $\frac{\frac{1}{x}+\frac{2}{x+1}}{\frac{1}{y}}$ is equal to $\frac{(3x+1)y}{x(x+1)}$ when $x \neq -1, 0$ and $y \neq 0$.

**USE PATTERNS AND STRUCTURE**
Remember that the fraction bar separating the numerator and denominator represents division.

**Try It!** **6.** Simplify each compound fraction.

**a.** $\dfrac{\frac{1}{x-1}}{\frac{x+1}{3}+\frac{4}{x-1}}$

**b.** $\dfrac{2-\frac{1}{x}}{x+\frac{2}{x}}$

## CONCEPT SUMMARY Find Sums and Differences of Rational Expressions

| | | |
|---|---|---|
| **WORDS** | To add or subtract rational expressions with common denominators, add the numerators and keep the denominator the same. | To add or subtract rational expressions with different denominators, rewrite each expression so that its denominator is the LCD, then add or subtract the numerators. |
| **NUMBERS** | $\frac{1}{5}+\frac{3}{5}=\frac{1+3}{5}=\frac{4}{5}$ | $\frac{1}{6}+\frac{1}{15}=\frac{1}{2\cdot 3}+\frac{1}{3\cdot 5}$ $=\frac{1\cdot 5+1\cdot 2}{2\cdot 3\cdot 5}$ |
| **ALGEBRA** | $\frac{x}{x+4}+\frac{5}{x+4}=\frac{x+5}{x+4}$ | $\frac{x+3}{x^2-1}+\frac{2}{x^2-3x+2}$ $=\frac{x+3}{(x+1)(x-1)}+\frac{2}{(x-1)(x-2)}$ $=\frac{(x+3)(x-2)}{(x+1)(x-1)(x-2)}+\frac{2(x+1)}{(x+1)(x-1)(x-2)}$ Rewrite the rational expressions using the LCD. |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do you rewrite rational expressions to find sums and differences?

2. **Vocabulary** In your own words, define **compound fraction** and provide an example of one.

3. **Error Analysis** A student added the rational expressions as follows:

$$\frac{5x}{x+7}+\frac{7}{x}=\frac{5x}{x+7}+\frac{7(7)}{x+7}=\frac{5x+49}{x+7}$$

Describe and correct the error the student made.

4. **Communicate and Justify** Explain why, when stating the domain of a sum or difference of rational expressions, not only should the simplified sum or difference be considered but the original expression should also be considered.

5. **Analyze and Persevere** In adding or subtracting rational expressions, why is the L in LCD significant?

## Do You KNOW HOW?

6. Find the sum of $\frac{3}{x+1}+\frac{11}{x+1}$.

**Find the LCM of the polynomials.**

7. $x^2-y^2$ and $x^2-2xy+y^2$

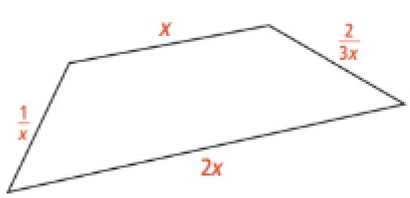
8. $5x^3y$ and $15x^2y^2$

**Find the sum or difference.**

9. $\frac{3x}{4y^2}-\frac{y}{10x}$

10. $\frac{9y+2}{3y^2-2y-8}+\frac{7}{3y^2+y-4}$

11. Find the perimeter of the quadrilateral in simplest form.



# PRACTICE & PROBLEM SOLVING

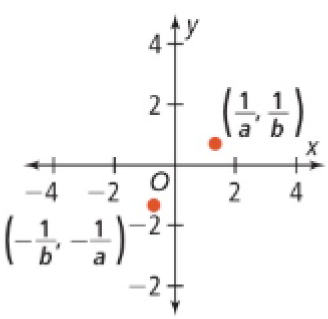
## UNDERSTAND

12. **Generalize** Explain how addition and subtraction of rational expressions is similar to and different from addition and subtraction of rational numbers.

13. **Error Analysis** Describe the error a student made in adding the rational expressions.

$$\frac{1}{x^2+3x+2}+\frac{x^2+4x}{4x+8}=\frac{1}{(x+1)(x+2)}+\frac{x(x+4)}{4(x+2)}$$
$$=\frac{4}{4(x+1)(x+2)}+\frac{x(x+4)}{4(x+1)(x+2)}$$
$$=\frac{4+x^2+4x}{4(x+1)(x+2)}$$
$$=\frac{x^2+4x+4}{4(x+1)(x+2)}$$
$$=\frac{(x+2)(x+2)}{4(x+1)(x+2)}$$
$$=\frac{x+2}{4(x+1)}\cdot\frac{(x+2)}{(x+2)}$$
$$=\frac{x+2}{4(x+1)}$$
✗

14. **Higher Order Thinking** Find the slope of the line that passes through the points shown. Express in simplest form.




15. **Communicate and Justify** For what values of $x$ is the sum of $\frac{x-5y}{x+y}$ and $\frac{x+7y}{x+y}$ undefined? Explain.

16. **Error Analysis** A student says that the LCM of $3x^2+7x+2$ and $9x+3$ is $(3x^2+7x+2)(9x+3)$. Describe and correct the error the student made.

## PRACTICE

**Find the sum.** SEE EXAMPLE 1

17. $\frac{4x}{x+7}+\frac{9}{x+7}$

18. $\frac{3y-1}{y^2+4y}+\frac{9y+6}{y(y+4)}$

**Find the LCM for each group of expressions.** SEE EXAMPLE 2

19. $x^2-7x+6,\ x^2-5x-6$

20. $y^2+2y-24,\ y^2-16,\ 2y$

**Find the sum.** SEE EXAMPLE 3

21. $\frac{6x}{x^2-8x}+\frac{4}{2x-16}$

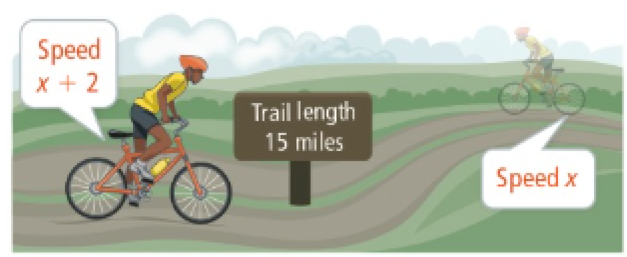
22. $\frac{3y}{2y^2-y}+\frac{2}{2y}$

**Find the difference.** SEE EXAMPLE 4

23. $\frac{4x}{x^2-1}-\frac{4}{x-1}$

24. $\frac{y-1}{3y+15}-\frac{y+3}{5y+25}$

25 On Saturday morning, Ahmed decided to take a bike ride from one end of the 15-mile bike trail to the other end of the bike trail and back. His average speed the first half of the ride was 2 mph faster than his speed on the second half. Find an expression for Ahmed's total travel time. If his average speed for the first half of the ride was 12 mph, how long was Ahmed's bike ride? SEE EXAMPLE 5




**Rewrite as a rational expression.** SEE EXAMPLE 6

26. $\dfrac{1+\frac{1}{x}}{x-\frac{1}{x}}$

27. $\dfrac{\frac{3}{y}+\frac{7}{x}}{\frac{1}{y}-\frac{2}{x}}$

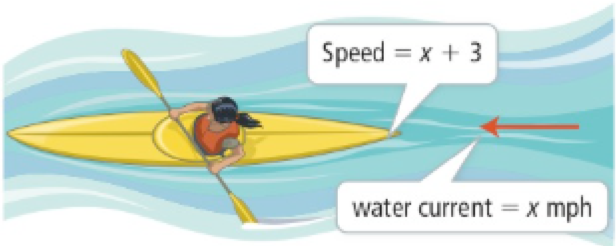
28. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{a^2-b^2}{ab}}$

29. $\dfrac{\frac{z^2-z-12}{z^2-2z-15}}{\frac{z^2+8z+12}{z^2-5z-14}}$

## PRACTICE & PROBLEM SOLVING

### APPLY

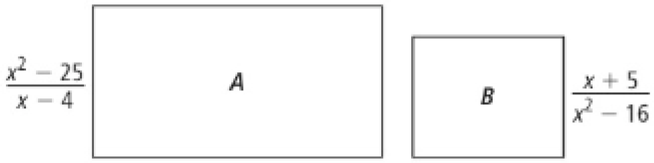
**30. Use Patterns and Structure** Aisha paddles a kayak 5 miles downstream at a rate 3 mph faster than the river's current. She then travels 4 miles back upstream at a rate 1 mph slower than the river's current. Hint: Let $x$ represent the rate of the river current.




**a.** Write and simplify an expression to represent the total time it takes Aisha to paddle the kayak 5 miles downstream and 4 miles upstream.

**b.** If the rate of the river current, $x$, is 2 mph, how long was Aisha's entire kayak trip?

**31. Apply Math Models** Rectangles A and B are similar. An expression that represents the width of each rectangle is shown. Find the scale factor of rectangle A to rectangle B in simplest form.




**32. Apply Math Models** The Taylor family drives 180 miles (round trip) to a professional basketball game. On the way to the game, their average speed is approximately 8 mph faster than their speed on the return trip home.

**a.** Let $x$ represent their average speed on the way home. Write and simplify an expression to represent the total time it took them to drive to and from the game.

**b.** If their average speed going to the game was 72 mph, how long did it take them to drive to the game and back?

### ASSESSMENT PRACTICE

**33.** What is the simplified form of the compound fraction? AR.1.9

$$\frac{\frac{x^2 + 5x + 4}{x^2 + 2x - 8}}{\frac{x^2 - 4x + 3}{x^2 - 3x + 2}}$$

Ⓐ $\frac{x + 1}{x - 3}$ Ⓑ $\frac{x - 3}{x + 1}$

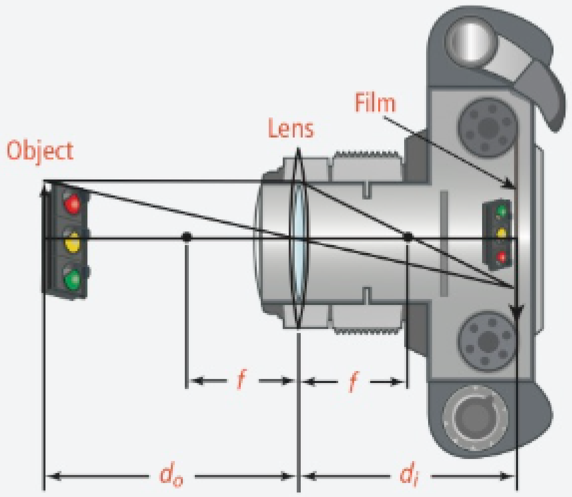
Ⓒ $\frac{(x + 1)(x - 3)}{(x - 2)^2}$ Ⓓ $\frac{x + 1}{x + 3}$

**34. SAT/ACT** What is the difference between $\frac{x}{9}$ and $\frac{x - y}{6}$?

Ⓐ $\frac{5x - y}{18}$ Ⓑ $\frac{5x + y}{18}$

Ⓒ $\frac{-x + 3y}{18}$ Ⓓ $\frac{-x - 3y}{18}$

**35. Performance Task** The lens equation $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$ represents the relationship between $f$, the focal length of a camera lens, $d_i$, the distance from the lens to the film, and $d_o$, the distance from the lens to the object.




**Part A** Find the focal length of a camera lens if an object that is 12 cm from a camera lens is in focus on the film when the lens is 6 cm from the film.

**Part B** Suppose the focal length of another camera lens is 3 inches, and the object to be photographed is 5 feet away. What distance (to the nearest tenth inch) should the lens be from the film?

# 4-5 Solving Rational Equations

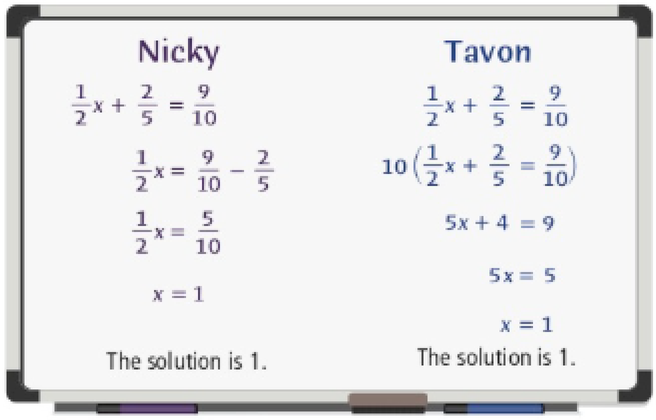
**I CAN…** solve rational equations and identify extraneous solutions.

**VOCABULARY**
- rational equation
- extraneous solution

MA.912.AR.8.1–Write and solve one-variable rational equations. Interpret solutions as viable in terms of the context and identify any extraneous solutions.
Also AR.8.3
MA.912.MTR.1.1, MTR.3.1, MTR.5.1

## CRITIQUE & EXPLAIN

Nicky and Tavon used different methods to solve the equation $\frac{1}{2}x + \frac{2}{5} = \frac{9}{10}$.




**A.** Explain the different strategies that Nicky and Tavon used and the advantages or disadvantages of each.

**B.** Did Nicky use a correct method to solve the equation? Did Tavon?

**C.** **Use Patterns and Structure** Why might Tavon have chosen to multiply both sides of the equation by 10? Could he have used another number? Explain.

## ESSENTIAL QUESTION

**How can you solve rational equations and identify extraneous solutions?**

## EXAMPLE 1 Solve a Rational Equation

**What is the solution to each rational equation?**

A **rational equation** is an equation that contains a rational expression.

**A.** $\frac{1}{x+4} = 2$

$$(x+4)\left(\frac{1}{x+4}\right) = 2(x+4)$$
$$1 = 2x + 8$$
$$x = -\frac{7}{2}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions. Then solve. Confirm that the solution is valid in the original equation.

The solution is $x = -\frac{7}{2}$.

**B.** $\frac{1}{x-3} = 5$

$$(x-3)\left(\frac{1}{x-3}\right) = 5(x-3)$$
$$1 = 5x - 15$$
$$x = \frac{16}{5}$$

The solution is $x = \frac{16}{5}$.

**STUDY TIP**
A fraction with a denominator equal to zero is undefined.

**Try It!** **1.** What is the solution to each equation?

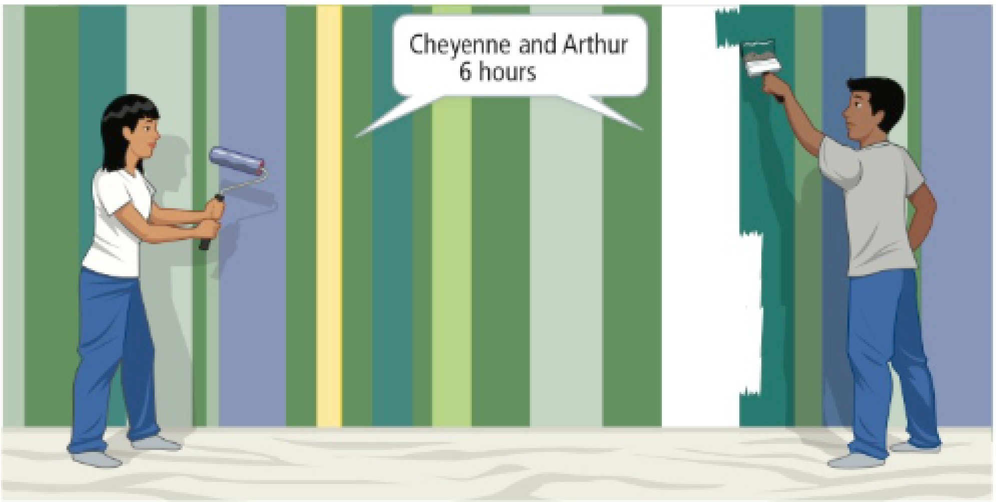
**a.** $\frac{2}{x+5} = 4$

**b.** $\frac{1}{x-7} = 2$

APPLICATION

## EXAMPLE 2 Solve a Work-Rate Problem

**Arthur and Cheyenne can paint a wall in 6 hours when working together. Cheyenne works twice as fast as Arthur. How long would it take Cheyenne to paint the wall if she were working alone?**




**Step 1** Determine the work-rates of Arthur and Cheyenne.

Let $x$ represent the number of hours Arthur needs to paint the wall himself.

The fraction of a job completed *per hour* is the work-rate.

Arthur can paint 1 wall in $x$ hours, or $\frac{1}{x}$ of a wall in 1 hour.

Cheyenne is twice as fast, so Cheyenne paints $\frac{2}{x}$ of a wall in 1 hour.

Together they paint 1 wall in 6 hours, or $\frac{1}{6}$ of a wall in 1 hour.

**COMMON ERROR**
You might have multiplied $x$ by 2 because Cheyenne works twice as fast. However, it takes Cheyenne *half* the time to paint the wall.

**Step 2** Write the equation for their rates working together.

$$\frac{1}{x} + \frac{2}{x} = \frac{1}{6}$$

$$6x\left(\frac{1}{x} + \frac{2}{x}\right) = 6x\left(\frac{1}{6}\right)$$

$$6 + 12 = x$$

$$18 = x$$

Write the equation for their rates working together. Then solve.

It takes Arthur 18 hours to paint the wall alone. Since Cheyenne works twice as fast as Arthur, it would take her 9 hours to paint the wall alone.

**Try It!** 2. It takes 12 hours to fill a pool with two pipes, where the water in one pipe flows three times as fast as the other pipe. How long will it take the slower pipe to fill the pool by itself?

CONCEPTUAL UNDERSTANDING

**HAVE A GROWTH MINDSET**
In what ways do you give your best effort and persist?

## EXAMPLE 3 Identify an Extraneous Solution

**What is the solution of the equation $\frac{1}{x-5}+\frac{x}{x-3}=\frac{2}{x^2-8x+15}$?**

**Step 1** Multiply each side of the equation by the common denominator, $(x-5)(x-3)$.

$$(x-5)(x-3)\left(\frac{1}{x-5}+\frac{x}{x-3}\right)=\frac{2(x-5)(x-3)}{x^2-8x+15}$$

**Step 2** Continue to simplify.

$$\frac{(x-5)(x-3)}{x-5}+\frac{x(x-5)(x-3)}{x-3}=\frac{2(x-5)(x-3)}{x^2-8x+15}$$

**Step 3** Divide out common factors in the numerator and the denominator.

$$\frac{\cancel{(x-5)}(x-3)}{\cancel{x-5}}+\frac{x(x-5)\cancel{(x-3)}}{\cancel{x-3}}=\frac{2\cancel{(x-5)(x-3)}}{\cancel{x^2-8x+15}}$$

You can divide out common factors under the assumption that $\frac{(x-5)}{(x-5)}=1$ and $\frac{(x-3)}{(x-3)}=1$. This is only true if $x \neq 5$ or 3.

**Step 4** Solve the equation.

$$(x-3)+x(x-5)=2$$
$$x-3+x^2-5x=2$$
$$x^2-4x-3=2$$
$$x^2-4x-5=0$$
$$(x-5)(x+1)=0$$
$$x=5 \text{ and } x=-1$$

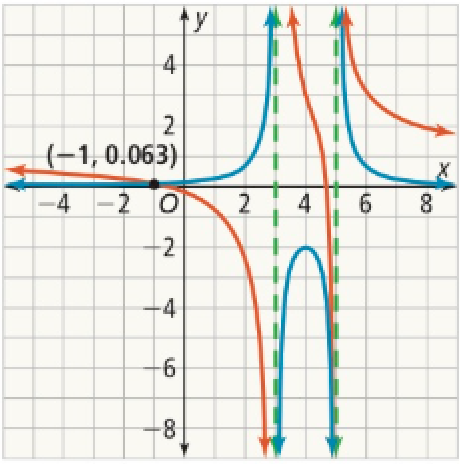
Consider both solutions. If either solution makes the value of the denominator 0, it is not valid.

The solution $x = 5$ is an **extraneous solution** because it makes the value of a denominator in the original equation equal to 0.

The solution of the equation $\frac{1}{x-5}+\frac{x}{x-3}=\frac{2}{x^2-8x+15}$ is –1.

Confirm with a graph.

Consider the graphs of $\frac{1}{x-5}+\frac{x}{x-3}$ and $\frac{2}{x^2-8x+15}$.




Note that each graph has a vertical asymptote at $x = 3$ and $x = 5$. Therefore, neither graph has a value at $x = 5$. The graphs only intersect in one point, at $x = -1$.

**USE PATTERNS AND STRUCTURE**
How are an extraneous solution and an asymptote related? Give an example when an extraneous solutions is not related to a vertical asymptote.

**Try It!** 3. What is the solution to the equation $\frac{1}{x+2}+\frac{1}{x-2}=\frac{4}{(x+2)(x-2)}$?

## EXAMPLE 4 Solve Problems With Extraneous Solutions

**What are the solutions to the following equations?**

**A.** $\frac{5x}{x-2} = 7 + \frac{10}{x-2}$

| | |
|---|---|
| $\frac{5x}{x-2} = 7 + \frac{10}{x-2}$ | Write the original equation. |
| $(x-2)\left(\frac{5x}{x-2}\right) = \left(7 + \frac{10}{x-2}\right)(x-2)$ | Multiply by the LCD. |
| $5x = 7(x-2) + 10$ | Distributive Property |
| $5x = 7x - 14 + 10$ | Distributive Property |
| $-2x = -4$ | Collect terms and simplify. |
| $x = 2$ | Solve for $x$. |

Check the solution in the original equation. The value 2 is an extraneous solution because it would cause the denominator in the original equation to be equal to 0. This equation has no solution.

**STUDY TIP**

Remember to multiply all terms on both sides of the equation by the least common denominator.

**B.** $\frac{3}{x-3} = \frac{x}{x-3} - \frac{x}{4}$

| | |
|---|---|
| $\frac{3}{x-3} = \frac{x}{x-3} - \frac{x}{4}$ | Write original equation. |
| $(4)(x-3)\left(\frac{3}{x-3}\right) = \left(\frac{x}{x-3} - \frac{x}{4}\right)(4)(x-3)$ | Multiply by LCD. |
| $4(3) = 4(x) - x(x-3)$ | Distributive Property |
| $12 = 4x - x^2 + 3x$ | Simplify. |
| $x^2 - 7x + 12 = 0$ | Write in standard form |
| $(x-3)(x-4) = 0$ | Factor. |
| $x - 3 = 0$ or $x - 4 = 0$ | Solve using the Zero Product Property. |
| $x = 3$ or $x = 4$ | Solve for $x$. |

Check the solutions in the original equation. The value 3 is an extraneous solution because it would cause the denominator of the original equation to be equal to zero. The only solution to the equation is $x = 4$.

**Try It!** **4.** What are the solutions to the following equations?

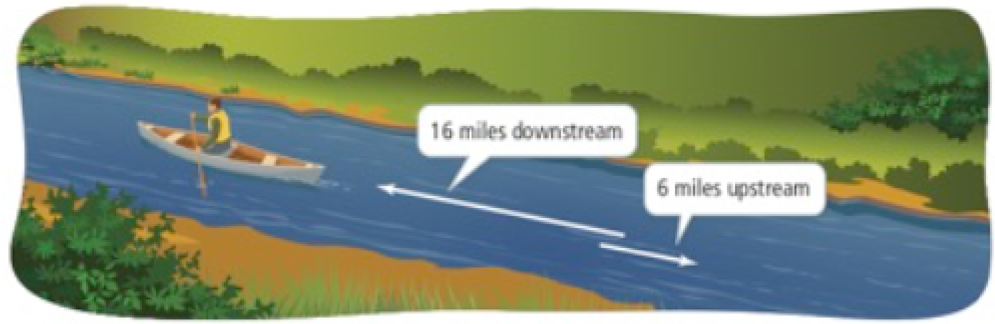
**a.** $x + \frac{6}{x-3} = \frac{2x}{x-3}$

**b.** $\frac{x^2}{x+5} = \frac{25}{x+5}$

APPLICATION

## EXAMPLE 5 Solve a Rate Problem

**Paddling with the current in a river, Jake traveled 16 miles. Even though he paddled upstream for an hour longer than the amount of time he paddled downstream, Jake could only travel 6 miles against the current. In still water, Jake paddles at a rate of 5 mph. What is the speed of the current in the river?**




**Formulate** Let $c$ be the rate of the river's current.

Recall that distance = rate • time so time $= \frac{\text{distance}}{\text{rate}}$.

$$\underbrace{\frac{16}{5+c} + 1 = \frac{6}{5-c}}_{}$$

Jake's paddle rate in still water + the rate of the current

Jake paddles 1 h longer upstream than downstream.

Jake's paddle rate in still water − the rate of the current

**Compute** Solve the equation for $c$.

$$\frac{16}{5+c} + 1 = \frac{6}{5-c}$$

$(5+c)(5-c)\left(\frac{16}{5+c} + 1\right) = \left(\frac{6}{5-c}\right)(5+c)(5-c)$ Multiply both sides by a common denominator.

$80 - 16c + 25 - c^2 = 30 + 6c$ Combine like terms.

$0 = c^2 + 22c - 75$ Write in standard form.

$0 = (c + 25)(c - 3)$ Factor.

$0 = c + 25 \quad 0 = c - 3$ Use the Zero Product Property.

$c = -25 \quad c = 3$ Solve.

**Interpret** The solution $c = -25$ is extraneous because the speed of the current cannot be negative.

The speed of the current is 3 mph.

**Try It!** 5. Three people are planting tomatoes in a community garden. Marta takes 50 minutes to plant the garden alone, Benito takes $x$ minutes and Tyler takes $x + 15$ minutes. If the three of them take 20 minutes to finish the garden, how long would it have taken Tyler alone?

## CONCEPT SUMMARY Solving Rational Equations

**WORDS** A rational equation is an equation that contains a rational expression. To solve, identify the domain for the variable. Then multiply both sides of the equation by a common denominator and solve. An extraneous solution is a solution that is not valid because that value is excluded from the domain of the original equation.

**ALGEBRA**

$$\frac{1}{x} + \frac{2}{x} = \frac{1}{6}$$ Domain: $x \neq 0$

$$6x\left(\frac{1}{x} + \frac{2}{x}\right) = 6x\left(\frac{1}{6}\right)$$

$$6 + 12 = x$$

$$18 = x$$

The domain includes $x = 18$, so the solution to the equation is 18.

$$\frac{x^2 + 4}{x - 1} = \frac{5}{x - 1}$$ Domain: $x \neq 1$

$$(x - 1)\left(\frac{x^2 + 4}{x - 1}\right) = (x - 1)\left(\frac{5}{x - 1}\right)$$

$$x^2 + 4 = 5$$

$$x^2 = 1$$

$$x = \pm 1$$

The domain does not include $x = 1$, so 1 is an extraneous solution. It does include $x = -1$, so the solution to the equation is $-1$.

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you solve rational equations and identify extraneous solutions?

2. **Vocabulary** Write your own example of a rational equation that, when solved, has at least one **extraneous solution**.

3. **Error Analysis** A student solved the rational equation as follows:

$$\frac{1}{2x} - \frac{2}{5x} = \frac{1}{10x} - 3;\ x = 0$$

Describe and correct the error the student made.

4. **Communicate and Justify** Yuki says, "*You can check the solution(s) of rational equations in any of the steps of the solution process.*" Explain why her reasoning is incorrect.

## Do You KNOW HOW?

**Solve.**

5. $\frac{4}{x + 6} = 2$

6. $\frac{x^2}{x + 3} = \frac{9}{x + 3}$

7. Organizing given information into a table can be helpful when solving rate problems. Use this table to solve the following problem.

| | Distance | Rate | Time |
|---|---|---|---|
| Upstream | | | |
| Downstream | | | |

The speed of a stream is 4 km/h. A boat can travel 6 km upstream in the same time it takes to travel 12 km downstream. Find the speed of the boat in still water.

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

8. **Check for Reasonableness** If you solve a work-rate problem and your solution, which represents the amount of time it would take *working together*, exceeds the individual *working alone* times that are given, then how do you know your solution is unreasonable? Explain.

9. **Communicate and Justify** Explain why a negative solution must be eliminated as an extraneous solution when solving a rational equation for an unknown rate.

10. **Error Analysis** Describe and correct the error Miranda made in solving the rational equation.

$$\frac{1}{x-2}+\frac{x-2}{x+2}=\frac{x-4}{x-2}$$

$$(x+2)(x-2)\left(\frac{1}{x-2}+\frac{x-2}{x+2}\right)=\left(\frac{x-4}{x-2}\right)(x+2)(x-2)$$

$$(x+2)(1)+(x-2)(x-2)=(x+4)(x+2)$$

$$x+2+\cancel{x^2}-4x+4=\cancel{x^2}+6x+8$$

$$-3x+6=6x+8$$

$$-2=9x;\ \text{or } x=-\frac{2}{9} \quad ✗$$

11. **Generalize** In addition to identifying extraneous solutions, why else is it useful to substitute your solution into the original equation?

12. **Mathematical Connections** Explain how solving rational equations is related to solving linear and quadratic equations.

13. **Higher Order Thinking** Write a rational equation that cannot have 2 or –6 as solutions.

14. **Analyze and Persevere** Solve the rational equation shown. Explain what is unique about the solution.

$$\frac{x^2-7x-18}{x+2}=x-9$$

## PRACTICE

**Solve the equation.** SEE EXAMPLE 1

15. $\frac{1}{x-3}=10$

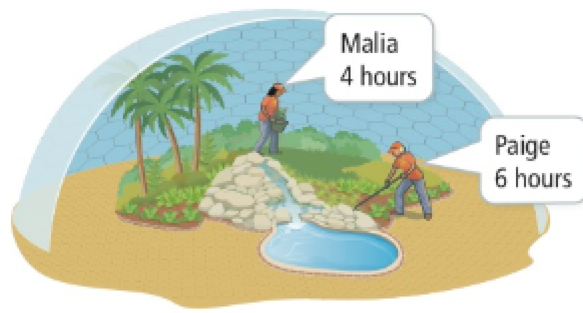
16. $\frac{15}{x+3}=3$

17. $\frac{12}{x-4}=9$

18. $\frac{5}{3-x}=1$

**Solve the problem.** SEE EXAMPLE 2

19. Paige can complete a landscaping job in 6 hours. Malia can complete the same job in 4 hours. Working together, how long would it take them to complete the job?




20. Russel and Aaron can build a shed in 8 hours when working together. Aaron works three times as fast as Russel. How long would it take Russel to build the shed if he were to work alone?


**Solve the equation.** SEE EXAMPLE 3

21. $\frac{x}{x-3}-4=\frac{3}{x-3}$

22. $\frac{x^2}{x-10}=\frac{100}{x-10}-10$

**Solve the equation.** SEE EXAMPLE 4

23. $\frac{4}{3(x+1)}=\frac{12}{x^2-1}$

24. $\frac{x}{x-3}+\frac{2x}{x+3}=\frac{18}{(x+3)(x-3)}$

**Solve the problem.** SEE EXAMPLE 5

25. A boat travels 8 miles upstream in the same amount of time it can travel 12 miles downstream. In still water the speed of the boat is 5 mi/h. What is the speed of the current?

## PRACTICE & PROBLEM SOLVING

### APPLY

**26.** **Analyze and Persevere** Kenji can finish a puzzle in 2 hours working alone. Oscar can finish the same puzzle in 3 hours working alone. How long would it take Oscar and Kenji to finish the puzzle if they worked on it together?

**27.** **Use Patterns and Structure** A commercial jet flies 1,500 miles with the wind. In the same amount of time it can fly 1,000 miles against the wind. The speed of the jet in still air is 550 mph. Find the speed of the wind.


a. Organize the given information and what you need to find in a table.

b. Write and solve a rational equation to find the wind speed.

**28.** **Analyze and Persevere** During their day at the beach, Jae and his friends rent a Jet Ski. They split the \$120 rental fee evenly among themselves. Then Jae, with only his friend Morgan, share the cost of a \$16 pizza. If Jae spends a total of \$48 for both, then find the number of friends, $n$, with whom he shared the cost of the Jet Ski rental.

**29.** **Analyze and Persevere** When driving to their family reunion, River's mom drove 10 miles at a rate of $x$ mph and then 25 miles at a rate of $x + 10$ mph. The total driving time was 45 minutes. What were the two driving speeds at which River's mom drove?

**30.** **Generalize** So far this baseball season, Philip has gotten a hit 8 times out of 40 at-bats. He wants to increase his batting average to 0.333. Calculate the number of consecutive hits, $h$, he would need in order to achieve this goal. Round your answer to the nearest whole number.

### ASSESSMENT PRACTICE

**31.** Which of the following rational equations have at least one extraneous solution? Select all that apply. AR.8.1

☐ A. $\frac{2}{x} = \frac{3}{x-4}$

☐ B. $\frac{x^2}{x-3} = \frac{9}{x-3}$

☐ C. $\frac{x-1}{x-5} = \frac{9}{x-5}$

☐ D. $x + \frac{3}{x} = 4$

☐ E. $\frac{x}{x-3} - \frac{3}{2} = \frac{3}{x-3}$

**32.** **SAT/ACT** Which of the following is the solution of $\frac{3}{x+1} = \frac{2}{x-3}$?

Ⓐ $x = -11$

Ⓑ $x = -\frac{7}{5}$

Ⓒ $x = \frac{7}{5}$

Ⓓ $x = 11$

**33.** **Performance Task** A chemist needs alcohol solution in the correct concentration for her experiment. She adds a 6% alcohol solution to 50 gallons of solution that is 2% alcohol. The function that represents the percent of alcohol in the resulting solution is $f(x) = \frac{50(0.02) + x(0.06)}{50 + x}$, where $x$ is the amount of 6% solution added.


**Part A** How much 6% solution should be added to create a solution that is 5% alcohol?

**Part B** **Choose Efficient Methods** Explain the steps you could take to use your graphing calculator to verify the correctness of your answer to part (A).

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.8.3–Solve and graph mathematical and real-world problems that are modeled with rational functions. Interpret key features and determine constraints in terms of the context. Also AR.1.9

MA.912.MTR.7.1

## Real Cool Waters

Nothing feels better on a hot day than jumping into a pool! Many cities have swimming pools that people can go to for a small fee. Some people have swimming pools in their backyards that they can enjoy any time.

If neither of these options are available, you can always create your own beach paradise! Get a kiddie pool, a lawn chair, and a beach umbrella. Think about your beach paradise during the Mathematical Modeling in 3 Acts lesson.

### ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. Write a number that you know is too small.
6. Write a number that you know is too large.
7. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

### ACT 2 Develop a Model

8. Use the math that you have learned in this Topic to refine your conjecture.

### ACT 3 Interpret the Results

9. Is your refined conjecture between the highs and lows you set up earlier?
10. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC 4

# Topic Review

## TOPIC ESSENTIAL QUESTION

1. How do you calculate with functions defined as quotients of polynomials, and what are the key features of their graphs?

## Vocabulary Review

**Choose the correct term to complete each sentence.**

2. The ________ can be represented by the equation $y = \frac{1}{x}$.
3. A(n) ________ is any function $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.
4. ________ can be modeled by the equation $y = \frac{k}{x}$.
5. A(n) ________ is the quotient of two polynomials.
6. A vertical ________ is a line that a graph approaches but may not touch.
7. A(n) ________ is a fraction that has one or more fractions in the numerator and/or the denominator.
8. A(n) ________ is a value that is a solution to an equation that is derived from an original equation but does not satisfy the original equation.

- inverse variation
- constant of variation
- reciprocal function
- asymptote
- rational function
- extraneous solution
- rational expression
- compound fraction

## Concepts & Skills Review

### LESSON 4-1 Inverse Variation and the Reciprocal Function

#### Quick Review

The equation $y = \frac{k}{x}$, or $xy = k$, $k \neq 0$, represents an **inverse variation**, where $k$ is the **constant of variation**. The parent **reciprocal function** is $y = \frac{1}{x}$.

#### Example

**In an inverse variation, $x = 9$ when $y = 2$. What is the value of $y$ when $x = 3$?**

$2 = \frac{k}{9}$ Substitute 9 and 2 for $x$ and $y$.

$18 = k$ Solve for $k$.

$y = \frac{18}{3}$ Substitute 18 and 3 for $k$ and $x$, respectively.

$y = 6$ Divide.

#### Practice & Problem Solving

9. In an inverse variation, $x = 2$ when $y = -4$. What is the value of $y$ when $x = 16$?
10. In an inverse variation, $x = 6$ when $y = \frac{1}{12}$. What is the value of $x$ when $y = 2$?
11. Graph the function $y = \frac{5}{x}$. What are the domain, range, and asymptotes of the function?
12. **Generalize** How is the parent reciprocal function related to an inverse variation?
13. **Analyze and Persevere** The volume, $V$, of a gas varies inversely with pressure, $P$. If the volume of a gas is 6 $cm^3$ with pressure 25 $kg/cm^2$, what is the volume of a gas with pressure 15 $kg/cm^2$?

## LESSON 4-2 Graphing Rational Functions

### Quick Review

Vertical asymptotes may occur when the denominator of a rational function is equal to 0.

Horizontal asymptotes guide the end behavior of a graph and depend on the degrees of the numerator and denominator.

### Example

**What is the graph of $f(x) = \frac{9x^2 - 25}{x^2 - 5x - 6}$?**

Find vertical asymptotes.

$x^2 - 5x - 6 = 0$ ········· Set denominator equal to 0.

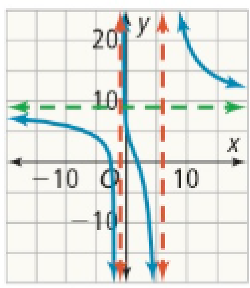
$(x + 1)(x - 6) = 0$ ········· Factor.

$x = -1 \quad x = 6$ ········· Solve.

Find horizontal asymptotes when the numerator and denominator have the same degree.

Find the ratio of leading terms.

$\frac{9x^2}{x^2} = 9$




### Practice & Problem Solving

**Identify the vertical and horizontal asymptotes of each rational function.**

**14.** $f(x) = \frac{x - 8}{x^2 + 9x + 14}$

**15.** $f(x) = \frac{2x + 1}{x^2 + 5x - 6}$

**16.** $f(x) = \frac{x^2 - 9}{2x^2 + 25}$

**17.** $f(x) = \frac{16x^2 - 1}{x^2 - 6x - 16}$

**Graph each function and identify the horizontal and vertical asymptotes.**

**18.** $f(x) = \frac{x}{x^2 - 1}$

**19.** $f(x) = \frac{3}{x - 2}$

**20.** $f(x) = \frac{2x^2 + 7}{x^2 + 2x + 1}$

**21.** $f(x) = \frac{3x^2 - 11x - 4}{4x^2 - 25}$

**22. Apply Math Models** The daily attendance at an amusement park after day $x$ is given by the function $f(x) = \frac{3{,}000x}{x^2 - 1}$. On approximately which day will the attendance be 1,125 people?

## LESSON 4-3 Multiplying and Dividing Rational Expressions

### Quick Review

To multiply **rational expressions**, divide out common factors and simplify. To divide rational expressions, multiply by the reciprocal of the divisor.

### Example

**What is the quotient of $\frac{x^2 + x - 2}{x + 3}$ and $\frac{x^2 + 3x - 4}{2x + 6}$?**

$= \frac{x^2 + x - 2}{x + 3} \cdot \frac{2x + 6}{x^2 + 3x - 4}$ ········· Multiply by reciprocal.

$= \frac{(x + 2)\cancel{(x - 1)}}{\cancel{x + 3}} \cdot \frac{2\cancel{(x + 3)}}{(x + 4)\cancel{(x - 1)}}$ ········· Divide out common factors.

$= \frac{2(x + 2)}{x + 4}$ ········· Simplify.

### Practice & Problem Solving

**Find the simplified product, and state the domain.**

**23.** $\frac{x^2 + x - 12}{x^2 - x - 6} \cdot \frac{x + 2}{x + 4}$

**24.** $\frac{x^2 + 8x}{x^3 + 5x^2 - 24x} \cdot (x^3 + 2x^2 - 15x)$

**Find the simplified quotient, and state the domain.**

**25.** $\frac{x^2 - 36}{x^2 - 3x - 18} \div \frac{x^2 + 2x - 24}{x^2 + 7x + 12}$

**26.** $\frac{2x^2 + 5x - 3}{x^2 - 4x - 21} \div \frac{2x^2 + 5x - 3}{3x + 9}$

**27. Analyze and Persevere** The volume, in cubic units, of a rectangular prism with a square base can be represented by $25x^3 + 200x^2$. The height, in units, can be represented by $x + 8$. What is the side length of the base of the rectangular prism, in units?

## LESSON 4-4 Adding and Subtracting Rational Expressions

### Quick Review

To add or subtract rational expressions, multiply each expression in both the numerator and denominator by a common denominator. Add or subtract the numerators. Then simplify.

### Example

**What is the sum of $\frac{x-2}{x^2-25}$ and $\frac{3}{x+5}$?**

$\frac{x-2}{(x+5)(x-5)} + \frac{3}{x+5}$ Factor denominators.

$= \frac{x-2}{(x+5)(x-5)} + \frac{3(x-5)}{(x+5)(x-5)}$ Find common denominator.

$= \frac{x-2+3(x-5)}{(x+5)(x-5)}$ Add numerators.

$= \frac{x-2+3x-15}{(x+5)(x-5)}$ Multiply.

$= \frac{4x-17}{(x+5)(x-5)}$ Simplify.

### Practice & Problem Solving

**Find the sum or difference.**

28. $\frac{2x}{x+6} + \frac{3}{x-1}$

29. $\frac{x}{x^2-4} - \frac{5}{x-2}$

**Simplify.**

30. $\frac{2+\frac{2}{x}}{2-\frac{2}{x}}$

31. $\frac{\frac{-1}{x}+\frac{3}{y}}{\frac{4}{x}-\frac{5}{y}}$

32. **Represent and Connect** Why is it necessary to consider the domain when adding and subtracting rational expressions?

33. **Analyze and Persevere** Mia paddles a kayak 6 miles downstream at a rate 4 mph faster than the river's current. She then travels 6 miles back upstream at a rate 2 mph faster than the river's current. Write and simplify an expression for the time it takes her to make the round trip in terms of the river's current $c$.

## LESSON 4-5 Solving Rational Equations

### Quick Review

A **rational equation** is an equation that contains a rational expression. An **extraneous solution** is a value that is a solution to an equation that is derived from an original equation but does not satisfy the original equation.

### Example

**What are the solutions to the equation $\frac{2}{x-2} = \frac{x}{x-2} - \frac{x}{4}$?**

$(4)(x-2)\left(\frac{2}{x-2}\right)$

$= \left(\frac{x}{x-2} - \frac{x}{4}\right)(4)(x-2)$ Multiply by the LCD.

$8 = 4x - x^2 + 2x$ Multiply.

$x^2 - 6x + 8 = 0$ Write in standard form.

$(x-2)(x-4) = 0$ Factor.

$x-2=0$ or $x-4=0$ Zero Product Property

$x=2$ or $x=4$ Solve to identify possible solutions.

The solution $x = 2$ is extraneous.
The only solution to the equation is $x = 4$.

### Practice & Problem Solving

**Solve the equation.**

34. $\frac{18}{x+4} = 6$

35. $\frac{9}{x-1} = 3$

36. $-\frac{4}{3} + \frac{2}{x} = 8$

37. $\frac{2x}{x+3} = 5 + \frac{6x}{x+3}$

38. $-8 + \frac{64}{x-8} = \frac{x^2}{x-8}$

39. $\frac{9}{x^2-9} = \frac{3}{6(x-3)}$

40. **Represent and Connect** Explain how to check if a solution to a rational equation is an extraneous solution.

41. **Apply Math Models** Diego and Stacy can paint a fence in 5 hours when working together. Diego works twice as fast as Stacy. Let $x$ be the number of hours it would take Diego to paint the fence and $y$ be the number of hours it would take Stacy to paint the fence. How long would it take Stacy to paint the fence if she worked alone? How long would it take Diego to paint the fence if he worked alone?

TOPIC 5

# Rational Exponents and Radical Functions

## TOPIC ESSENTIAL QUESTION

**How are rational exponents and radical equations used to solve real-world problems?**

## Topic Overview

**enVision® STEM Project:**
Tune a Piano

**5-1** *n*th Roots, Radicals, and Rational Exponents
NSO.1.3, MTR.3.1, MTR.4.1, MTR.5.1

**5-2** Properties of Exponents and Radicals
NSO.1.3, NSO.1.5, MTR.3.1, MTR.5.1, MTR.6.1

**5-3** Graphing Radical Functions
AR.7.2, AR.7.3, F.1.7, F.2.2, F.2.3, F.2.5, MTR.2.1, MTR.5.1, MTR.7.1

**5-4** Solving Radical Equations
AR.7.1, AR.7.3, MTR.1.1, MTR.4.1, MTR.7.1

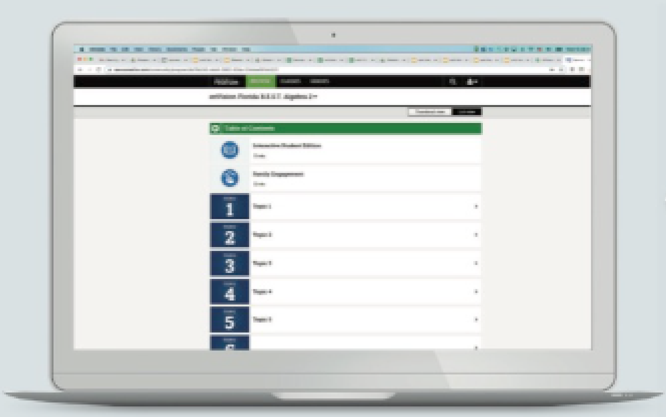
**Mathematical Modeling in 3 Acts:**
The Snack Shack
AR.7.1, MTR.7.1

**5-5** Function Operations
F.3.2, F.3.4, MTR.1.1, MTR.2.1, MTR.6.1

**5-6** Inverse Relations and Functions
F.3.4, F.3.6, F.3.7, MTR.1.1, MTR.2.1, MTR.4.1

## Topic Vocabulary

- composite function
- composition of functions
- extraneous solution
- index
- inverse function
- inverse relation
- like radicals
- *n*th root
- radical function
- radical symbol
- radicand
- reduced radical form



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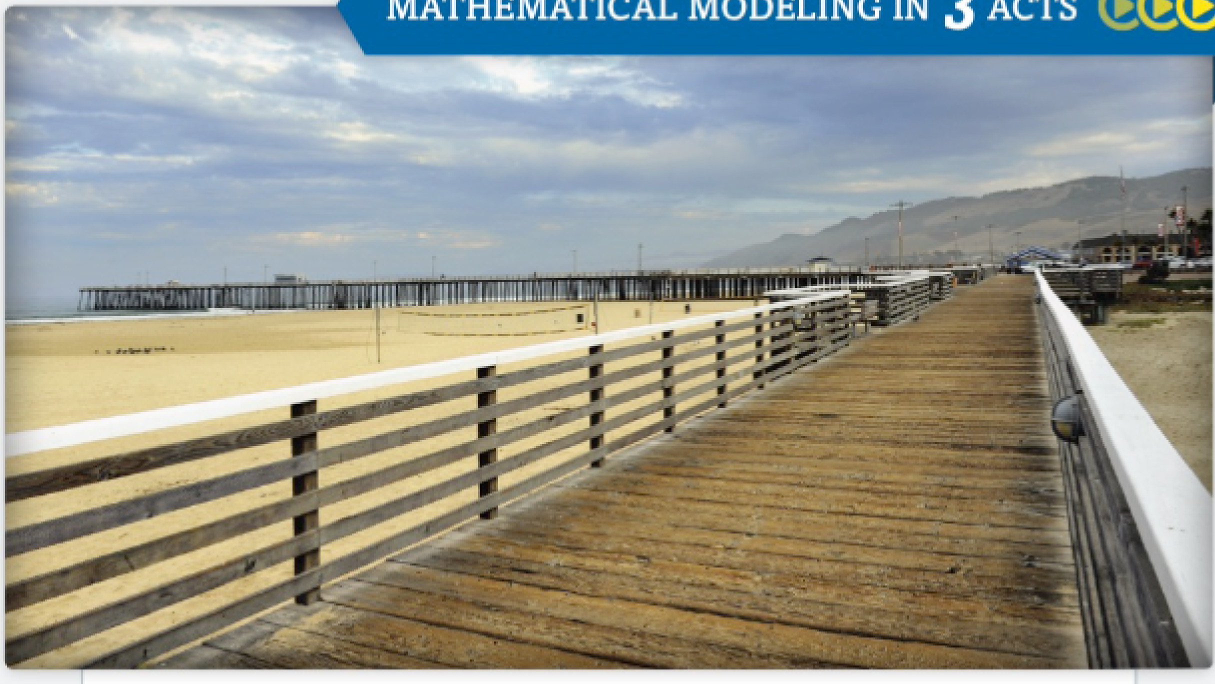
MATHEMATICAL MODELING IN 3 ACTS



## The Snack Shack

Many Americans love the beach! When visiting the beach, some people bring coolers packed with food and drinks. Others prefer to take advantage of snack bars and shops set up along the beach.

Some beachside communities have built long wooden walkways, or boardwalks, to make it easier for beachgoers to walk to the snack bars and stores. How easy do you find walking in the sand? Think about this during the Mathematical Modeling in 3 Acts lesson.

**VIDEOS** Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

**ADAPTIVE PRACTICE** Practice that is *just right* and *just for you*.

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**CONCEPT SUMMARY** Review key lesson content through multiple representations.

**ASSESSMENT** Show what you've learned.

**TUTORIALS** Get help from ***Virtual Nerd***, right when you need it.

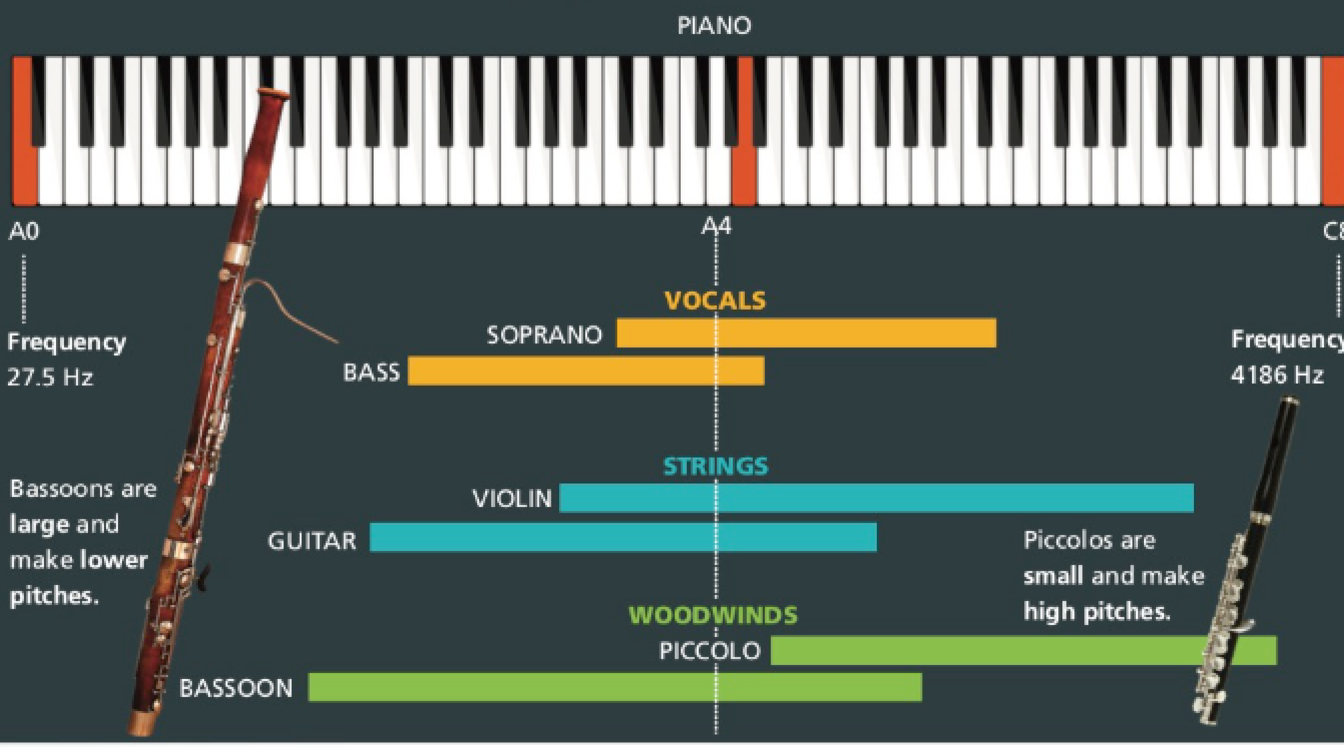
**MATH TOOLS** Explore math with digital tools and manipulatives.

**DESMOS** Use Anytime and as embedded Interactives in Lesson content.

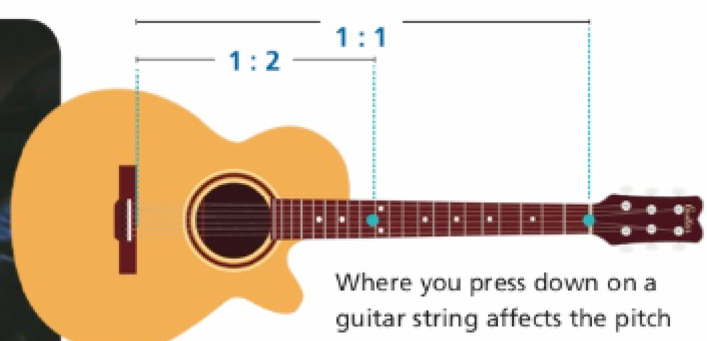
**QR CODES** Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

## Did You Know?

The **size** of an instrument can affect the **range of pitches** it can produce.




**Digital music programs** allow musicians, audio technicians, and music producers to **alter a tone or pitch,** edit music, and visualize sounds.




Where you press down on a guitar string affects the pitch of the string when plucked. Pressing halfway down the string **produces a pitch an octave higher** than pressing the top of the string.

## Your Task: Tune a Piano

You and your classmates will investigate different ways to tune a piano. You will select a musical piece and then decide what tuning sounds best for it.

# 5-1

## $n$th Roots, Radicals, and Rational Exponents

**I CAN…** relate roots and rational exponents and use them to simplify expressions and solve equations.

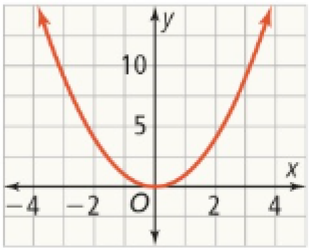
**VOCABULARY**

- index
- $n$th root
- radical symbol
- radicand

MA.912.NSO.1.3–Generate equivalent algebraic expressions involving radicals or rational exponents using the properties of exponents.
MA.K12.MTR.3.1, MTR.4.1, MTR.5.1

## EXPLORE & REASON

**The graph shows $y = x^2$.**




**A.** Find *all* possible values of $x$ or $y$ so that the point is on the graph.

(a) (2, ___) (b) (3, ___)
(c) (−3, ___) (d) (5, ___)
(e) (___ , 4) (f) (___, −16)
(g) (___ , 7) (h) (___, 5)

**B.** **Communicate and Justify** Write a precise set of instructions that show how to find an approximate value of $\sqrt{13}$ using the graph.

**C.** Draw a graph of $y = x^3$. Use the graph to approximate each value.

(a) $\sqrt[3]{5}$ (b) $\sqrt[3]{-5}$
(c) $\sqrt[3]{8}$ (d) A solution to $x^3 = 5$
(e) A solution to $x^3 = -5$ (f) A solution to $x^3 = 8$

**ESSENTIAL QUESTION** How are exponents and radicals used to represent roots of real numbers?

## EXAMPLE 1 Find All Real $n$th Roots

**A. What are all the real cube roots of 125?**

An **$n$th root** of a number $c$ is $x$, such that $x^n = c$. An $n$th root can be denoted by a **radical symbol** with an **index** of $n$: $\sqrt[n]{c}$ ; $c$ is called the **radicand**.

To represent the real cube root of 125, write $x = \sqrt[3]{125}$.

Find the value of $x$ such that $x^3 = 125$.

To solve $x^3 = 125$, note that $5^3 = 125$, so 5 is a root.

Recall that complex solutions to polynomial equations come in pairs. The number of real roots depends on the degree of the equation. The number of real solutions must be counted according to multiplicity.
- odd degree → odd number of real roots
- even degree → even number of real roots

Consider if there are others.
To determine this, set the expression equal to 0 and factor.

$x^3 - 125 = 0$

$(x - 5)(x^2 + 5x + 25)$

$x - 5 = 0$ gives the root 5

$x^2 + 5x + 25 = 0$

**ANALYZE AND PERSEVERE**
Recall that third degree equations have three solutions; here, one solution is real and two are complex.

The discriminant of the expression is $5^2 - 4(1)(25)$, or −75. The roots are not real.

Therefore, 5 is the only real cube root of 125.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

**B. What are all the real fourth roots of 16?**

The index is 4 and the radicand is 16. Write $x = \sqrt[4]{16}$.

Find the value of $x$ such that $x^4 = 16$.

When written as $x^4 = 16$, find the real roots, unless told otherwise. The radical symbol, $\sqrt[2n]{x}$, represents the positive (principal) root.

Set the expression equal to 0 and factor.

$$x^4 - 16 = 0$$
$$(x^2 - 4)(x^2 + 4) = 0$$
$$(x + 2)(x - 2)(x^2 + 4) = 0$$
$$x = 2, -2$$

$x^2 + 4 = 0$ leads to imaginary roots.

The real roots of $x^4 = 16$ are 2 and –2.

**Try It!** **1.** Find the specified roots of each number.

**a.** real fourth roots of 81 **b.** real cube roots of 64

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Understand Rational Exponents

**A. What is the meaning of the exponent in the expression $16^{\frac{1}{4}}$?**

To interpret a rational exponent, look at what happens if you extend the properties of *integer* exponents to *rational* exponents.

Assume the Power of a Power Property applies to exponents in the form $\frac{1}{n}$.

**STUDY TIP**
It is important to recognize multiple representations of the same value so you can select the form that is most efficient for a given situation.

$$\left(16^{\frac{1}{4}}\right) = (2^4)^{\frac{1}{4}} = 2 = \sqrt[4]{16}$$

$16^{\frac{1}{4}}$ is a number you raise to the 4th power to get 16.

This means you can define $16^{\frac{1}{4}}$ to be $\sqrt[4]{16}$, or 2.

In general, for positive integer $n$, $x^{\frac{1}{n}}$ is defined to be $\sqrt[n]{x}$.

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

Both the exponent $\frac{1}{n}$ and the radical symbol $\sqrt[n]{x}$ indicate the principal, or positive, $n$th root when $n$ is even.

**B. What is the meaning of the exponent in the expression $27^{\frac{2}{3}}$?**

$27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2$ Rewrite using the Power of a Power Property.

$= (\sqrt[3]{27})^2$ Use the definition of the exponent $\frac{1}{n}$, $x^{\frac{1}{n}} = \sqrt[n]{x}$.

$= 3^2$

$= 9$

This means that you can define $27^{\frac{2}{3}}$ to be the $(\sqrt[3]{27})^2$, or 9.

**Try It!** **2.** Explain what each fractional exponent means, then evaluate.

**a.** $25^{\frac{1}{2}}$ **b.** $32^{\frac{2}{5}}$

**CONCEPT** Interpreting Fractional Exponents

The index of a radical is equivalent to the denominator of a fractional exponent.

In general, if the $n$th root of $c$ is a real number, $\sqrt[n]{c} = c^{\frac{1}{n}}$.

Furthermore, if $m$ is an integer and $\frac{m}{n}$ is in lowest terms, then $c^{\frac{m}{n}} = \left(c^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{c}\right)^m$ and $\sqrt[n]{c^m} = \left(c^m\right)^{\frac{1}{n}} = c^{\frac{m}{n}}$.

APPLICATION

## EXAMPLE 3 Evaluate Expressions With Rational Exponents

**CHOOSE EFFICIENT METHODS**
Many expressions with integer roots can be evaluated using mental math. However, when the root is not an integer, it can be efficient to use the power function of a calculator to compute rational roots.

**A. Evaluate the expressions $32^{\frac{3}{5}}$, $27^{-\frac{2}{3}}$, $50^{\frac{3}{4}}$.**

$$32^{\frac{3}{5}} = \left(32^{\frac{1}{5}}\right)^3 = 2^3 = 8$$

$$27^{-\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^{-2} = (3)^{-2} = \frac{1}{9}$$

Since 50 does not have a perfect 4th root, use a calculator to approximate: $50^{\frac{3}{4}} \approx 18.80$

**B. The Fujita scale rating, $F$, of a tornado is represented by $F = \sqrt[3]{\left(\frac{W}{14.1}\right)^2} - 2$, where $W$ is the estimated wind speed of the tornado in miles per hour. What is the Fujita scale rating of a tornado with estimated wind speeds of 100 mph?**


$F = \sqrt[3]{\left(\frac{100}{14.1}\right)^2} - 2$ Substitute 100 for $W$ in the formula.

$= \left(\frac{100}{14.1}\right)^{\frac{2}{3}} - 2$ Rewrite the radical using an exponent.

$\approx (7.09)^{\frac{2}{3}} - 2$ Divide 100 by 14.1.

$\approx 3.69 - 2$ Evaluate the exponent.

$\approx 1.69$ Subtract.

A tornado with estimated wind speeds of 100 mph is classified as F1 according to the Fujita scale.

**Try It!** 3. What is the value of each expression? Round to the nearest hundredth if necessary.

**a.** $-\left(16^{\frac{3}{4}}\right)$ **b.** $\sqrt[5]{3.5^4}$

## EXAMPLE 4 Simplify *n*th Roots

**Simplify each expression. Assume all variables are positive.**

**A.** $\sqrt[5]{32m^{15}}$

$$\sqrt[5]{2^5m^{15}} = \sqrt[5]{(2m^3)^5}$$
$$= 2m^3$$

Write the radicand as an expression to the 5th power.

So, $\sqrt[5]{32m^{15}} = 2m^3$.

**B.** $\sqrt[4]{x^{20}y^8}$

$$\sqrt[4]{x^{20}y^8} = \sqrt[4]{(x^5y^2)^4}$$
$$= x^5y^2$$

Write the radicand as an expression to the 4th power.

So, $\sqrt[4]{x^{20}y^8} = x^5y^2$.

**COMMON ERROR**
You may think that because 20 and 8 do not have perfect fourth roots this expression cannot be simplified. However, each exponent is a multiple of 4, and: $(n^4)^m = n^{4m}$.

**Try It!** **4.** Simplify each expression.

**a.** $\sqrt[3]{-8a^3b^9}$

**b.** $\sqrt[4]{256x^{12}y^{24}}$

## EXAMPLE 5 Use *n*th Roots to Solve Equations

**Solve the equations over the real numbers.**

**A.** $2x^5 = 64$

$$2x^5 = 64$$
$$x^5 = 32$$
$$(x^5)^{\frac{1}{5}} = 32^{\frac{1}{5}}$$
$$x = 2$$

Raise each side of the equation to the $\frac{1}{5}$ power.

Use Power of a Power Property on the left, and simplify the rational exponent on the right.

The solution to the equation is $x = 2$.

**GENERALIZE**
Raise both sides of the equation to a power so that the exponent of the variable becomes 1. Think reciprocal. Using the Power of a Power Property, $(x^n)^{\frac{1}{n}} = x^1$.

**B.** $3x^4 = 1875$

$$x^4 = 625$$
$$(x^4)^{\frac{1}{4}} = \pm(625)^{\frac{1}{4}}$$
$$x = \pm 5$$

A negative integer raised to an even power is positive, so $-5$ is also a real solution.

The solutions are $x = \pm 5$.

**Try It!** **5.** Solve the equations over the real numbers.

**a.** $5x^3 = 320$.

**b.** $2p^4 = 162$.

## CONCEPT Solving an Equation in the Form $x^n = c$

To solve an equation in the form $x^n = c$ over the real numbers, find the $n$th root of both sides by raising each expression to the $\frac{1}{n}$ power.

$$(x^n)^{\frac{1}{n}} = (c)^{\frac{1}{n}}$$

$x = c^{\frac{1}{n}}$ when $n$ is odd $\qquad$ $x = \pm c^{\frac{1}{n}}$ when $n$ is even

APPLICATION

## EXAMPLE 6 Use $n$th Roots to Solve Problems

**One cube-shaped container has an edge length 2 cm longer than the edge length of a second cube. The volume of the larger cube is 729 cm$^3$. Assuming that the larger cube fills completely before emptying into the smaller cube, how much water will spill?**


**APPLY MATH MODELS**
Use the information in the problem to write an equation that models the situation.

$$(x+2)^3 = 729$$
$$\left[(x+2)^3\right]^{\frac{1}{3}} = (729)^{\frac{1}{3}}$$
$$x + 2 = 9$$
$$x = 7$$

Raise both sides to the $\frac{1}{3}$ power and solve.

The volume of the smaller cube is $7^3$, or 343 cm$^3$.

The amount of water that spills is $729 - 343 = 386$.

When the larger cube empties into the smaller cube, 386 cm$^3$ of water will spill out.

**Try It!** 6. One cube has an edge length 3 cm shorter than the edge length of a second cube. The volume of the smaller cube is 200 cm$^3$. What is the volume of the larger cube?

## CONCEPT SUMMARY *n*th Roots, Radicals, and Rational Exponents

| | Relating Radical and Exponential Forms | Solving an Equation in the Form $x^n = c$ |
|---|---|---|
| **WORDS** | The index of a radical is equivalent to the denominator of a fractional exponent. The exponent of the radicand is equivalent to the numerator of a fractional exponent. | To solve an equation in the form $x^n = c$ over the real numbers, find the $n$th root of both sides of the equation by raising each expression to the $\frac{1}{n}$ power. |
| **NUMBERS** | Radical Form $\sqrt[5]{32^4} = (32^4)^{\frac{1}{5}} = 32^{\frac{4}{5}} = 16$<br>Exponential Form $729^{\frac{5}{6}} = \left(729^{\frac{1}{6}}\right)^5 = \left(\sqrt[6]{729}\right)^5 = 243$ | $x^3 = 1{,}728$<br>$(x^3)^{\frac{1}{3}} = (1{,}728)^{\frac{1}{3}}$<br>$x = 12$ |
| **ALGEBRA** | Radical Form $\sqrt[n]{c^m} = (c^m)^{\frac{1}{n}} = c^{\frac{m}{n}}$<br>Exponential Form $c^{\frac{m}{n}} = \left(c^{\frac{1}{n}}\right)^m = \sqrt[n]{c^m}$ | $x^n = c$<br>$(x^n)^{\frac{1}{n}} = (c)^{\frac{1}{n}}$<br>$x = c^{\frac{1}{n}}$ if $n$ is odd, $x = \pm c^{\frac{1}{n}}$ if $n$ is even |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are exponents and radicals used to represent roots of real numbers?

2. **Error Analysis** Kaitlyn said $\sqrt[3]{10} = 10^3$. Explain Kaitlyn's error.

3. **Vocabulary** In the radical expression $\sqrt[5]{125}$, what is the index? What is the radicand?

4. **Use Patterns and Structure** Why is $75^{\frac{3}{5}}$ equal to $\left(75^{\frac{1}{5}}\right)^3$?

5. **Communicate and Justify** Anastasia said that $(x^8)^{\frac{1}{4}} = \frac{x^8}{x^4} = x^4$. Is Anastasia correct? Explain.

6. **Analyze and Persevere** Is it possible for a rational exponent to be an improper fraction? Explain how $27^{\frac{4}{3}}$ is evaluated or why it cannot be evaluated.

## Do You KNOW HOW?

**Write each expression in radical form.**

7. $a^{\frac{1}{5}}$

8. $7^{\frac{2}{3}}$

**Write each expression in exponential form.**

9. $\sqrt[3]{b}$

10. $\sqrt[4]{p^7}$

11. How many real third roots does 1,728 have?

12. How many real sixth roots does 15,625 have?

13. Solve the equation $4x^3 = 324$.

14. Solve the equation $2x^4 = 2{,}500$.

**Simplify each expression. Assume all variables are positive.**

15. $\sqrt[3]{27x^{12}y^6}$

16. $\sqrt[5]{-32x^5y^{30}}$

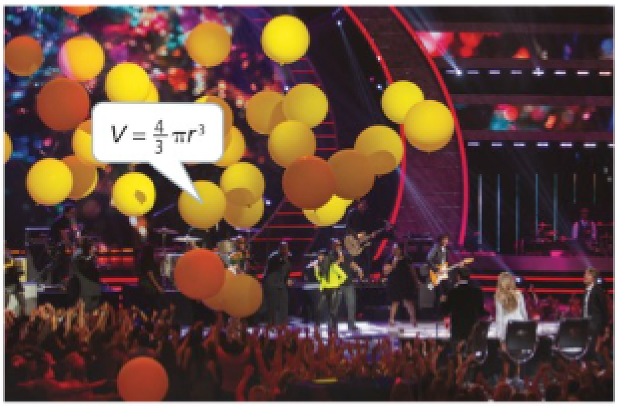
17. A snow globe is packaged in a cubic container that has volume 64 in.$^3$ A large shipping container is also a cube, and its edge length is 8 inches longer than the edge length of the snow globe container. How many snow globe containters can fit into the larger shipping container?

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

18. **Communicate and Justify** Justice found that the fifth root of $243x^{15}y^5$ is $3x^3y$. Is Justice correct? Explain your reasoning.

19. **Analyze and Persevere** For a show, each sphere was inflated to have a volume of $4{,}186\frac{2}{3}$ in.$^3$ Explain how to find the radius $r$ of one of the inflated spheres. Use technology to compute your answer.




20. **Error Analysis** Describe and correct the error a student made in writing this exponential expression in radical form.

$$x^{\frac{4}{3}} = (x^4)^{\frac{1}{3}}$$
$$(x^4)^{\frac{1}{3}} = \sqrt[4]{x^3} \quad ✗$$

21. **Communicate and Justify** Determine whether $\sqrt[3]{x^2}$ is equal to $(\sqrt[3]{x})^2$. Explain your reasoning.

22. **Use Patterns and Structure** How many real third roots does –512 have? Explain your reasoning.

23. **Higher Order Thinking** The annual interest formula below calculates the final balance of an account, $F$, given a starting balance, $S$, and an interest rate, $r$, after 10 years.

$$F = S(1 + r)^{10}$$

When solving for $r$, why can the negative root be ignored?

24. **Mathematical Connections** The lengths of the two legs of a right triangle are 4 and 8. What is the length of the hypotenuse, in simplest radical form?

### PRACTICE

**Find the specified roots of each number.**
SEE EXAMPLE 1

25. the real fourth roots of 81

26. the real third roots of 343

27. the real fifth roots of 1,024

28. the real square roots of 25

**Rewrite each expression using a fractional exponent.** SEE EXAMPLE 2

29. $\sqrt[4]{16^2}$

30. $\sqrt[6]{729}$

31. $\sqrt[7]{x^2}$

32. $\sqrt[4]{ab}$

**What is the value of each expression? Round to the nearest hundredth if necessary.** SEE EXAMPLE 3

33. $\sqrt[4]{25^2}$

34. $-\sqrt[3]{125^5}$

**Simplify each expression. Assume all variables are positive.** SEE EXAMPLE 4

35. $\sqrt[3]{8y^9}$

36. $\sqrt[4]{q^{12}z^4}$

37. $\sqrt[6]{729a^{24}b^{18}}$

38. $\sqrt[8]{v^8g^{40}}$

**Solve each equation over the real numbers.**
SEE EXAMPLE 5

39. $1{,}125 = 9x^3$

40. $6{,}480 = 5w^4$

41. $270 = 10q^3$

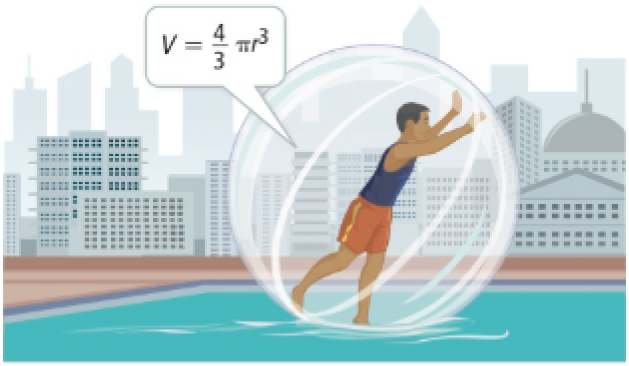
42. $256 = 4h^6$

43. A small cube has the volume shown. Its side length is 1.5 in. less than a second, larger cube. What is the volume of the larger cube?
SEE EXAMPLE 6

## PRACTICE & PROBLEM SOLVING

**APPLY**

**44. Apply Math Models** A water-walking ball has a volume of approximately 4.19 $m^3$. What is the radius, $r$, of the ball?




**45. Analyze and Persevere** Ahmed received a box of gifts. The box is a rectangular prism with the same height and width, and the length is twice the width. The volume of the box is 3,456 $in.^3$ What is the height of the box?


**46. Analyze and Persevere** Amelia's bank account earns interest annually. The equation shows her starting balance of $200 and her balance at the end of four years, $220.82. At what rate, $r$, did Amelia earn interest?

$$220.82 = 200(1 + r)^4$$

**47. Apply Math Models** One measure of a patient's body surface area is found using the expression $\sqrt{\frac{H \cdot W}{3{,}600}}$. Write this with a fractional exponent.

**ASSESSMENT PRACTICE**

**48.** Select all the expressions that are equivalent to $(16b)^{\frac{3}{4}}$. NSO.1.3

☐ **A.** $8\sqrt[4]{b^3}$

☐ **B.** $8(b^3)^{\frac{1}{4}}$

☐ **C.** $(16b)^{\frac{4}{3}}$

☐ **D.** $8\sqrt[3]{b^4}$

☐ **E.** $\frac{(16b)^3}{(16b)^4}$

**49. SAT/ACT** Which of the following is equivalent to $\sqrt[6]{4{,}096x^{18}y^{30}}$? Assume $x > 0$, $y > 0$.

Ⓐ $682.7x^{15}y^{24}$

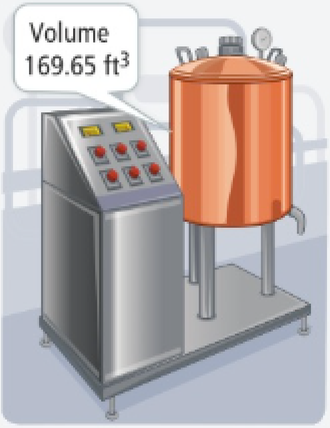
Ⓑ $4x^{1.6}y^{1.8}$

Ⓒ $4{,}096x^3y^5$

Ⓓ $4x^3y^5$

Ⓔ $682.7x^3y^5$

**50. Performance Task** A milk processing company uses cylindrical-shaped containers. The height of the container is equal to the diameter of the base.




**Part A** The volume of one container is about 169.65 $ft^3$. How much material is needed to make the lateral surface of the shipping container?

**Part B** The cargo hold of a ship is 20 ft high. What is the largest number of these shipping containers that could be stacked on top of each other inside the cargo hold?

# 5-2 Properties of Exponents and Radicals

**I CAN…** use properties of exponents and radicals to simplify radical expressions.

**VOCABULARY**

- like radicals
- reduced radical form

MA.912.NSO.1.5–Add, subtract, multiply and divide algebraic expressions involving radicals. Also NSO.1.3

MA.K12.MTR.3.1, MTR.5.1, MTR.6.1

## CRITIQUE & EXPLAIN

**Olivia was practicing evaluating and simplifying expressions. Her work for three expressions is shown.**

1. $24^2 = 400 + 16 = 416$
2. $3^6 = 9(27) = 270 - 27 = 243$
3. $\sqrt{625} = \sqrt{400} + \sqrt{225} = 20 + 15 = 35$

**A.** Is Olivia's work in the first example correct? Explain your thinking.

**B.** Is Olivia's work in the second example correct? Explain your thinking.

**C.** Is Olivia's work in the third example correct? Explain your thinking.

**D.** **Analyze and Persevere** What advice would you give Olivia on simplifying expressions?

## ESSENTIAL QUESTION

**How can properties of exponents and radicals be used to rewrite radical expressions?**

## CONCEPT Properties of Rational Exponents

The properties of exponents apply not only to integer exponents, but to *rational* exponents as well. Now let *m* and *n* represent *rational* numbers, with *a*, *b* positive real numbers.

| | Property | Example |
|---|---|---|
| Product of Powers | $a^m \cdot a^n = a^{m+n}$ | $4^{\frac{2}{3}} \cdot 4^{-\frac{1}{3}} = 4^{\frac{1}{3}}$ |
| Quotient of Powers | $\frac{a^m}{a^n} = a^{m-n}$ | $\frac{3^4}{3^2} = 3^{4-2} = 3^2 = 9$ |
| Power of Power | $(a^m)^n = a^{mn}$ | $(7^3)^{\frac{2}{3}} = 7^2$ |
| Power of Product | $(ab)^m = a^m b^m$ | $(16x)^{\frac{1}{2}} = (16^{\frac{1}{2}}x^{\frac{1}{2}}) = 4x^{\frac{1}{2}}$ |
| Negative Exponent | $a^{-m} = \frac{1}{a^m}$ | $5^{-\frac{1}{2}} = \frac{1}{5^{\frac{1}{2}}}$ |

## EXAMPLE 1 Use Properties of Exponents

**USE PATTERNS AND STRUCTURE**
When multiplying numbers with the same base, adding a negative exponent gives the same result as subtracting its opposite.

**How can you rewrite each expression using the properties of exponents?**

**A.** $81^{\frac{5}{6}} \cdot 81^{-\frac{1}{3}}$

$81^{\frac{5}{6}} \cdot 81^{-\frac{1}{3}} = 81^{\frac{5}{6} - \frac{1}{3}}$ ........ Use the Product of Powers Property.

$= 81^{\frac{1}{2}}$ ........ Simplify the exponent.

$= 9$ ........ Evaluate.

You can rewrite $81^{\frac{5}{6}} \cdot 81^{-\frac{1}{3}}$ as 9.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

**B.** $\left(\dfrac{xy^3}{x^{\frac{1}{2}}}\right)^{\frac{2}{3}}$

**COMMON ERROR**
Remember to subtract the exponents, not divide them.

$\left(\dfrac{xy^3}{x^{\frac{1}{2}}}\right)^{\frac{2}{3}} = \left(x^{1-\frac{1}{2}}y^3\right)^{\frac{2}{3}}$ Use the Quotient of Powers Property.

$= \left(x^{\frac{1}{2}}y^3\right)^{\frac{2}{3}}$ Simplify.

$= x^{\frac{1}{3}}y^2$ Use the Power of Product and Power of Power Properties.

$= y^2\sqrt[3]{x}$ Write in radical form.

You can rewrite $\left(\dfrac{xy^3}{x^{\frac{1}{2}}}\right)^{\frac{2}{3}}$ as $y^2\sqrt[3]{x}$.

**Try It!** **1.** How can you rewrite each expression using the properties of exponents?

**a.** $\left(\dfrac{3}{32^{\frac{2}{5}}}\right)^{\frac{1}{2}}$  **b.** $2a^{\frac{1}{3}}\left(ab^{\frac{1}{2}}\right)^{\frac{2}{3}}$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Use Properties of Exponents to Rewrite Radicals

**How can you extend the properties of exponents to derive the properties of radicals?**

**COMMUNICATE AND JUSTIFY**
Why do we assume $a$ and $b$ are positive? The exponent laws cannot be assumed for complex numbers.
For example: $\sqrt{-1} \cdot \sqrt{-1} = (i)(i) = -1$ while $\sqrt{-1 \cdot -1} = \sqrt{1} = 1$.

**A. How can you rewrite $\sqrt[n]{ab}$ using the properties of exponents? Assume $a$ and $b$ are positive.**

$\sqrt[n]{ab} = (ab)^{\frac{1}{n}}$ Rewrite the radical as a rational exponent.

$= a^{\frac{1}{n}}b^{\frac{1}{n}}$ Rewrite using the Power of a Product Property.

$= \sqrt[n]{a}\sqrt[n]{b}$ Rewrite the rational exponents as radicals.

So, $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

You can use a similar method to show that $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

**B. How can you rewrite $\sqrt[3]{16x^5}$ using the properties of exponents? Assume $x > 0$.**

$\sqrt[3]{16x^5} = \sqrt[3]{8x^3 \cdot 2x^2}$

$= \sqrt[3]{8x^3} \cdot \sqrt[3]{2x^2}$

$= 2x\sqrt[3]{2x^2}$

Factors of 16 and $x^5$ that are perfect cubes

Remaining factors of 16 and $x^5$ that are not perfect cubes

So, $\sqrt[3]{16x^5} = 2x\sqrt[3]{2x^2}$.

Writing the expression as $2x\sqrt[3]{2x^2}$ may be referred to as the **reduced radical form** of the expression because all $n$th roots of perfect $n$th powers in the radicand have been simplified, and no radicals remain in the denominator.

**Try It!** **2.** How can you rewrite each expression? Assume all variables are positive.

**a.** $\sqrt[4]{81a^8b^5}$  **b.** $\sqrt[3]{\dfrac{x^4y^2}{125x}}$

## CONCEPT Properties of Radicals

**USE PATTERNS AND STRUCTURE**
The Product of Powers and Quotient of Powers Properties lead to the Product and Quotient Properties for Radicals.

**Product Property of Radicals**
The $n$th root of a product of positive real numbers is equal to the product of the $n$th roots of those numbers.

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $\qquad (ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$

**Quotient Property of Radicals**
The $n$th root of a quotient of positive real numbers is equal to the quotient of the $n$th roots of those numbers.

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $\qquad \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$

## EXAMPLE 3 Rewrite the Product or Quotient of a Radical

**A. What is $\sqrt[5]{16} \cdot \sqrt[5]{8}$ in reduced radical form?**

$\sqrt[5]{16} \cdot \sqrt[5]{8} = \sqrt[5]{16 \cdot 8}$ ........ Use the Product Property of Radicals.

$= \sqrt[5]{128}$ ........ Multiply radicands.

$= \sqrt[5]{32} \cdot \sqrt[5]{4}$ ........ Use the Product Property of Radicals.

$= 2\sqrt[5]{4}$ ........ Simplify.

In reduced radical form $\sqrt[5]{16} \cdot \sqrt[5]{8}$ is $2\sqrt[5]{4}$.

**B. What is $\sqrt[6]{8x} \cdot \sqrt[3]{2x}$ in reduced radical form? Assume $x > 0$.**

Use rational exponents.

$\sqrt[6]{8x} \cdot \sqrt[3]{2x} = (8x)^{\frac{1}{6}} \cdot (2x)^{\frac{1}{3}}$ ........ Rewrite using rational exponents.

$= (8x)^{\frac{1}{6}} \cdot (2x)^{\frac{2}{6}}$ ........ Write with common index (denominator).

$= \left(8x \cdot (2x)^2\right)^{\frac{1}{6}}$ ........ Use the Product Property of Radicals.

$= \sqrt[6]{32x^3}$ ........ Simplify.

In reduced radical form $\sqrt[6]{8x} \cdot \sqrt[3]{2x}$ is $\sqrt[6]{32x^3}$.

**C. What is $\sqrt[3]{\frac{2n}{9m}}$ in reduced radical form? Assume $m$ and $n$ are positive.**

To rationalize the denominator of an expression, rewrite it so there are no radicals in any denominator and no denominators in any radical.

**CHOOSE EFFICIENT METHODS**
Think about the simplest factor by which you can multiply the denominator to eliminate the radical. There are many options, but it is more efficient to choose the simplest factor.

$\sqrt[3]{\frac{2n}{9m}} = \frac{\sqrt[3]{2n}}{\sqrt[3]{9m}}$

To rationalize the denominator, find a factor so that a cube root of a perfect cube is created.

$= \frac{\sqrt[3]{2n}}{\sqrt[3]{9m}} \cdot \frac{\sqrt[3]{3m^2}}{\sqrt[3]{3m^2}}$

9 is a factor of the perfect cube 27, and $m^1$ is a factor of $m^3$. Multiply the denominator and numerator by $\sqrt[3]{3m^2}$ to create a perfect cube.

$= \frac{\sqrt[3]{6nm^2}}{\sqrt[3]{27m^3}}$

$= \frac{\sqrt[3]{6nm^2}}{3m}$

In reduced radical form $\sqrt[3]{\frac{2n}{9m}}$ is $\frac{\sqrt[3]{6nm^2}}{3m}$.

**Try It!** **3. What is the reduced radical form of each expression? Assume $x$ is positive.**

**a.** $\sqrt[5]{\frac{7}{16x^3}}$ $\qquad$ **b.** $\sqrt[4]{27x^2} \cdot \sqrt{3x}$

## EXAMPLE 4 Add and Subtract Radical Expressions

**A. What is the sum of $\sqrt{20} - \sqrt[3]{16} + \sqrt[3]{250} - \sqrt{5}$?**

**Like radicals** have the same index and the same radicand. Only like radicals can be combined with addition and subtraction.

$\sqrt{20} - \sqrt[3]{16} + \sqrt[3]{250} - \sqrt{5}$

$\sqrt{20} - \sqrt{5} - \sqrt[3]{16} + \sqrt[3]{250}$ ······ Group radical terms with like indices.

$2\sqrt{5} - \sqrt{5} - 2\sqrt[3]{2} + 5\sqrt[3]{2}$ ······ Simplify each radical term.

$(2 - 1)\sqrt{5} + (-2 + 5)\sqrt[3]{2}$ ······ Factor out the radicals with the inverse of the Distributive Property.

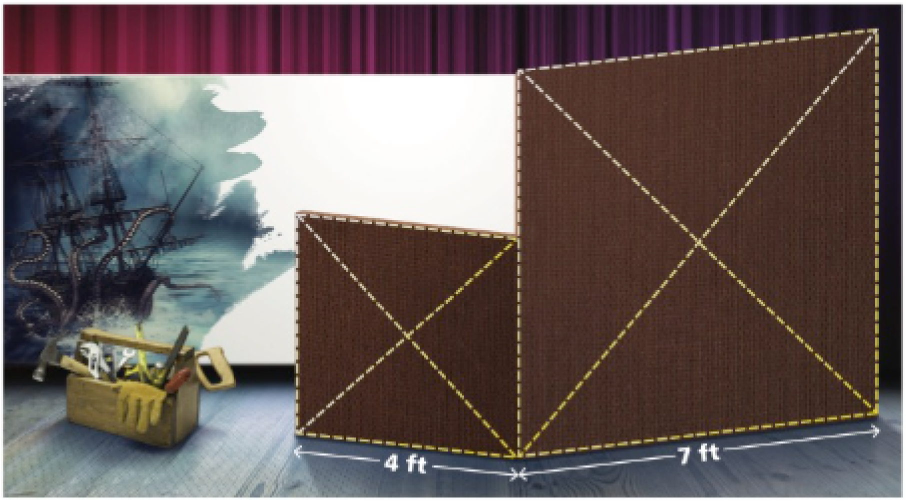
$\sqrt{5} + 3\sqrt[3]{2}$ ······ Combine like radical terms.

The expression $\sqrt{20} - \sqrt[3]{16} + \sqrt[3]{250} - \sqrt{5}$ is equivalent to $\sqrt{5} + 3\sqrt[3]{2}$.

**STUDY TIP**
Use the inverse of the Distributive Property to combine like radicals in the same way that you would combine like terms.

**APPLICATION**

**B. The design shows the boards needed for bracing the back of some set scenery. Will 75 ft of wood be enough for all of the bracing?**




Let $a$ represent the length of the diagonal for the smaller square and $b$ represent the length of the diagonal for the larger square. Determine the lengths of the diagonals:

$$4^2 + 4^2 = a^2 \qquad 7^2 + 7^2 = b^2$$
$$32 = a^2 \qquad 98 = b^2$$
$$4\sqrt{2} = a \qquad 7\sqrt{2} = b$$

Take the square root of both sides of the equation and simplify the radical. Since the context is length, the negative solution may be disregarded.

There are 4 edges that are 7 ft, 3 more edges that are 4 ft, and 2 diagonals each of $4\sqrt{2}$ ft and $7\sqrt{2}$ ft in length. Determine the total length of the boards:

$$4(7) + 3(4) + 2(4\sqrt{2}) + 2(7\sqrt{2}) = 40 + 22\sqrt{2}$$

Evaluating the expression, $40 + 22\sqrt{2} \approx 71.1$.
So 75 ft of wood is enough to make the bracing for the set scenery.

**Try It!** 4. How can you rewrite each expression in a simpler form?

a. $\sqrt[3]{2{,}000} + \sqrt{2} - \sqrt[3]{128}$

b. $\sqrt{20} - \sqrt{600} - \sqrt{125}$

## EXAMPLE 5 Multiply Binomial Radical Expressions

**What is the reduced radical form of each product?**

**A.** $\sqrt[3]{7}(2 - \sqrt[3]{49})$

$\sqrt[3]{7}(2) - \sqrt[3]{7}\sqrt[3]{49}$ ........ Use the Distributive Property.

$\sqrt[3]{7}(2) - \sqrt[3]{343}$ ........ Multiply radicands with like indices.

$2\sqrt[3]{7} - 7$ ........ Simplify each radical term.

The product is $2\sqrt[3]{7} - 7$.

**B.** $(2x - \sqrt{3})\,(2x - \sqrt{3})$

$4x^2 - 2x\sqrt{3} - 2x\sqrt{3} + \sqrt{9}$ ........ Expand the product.

$4x^2 - 4x\sqrt{3} + 3$ ........ Combine like terms.

The product is $4x^2 - 4x\sqrt{3} + 3$.

**STUDY TIP**
Recall that there are different methods for expanding the product of binomial factors.

**Try It!** **5.** Multiply.

**a.** $(x - \sqrt{10})(x + \sqrt{10})$

**b.** $\sqrt{6}(5 + \sqrt{3})$

## EXAMPLE 6 Rationalize a Binomial Denominator

**How can you rewrite $\frac{1}{2 + \sqrt{5}}$ without a radical in the denominator?**

**STUDY TIP**
The product of conjugates is $a^2 - b^2$, which eliminates radicals from the denominator.

To rationalize a denominator that has a binomial denominator, multiply by the conjugate of the denominator.

$\frac{1}{2 + \sqrt{5}} \cdot \frac{2 - \sqrt{5}}{2 - \sqrt{5}}$ ........ Multiply the numerator and denominator by the conjugate of the denominator.

$\frac{2 - \sqrt{5}}{4 - 5}$ ........ Multiply the numerators and the denominators.

$\frac{2 - \sqrt{5}}{-1}$ ........ Subtract the terms in the denominators.

$\sqrt{5} - 2$ ........ Simplify.

$\frac{1}{2 + \sqrt{5}}$ can be rewritten as $\sqrt{5} - 2$.

**Try It!** **6.** What is the reduced radical form of each expression?

**a.** $\frac{5 - \sqrt{2}}{2 - \sqrt{3}}$

**b.** $\frac{-4x}{1 - \sqrt{x}}$

## CONCEPT SUMMARY Properties of Radicals

| | Product Property of Radicals | Quotient Property of Radicals | Rationalize the Denominator |
|---|---|---|---|
| **WORDS** | The *n*th root of a product is equal to the product of the *n*th roots of the factors. | The *n*th root of a quotient is equal to the quotient of the *n*th roots of the factors. | To rationalize the denominator of an expression, multiply by the conjugate of the denominator. |
| **ALGEBRA** | $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ | $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ | $\frac{3}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{3\sqrt{x}}{x}$ |
| **NUMBERS** | $\sqrt[3]{2} \cdot \sqrt[3]{20} = \sqrt[3]{40}$ <br> $\sqrt[3]{40} = \sqrt[3]{8} \cdot \sqrt[3]{5} = 2\sqrt[3]{5}$ | $\sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$ | $\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$ |

Using Properties of Radicals

**SIMPLIFY**

$$\sqrt[4]{32x^9} = \sqrt[4]{16x^8} \cdot \sqrt[4]{2x}$$
$$= 2x^2\sqrt[4]{2x}$$

Find factors that have a perfect 4th root.

$$\frac{5}{x-\sqrt{8}} \cdot \frac{x+\sqrt{8}}{x+\sqrt{8}} = \frac{5(x+\sqrt{8})}{x^2 + x\sqrt{8} - x\sqrt{8} - 8} = \frac{5x + 5\sqrt{8}}{x^2 - 8}$$

Since the denominator is a binomial, multiply the numerator and the denominator by the conjugate of the denominator.

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can properties of exponents and radicals be used to rewrite radical expressions?

2. **Vocabulary** How can you determine if a radical expression is in reduced form?

3. **Use Patterns and Structure** Explain why $(-64)^{\frac{1}{3}}$ equals $-64^{\frac{1}{3}}$ but $(-64)^{\frac{1}{2}}$ does not equal $-64^{\frac{1}{2}}$.

4. **Error Analysis** Explain the error in Julie's work in rewriting the radical expression.
$\sqrt{-3} \cdot \sqrt{-12} = \sqrt{-3(-12)} = \sqrt{36} = 6$

## Do You KNOW HOW?

**What is the reduced radical form of each expression? Assume variables are positive.**

5. $49^{\frac{3}{4}} \cdot 49^{\frac{-1}{4}}$

6. $\left(\frac{a^2b^8}{a^{\frac{1}{3}}}\right)^{\frac{3}{4}}$

7. $\sqrt[4]{1{,}024x^9y^{12}}$

8. $\sqrt[3]{\frac{4}{9m^2}}$

9. $\sqrt{63} - \sqrt{700} - \sqrt{112}$

10. $\sqrt{5}(6 + \sqrt{2})$

11. $\frac{3}{\sqrt{6}}$

12. $\frac{\sqrt{7}}{\sqrt{5} + 3}$

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

13. **Apply Math Models** In the expression $PV^{\frac{4}{3}}$, $P$ represents the pressure and $V$ represents the volume of a sample of a gas. Evaluate the expression for $P = 7$ and $V = 8$.

14. **Check for Reasonableness** Describe the possible values of $k$ such that $\sqrt{32} + \sqrt{k}$ can be rewritten as a single term.

15. **Error Analysis** Explain why the following work is incorrect. Find the correct answer.

$$5\left(4 - 5^{\frac{1}{2}}\right) = 5(4) - 5\left(5^{\frac{1}{2}}\right)$$
$$= 20 - 25^{\frac{1}{2}}$$
$$= 15$$

16. **Communicate and Justify** Discuss the advantages and disadvantages of first rewriting $\sqrt{27} + \sqrt{48} + \sqrt{147}$ in order to estimate its decimal value.

17. **Higher Order Thinking** Write $\sqrt{\frac{4}{5}}$ in two different ways, one where the numerator is simplified and another where the denominator is rationalized.

18. **Communicate and Justify** Justify each step used in simplifying the expression below.

$$\left(\frac{a^2}{a^{\frac{3}{4}}}\right)^{\frac{1}{5}} = \left(a^{2-\frac{3}{4}}\right)^{\frac{1}{5}}$$
$$= \left(a^{\frac{5}{4}}\right)^{\frac{1}{5}}$$
$$= a^{\frac{1}{4}}$$
$$= \sqrt[4]{a}$$

### PRACTICE

**What is the reduced radical form of each expression? Assume variables are positive.** SEE EXAMPLE 1

19. $\left(3x^{\frac{1}{2}}\right)\left(4x^{\frac{2}{3}}\right)$

20. $2b^{\frac{1}{2}}\left(3b^{\frac{1}{2}}c^{\frac{1}{3}}\right)^2$

21. $\left(x^{\frac{1}{2}} \cdot x^{\frac{5}{12}}\right)^4 \div x^{\frac{2}{3}}$

22. $\left(\frac{16c^{14}}{81d^{18}}\right)^{\frac{1}{2}}$

**What is the reduced radical form of each expression? Assume variables are positive.** SEE EXAMPLE 2

23. $\sqrt[3]{250y^2z^4}$

24. $\sqrt[4]{256v^7w^{12}}$

25. $\sqrt{\frac{48x^3}{3xy^2}}$

26. $\sqrt{\frac{56x^5y^5}{7xy}}$

27. $\sqrt[3]{216m}$

28. $\sqrt[3]{\frac{250f^7g^3}{2f^2g}}$

**What is the reduced radical form of each expression? Assume variables are positive.** SEE EXAMPLE 3

29. $\sqrt{x^5y^5} \cdot 3\sqrt{2x^7y^6}$

30. $\sqrt[3]{\frac{18n^2}{24n}}$

31. $\sqrt[3]{3x^2} \cdot \sqrt[3]{x^2} \cdot \sqrt[3]{9x^3}$

32. $\sqrt{\frac{162a}{6a^3}}$

33. $\sqrt[5]{2pq^6} \cdot 2\sqrt{2p^3q}$

34. $\sqrt[3]{\frac{x^2}{9y}}$

35. $\sqrt[3]{6} \cdot \sqrt[3]{16}$

36. $\sqrt[4]{\frac{2}{5x}}$

**What is the reduced radical form of each expression? Assume variables are positive.** SEE EXAMPLE 4

37. $4\sqrt[3]{81} - 2\sqrt[3]{72} - \sqrt[3]{24}$

38. $6\sqrt{45y^2} - 4\sqrt{20y^2}$

39. $3\sqrt{12} - \sqrt{54} + 7\sqrt{75}$

40. $\sqrt{32h} + 4\sqrt{98h} - 3\sqrt{50h}$

**Multiply.** SEE EXAMPLE 5

41. $(3\sqrt{p} - \sqrt{5})(\sqrt{p} + 5\sqrt{5})$

42. $(4m - \sqrt{3})(4m - \sqrt{3})$

43. $(3\sqrt{2} + 8)(3\sqrt{2} - 8)$

44. $\sqrt[3]{3}(5\sqrt[3]{9} - 4)$

**What is the reduced radical form of each expression?** SEE EXAMPLE 6

45. $\frac{4}{1 - \sqrt{3}}$

46. $\frac{20}{3 + \sqrt{2}}$

47. $\frac{3 + \sqrt{8}}{2 - 2\sqrt{8}}$

48. $\frac{-2x}{3 + \sqrt{x}}$

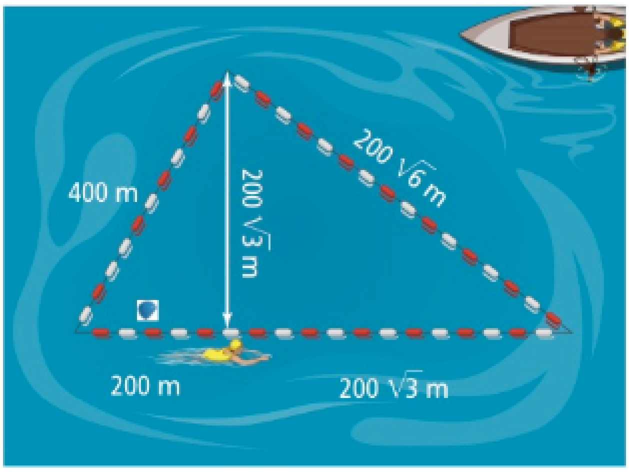
# PRACTICE & PROBLEM SOLVING

## APPLY

**49. Apply Math Models** A triangular swimming area is marked off by a rope.

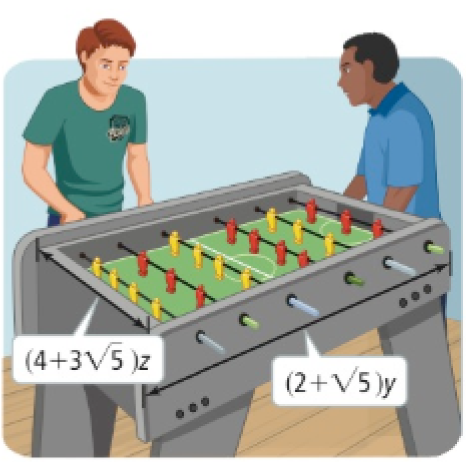
a. If a woman swims around the perimeter of the swimming area, how far will she swim?

b. What is the area of the roped off section?




**50. Use Patterns and Structure** The interest rate $r$ required to increase your investment $p$ to the amount $a$ in $m$ months is found by $r = \left(\frac{a}{p}\right)^{\frac{1}{m}} - 1$. What interest rate would be required to increase your investment of \$3,600 to \$6,400 over 7 months? Round your answer to the nearest tenth of a percent.

**51. Use Patterns and Structure** The length of a rectangle is $(2 + \sqrt{5})y$. The width is $(4 + 3\sqrt{5})z$. What is the area of the rectangle?




**52. Apply Math Models** A rectangular boardroom table is $\sqrt{440}$ ft by $\sqrt{20}$ ft. Find its area.

## ASSESSMENT PRACTICE

**53.** Aaron is rewriting $\frac{1+\sqrt{3}}{5-\sqrt{3}}$ into reduced radical form. Select all the steps that Aaron would have used to show his work. NS0.1.3

☐ A. $\frac{6 + 4\sqrt{3} - 3}{25 + 9}$

☐ B. $\frac{5 + \sqrt{3} + 5\sqrt{3} + \sqrt{9}}{25 + 5\sqrt{3} - 5\sqrt{3} - \sqrt{9}}$

☐ C. $\frac{4 + 3\sqrt{3}}{11}$

☐ D. $\frac{8 + 6\sqrt{3}}{28}$

☐ E. $\frac{5 + 6\sqrt{3} + 3}{25 - 3}$

**54. SAT/ACT** Which expression cannot be rewritten as $-10$?

Ⓐ $\sqrt{25} \cdot \sqrt[3]{-8}$

Ⓑ $\sqrt[3]{-125} \cdot \sqrt[4]{16}$

Ⓒ $-\sqrt[3]{1{,}000}$

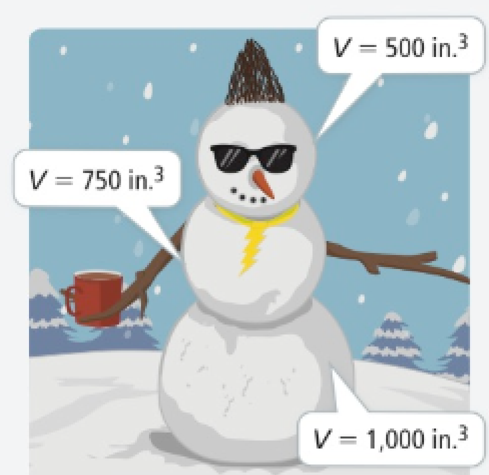
Ⓓ $-\sqrt{25} \cdot \sqrt[5]{-32}$

Ⓔ $\sqrt{4} \cdot -\sqrt[3]{125}$

**55. Performance Task** The volume of a sphere of radius $r$ is $V = \frac{4}{3}\pi r^3$.

**Part A** Use the formula to find $r$ in terms of $V$. Rationalize the denominator.

**Part B** A snowman is made using three spherical snowballs. The top snowball for the head has a volume of 500 in.$^3$. What is the diameter of the top snowball?




**Part C** The volumes of the other two snowballs are 750 in.$^3$ and 1,000 in.$^3$. How tall is the snowman?

# 5-3 Graphing Radical Functions

**I CAN…** graph and transform radical functions.

**VOCABULARY**

- radical function

MA.912.AR.7.2–Solve and graph mathematical and real-world problems that are modeled with square root or cube root functions. Interpret key features and determine constraints in terms of the context. Also AR.7.3, F.1.7, F.2.2, F.2.3, F.2.5

MA.K12.MTR.2.1, MTR.5.1, MTR.7.1

## EXPLORE & REASON

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**Consider the formula for the area of a square: $A = s^2$**

**A.** Graph the function that represents area as a function of side length.

**B.** On the same set of axes, graph the function that represents side length as a function of area.

**C.** Use Patterns and Structure How are the two graphs related?

## ESSENTIAL QUESTION

**How can you use what you know about transformations of functions to graph radical functions?**

## EXAMPLE 1 Graph Square Root and Cube Root Functions

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**Graph the following functions. What are the domain and range of each function? Is the function increasing or decreasing?**

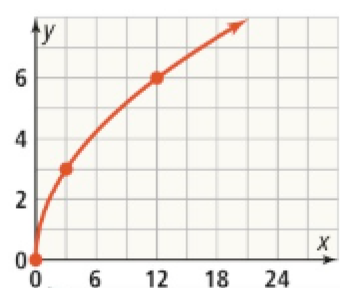
**A.** $f(x) = \sqrt{3x}$

Make a table of values and graph.

| $x$ | 0 | 3 | 12 | 27 |
|---|---|---|---|---|
| $y$ | 0 | 3 | 6 | 9 |

For ease, choose $x$-values that make the radicand a perfect square.

For a square root function, the radicand cannot be negative, so the domain of the function is $\{x \mid x \geq 0\}$.




Radical functions do not have a horizontal asymptote; they grow without limit.

The graph has a minimum at $x = 0$.

**USE PATTERNS AND STRUCTURE**
How does the graph of the function shown compare to the graph of its parent function?

A square root always returns a positive value or 0, so the range is $\{y \mid y \geq 0\}$. As $x$ increases, $y$ increases, so the function is increasing.

**B.** $g(x) = \sqrt[3]{2x}$

Make a table of values, and graph.

| $x$ | −13.5 | −4 | 0 | 4 | 13.5 |
|---|---|---|---|---|---|
| $y$ | −3 | −2 | 0 | 2 | 3 |

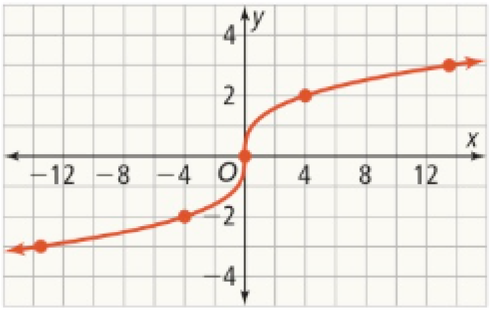
The radicand in a cube root function can be positive or negative.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

There are no restrictions on the radicand of a cube root function, so the domain and range of the function is all real numbers.

The function is increasing over the entire domain.




**GENERALIZE**
*Odd functions* are symmetric about the origin. The cube root parent is an odd function. *Even functions* are symmetric about the $y$-axis.

**Try It!** **1.** Graph the following functions. What are the domain and range of each function? Is the function increasing or decreasing?

**a.** $f(x) = \sqrt{x-5}$ **b.** $g(x) = \sqrt[3]{x+1}$

## CONCEPT Radical Function

A **radical function** is a function of the form $f(x) = a\sqrt[n]{x-h} + k$, where

*a* determines a vertical stretch or compression.

*h* determines a horizontal translation.

*k* determines a vertical translation.

## EXAMPLE 2 Graph a Transformation of a Radical Function

**Graph $g(x) = 2\sqrt{x+3} + 5$. What transformations map the graph of $f(x) = \sqrt{x}$ to the graph of $g$? How do the domain and range of $g$ differ from those of $f$?**

**STUDY TIP**
The radicand $x + 3$ can be written as $x - (-3)$, which relates more directly to the general form of a radical function. The horizontal translation, $h$, is $-3$, so the graph of the function will shift (or translate) *left* 3 units.

**Step 1** Identify the parameters in $g(x) = 2\sqrt{x+3} + 5$.

$$g(x) = 2\sqrt{x-(-3)} + 5$$

$a = 2$, so the function is stretched vertically by a factor of 2.

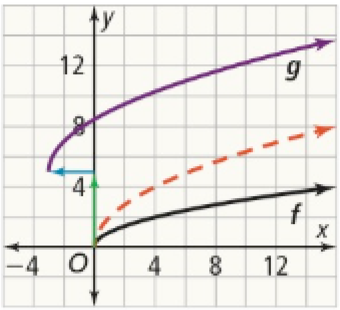
$h = -3$, so the graph is translated left 3 units.

$k = 5$, so the graph is translated up 5 units.

**Step 2** Graph the parent function $f$, and use it as a guide to graph $g$.

| $x$ | 0 | 1 | 4 | 9 | 16 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 1 | 2 | 3 | 4 |

This stretched graph is then translated 3 units left and 5 units up to show the graph of $g$.




The graph of $f$ is stretched vertically by a factor of 2, so each $y$-value is twice as far from the $x$-axis.

The domain of $f$ is $\{x \mid x \geq 0\}$, while the domain of $g$ is $\{x \mid x \geq -3\}$.
The range of $f$ is $\{y \mid y \geq 0\}$, while the range of $g$ is $\{y \mid y \geq 5\}$.

CONTINUED ON THE NEXT PAGE

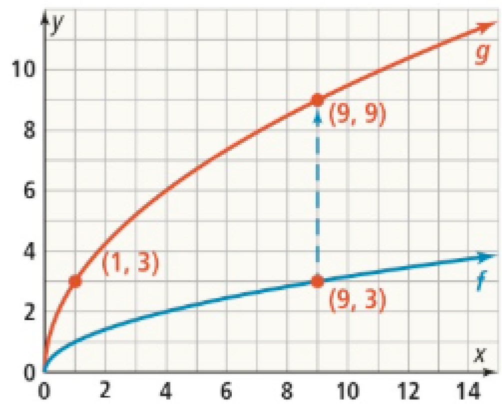
**Try It!** 2. Graph $g(x) = \frac{1}{2}\sqrt{x-1} - 3$. What transformations of the graph of $f(x) = \sqrt{x}$ produce the graph of $g$? What is the effect of the transformations on the domain and range of $g$?

CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Rewrite Radical Functions to Identify Transformations

**STUDY TIP**
You may recall that the parameter $k$ in $f(kx)$ determines horizontal stretch or compression. In this example, $g(x)$ can be described as either a horizontal compression or a vertical stretch. Both transformations of the parent function result in the same graph.

**How can you rewrite the following radical functions to identify their transformations from the parent graph of $f(x) = \sqrt{x}$?**




**A.** $g(x) = \sqrt{9x}$

Use the properties of radicals to rewrite the function and identify a vertical change.

$$g(x) = \sqrt{9x} \quad \text{Write the original equation.}$$
$$= \sqrt{9} \cdot \sqrt{x} \quad \text{Use the Product Property of Radicals.}$$
$$= 3\sqrt{x} \quad \text{Simplify.}$$

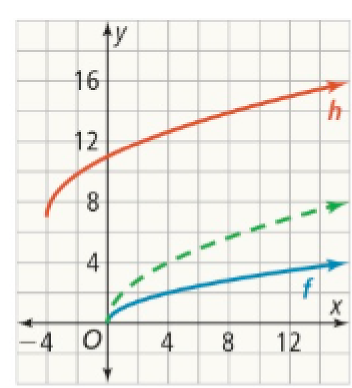
$a = 3$, so the graph of $g$ is stretched vertically by a factor of 3 from the parent graph. Both $h$ and $k$ are 0, so there is no translation of the graph.

**B.** $h(x) = \sqrt{4x + 16} + 7$

Rewrite the function in the form $h(x) = a\sqrt{x - h} + k$.

$$h(x) = \sqrt{4x + 16} + 7 \quad \text{Write the original equation.}$$
$$= \sqrt{4(x + 4)} + 7 \quad \text{Factor the radicand.}$$
$$= \sqrt{4} \cdot \sqrt{x + 4} + 7 \quad \text{Use the Product Property of Radicals.}$$
$$= 2\sqrt{x + 4} + 7 \quad \text{Simplify.}$$

The graph of $h(x)$ is a vertical stretch of the parent function by a factor of 2, followed by a translation of 4 units to the left, and a translation of 7 units up.

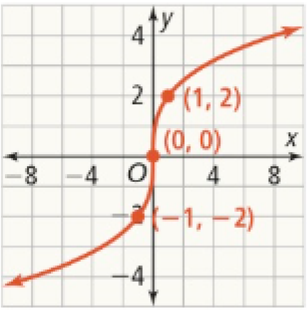



**Try It!** 3. What transformations of the parent graph of $f(x) = \sqrt{x}$ produce the graphs of the following functions?

**a.** $m(x) = \sqrt{7x - 3.5} - 10$ **b.** $j(x) = -2\sqrt{12x} + 4$

## EXAMPLE 4 Write an Equation of a Transformation

**What radical function is represented in the graph?**

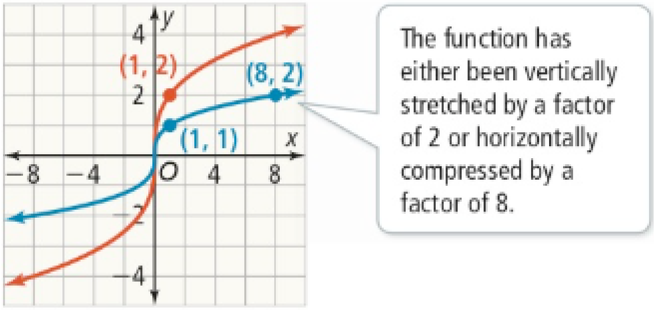



Compare the graph to the parent graph of $f(x) = \sqrt[3]{x}$.

**Step 1** Check to see if any vertical or horizontal translations have been performed.

Since $f(-x) = -f(x)$, you know that the function is odd. Like the graph of the parent function, the graph of this function is symmetric about the origin, so no translation has been performed.

**Step 2** Check for a vertical and horizontal stretch.




**STUDY TIP**
Vertical stretch is written as $f(x) = k\sqrt{x}$, and horizontal compression is written as $f(x) = \sqrt{kx}$.

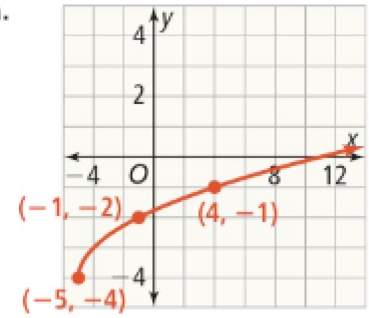
Either transformation maps the initial graph to the same resulting graph.

**Step 3** Identify the transformation.

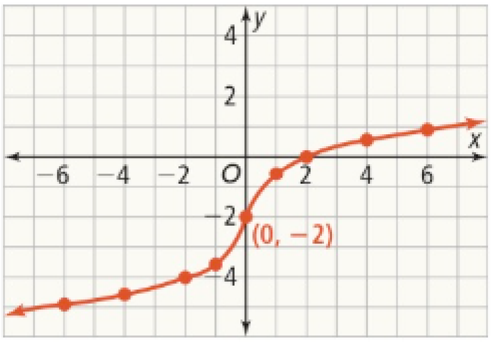
The function $g$ can be written as $g(x) = 2\sqrt[3]{x}$ or $g(x) = \sqrt[3]{8x}$.

**Try It!** **4.** What radical function is represented in each graph below?

a.



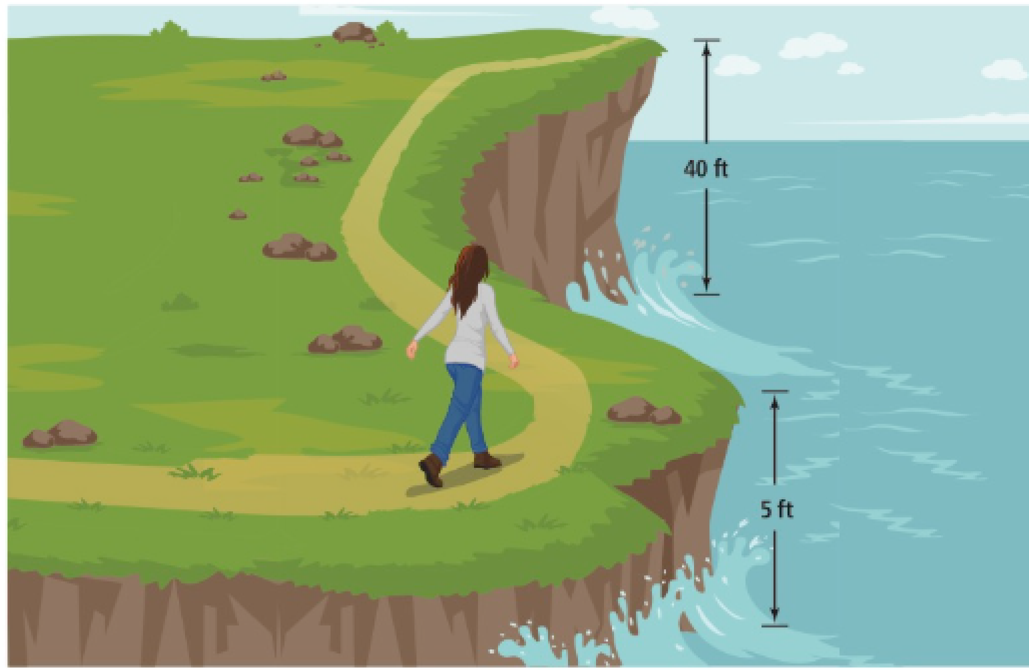

b.



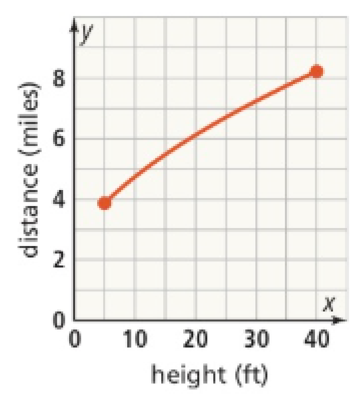
APPLICATION

## EXAMPLE 5 Interpret a Radical Function Model

**When Sasha looks out to the sea, the visibility in miles from a path on a cliff can be calculated using the function $d(x) = \sqrt{1.5(x + 5)}$, where $x$ is the height in feet above sea level of the path. Sasha walks along through elevations ranging from 5 ft to 40 ft above sea level. Will Sasha be able to see a ship that is anchored 8.5 miles offshore?**




**Formulate** Graph the function. Since the square root function is increasing, the maximum distance that Sasha can see occurs at the maximum value of $x$, $x = 40$.




**Compute** Find $d(40)$ to find the maximum distance Sasha can see.

$$\begin{aligned} d(40) &= \sqrt{1.5(45)} \\ &= \sqrt{67.5} \\ &\approx 8.2 \end{aligned}$$

**Interpret** At the highest point along the path, Sasha can see about 8.2 miles offshore. Sasha will not be able to see the ship anchored 8.5 miles away.

**Try It!** **5.** Sasha's brother walks along a separate path ranging from 8 ft to 46 ft in elevation. His visibility can be calculated by the function $d(x) = \sqrt{(1.5(x + 5.7))}$. Will Sasha's brother be able to see a ship 8.5 miles offshore?

## CONCEPT SUMMARY Use Transformations to Graph Radical Functions

**ALGEBRA** Understand how the values of a radical function transform the graph of the parent function.

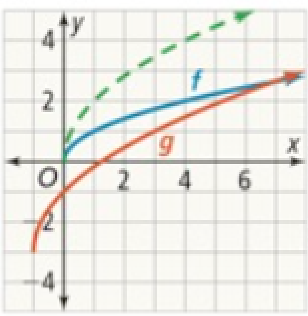
$$f(x) = a\sqrt[n]{x-h} + k$$

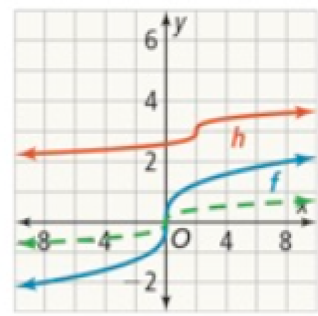
$a$ determines a vertical stretch or compression.

$h$ determines a horizontal translation.

$k$ determines a vertical translation.

**GRAPHS** Understand the relationship between the graph of the parent function and the graph of the radical function.




$f(x) = \sqrt{x}$ is the parent square root function.

$g(x) = 2\sqrt{x+1} - 3$ is the result of a vertical stretch by a factor of 2, a translation 1 unit left, and a translation 3 units down from the parent function.

$f(x) = \sqrt[3]{x}$ is the parent cube root function.

$h(x) = \frac{1}{3}\sqrt[3]{x-2} + 3$ is the result of a vertical compression by a factor of 3, a translation 2 units right, and a translation 3 units up from the parent function.

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you use what you know about transformations of functions to graph radical functions?

2. **Error Analysis** Parker said the graph of the radical function $g(x) = -\sqrt{x+2} - 1$ is a translation 2 units left and 1 unit down from the parent function $f(x) = \sqrt{x}$. Describe and correct the error.

3. **Generalize** What effect does $a$ have on the graph of $f(x) = a\sqrt{x}$?

4. **Represent and Connect** Sketch the graph represented by the values in the table. How does end behavior effect your graph?

| $x$ | −14 | −7 | −6 | −5 | 2 |
|---|---|---|---|---|---|
| $h(x)$ | 6 | 5 | 4 | 3 | 2 |

## Do You KNOW HOW?

**Graph each function. Then identify its domain and range.**

5. $f(x) = \sqrt{x-2}$

6. $f(x) = \sqrt[3]{x+2}$

7. $f(x) = \sqrt{x+1} - 2$

8. $f(x) = \sqrt[3]{x-3} + 2$

9. $f(x) = 3\sqrt{x-5}$

10. $f(x) = \frac{1}{2}\sqrt[3]{x} + 1$

11. The volume of a cube is a function of the cube's side length. The function can be written as $V(s) = s^3$, where $s$ is the length of the cube's edge and $V$ is the volume.

    a. Express a cube's edge length as a function of its volume, $s(V)$.

    b. Graph $V(s)$ and $s(V)$. What are the domain and range of the functions? Explain.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

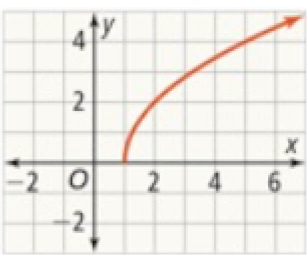
12. **Represent and Connect** What is the domain and range of the radical function $h(x) = \sqrt{x + a} + b$? Is the function increasing or decreasing? Explain.

13. **Apply Math Models** The graph of a cube root function has a horizontal translation that is three times the vertical translation. The vertical translation is negative.

    a. Write a function, $g$, that has these attributes.

    b. Graph your function and the parent function, $f$, to verify it is correct.

14. **Error Analysis** Helena is trying to write a radical function that is represented by the graph below. Describe and correct the error Helena made in writing the radical function.




$f(x) = \sqrt{x - 1}$ ✗

15. **Higher Order Thinking** Rewrite the radical function $g(x) = \sqrt[3]{8x + 64} - 3$ to identify the transformations from the parent graph of $f(x) = \sqrt[3]{x}$. Explain how you rewrote the radical function.

16. **Communicate and Justify** The parent function $f(x) = \sqrt{x}$ and a transformation of the parent function, $g(x)$, are reflections of each other over the $x$-axis. Write the function $g(x)$.

17. **Mathematical Connections** How do the transformations of a radical function compare to the transformations of an absolute value function?

### PRACTICE

**Graph the following functions. State the domain and range. Is the function increasing or decreasing?** SEE EXAMPLE 1

18. $f(x) = \sqrt{x} + 2$

19. $f(x) = \sqrt[3]{x} - 4$

20. $f(x) = \sqrt[3]{x - 8}$

21. $f(x) = \sqrt{x + 6}$

22. Graph $f(x) = \sqrt[3]{x}$ and $g(x) = 3\sqrt[3]{x + 9} - 8$. What transformations of the graph of $f$ produce the graph of $g$? What effect do the transformations have on the domain and range of $g$? SEE EXAMPLE 2

**Rewrite the following radical functions to identify their transformations from the parent graph $f(x) = \sqrt{x}$.** SEE EXAMPLE 3

23. $f(x) = \sqrt{16x}$

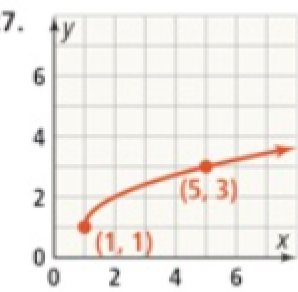
24. $f(x) = \sqrt{25x + 75}$

25. $f(x) = \sqrt{9x - 45}$

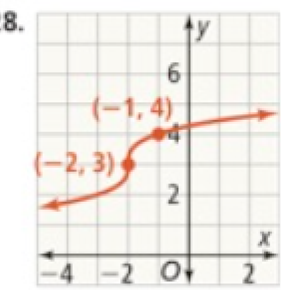
26. $f(x) = \sqrt{4x - 24} - 6$

**What radical function is represented in each graph?** SEE EXAMPLE 4

27.




28.




29. The hull speed, $y$, measured in knots, of a sailboat can be estimated by the function $y = 1.34\sqrt{x}$, where $x$ is the waterline length of the sailboat, in feet. Luis works at a sailboat rental business with boats that have a waterline length between 25 ft and 64 ft. SEE EXAMPLE 5

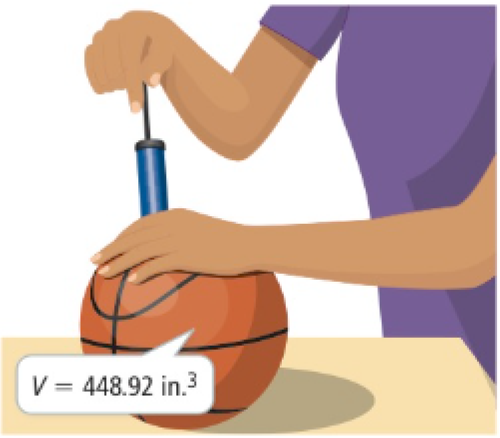
    a. Graph the relationship between the hull speed of a sailboat and its waterline length.

    b. What are the minimum and maximum hull speeds of the sailboats at the rental business?

## PRACTICE & PROBLEM SOLVING

### APPLY

30. **Analyze and Persevere** The radius of a sphere can be found using the function $r = \sqrt[3]{\frac{3V}{4\pi}}$, where $V$ is the volume of the sphere. Rim filled a basketball with 448.92 in.$^3$ of air.




a. Graph the function.

b. Identify the domain and range of the graph.

c. Do the domain and range make sense in this context? Explain.

d. What is the length of the radius of the basketball?

31. A formula for calculating the distance to the horizon is $d = \sqrt{\frac{h}{0.078}}$, where $d$ is the distance to the horizon, in kilometers, and $h$ is the height above the surface, in meters.


a. Graph the function.

b. **Represent and Connect** What is your height above the surface if you can see a distance of 25.3 km to the horizon?

### ASSESSMENT PRACTICE

32. Select the function that is odd. F.1.9

Ⓐ $f(x) = 5\sqrt{x - 10} - 12$

Ⓑ $f(x) = \frac{1}{4}\sqrt[3]{x}$

Ⓒ $f(x) = \frac{1}{2}\sqrt{x + 8} - 1$

Ⓓ $f(x) = 9\sqrt[3]{x - 7} + 8$

33. **SAT/ACT** Which function has a graph with domain $x \geq -1$ and range $y \geq -2$?

Ⓐ $f(x) = \sqrt{x - 1} + 2$ Ⓑ $f(x) = \sqrt[3]{x + 1} - 2$

Ⓒ $f(x) = \sqrt[3]{x - 1} + 2$ Ⓓ $f(x) = \sqrt{x + 1} - 2$

34. **Performance Task** The table shows the domain and range of the function $f(x) = \sqrt[n]{x}$ for different values of $n$, where $x$ is a positive real number.

| $n$ | Domain of $f(x) = \sqrt[n]{x}$ | Range of $f(x) = \sqrt[n]{x}$ |
|---|---|---|
| 1 | All real numbers | All real numbers |
| 2 | $x \geq 0$ | $y \geq 0$ |
| 3 | All real numbers | All real numbers |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |

**Part A** Identify the domain and range of the function $f(x) = \sqrt[n]{x}$ when $n = 4, 5, 6, 7$, and 8.

**Part B** Make a conjecture about the values of $n$ that gives a domain and range of all real numbers.

**Part C** Make a conjecture about the values of $n$ that gives a domain of $x \geq 0$ and a range of $y \geq 0$.

# 5-4 Solving Radical Equations

**I CAN…** solve radical equations and inequalities.

**VOCABULARY**

- extraneous solution

MA.912.AR.7.1–Solve one-variable radical equations. Interpret solutions as viable in terms of context and identify any extraneous solutions. Also AR.7.3

MA.K12.MTR.1.1, MTR.4.1, MTR.7.1

## EXPLORE & REASON

**A.** Solve $3(a + 1)^2 + 2 = 11$. Use at least two different methods.

**B.** Try each of the methods you used in part (a) to solve $3\sqrt{(a + 1)} + 2 = 11$.

**C. Generalize** Which of the methods is better suited for solving an equation with a radical? What problems arise when using the other method?

## ESSENTIAL QUESTION

**How can you solve equations that include radicals or rational exponents?**

## EXAMPLE 1 Solve an Equation With One Radical

**A. Solve the radical equation $\sqrt{x + 5} - 1 = 3$.**

To solve this equation, you can isolate the radical. Then you can square both sides of the equation to eliminate the radical and solve for $x$.

$\sqrt{x + 5} - 1 = 3$ ........ Write the original equation.

$\sqrt{x + 5} = 4$ ........ Add 1 to each side.

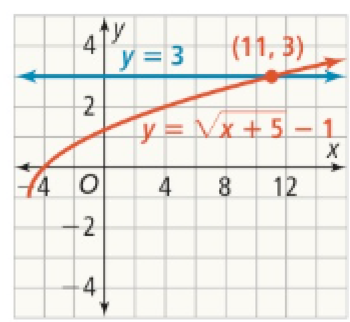
$(\sqrt{x + 5})^2 = 4^2$ ........ Square both sides to eliminate the radical.

$x + 5 = 16$ ........ Simplify.

$x = 11$ ........ Subtract 5 from each side.

So the solution to the radical equation is $x = 11$.

Check your answer by substituting 11 for $x$ in the original equation:
$\sqrt{11 + 5} - 1 = \sqrt{16} - 1 = 4 - 1 = 3$ ✓
You can also check by graphing $y = \sqrt{x + 5} - 1$ and $y = 3$ on the same coordinate axes. The graphs of the equations intersect at (11, 3).




**GENERALIZE**
When you square both sides of an equation, you are multiplying each side by the same quantity. The expression for the quantity differs, but $\sqrt{x + 5}$ and 4 are equal.

**B. Solve the radical equation $\sqrt[3]{x} + 2 = 4$.**

$\sqrt[3]{x} + 2 = 4$ ........ Write the original equation.

$\sqrt[3]{x} = 2$ ........ Subtract 2 from each side to isolate the radical.

$(\sqrt[3]{x})^3 = 2^3$ ........ Cube both sides to eliminate the cube root.

$x = 8$ ........ Simplify.

So the solution to the radical equation is $x = 8$.

Check your answer by substituting 8 for $x$ in the original equation:
$\sqrt[3]{8} + 2 = 2 + 2 = 4$ ✓

**Try It!** **1.** Solve each radical equation.

**a.** $\sqrt{x - 2} + 3 = 5$ **b.** $\sqrt[3]{x - 1} = 2$

APPLICATION

## EXAMPLE 2 Rewrite a Formula

**The suspension cables from the Golden Gate Bridge's towers are farther above the roadway near the towers and closer to the roadway near the middle of the bridge. The distance from the middle of the bridge, *x*, in feet, is related to the height of the suspension cable, *y*, in feet, by the equation $x = \frac{\sqrt{y - 220}}{0.010583}$. Where are the height of the cable and the distance from the middle of the bridge equal to each other?**


To find where the height of the cable and the distance from the middle of the bridge are equal, ***x*** and ***y*** must be equal.

| | |
|---|---|
| $x = \frac{\sqrt{y - 220}}{0.010583}$ | Write the original function. |
| $y = \frac{\sqrt{(y - 220)}}{0.010583}$ | Substitute. |
| $0.010583y = \sqrt{y - 220}$ | Multiply both sides by 0.010583. |
| $(0.010583y)^2 = \left(\sqrt{y - 220}\right)^2$ | Square both sides. |
| $0.000112y^2 \approx y - 220$ | Simplify. |
| $0.000112y^2 - y + 220 \approx 0$ | Set equal to zero. |
| $y \approx \frac{1 \pm \sqrt{(-1)^2 - 4(0.000112)(220)}}{2(0.000112)}$ | Use the quadractic equation. |
| $y \approx 8702 \text{ or } 226$ | Simplify. |

**COMMON ERROR**

When squaring a term that includes a coefficient and a variable, remember to square both parts of the term. If you have to round the coefficient after squaring, be sure to use the approximately equal sign.

Since the height of each tower is 746 above the water, 8,700 feet is not a viable solution.

The cable is about 226 ft above the bridge's roadway when you are about 226 ft from the middle of the bridge.

**Try It!** **2.** The speed, $v$, of a vehicle in relation to its stopping distance, $d$, is represented by the equation $v = 3.57\sqrt{d}$. What is the equation for the stopping distance in terms of the vehicle's speed?

CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Identify an Extraneous Solution

**A. Solve the radical equation $\sqrt{3x-2} = x - 4$.**

| | |
|---|---|
| $\sqrt{3x-2} = x - 4$ | Write the original equation. |
| $\left(\sqrt{3x-2}\right)^2 = (x-4)^2$ | Square both sides. |
| $3x - 2 = x^2 - 8x + 16$ | Simplify the left side; expand the right side using the Distributive Property. |
| $0 = x^2 - 11x + 18$ | Write in standard form. |
| $0 = (x-9)(x-2)$ | Factor. |
| $x = 9$ or $x = 2$ | Use the Zero-Product Property to solve. |

Check the potential solutions by substituting the solutions for $x$ in the original equation:

| | |
|---|---|
| $\sqrt{3x-2} = x - 4$ | $\sqrt{3x-2} = x - 4$ |
| $\sqrt{3(9)-2} = 9 - 4$ | $\sqrt{3(2)-2} = 2 - 4$ |
| $\sqrt{25} = 5$ ✓ | $\sqrt{4} \neq -2$ ✗ |

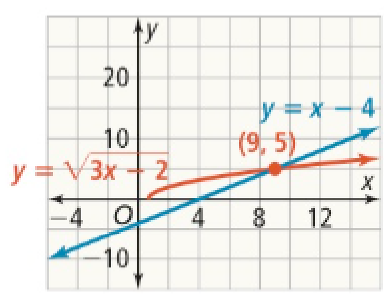
So 9 is the only solution to the equation and 2 is an extraneous solution.

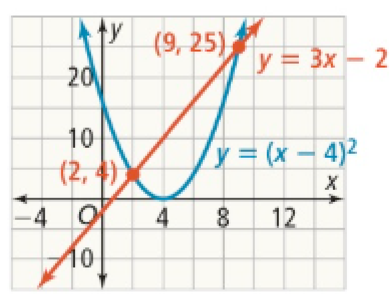
**VOCABULARY**
An **extraneous solution** is a solution of an equation *derived* from an original equation, but it is *not* a solution of the original equation.

**B. Why does this extraneous solution arise?**

Your first step in solving the equation in part (a) was to square both sides. You now have two equations. Are they equivalent equations?

Use graphs to represent both the original equation from part (a) and the equation you got after squaring both sides of the original equation:




The graphs of $y = \sqrt{3x-2}$ and $y = x - 4$ intersect at (9, 5), so the solution to $\sqrt{3x-2} = x - 4$ is $x = 9$.

The graphs of $3x - 2$ and $(x-4)^2$ intersect at (2, 4) and (9, 25), so the solution to $\left(\sqrt{3x-2}\right)^2 = (x-4)^2$ is $x = 2$ and $x = 9$.

This means that the equations $\sqrt{3x-2} = x - 4$ and $\left(\sqrt{3x-2}\right)^2 = (x-4)^2$ are not equivalent. Both graphs have an intersection point at $x = 9$, but the second graph also has an intersection point at $x = 2$. Squaring both sides of the original equation created an extraneous solution.

**Try It!** 3. Solve each radical equation. Identify any extraneous solutions.

a. $x = \sqrt{7x+8}$

b. $x + 2 = \sqrt{x+2}$

## EXAMPLE 4 Solve Equations With Rational Exponents

**A. What are the solutions to the equation $(x^2 + 5x + 5)^{\frac{5}{2}} = 1$?**

$$(x^2 + 5x + 5)^{\frac{5}{2}} = 1$$

$$\left((x^2 + 5x + 5)^{\frac{5}{2}}\right)^{\frac{2}{5}} = (1)^{\frac{2}{5}}$$

Raise both sides to the reciprocal power.

$$x^2 + 5x + 5 = 1$$

$$x^2 + 5x + 4 = 0$$

$$(x + 4)(x + 1) = 0$$

$$x + 4 = 0 \text{ or } x + 1 = 0$$

Use the Zero-Product Property.

$$x = -4 \text{ or } x = -1$$

**STUDY TIP**
An equation can have one solution, multiple solutions, or no solutions, so it's important to check all potential solutions.

Check for extraneous solutions.

$$((-4)^2 + 5(-4) + 5)^{\frac{5}{2}} \stackrel{?}{=} 1$$

$$(16 - 20 + 5)^{\frac{5}{2}} \stackrel{?}{=} 1$$

$$(1)^{\frac{5}{2}} \stackrel{?}{=} 1$$

$$1 = 1 \checkmark$$

$$((-1)^2 + 5(-1) + 5)^{\frac{5}{2}} \stackrel{?}{=} 1$$

$$(1 - 5 + 5)^{\frac{5}{2}} \stackrel{?}{=} 1$$

$$(1)^{\frac{5}{2}} \stackrel{?}{=} 1$$

$$1 = 1 \checkmark$$

This equation has two solutions, $x = -4$ and $x = -1$. There are no extraneous solutions.

**B. What is the solution to $(x + 18)^{\frac{3}{2}} = (x - 2)^3$?**

$$(x + 18)^{\frac{3}{2}} = (x - 2)^3$$

$$\left((x + 18)^{\frac{3}{2}}\right)^{\frac{2}{3}} = ((x - 2)^3)^{\frac{2}{3}}$$

Raise both sides to the reciprocal power.

$$x + 18 = (x - 2)^2$$

$$x + 18 = x^2 - 4x + 4$$

$$x^2 - 5x - 14 = 0$$

$$(x + 2)(x - 7) = 0$$

$$x + 2 = 0 \qquad x - 7 = 0$$

Use the Zero-Product Property.

$$x = -2 \qquad x = 7$$

Check for extraneous solutions.

$$(-2 + 18)^{\frac{3}{2}} \stackrel{?}{=} (-2 - 2)^3$$

$$16^{\frac{3}{2}} \stackrel{?}{=} (-4)^3$$

$$64 \neq -64 \text{ ✗}$$

$$(7 + 18)^{\frac{3}{2}} \stackrel{?}{=} (7 - 2)^3$$

$$25^{\frac{3}{2}} \stackrel{?}{=} 5^3$$

$$125 = 125 \checkmark$$

So 7 is the only solution to the equation, and $-2$ is an extraneous solution.

**Try It!** **4.** Solve each equation.

**a.** $(x^2 - 3x - 6)^{\frac{3}{2}} - 14 = -6$ **b.** $(x + 8)^2 = (x - 10)^{\frac{5}{2}}$

## EXAMPLE 5 Solve an Equation With Two Radicals

**Solve the radical equation $\sqrt{x+9} - \sqrt{2x} = 3$.**

When an equation has two radicals, start the solution process by isolating one radical.

$\sqrt{x+9} - \sqrt{2x} = 3$ ........ Write the original equation.

$\sqrt{x+9} = \sqrt{2x} + 3$ ........ Add $\sqrt{2x}$ to both sides to isolate $\sqrt{x+9}$.

$(\sqrt{x+9})^2 = (\sqrt{2x} + 3)^2$ ........ Square both sides.

$x + 9 = 2x + 6\sqrt{2x} + 9$ ........ Simplify the left side; expand the right side using the Distributive Property.

$0 = x + 6\sqrt{2x}$ ........ Simplify.

$-x = 6\sqrt{2x}$ ........ Isolate the radical expression.

$(-x)^2 = (6\sqrt{2x})^2$ ........ Square both sides again.

$x^2 = 72x$ ........ Simplify

$x^2 - 72x = 0$ ........ Write in standard form.

$x(x - 72) = 0$ ........ Factor completely.

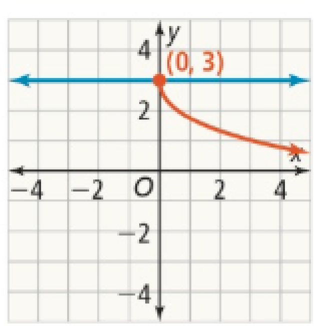
$x = 0$ or $x = 72$ ........ Use the Zero-Product Property.

Check the potential solutions by substituting 0 and 72 for $x$ in the original equation:

| | |
|---|---|
| $\sqrt{x+9} - \sqrt{2x} \stackrel{?}{=} 3$ | $\sqrt{x+9} - \sqrt{2x} \stackrel{?}{=} 3$ |
| $\sqrt{(0)+9} - \sqrt{2(0)} \stackrel{?}{=} 3$ | $\sqrt{(72)+9} - \sqrt{2(72)} \stackrel{?}{=} 3$ |
| $\sqrt{9} - \sqrt{0} \stackrel{?}{=} 3$ | $\sqrt{81} - \sqrt{144} \stackrel{?}{=} 3$ |
| $3 - 0 \stackrel{?}{=} 3$ | $9 - 12 \stackrel{?}{=} 3$ |
| $3 = 3$ ✓ | $-3 \neq 3$ ✗ |

The only solution is 0. The value 72 does not make the original equation true, so it is an extraneous solution.

You can also see this by graphing $y = \sqrt{x+9} - \sqrt{2x}$ and $y = 3$ together. They intersect only at $x = 0$.




**COMMON ERROR**
A common mistake when squaring an expression like $\sqrt{2x} + 3$ is to only square the radical portion. Square this expression as you would any binomial, by using the Distributive Property.

**Try It!** 5. Solve each radical equation. Check for extraneous solutions.

a. $\sqrt{x+4} - \sqrt{3x} = -2$

b. $\sqrt{15 - x} - \sqrt{6x} = -3$

APPLICATION

## EXAMPLE 6 Solve a Radical Inequality

**The body surface area (BSA) of a human being is used to determine doses of medication. The formula for finding BSA is $BSA = \sqrt{\frac{H \cdot M}{3{,}600}}$, where $H$ is the height in centimeters and $M$ is the mass in kilograms.**

**A doctor calculates a particular dose of medicine for a patient whose BSA is less than 1.9. If the patient is 160 cm tall, what must the mass of the person be for the dose to be appropriate?**


**GENERALIZE**
Recall that you solve an inequality similarly to how you do an equation, using an inequality sign instead of an equal sign.

$BSA = \sqrt{\frac{H \cdot M}{3{,}600}}$ Write the BSA model.

$\sqrt{\frac{160 \cdot M}{3{,}600}} < 1.9$ Write an inequality to represent the situation. Substitute 160 for $H$.

$\left(\sqrt{\frac{160 \cdot M}{3{,}600}}\right)^2 < (1.9)^2$ Square both sides to remove the radical sign.

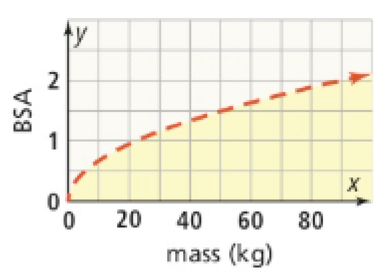
$\frac{160 \cdot M}{3{,}600} < 3.61$ Simplify.

$160 \cdot M < 12{,}996$ Multiply both sides by 3,600.

$M < 81.225$ Divide both sides by 160.

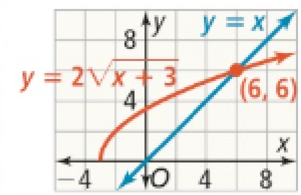
The mass of the individual must be less than 81.225 kg for the dose to be appropriate.

The graph of the inequality $y < \sqrt{\frac{160 \cdot M}{3{,}600}}$ shows that when the BSA is 1.9, the mass of the individual must be less than approximately 81 kg.




**Try It!** 6. A doctor calculates that a particular dose of medicine is appropriate for an individual whose BSA is less than 1.8. If the mass of the individual is 75 kg, how many cm tall can he or she be for the dose to be appropriate?

## CONCEPT SUMMARY Solving Radical Equations

| | WORDS | ALGEBRA | GRAPH |
|---|---|---|---|
| **Step 1** | Isolate the radical term. | $2\sqrt{x+3} - x = 0$<br>$2\sqrt{x+3} = x$ | |
| **Step 2** | Square both sides to remove the radical. | $\left(2\sqrt{x+3}\right)^2 = (x)^2$ | |
| **Step 3** | Solve the equation. | $4(x+3) = x^2$<br>$x^2 - 4x - 12 = 0$<br>$(x-6)(x+2) = 0$<br>$x = 6$ or $x = -2$ |  |
| **Step 4** | Eliminate extraneous solutions. | $2\sqrt{6+3} - 6 \stackrel{?}{=} 0$   $2\sqrt{-2+3} - (-2) \stackrel{?}{=} 0$<br>$2\sqrt{9} \stackrel{?}{=} 6$   $2\sqrt{1} \stackrel{?}{=} -2$<br>$6 \stackrel{?}{=} 6$   $2 \stackrel{?}{=} -2$<br>$6 = 6$ ✓   $2 \neq -2$ ✗ | |



## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you solve equations that include radicals or rational exponents?

2. **Communicate and Justify** How can you use a graph to confirm that the solution to $\sqrt[3]{84x+8} = 8$ is 6?

3. **Vocabulary** Why does solving a radical equation sometimes result in an extraneous solution?

4. **Error Analysis** Neil said that −3 and 6 are the solutions to $\sqrt{3x+18} = x$. What error did Neil make?

5. **Communicate and Justify** Describe how you would solve the equation $x^{\frac{2}{3}} = n$. How is this solution method to be interpreted if the equation had been written in radical form instead?

## Do You KNOW HOW?

**Solve for $x$.**

6. $3\sqrt{x+22} = 21$

7. $\sqrt[3]{5x} = 25$

**In 8 and 9, find the extraneous solution.**

8. $\sqrt{8x+9} = x$

9. $x = \sqrt{24-2x}$

10. Rewrite the equation $y = \sqrt{\frac{x-48}{6}}$ to isolate $x$.

11. Use a graph to find the solution to the equation $9 = \sqrt{3x+11}$.

**Solve each equation.**

12. $(3x+2)^{\frac{2}{5}} = 4$

13. $\sqrt{2x-5} - \sqrt{x-3} = 1$

14. $\sqrt{x+2} + \sqrt{3x+4} = 2$

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

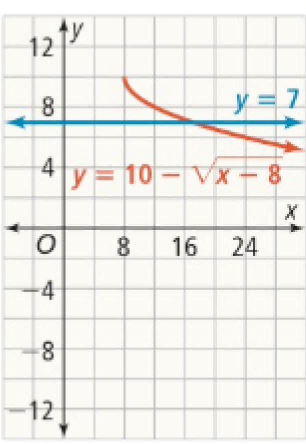
**15.** **Generalize** Explain how to identify an extraneous solution for an equation containing a radical expression.

**16.** **Use Patterns and Structure** Write a radical equation that relates a square's perimeter to its area. Explain your reasoning. Use $s$ to represent the side length of the square.

**17.** **Error Analysis** Describe and correct the error a student made in rewriting the equation to isolate $y$.

$$x = \frac{\sqrt{58 + y}}{1.98}$$
$$1.98x = \sqrt{58 + y}$$
$$1.98x^2 = 58 + y$$
$$1.98x^2 - 58 = y$$

**18.** **Use Patterns and Structure** Use the equations represented in the graph below to find the point of intersection.




**19.** **Higher Order Thinking** Some, but not all, equations with rational exponents have extraneous solutions. What is the relationship between the exponents and the possibility of having extraneous solutions for equations with rational exponents? Explain your reasoning.

**20.** **Communicate and Justify** Describe the process used to solve an equation with two radical expressions. How is this process different from solving an equation with only one radical expression?

## PRACTICE

**Solve each radical equation.** SEE EXAMPLE 1

**21.** $\sqrt[3]{x} + 8 = 13$

**22.** $\sqrt{4x} = 11$

**23.** $\sqrt{75 + x} - 6 = 14$

**24.** $25 - \sqrt[4]{x} = 22$

**Solve for *y*.** SEE EXAMPLE 2

**25.** $x = 3(\sqrt[3]{15 + y})$

**26.** $x = \frac{\sqrt{2y}}{26}$

**27.** $x = \frac{\sqrt{y - 14.2}}{0.05}$

**28.** $x = \frac{1}{3}(\sqrt[4]{y})$

**Solve each radical equation. Check for extraneous solutions.** SEE EXAMPLE 3

**29.** $x = \sqrt{x + 6}$

**30.** $2x = \sqrt{17x - 15}$

**31.** $4x = \sqrt{6x + 10}$

**32.** $x = \sqrt{56 - x}$

**Solve.** SEE EXAMPLE 4

**33.** $0.5(x^2 + 5x + 136)^{\frac{2}{3}} = 50$

**34.** $2(x^2 - 12x - 4)^{\frac{1}{2}} - 3 = 15$

**35.** $(x^2 + 4x + 5)^{\frac{3}{2}} + 1 = 0$

**Solve each radical equation. Check for extraneous solutions.** SEE EXAMPLE 5

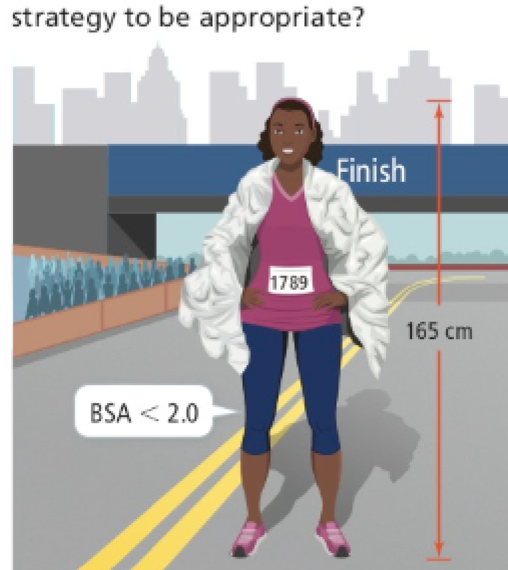
**36.** $\sqrt{6 + x} - \sqrt{x - 5} = 2$

**37.** $\sqrt{4x + 5} - \sqrt{x + 1} = 1$

**38.** $\sqrt{x + 1} + 1 = \sqrt{x + 3}$

**Solve using the formula $BSA = \sqrt{\frac{H \cdot M}{3{,}600}}$.** SEE EXAMPLE 6

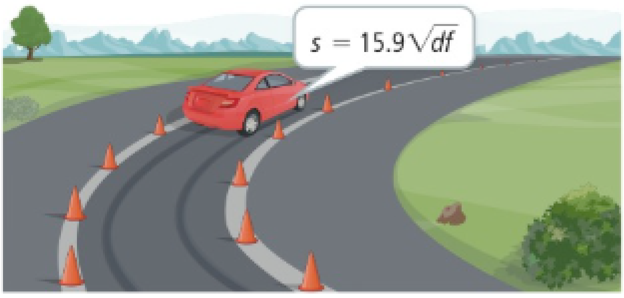
**39.** A sports medicine specialist determines that a hot-weather training strategy is appropriate for a 165 cm tall individual whose BSA is less than 2.0. To the nearest hundredth, what can the mass of the individual be for the training strategy to be appropriate?



## PRACTICE & PROBLEM SOLVING

### APPLY

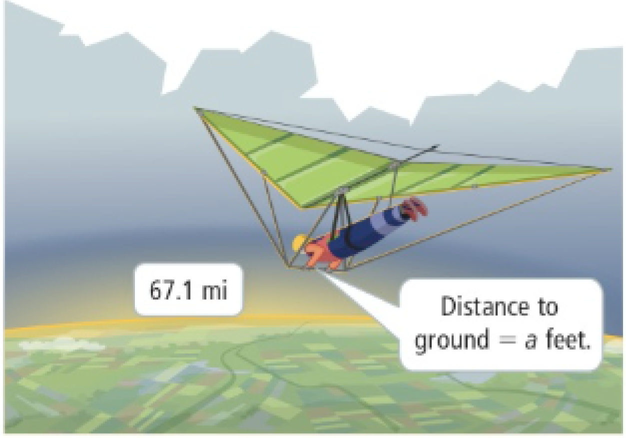
**40. Use Patterns and Structure** Specialists can determine the speed a vehicle was traveling from the length of its skid marks, $d$, and the coefficient of friction, $f$. The formula for calculating the speed, $s$, is $s = 15.9\sqrt{df}$. Rewrite the formula to solve for the length of the skid marks.




**41. Analyze and Persevere** The half-life of a certain type of soft drink is 5 h. If you drink 50 mL of this drink, the formula $y = 50(0.5)^{\frac{t}{5}}$ tells the amount of the drink left in your system after $t$ hours. How much of the soft drink will be left in your system after 16 hours?

**42. Apply Math Models** Big Ben's pendulum takes 4 s to swing back and forth. The formula $t = 2\pi\sqrt{\frac{L}{32}}$ gives the swing time, $t$, in seconds, based on the length of the pendulum, $L$, in feet. What is the minimum length necessary to build a clock with a pendulum that takes longer than Big Ben's pendulum to swing back and forth?

**43. Analyze and Persevere** Derek is hang gliding on a clear day at an altitude of $a$ feet. His visibility, $v$, is 67.1 mi. Use the formula $v = 1.225\sqrt{a}$ to find the altitude at which Derek is hang gliding.




### ASSESSMENT PRACTICE

**44.** Complete the table to solve for the unknown value in the equation $y = \sqrt[3]{2x + z} - 12$, using the given values in each row. AR.7.1

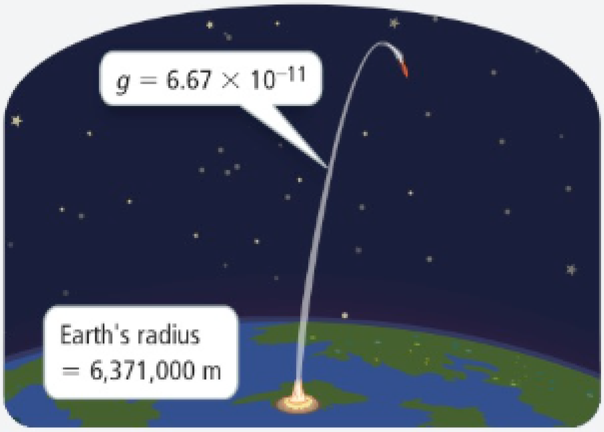
| $y$ | $x$ | $z$ |
|---|---|---|
| 0 | 462 | |
| –3 | | 439 |
| –10 | 1.25 | |
| 3 | | 16 |

**45. SAT/ACT** What is the solution to the equation $(x^2 + 5x + 25)^{\frac{3}{2}} = 343$?

Ⓐ –8 only
Ⓑ 3 only
Ⓒ 77 only
Ⓓ –8 and 3
Ⓔ There are no solutions.

**46. Performance Task** Escape velocity is the velocity at which an object must be traveling to leave a star or planet without falling back to its surface or into orbit. Escape velocity, $v$, depends on the gravitational constant, $G$, the mass, $M$, and radius, $r$, of the star or planet.

$v = \sqrt{\frac{2GM}{r}}$




**Part A** Rewrite the equation to solve for mass.

**Part B** The escape velocity of Earth is 11,200 m/s and its radius is 6,371,000 m. The gravitational constant is $6.67 \times 10^{-11}$. What is Earth's mass in kilograms?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.7.1–Solve one-variable radical equations. Interpret solutions as viable in terms of context and identify any extraneous solutions.
MA.K12.MTR.7.1

## The Snack Shack

Americans seem to love the beach! When the weather is warm, people flock to the beach. Some people bring coolers packed with food and drinks. Others prefer to take advantage of snack bars and shops set up along the beach.

Some beachside communities have built long wooden walkways, or boardwalks, to make it easier for beachgoers to walk to the snack bars and stores. How easy do you find walking in the sand? Think about this during the Mathematical Modeling in 3 Acts lesson.

### ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

### ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

### ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

# 5-5 Function Operations

**I CAN…** perform operations on functions to answer real-world questions.

**VOCABULARY**

- composite function
- composition of functions

MA.912.F.3.2–Given a mathematical or real-world context, combine two or more functions, limited to linear, quadratic, exponential and polynomial, using arithmetic operations. When appropriate, include domain restrictions for the new function. Also F.3.4

MA.K12.MTR.1.1, MTR.2.1, MTR.6.1

## MODEL & DISCUSS

**In business, the term *profit* is used to describe the difference between the money the business earns (revenue) and the money the business spends (cost).**


**A.** Grooming USA charges \$25 for every pet that is groomed. Let $x$ represent the number of pets groomed in a month. Define a revenue function for the business.

**B.** Materials and labor for each pet groomed cost \$15. The business also has fixed costs of \$1,000 each month. Define a cost function for this business.

**C.** Last month, Grooming USA groomed 95 pets. Did they earn a profit? What would the profit be if the business groomed 110 pets in a month?

**D.** **Generalize** Explain your procedure for calculating the profit for Grooming USA. Suppose you wanted to calculate the profit for several different scenarios. How could you simplify your process?

## ESSENTIAL QUESTION

**How do you combine, multiply, divide, and compose functions, and how do you find the domain of the resulting function?**

## EXAMPLE 1 Add and Subtract Functions

**How do you define the sum, $f + g$, and the difference, $f - g$, of the functions $f(x) = 5x^3 - \frac{3}{2}x^2 + 4x + 12$ and $g(x) = 3x^3 - 2x^2 - \frac{7}{4}x - 6$?**

**A. What is the sum of $f(x) = 3x + 4$ and $g(x) = x^2 - 5x + 2$?**

To define the sum of two functions with known rules, add their rules.

$$(f+g)(x) = f(x) + g(x)$$

$$= \left(5x^3 - \tfrac{3}{2}x^2 + 4x + 12\right) + \left(3x^3 - 2x^2 - \tfrac{7}{4}x - 6\right)$$ Substitute the rule of each function.

$$= (5x^3 + 3x^3) + \left(-\tfrac{3}{2}x^2 - 2x^2\right) + \left(4x - \tfrac{7}{4}x\right) + (12 - 6)$$ Group like terms.

$$(f+g)(x) = 8x^3 - \tfrac{7}{2}x^2 + \tfrac{9}{4}x + 6$$ Combine like terms.

The domain of $f$ is $\{x \mid x \text{ is a real number}\}$.

The domain of $g$ is $\{x \mid x \text{ is a real number}\}$.

So the domain of $f + g$ is $\{x \mid x \text{ is a real number}\}$.

The sum of the two functions is $(f+g)(x) = 8x^3 - \frac{7}{2}x^2 + \frac{9}{4}x + 6$.

**REPRESENT AND CONNECT**
Defining a function includes describing its domain. The domain of $f \pm g$ is the intersection of the domains of $f$ and $g$.

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EXAMPLE 1 CONTINUED

**B. What is the difference of $f(x) = 5x^3 - \frac{3}{2}x^2 + 4x + 12$ and $g(x) = 3x^3 - 2x^2 - \frac{7}{4}x - 6$?**

To define the difference of two functions with known rules, subtract their rules.

$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= \left(5x^3 - \tfrac{3}{2}x^2 + 4x + 12\right)\\ &\quad - \left(3x^3 - 2x^2 - \tfrac{7}{4}x - 6\right) && \text{Substitute the rule of each function.}\\ &= 5x^3 - \tfrac{3}{2}x^2 + 4x + 12\\ &\quad - 3x^3 + 2x^2 + \tfrac{7}{4}x + 6 && \text{Use the Distributive Property.}\\ &= 2x^3 + \tfrac{1}{2}x^2 + \tfrac{23}{4}x + 18 && \text{Combine like terms.}\end{aligned}$$

The domain of $f$ is $\{x \mid x \text{ is a real number}\}$. The domain of $g$ is $\{x \mid x$ is a real number$\}$. So the domain of $f - g$ is $\{x \mid x$ is a real number$\}$.

The difference of the two functions is $(f-g)(x) = 2x^3 + \frac{1}{2}x^2 + \frac{23}{4}x + 18$.

**Try It!** 1. Let $f(x) = 2x^2 + 3.7x - 0.05$ and $g(x) = 3 - 2.6x^2 + 1.29x$. Identify rules for the following functions.

**a.** $f + g$ **b.** $f - g$

APPLICATION

## EXAMPLE 2 Multiply Functions

**The demand $d$, in units sold, for a company's new brand of cell phone at price $x$, in dollars, is $d(x) = 5{,}000 - 10x$. The cost to manufacture the cell phone is $c(x) = 1400 + 80x$. What is the company's expected profit from cell phone sales in terms of the price, $x$?**

Profit = Price × Demand − Cost

The company's profit will equal the price of its cell phones multiplied by the demand for its phones less the cost to manufacture.

Profit = price × demand − cost

The demand is the function $d(x)$. The price is the function $p(x)$.

The product of two functions is the product of their rules: $(p \cdot d) = p(x) \cdot d(x)$.

| | | Domains |
|---|---|---|
| Price: | $p(x) = x$ | $p(x)$: $0 \le x$ |
| Demand: | $d(x) = 5000 - 10x$ | $d(x)$: $x \le 500$ |
| Cost: | $c(x) = 2200 + 80x$ | $c(x)$: $x \le 500$ |
| Profit: | $P(x) = p(x) \cdot d(x) - c(x)$ | $P(x)$: $0 \le x \le 500$ |

$$\begin{aligned}&= x(5{,}000 - 10x) - (2200 + 80x)\\ &= 5000x - 10x^2 - 2200 - 80x\\ &= -10x^2 + 4920x - 2200\end{aligned}$$

Neither price nor demand can be negative.

Restricted by demand.

Domain is the intersection of the domains of $p$, $d$ and $c$.

**USE PATTERNS AND STRUCTURE**
You can use the Associative and Commutative Properties to add and multiply functions, since these operations are based on addition and multiplication of real numbers.

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EXAMPLE 2 CONTINUED

The profit the company will earn in terms of the cell phone price $x$ is represented by $P(x) = -10x^2 + 4020x - 2200$

**Try It!** 2. Suppose demand, $d$, for a company's product at cost, $x$, is predicted by the function $d(x) = -0.25x^2 + 1{,}000$, and the price, $p$, that the company can charge for the product is given by $p(x) = x + 16$. Find the company's revenue function.

## EXAMPLE 3 Divide Functions

**How do you define the quotient $\frac{f}{g}$ of the functions $f(x) = x - 7$ and $g(x) = x^3 - 343$?**

To define the quotient of two functions, take the quotient of their rules: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x-7}{x^3-343} \quad \text{Substitute the rule of each function.}$$

$$= \frac{x-7}{(x-7)(x^2+7x+49)} \quad \text{Factor the denominator.}$$

$$= \frac{1}{x^2+7x+49} \quad \text{Simplify.}$$

**COMMON ERROR**
You may think that the domain of $\frac{f}{g}$ is the set of real numbers since the denominator is never zero. However, $x \neq 7$. Remember to identify the domain *before* simplifying the rational function.

The quotient of $\frac{f}{g}$ is $\frac{1}{x^2+7x+49}$. The domain of $\frac{f}{g}$ is the set of all values for which $f$, $g$, and $\frac{f}{g}$ are defined, so $g(x)$ cannot be 0. This is the set of all real numbers $x$ such that $x \neq 7$.

**Try It!** 3. Identify the rule and domain for $\frac{f}{g}$ for each pair of functions.

**a.** $f(x) = x^2 - 4$, $g(x) = 3x^3 - 5x^2 - 2x$

**b.** $f(x) = 4x^2 - 36$, $g(x) = 2x^3 - 4x^2 - 42x - 36$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 4 Compose Functions

**Let $f(x) = x^2$ and let $g(x) = x + 1$. Investigate what happens when you apply the rule for $g$ and then the rule for $f$ to a number or variable.**

**A. Find the values of $f(g(0))$, $f(g(3))$, $f(g(-2))$.**

For each value of $x$, first apply the rule for $g$. Then apply the rule for $f$ to the result.

| $x$ | $g(x)$ | $f(g(x))$ |
|---|---|---|
| 0 | $g(0) = 0 + 1 = 1$ | $f(g(1)) = 1^2 = 1$ |
| 3 | $g(3) = 3 + 1 = 4$ | $f(g(3)) = 4^2 = 16$ |
| −2 | $g(-2) = -2 + 1 = -1$ | $f(g(-2)) = (-1)^2 = 1$ |

**USE PATTERNS AND STRUCTURE**
When finding the rule for $f(g(x))$, work from the inside out. Notice that $g(x)$ takes the place of the variable $x$ in the function $f(x)$.

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EXAMPLE 4 CONTINUED

**B. Find the rule for $f(g(x))$.**

$g(x) = x + 1$ ········ Apply the rule for $g$.

$f(g(x)) = f(x + 1)$ ········ Apply the rule for $f$ to the result.

$= (x + 1)^2$ ········ Use the rule for $f$.

$= x^2 + 2x + 1$ ········ Square the binomial.

$f(g(x)) = x^2 + 2x + 1$

When you apply the rule for one function to the rule of another function, you create an entirely new function.

**Try It!** **4.** Let $f(x) = 2x - 1$ and $g(x) = 3x$. Identify the rule for the following functions.

**a.** $f(g(2))$ **b.** $f(g(x))$

## CONCEPT Composite Function

A **composite function** is the result of applying the rule for one function, $f$, to the rule of another function, $g$. The new rule is denoted as $f \circ g$.

$$(f \circ g)(x) = f(g(x))$$

The operation $\circ$ that forms a composite functions is called **composition of functions**.

The domain of $f \circ g$ is the set of all real numbers $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$.

## EXAMPLE 5 Write a Rule for a Composite Function

**HAVE A GROWTH MINDSET**
After receiving constructive feedback, how do you use it as an opportunity to improve?

**What are the rules for compositions: $(f \circ g)(x)$ and $(g \circ f)(x)$?**

**A.** $f(x) = x^2 + x + 2$ and $g(x) = 4 - x$

$(f \circ g)(x) = f(g(x))$

$= f(4 - x)$ ········ Apply the rule for $g$.

$= (4 - x)^2 + (4 - x) + 2$ ········ Apply the rule for $f$.

$= 16 - 8x + x^2 + 4 - x + 2$ ········ Expand the binomial.

$= x^2 - 9x + 22$ ········ Combine like terms.

The rule for the composition $(f \circ g)(x)$ is $x^2 - 9x + 22$, and the domain is all real numbers.

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**ANALYZE AND PERSEVERE**
When finding the rule for the composition of functions, $f \circ g$, the rule for $g$ is substituted into the rule for $f$ just like substituting a value in for a variable. How might you find the rule for $g \circ f$?

EXAMPLE 5 CONTINUED

**B.** $f(x) = x^2 + x + 2$ and $g(x) = 4 - x$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2 + x + 2) && \text{Apply the rule for } f. \\ &= 4 - (x^2 + x + 2) && \text{Apply the rule for } g. \\ &= -x^2 - x + 2 && \text{Simplify.}\end{aligned}$$

The rule for the composition $(g \circ f)(x)$ is $= -x^2 - x + 2$, and the domain is all real numbers.

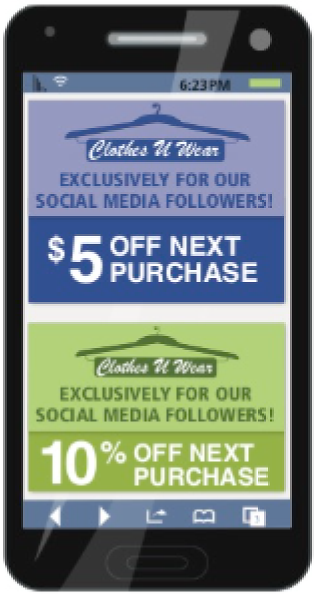
**Try It!** **5.** Identify the rules for $f \circ g$ and $g \circ f$.

**a.** $f(x) = x^3$, $g(x) = x + 1$

**b.** $f(x) = \sqrt{(x + 7)}$, $g(x) = 2x - 5$

**APPLICATION**

## EXAMPLE 6 Use a Composite Function Model




**Clothes U Wear posts discounts on social media. The store allows customers to use multiple discounts. They simply need to tell the cashier in which order they would like the discounts to be applied.**

**On your next trip to Clothes U Wear, in which order should you ask for the discounts?**

**STUDY TIP**
Using a model can help solve a problem. By writing functions to model the discounts, determine which sequence of functions provides a better deal for the customer.

Write functions to model the discounts, letting $x$ represent the price of the purchase.

$$f(x) = x - 5$$

$$\begin{aligned}g(x) &= x - 0.1x && \text{price} - 10\% \text{ of price} \\ &= 0.9x\end{aligned}$$

10% off, then \$5 off ⇒ Identify the rule for $f \circ g$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(0.9x) \\ &= 0.9x - 5 && \text{This is the better deal.}\end{aligned}$$

\$5 off, then 10% off ⇒ Identify the rule for $g \circ f$.

$$\begin{aligned}(f \circ g)(x) &= g(f(x)) \\ &= g(x - 5) \\ &= 0.9(x - 5) \\ &= 0.9x - 4.5\end{aligned}$$

Suppose a shirt costs \$30.00 before the discounts. Applying the discounts as $(f \circ g)(x)$ means the new cost is $(0.9)(30) - 5 = 22$, or \$22.00. Applying the discounts as $(f \circ g)(x)$ means the new cost is $(0.9)(30) - 4.5 = 22.5$, or \$22.50.

The first method yields the better deal.

**Try It!** **6.** As a member of the Games Shop rewards program, you get a 20% discount on purchases. All sales are subject to a 6% sales tax. Write functions to model the discount and the sales tax, then identify the rule for the composition function that calculates the final price you would pay at Games Shop.

## CONCEPT SUMMARY Function Operations

| | Add or Subtract Functions | Multiply or Divide Functions | Compose Functions |
|---|---|---|---|
| ALGEBRA | $(f+g)(x) = f(x) + g(x)$<br>$(f-g)(x) = f(x) - g(x)$ | $(f \cdot g)(x) = f(x) \cdot g(x)$<br>$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ | $(f \circ g)(x) = f(g(x))$<br>$(g \circ f)(x) = g(f(x))$ |
| WORDS | The domain of the sum or difference of $f$ and $g$ is the intersection of the domain of $f$ and the domain of $g$. | The domain is the set of all real numbers for which $f$ and $g$ and the new function are defined. | The **domain** of $f \circ g$ is the set of all real numbers $x$, in the domain of $g$, such that $g(x)$ is in the domain of $f$. |
| EXAMPLES | For $f(x) = 3x + 5$ and $g(x) = x - 3$, $(f+g)(x) = (3x+5) + (x-3) = 4x + 2$ and $(f-g)(x) = (3x+5) - (x-3) = 2x + 8$ | For $f(x) = 3x + 5$ and $g(x) = x - 3$, $(f \cdot g)(x) = (3x+5)(x-3) = 3x^2 - 4x - 15$ and $\left(\frac{f}{g}\right)(x) = \frac{3x+5}{x-3}$ for $x \neq 3$ | For $f(x) = 3x + 5$ and $g(x) = x - 3$, $(f \circ g)(x) = 3(x-3) + 5 = 3x - 4$ and $(g \circ f)(x) = (3x+5) - 3 = 3x + 2$ |

## Do You UNDERSTAND?

1. ESSENTIAL QUESTION How do you combine, multiply, divide, and compose functions, and how do you find the domain of the resulting function?

2. **Vocabulary** In your own words, define and provide an example of a composite function.

3. **Error Analysis** Reagan said the domain of $\frac{f}{g}$ when $f(x) = 5x^2$ and $g(x) = x + 3$ is the set of real numbers. Explain why Reagan is incorrect.

4. **Use Patterns and Structure** Explain why changing the order in which two functions occur affects the result when subtracting and dividing the functions.

## Do You KNOW HOW?

**Let $f(x) = 3x^2 + 5x + 1$ and $g(x) = 2x - 1$.**

5. Identify the rule for $f + g$.

6. Identify the rule for $f - g$.

7. Identify the rule for $g - f$.

**Let $f(x) = x^2 + 2x + 1$ and $g(x) = x - 4$.**

8. Identify the rule for $f \cdot g$.

9. Identify the rule for $\frac{f}{g}$, and state the domain.

10. Identify the rule for $\frac{g}{f}$, and state the domain.

11. If $f(x) = 2x^2 + 5$ and $g(x) = -3x$, what is $f(g(x))$?

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

12. **Generalize** Does $f \circ g$ always equal $g \circ f$? Justify your response.

13. **Communicate and Justify** Explain why the domain for the quotient of functions might not be the set of all real numbers.

14. **Error Analysis** Describe and correct the error a student made in finding the rule for the composition $f \circ g$ of the functions $f(x) = 3x^2 - x + 2$ and $g(x) = 2x + 1$.

$$\begin{aligned} f \cdot g &= f(g(x)) \\ &= 3(2x + 1)^2 - 2x + 1 + 2 \\ &= 3(4x^2 + 4x + 1) - 2x + 1 + 2 \\ &= 12x^2 + 12x + 3 - 2x + 1 + 2 \\ &= 12x^2 + 10x + 6 \end{aligned}$$

15. **Analyze and Persevere** Identify the rules for two functions, $f(x)$ and $g(x)$, for which $f \circ g = g \circ f$.

16. **Higher Order Thinking** Suppose two functions, $f$ and $g$ are only defined by the ordered pairs listed below.

    $f = \{(6, 7), (5, 2), (4, 1), (10, 8)\}$
    $g = \{(5, 4), (3, 6), (1, 5), (2, 10)\}$
    Find the ordered pairs that comprise $(f \circ g)$.

17. **Mathematical Connections** How is the process of finding the rule for the composition of functions related to the order of operations in arithmetic?

18. **Analyze and Persevere** Recalling that the identity function is $f(x) = x$, identify the rules for two functions $f(x)$ and $g(x)$, for which $f(g(x)) = x$.

19. **Communicate and Justify** Is it possible that the result of subtracting two linear functions is a horizontal line? If so, give an example. What must be true about the two linear functions? If not, explain why it is not possible.

### PRACTICE

**Let $f(x) = \frac{5}{3}x^2 + 2x - \frac{5}{8}$ and $g(x) = 3x^2$. Identify the rules for the following functions.** SEE EXAMPLE 1

20. $f + g$

21. $f - g$

22. Suppose the demand $d$, in units sold, for a company's jeans at price $x$, in dollars, is $d(x) = 6500 - 6.83x$.

    a. If revenue = price × demand, write the rule for the function $R(x)$, which represent the company's expected revenue in jean sales. Then state the domain of this function.

    b. If the cost to manufacture the jeans is $c(x) = 386 + 1.27x$, find the equation for the company's profit. How much does the company earn if the price is \$79?
    SEE EXAMPLE 2

23. Identify the rule and domain for $\frac{f}{g}$ when $f(x) = 10x^2 + 3x - 18$ and $g(x) = 2x + 3$.
    SEE EXAMPLE 3

**Let $f(x) = 4x - 5$ and $g(x) = -7x$. Evaluate each expression.** SEE EXAMPLE 4

24. $f(g(3))$

25. $f(g(x))$

26. $g(f(2))$

27. $g(f(x))$

**Let $f(x) = x^2 + x$ and $g(x) = 9 - 2x$. Identify the rules for the following functions.** SEE EXAMPLE 5

28. $f \circ g$

29. $g \circ f$

30. A store is running a sale on surfboards. Kayden wants a board that costs \$400. The store is offering a \$50 instant rebate, as well as a 10% discount.

    In which order should these special offers be applied to the cost of the surfboard in order to benefit Kayden? Explain. SEE EXAMPLE 6

## PRACTICE & PROBLEM SOLVING

### APPLY

**31. Apply Math Models** The cost (in dollars) to produce $x$ surfboards in a factory is given by the function $C(x) = 20x + 500$. The number of surfboards that can be produced in $h$ hours is given by the function $x(h) = 30h$.

a. Find the rule for $C(x(h))$.

b. Find the cost when $h = 8$ hours.

c. Explain what the answer to part (b) represents.

**32. Use Patterns and Structure** A music store is running the following promotions.


a. Use composition of functions to find the sale price of a \$90 purchase when the \$5 off discount is applied prior to the 15% off discount.

b. Use composition of functions to find the sale price of a \$90 purchase when the 15% off discount is applied prior to the \$5 off discount.

c. In which order is the deal better for the consumer? Explain.

**33. Check for Reasonableness** From 2000 to 2015, the number of births, $b$, (in the hundreds) in Fairfield County can be modeled by the function $b(x) = 300 - 5x$. The number of deaths, $d$, (in the hundreds) can be modeled by the function $d(x) = 10x + 5$. The variable $x$ represents the number of years since 2000.

a. Which function operation can be used to represent the net increase in the population?

b. Write and simplify a function which represents the net increase in the population, $p(x)$. State the domain of this function.

### ASSESSMENT PRACTICE

**34.** Mia makes necklaces to sell at an art fair. She can set the price however she wants, but the number of items she can sell decreases as the price increases. In particular, if the price is given by $P(x) = x$, in dollars, then the quantity she sells is $Q(x) = 30 - 5x$. Her total revenue is the product of the price and the quantity she sells, and her profit is revenue minus cost. If it costs Mia \$2 to make each item that she sells, which function gives Mia's total profit? F.3.2

Ⓐ $P(x) - 2Q(x) = 11x - 60$

Ⓑ $P(Q(x)) - 2 = -5x + 28$

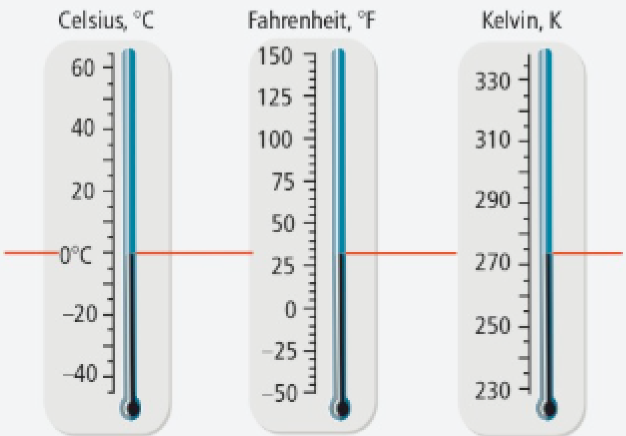
Ⓒ $P(x)Q(x) - 2Q(x) = -5x^2 + 40x - 60$

Ⓓ $Q(x) - 2P(x)Q(x) = 10x^2 - 65x + 30$

**35. SAT/ACT** Find the value of $f(g(5))$ if $f(x) = 4x + 1$ and $g(x) = x^2 + 6$.

Ⓐ 101 Ⓑ 124 Ⓒ 125 Ⓓ 676 Ⓔ 682

**36. Performance Task** The temperature in degrees Celsius is 32 less than the Fahrenheit temperature, multiplied by five ninths. The temperature in degrees Kelvin is the number of degrees Celsius plus 273.




**Part A** Derive a conversion formula for finding the number of degrees Kelvin, given the temperature in Fahrenheit.

**Part B** Using your conversion formula from part (a), find the temperature in degrees Kelvin when the temperature is 27°F. Round to the nearest whole number if necessary.

# 5-6 Inverse Relations and Functions

**I CAN…** represent the inverse of a relation using tables, graphs, and equations.

**VOCABULARY**

- inverse function
- inverse relation

MA.912.F.3.7–Represent the inverse of a function algebraically, graphically or in a table. Use composition of functions to verify that one function is the inverse of the other. Also F.3.4, F.3.6
MA.K12.MTR.1.1, MTR.2.1, MTR.4.1

## EXPLORE & REASON

**Each number path will lead you from a number in the domain, the set of all real numbers, to a number in the range.**

Number Path $f: x \rightarrow f(x)$

- Start with $x$.
- Subtract 3.
- Multiply by $-2$.
- Add 5.

Number Path $g: x \rightarrow g(x)$

- Start with $x$.
- Add 1.
- Square the value.
- Subtract 2.

**A.** Follow the number paths to find $f(1)$ and $g(1)$.

**B.** Identify all possible values of $x$ that lead to $f(x) = 7$ and all values that lead to $g(x) = 7$.

**C. Analyze and Persevere** Based on the two Number Paths, under what conditions can you follow a path back to a unique value in the domain?

## ESSENTIAL QUESTION

**How can you find the inverse of a function and verify the two functions are inverses?**

CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Represent the Inverse of a Relation

**What is the inverse of the relation represented in the table?**

independent variable

dependent variable

| $x$ | $y$ |
|---|---|
| −2 | 1 |
| −1 | 4 |
| 0 | 3 |
| 1 | 1 |

Recall that a relation is any set of ordered pairs $(x, y)$, where $x$ is the independent variable and $y$ is the dependent variable. An **inverse relation** is formed when the roles of the independent and dependent variables are reversed.

Relation

This relation is a function.

| $x$ | $y$ |
|---|---|
| −2 | 1 |
| −1 | 4 |
| 0 | 3 |
| 1 | 1 |

Inverse Relation

| $x$ | $y$ |
|---|---|
| 1 | −2 |
| 4 | −1 |
| 3 | 0 |
| 1 | 1 |

Switch the values in the columns for $x$ and $y$.

| $x$ | $y$ |
|---|---|
| 1 | 1 |
| 1 | −2 |
| 3 | 0 |
| 4 | −1 |

Reorder the pairs based on the new independent variable.

This relation is *not* a function.

**GENERALIZE**
If two distinct values in the domain of $f$ have the same image, then the inverse of $f$ is not a function.

If an inverse relation of a function, $f$, is itself a function, it is called the **inverse function** of $f$, which is written $f^{-1}(x)$.

CONTINUED ON THE NEXT PAGE

**Try It!** **1.** Identify the inverse relation. Is it a function?

| $x$ | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $y$ | 9 | 7 | 5 | 3 | 1 | −1 |

## EXAMPLE 2 Find an Equation of an Inverse Relation

Let $f(x) = x^2$.

**A. How can you represent the inverse relation of $f$ algebraically?**

$f(x) = x^2 \rightarrow y = x^2$

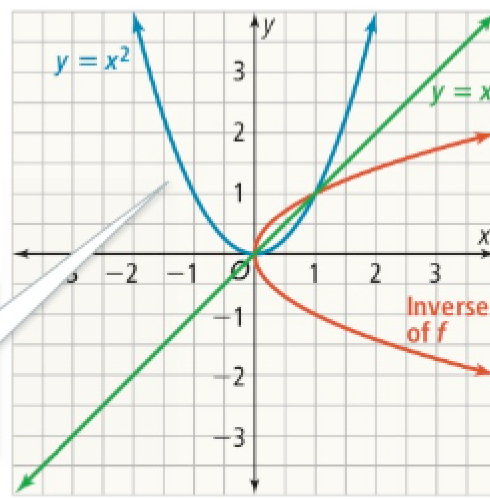
$x = y^2$ Switch the roles of $x$ and $y$.

$y = \pm\sqrt{x}$ Solve for $y$.

The inverse of $f$ can be represented algebraically by the equation $y = \pm\sqrt{x}$.

**B. How are the graphs of $y = x^2$ and $y = \pm\sqrt{x}$ related?**

**GENERALIZE**
The domain of a relation becomes the range of its inverse, and the range of a relation becomes the domain of its inverse.




The range of $f$ is $y \geq 0$, so the domain of the inverse relation is $x \geq 0$.

The inverse of $f$ is ***not*** a function.

The graph of the inverse of $f$ is the reflection of the graph of $y = x^2$ across the line $y = x$.

**Try It!** **2.** Let $f(x) = 2x + 1$.

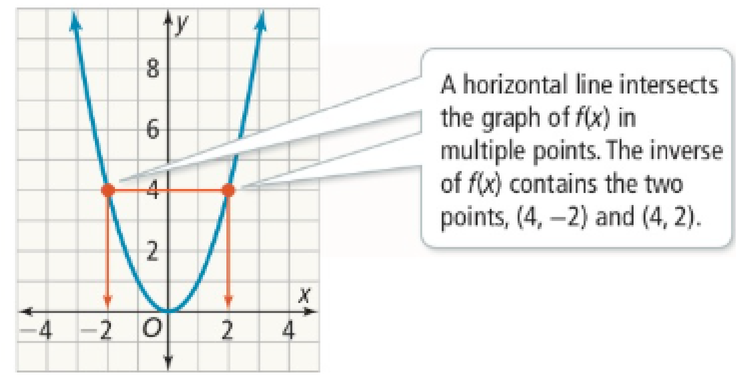
**a.** Write an equation to represent the inverse of $f$.

**b.** How can you use the graph of $f$ to determine if the inverse of $f$ is a function? Explain your answer.

## EXAMPLE 3 When Is an Inverse Also a Function?

**LEARN TOGETHER**
How can you respectfully disagree and manage your emotions?

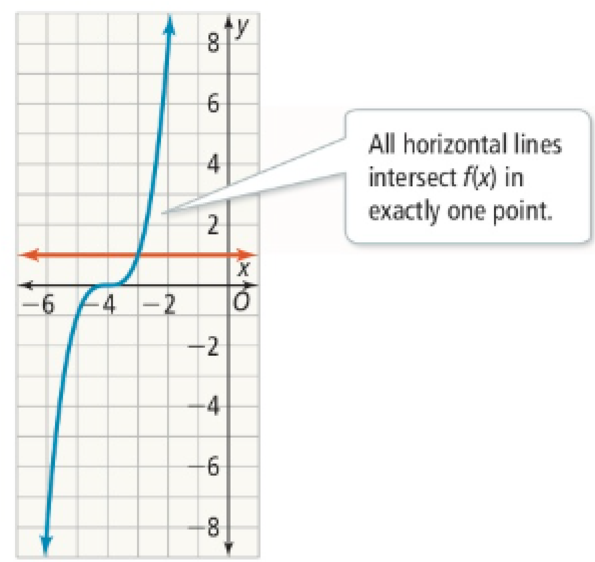
**A. Consider again the function $f(x) = x^2$. Is the inverse relation also a function?**




When any horizontal line intersects the graph of a function in two points, the function has two $x$-values for the same $y$-value. In this case, its inverse will not be a function. If a function has exactly one $x$-value for all $y$-values, its inverse is also a function.

**B. Is the inverse of $f(x) = (x + 4)^3$ an inverse function?**

Look at the graph of $f(x)$. Any horizontal line intersects the graph in one point. So, all of the $y$-values have exactly one $x$-value.




Therefore, the inverse of $f(x)$ is a function.

**Try It!** 3. Graph each function to determine if its inverse is also a function. Explain.

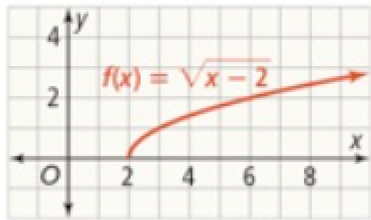
a. $g(x) = 3x^4 - 6$

b. $h(x) = (x - 1)^5 + 4$

## EXAMPLE 4 Find an Equation of an Inverse Function

**A. Find an equation of the inverse function of $f(x) = \sqrt{x-2}$.**

The graph shows that no horizontal line intersects the graph in more than one point. When the graph is reflected over the line $y = x$ to produce an inverse, there will be no vertical line that will intersect the graph more than once. Since the inverse will pass the vertical line test, the inverse relation will be a function.




Use the graph to identify the domain and range of $f$ and $f^{-1}$.

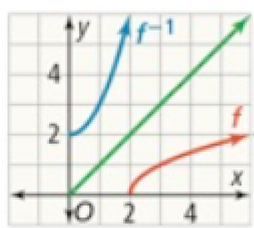
| | domain | range |
|---|---|---|
| $f$ | $x \geq 2$ | $y \geq 0$ |
| $f^{-1}$ | $x \geq 0$ | $y \geq 2$ |

$$y = \sqrt{x-2}$$
$$x = \sqrt{y-2}$$
$$x^2 = y - 2$$
$$x^2 + 2 = y$$

Switch the roles of $x$ and $y$ and solve for $y$.

**COMMON ERROR**
Don't apply $f^{-1}$ notation too quickly. This notation is only used when the inverse of $f$ is a function.

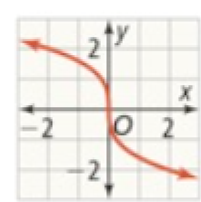
So the inverse of $f(x) = \sqrt{x-2}$ is a function, $f^{-1}(x) = x^2 + 2$, $x \geq 0$. You can verify this on a graph.




The graphs of $f$ and $f^{-1}$ are both functions and are reflections over the line $y = x$.

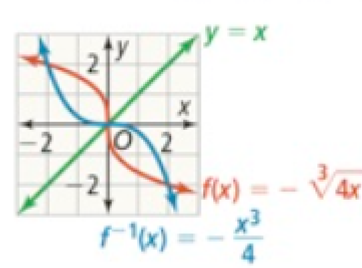
**B. Find an equation of the inverse function of $f(x) = -\sqrt[3]{4x}$.**

The graph shows that no horizontal line intersects the graph in more than one point, so the inverse relation will be a function.




$$y = -\sqrt[3]{4x}$$
$$x = -\sqrt[3]{4y}$$
$$x^3 = -4y$$
$$\frac{-x^3}{4} = y$$

Since the inverse is a function, $f^{-1}(x) = \frac{-x^3}{4}$ for all real $x$. You can verify this on a graph.




**Try It!** 4. Let $f(x) = 2 - \sqrt[3]{x+1}$.

a. Sketch the graph of $f$.

b. Verify that the inverse will be a function and write an equation for $f^{-1}(x)$.

## EXAMPLE 5 Use Composition to Verify Inverse Functions

**A. What is the inverse of $f(x) = 2x + 5$, and how can you verify this?**

$y = 2x + 5 \rightarrow x = 2y + 5$ ........ Switch $x$ and $y$.

$2y = x - 5$ ........ Isolate $y$-term.

$y = \frac{1}{2}x - \frac{5}{2}$ ........ Divide by 2.

So the inverse of $f$ is $g(x) = \frac{1}{2}x - \frac{5}{2}$. You can verify this with function composition: To be inverse functions, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

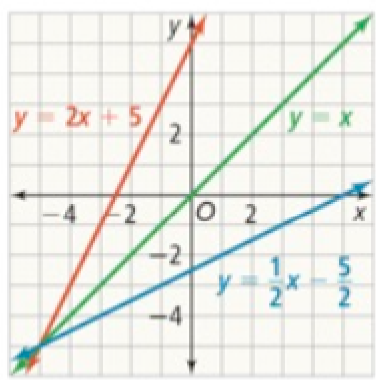
**GENERALIZE**
Because a function's inverse reverses the action of the original function, the composition of the two (in either order) simplifies to the identity function.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= 2(g(x)) + 5\\ &= 2\left(\tfrac{1}{2}x - \tfrac{5}{2}\right) + 5\\ &= x - 5 + 5\\ &= x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= \tfrac{1}{2}(f(x)) - \tfrac{5}{2}\\ &= \tfrac{1}{2}(2x + 5) - \tfrac{5}{2}\\ &= x + \tfrac{5}{2} - \tfrac{5}{2}\\ &= x\end{aligned}$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, the functions $f(x) = 2x + 5$ and $g(x) = \frac{1}{2}x - \frac{5}{2}$ are inverses.

You can also verify $g(x)$ is the inverse of $f(x)$ by graphing.




The graph of $y = \frac{1}{2}x - \frac{5}{2}$ is a reflection of the graph of $y = 2x + 5$ across the line $y = x$.

**B. Are the two functions $f(x) = x^2 + 5$ and $g(x) = \sqrt{x} - 5$ inverses of each other?**

To be inverse functions, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= (g(x))^2 + 5\\ &= (\sqrt{x} - 5)^2 + 5\\ &= x - 10\sqrt{x} + 25 + 5\\ &= x - 10\sqrt{x} + 30\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= \sqrt{f(x)} - 5\\ &= \sqrt{x^2 + 5} - 5\end{aligned}$$

Since neither $(f \circ g)(x)$ nor $(g \circ f)(x)$ simplify to the identity function, the functions are not inverses.

**Try It!** 5. Use composition to determine whether $f$ and $g$ are inverse functions.

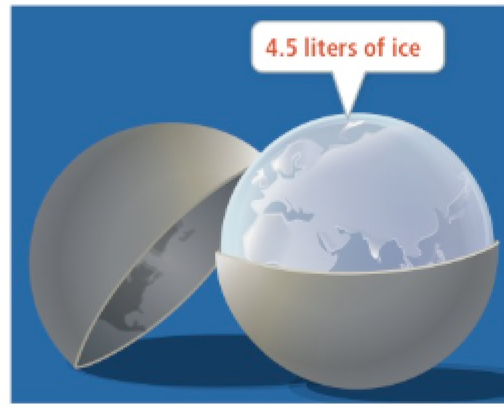
a. $f(x) = \frac{1}{4}x + 7$, $g(x) = 4x - 7$

b. $f(x) = \sqrt[3]{x - 1}$, $g(x) = x^3 + 1$

APPLICATION

## EXAMPLE 6 Rewrite a Formula

**A sculpture artist is making an ice sculpture of Earth for a display. He created a mold that can hold 4.5 L of ice. What will the radius of the ice sculpture mold be?**




The volume of a sphere is calculated using the formula $V = \frac{4}{3}\pi r^3$.

Rewrite the formula to find the length of the radius.

$$\frac{4}{3}\pi r^3 = V$$
$$\pi r^3 = \frac{3}{4}V$$
$$r^3 = \frac{3}{4\pi}V$$
$$r = \sqrt[3]{\frac{3}{4\pi}V}$$

Since a liter is a measure of capacity, or volume, it can be expressed using cubic units of length. When you take the cube root of an expression involving volume in cubic units of length, the units of the result will be in the correct units to describe length. One liter is equivalent to 1,000 $cm^3$, so 4.5 L is equivalent to 4,500 $cm^3$.

**ANALYZE AND PERSEVERE**
If this seems smaller than expected, remember that this is the radius—the diameter of the ice sculpture mold is about 20.4 cm.

$$r = \sqrt[3]{\frac{3}{4\pi}V}$$
$$= \sqrt[3]{\frac{3}{4\pi} \cdot 4{,}500 \text{ cm}^3}$$
$$\approx 10.2 \text{ cm}$$

Substitute 4,500 $cm^3$ for $V$.

Rewriting the equation to show $r$ in terms of $V$ is similar to finding the inverse. In effect, you are exchanging the roles of the dependent and independent variables.

$$V = \frac{4}{3}\pi r^3$$

In the original equation, you can see how the value of $V$ *depends* on the value of $r$.

$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

In this form of the equation, you can see how the value of $r$ can be determined by, and depends on, a given value of $V$.

The ice sculpture mold will have a radius of about 10 cm.

**Try It!** **6.** The manufacturer of a gift box designs a box with length and width each twice as long as its height. Find a formula that gives the height $h$ of the box in terms of its volume $V$. Then give the length of the box if the volume is 640 $cm^3$.

## CONCEPT SUMMARY Inverse Functions

To find the **inverse** of a function, exchange the roles of the independent and dependent variables.

**TABLES** Switch the columns.

| *f* | |
|---|---|
| $x$ | $y$ |
| 0 | 2 |
| 1 | 4 |
| 2 | 6 |
| 3 | 8 |
| 4 | 10 |

| $f^{-1}$ | |
|---|---|
| $x$ | $y$ |
| 2 | 0 |
| 4 | 1 |
| 6 | 2 |
| 8 | 3 |
| 10 | 4 |

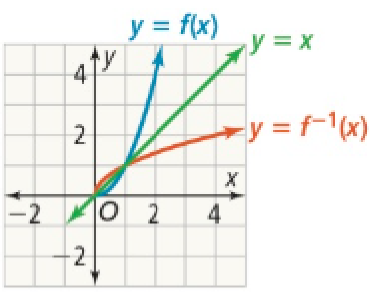
**ALGEBRA**

Exchange the roles of the independent and dependent variables. Solve for the new dependent variable.

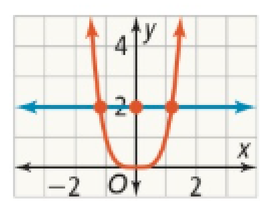
$$y = x + 9$$
$$x = y + 9$$
$$x - 9 = y$$
$$y = x - 9$$

**GRAPHS**

Reflect the graph across the line $y = x$.




If a horizontal line intersects a graph of a function in more than one point, then the inverse is not a function.



**Composition verifies inverses:** $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you find the inverse of a function and verify the two functions are inverses?

2. **Error Analysis** Abi said the inverse of $f(x) = 3x + 1$ is $f^{-1}(x) = \frac{1}{3}x - 1$. Is she correct? Explain.

3. **Represent and Connect** Is the inverse of a function always a function? Explain.

## Do You KNOW HOW?

**Consider the function $f(x) = -\frac{1}{2}x + 5$.**

4. Write an equation for the inverse of $f$.

5. Use composition to show that $f$ and the equation you wrote are inverses.

6. Sketch a graph of $f$ and its inverse.

7. How can you verify by the graph of $f$ and its inverse that they are indeed inverses?

8. Is the inverse of $f$ a function? Explain.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

9. **Represent and Connect** Explain how to find the range of the inverse of $f(x) = \sqrt{2x - 3}$ without finding $f^{-1}(x)$.

10. **Error Analysis** Describe and correct the error a student made in finding the inverse of the function $f(x) = x^2 - 4$.

$$\begin{aligned} f(x) &= x^2 - 4 \\ x &= y^2 - 4 \\ \sqrt{x} &= \sqrt{y^2 - 4} \\ \sqrt{x} &= y - 2 \\ \sqrt{x} + 2 &= y \\ f^{-1}(x) &= \sqrt{x} + 2 \quad ✗ \end{aligned}$$

11. **Higher Order Thinking** What is the inverse operation of raising a number to the 4th power? How can you use the inverse operation of a number raised to the 4th power to find the inverse of the function $f(x) = x^4 - 1$? Is the inverse of $f$ a function? Explain.

12. **Communicate and Justify** A function has the ordered pairs (1, 3), (7, 4), (8, 6), and (9, $y$). What restrictions are there on the value of $y$ so that the inverse of the function is also a function? Explain.

13. **Communicate and Justify** What is the inverse of the function $a(b) = \frac{1}{4}b^2$? Show how to use composition of functions to prove you found the correct inverse.

14. **Communicate and Justify** A relation has one element in its domain and two elements in its range. Is the relation a function? Is the inverse of the relation a function? Explain.

15. **Mathematical Connections** Find the $x$- and $y$-intercepts of the function $y = 2x + 1$. What are the intercepts of the inverse function? How are the intercepts related?

### PRACTICE

**Identify the inverse relation. Is it a function?** SEE EXAMPLE 1

16.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | −2 | −1 | 0 | 1 | 2 | 3 |
| $y$ | 9 | 3 | −4 | 8 | −6 | 3 |

17.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | −2 | 1 | 0 | 1 | 2 | 3 |
| $y$ | −7 | 6 | 8 | −1 | 3 | 7 |

**Write an equation to represent the inverse of $f$. Sketch the graphs of $f$, $y = x$, and the inverse of $f$ on the same coordinate axes. Is the inverse of $f$ a function?** SEE EXAMPLE 2

18. Let $f(x) = x + 3$.

19. Let $f(x) = 4x - 1$.

20. Let $f(x) = x^2 + 1$.

21. Let $f(x) = \sqrt{x + 5}$.

**Is the inverse of the given function also a function?** SEE EXAMPLE 3

22. $f(x) = x^2 + 4x + 4$

23. $f(x) = x^2 - 6x + 9$

24. $f(x) = \sqrt{x^2 - 2}$

25. $f(x) = x^3 + 5$

**Find an equation of the inverse function, and state the domain of the inverse.** SEE EXAMPLE 4

26. $f(x) = 2x^2 - 5$

27. $f(x) = \sqrt{x + 6}$

28. $f(x) = 3x + 10$

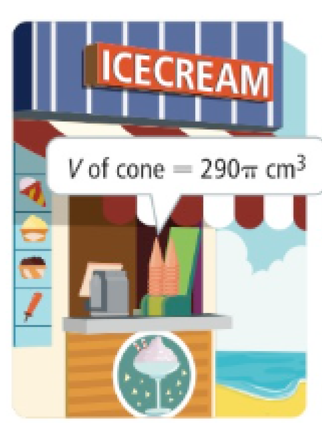
29. $f(x) = \sqrt{x - 9}$

**Use composition to determine whether $f$ and $g$ are inverse functions.** SEE EXAMPLE 5

30. $f(x) = 2x - 9$, $g(x) = \frac{1}{2}x + 9$

31. $f(x) = \sqrt{\frac{x + 4}{3}}$, $g(x) = 3x^2 - 4$

32. A manager purchased cones for ice cream. Find a formula for the length of the radius, $r$, of a cone in terms of its volume, $V$. Then find the length of the radius of a cone if the volume is $290\pi$ cm$^3$ and the height is 15 cm. SEE EXAMPLE 6



## PRACTICE & PROBLEM SOLVING

### APPLY

**33. Apply Math Models** The formula for converting Celsius to Fahrenheit is $F = \frac{5}{9}(C - 32)$. Find the inverse formula, and use it to find the Celsius temperature when the Fahrenheit temperature is 56° F.

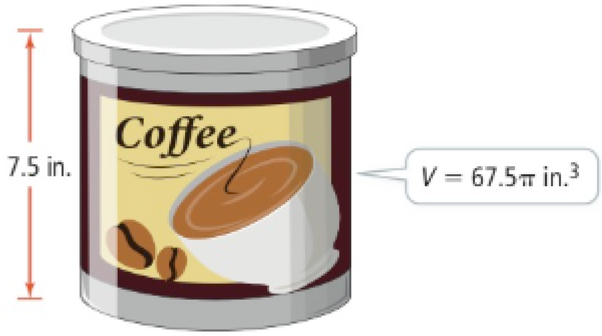
**34. Analyze and Persevere** A DJ charges an hourly fee and an equipment setup fee.


**a.** Write a function for the cost, $C$, of hiring a DJ for $n$ hours.

**b.** Find the inverse of the cost function. What does the function represent?

**c.** If the DJ charged \$550, for how many hours was she hired? Use the inverse function.

**35. Analyze and Persevere** A coffee can is in the shape of a cylinder.




**a.** Find the formula that gives the radius of the coffee can $r$ in terms of the volume $V$ and height $h$.

**b.** Describe any restrictions on the formula.

**c.** What is the radius of a coffee can given the volume is $67.5\pi$ in.$^3$ and the height is 7.5 in.?

### ASSESSMENT PRACTICE

**36.** If functions $f$ and $g$ are inverses, then what is $(f \circ g)(x)$? F.3.7

Ⓐ all real numbers

Ⓑ $(f \circ g)(x) = x$

Ⓒ $(f \circ g)(x) = f(x)$

Ⓓ $(f \circ g)(x) = g(x)$

**37. SAT/ACT** What is the range of the inverse of $f(x) = \sqrt{-ax + b} - c$, where $a$, $b$, and $c$ are real numbers?

Ⓐ $y \geq \frac{a}{b}$

Ⓑ $y \leq \frac{b}{a}$

Ⓒ $y \geq -\frac{a}{b}$

Ⓓ $y \geq -\frac{b}{a}$

Ⓔ $y \geq c$

**38. Performance Task** The table shows several functions and some of the inverses of those functions. The table also shows whether some of the inverses are functions.

| Function | Inverse | Is the inverse a function? |
|---|---|---|
| $f(x) = x$ | $f^{-1}(x) = x$ | yes |
| $g(x) = x^2$ | $g^{-1}(x) = \pm\sqrt{x}$ | no |
| $h(x) = x^3$ | $h^{-1}(x) = \sqrt[3]{x}$ | yes |
| $k(x) = x^4$ | | |
| $m(x) = x^5$ | | |
| $n(x) = x^6$ | | |

**Part A** Determine the inverses of the remaining functions in the table.

**Part B** Determine if the inverses of the remaining functions in the table are functions.

**Part C** Make a conjecture about the power of a function if the inverse of that function is a function.

TOPIC 5

# Topic Review

## TOPIC ESSENTIAL QUESTION

1. How are rational exponents and radical equations used to solve real-world problems?

## Vocabulary Review

**Choose the correct term to complete each sentence.**

2. In the expression $\sqrt[n]{c}$, $n$ is the ________.
3. In the expression $\sqrt[n]{c}$, $c$ is the ________.
4. Radicals with the same index and the same radicand are ________.
5. A(n) ________ is a function defined by $f(x) = a\sqrt[n]{x-h} + k$.
6. A(n) ________ of a number $c$ is $x$, such that $x^n = c$.
7. A(n) ________ is a potential solution that must be rejected because it does not satisfy the original equation.
8. When all $n$th roots of perfect $n$th powers have been simplified and no radicals remain in the denominator, an expression is in ________.
9. A(n) ________ results from the application of one function to the output of another function.

- composite function
- extraneous solution
- index
- inverse function
- like radicals
- $n$th root
- radical function
- radicand
- reduced radical form

## Concepts & Skills Review

### LESSON 5-1 $n$th Roots, Radicals, and Rational Exponents

#### Quick Review

An **$n$th root** of a number $c$ is $x$, such that $x^n = c$. The $n$th root of $c$ can be represented as $\sqrt[n]{c}$, where $n$ is the **index** and $c$ is the **radicand**.

#### Example

**Solve the equation $2x^4 = 162$.**

$2x^4 = 162$ Write the original equation.

$x^4 = 81$ Divide both sides by 2.

$(x^4)^{\frac{1}{4}} = (81)^{\frac{1}{4}}$ Raise both sides to the reciprocal of the exponent of $x$.

$x = \pm 3$ Use the Power of a Power Property.

#### Practice & Problem Solving

**What is the value of each expression? Round to the nearest hundredth, if necessary.**

10. $\sqrt[4]{16^2}$
11. $-\sqrt[3]{25^6}$

**Simplify each expression. Assume variables are positive.**

12. $\sqrt[3]{27x^{12}}$
13. $\sqrt[4]{16a^{24}b^8}$

**Solve each equation.**

14. $750 = 6y^3$
15. $1{,}280 = 5z^4$

16. **Communicate and Justify** Describe the relationship between a rational exponent and a root of a number $x$.

17. **Analyze and Persevere** The function $d(t) = 9.8t^2$ represents how far an object falls, in meters, in $t$ seconds. How long would it take a rock to fall from a height of 300 m? Round to the nearest hundredth of a second.

# LESSON 5-2 Properties of Exponents and Radicals

## Quick Review

To simplify radical expressions, look for factors of the radicand that are perfect $n$th power factors.

The Product Property of Radicals and Quotient Property of Radicals can also be used to rewrite radical expressions.

**Product Property of Radicals** $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

**Quotient Property of Radicals** $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

## Example

**What is $\sqrt[4]{64} \cdot \sqrt[4]{2}$ in reduced radical form?**

$\sqrt[4]{64} \cdot \sqrt[4]{2}$ Write the original expression.

$\sqrt[4]{64 \cdot 2}$ Use the Product Property of Radicals.

$\sqrt[4]{128}$ Multiply.

$\sqrt[4]{16} \cdot \sqrt[4]{8}$ Rewrite using the Product Property of Radicals.

$2\sqrt[4]{8}$ Simplify.

## Practice & Problem Solving

**What is the reduced radical form of each expression? Assume variables are positive.**

18. $\sqrt{x^6y^4} \cdot \sqrt{x^8y^6}$

19. $\sqrt[3]{\frac{243m^4}{3m}}$

20. $\sqrt[3]{5x^4} \cdot \sqrt[3]{x^2} \cdot \sqrt[3]{25x^3}$

21. $\sqrt{\frac{98a^{10}}{2a^4}}$

**Multiply.**

22. $(\sqrt{n} - \sqrt{7})(\sqrt{n} + 3\sqrt{7})$

23. $(9x + \sqrt{2})(9x + \sqrt{2})$

24. $(5\sqrt{3} + 6)(5\sqrt{3} - 6)$

25. $\sqrt[3]{4}\,(6\sqrt[3]{2} - 1)$

**How can you rewrite each expression so there are no radicals in the denominator?**

26. $\frac{6}{1 + \sqrt{2}}$

27. $\frac{5}{2 - \sqrt{5}}$

28. $\frac{4 + \sqrt{6}}{3 - 3\sqrt{6}}$

29. $\frac{-9x}{\sqrt{x}}$

30. **Error Analysis** Describe and correct the error made in rewriting the radical expression.
$5\sqrt{18} - \sqrt{27} = 7\sqrt{2}$

31. **Represent and Connect** A rectangular wall is $\sqrt{240}$ ft by $\sqrt{50}$ ft. You need to paint the wall twice to cover the area with two coats of paint. If each can of paint can cover 60 square feet, how many cans of paint will you need?

## LESSON 5-3 Graphing Radical Functions

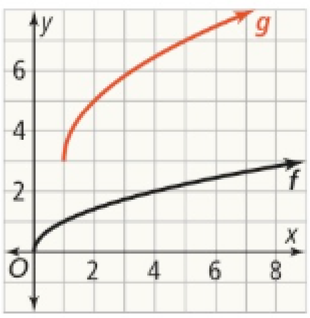
### Quick Review

A **radical function** is a function using a radical expression. To determine transformations of a radical function, write the radical function in the form $h(x) = a\sqrt[n]{x-h} + k$ and compare it to the parent function.

### Example

**Graph $g(x) = 2\sqrt{x-1} + 3$.**

$g(x) = 2\sqrt{x-1} + 3$ is a vertical stretch by a factor of 2, a horizontal shift 1 unit to the right, and a vertical shift 3 units up from the parent function $f(x) = \sqrt{x}$.




### Practice & Problem Solving

**Graph the following functions. What are the domain and range? Is the function increasing or decreasing?**

**32.** $f(x) = \sqrt{x} - 1$

**33.** $f(x) = \sqrt[3]{x} + 2$

**34.** $f(x) = \frac{1}{2}\sqrt{x+1}$

**35.** $f(x) = 2\sqrt[3]{x} - 1$

**36.** $f(x) = \sqrt[3]{x-3}$

**37.** $f(x) = \sqrt{x+4} - 2$

**38.** **Communicate and Justify** Explain how to rewrite the function $g(x) = \sqrt[3]{8x-24} + 1$ to identify the transformations from the parent graph $f(x) = \sqrt[3]{x}$.

**39.** **Represent and Connect** The speed $s$, in miles per hour, of a car when it starts to skid can be estimated using the formula $s = \sqrt{30 \cdot 0.5d}$, where $d$ is the length of the skid marks, in feet. Graph the function. If a car's skid marks measure 40 ft in a zone where the speed limit is 25 mph, was the car speeding? Explain.

## LESSON 5-4 Solving Radical Equations

### Quick Review

To solve a radical equation, isolate the radical. Raise both sides of the equation to the appropriate power to eliminate the radical and solve for $x$. Then check for **extraneous solutions**. If the equation includes more than one radical, eliminate one radical at a time using a similar process.

### Example

**Solve the radical equation $\sqrt{6-x} = x$.**

| | |
|---|---|
| $\sqrt{6-x} = x$ | Write the original equation. |
| $(\sqrt{6-x})^2 = (x)^2$ | Square both sides. |
| $6 - x = x^2$ | Simplify. |
| $0 = x^2 + x - 6$ | Write in standard form. |
| $0 = (x+3)(x-2)$ | Factor. |
| $x = -3$ or $x = 2$ | Use the Zero-Product Property. |

Check the solutions to see if they both make the original equation true.

### Practice & Problem Solving

**Solve each radical equation. Check for extraneous solutions.**

**40.** $\sqrt[3]{x} - 2 = 7$

**41.** $\sqrt{2x} = 12$

**42.** $\sqrt{25+x} + 5 = 9$

**43.** $13 - \sqrt[4]{x} = 10$

**44.** $\sqrt{5x+1} + 1 = x$

**45.** $\sqrt{6x-20} - x = -6$

**46.** **Communicate and Justify** Give an example of a radical equation that has no real solutions. Explain your reasoning.

**47.** **Analyze and Persevere** The formula $d = \frac{\sqrt{15w}}{3.14}$ gives the diameter $d$, in inches, of a rope needed to lift a weight of $w$, in tons. How much weight can be lifted with a rope that has a diameter of 4 in?

## LESSON 5-5 Function Operations

### Quick Review

You can add, subtract, multiply, or divide functions. When adding, subtracting, and multiplying functions, the domain is the intersection of the domains of the two functions. When dividing functions, the domain is the set of all real numbers for which both original functions and the new function are defined. You can also compose functions, by using one function as the input for another function. These are called **composite functions**.

### Example

**Let $f(x) = 5x$ and $g(x) = 3x - 1$. What is the rule for the composition $f \circ g$?**

$f \circ g = f(g(x))$ ········ Apply the definition.

$= f(3x - 1)$ ······ Apply the rule for $g$.

$= 5(3x - 1)$ ····· Apply the rule for $f$.

$= 15x - 5$ ······· Distribute.

### Practice & Problem Solving

**Let $f(x) = -x + 6$ and $g(x) = 5x$. Identify the rule for the following functions.**

48. $f + g$

49. $f - g$

50. $g(f(2))$

51. $f(g(-1))$

52. **Analyze and Persevere** For the functions $f$ and $g$, what is the domain of $f \circ g$? $\frac{f}{g}$? $\frac{g}{f}$?

53. **Analyze and Persevere** A test has a bonus problem. If you get the bonus problem correct, you will receive 2 bonus points and your test score will increase by 3% of your score. Let $f(x) = x + 2$ and $g(x) = 1.03x$, where $x$ is the test score without the bonus problem. Find $g(f(78))$. What does $g(f(78))$ represent?

## LESSON 5-6 Inverse Relations and Functions

### Quick Review

An **inverse relation** is formed when the roles of the independent and dependent variables are reversed. If an inverse relation of a function, $f$, is itself a function, it is called the **inverse function** of $f$, which is written $f^{-1}(x)$.

### Example

**What is the inverse of the relation represented in the table?**

| $x$ | $y$ |
|---|---|
| −2 | 0 |
| −1 | 6 |
| 0 | 5 |
| 1 | 3 |
| 3 | −1 |

Switch the values of $x$ and $y$. Then reorder the ordered pairs.

| $x$ | $y$ |
|---|---|
| −1 | 3 |
| 0 | −2 |
| 3 | 1 |
| 5 | 0 |
| 6 | −1 |

### Practice & Problem Solving

**Find an equation of the inverse function.**

54. $f(x) = -4x^2 + 3$

55. $f(x) = \sqrt{x - 4}$

56. $f(x) = 9x + 5$

57. $f(x) = \sqrt{x + 7} - 1$

58. **Error Analysis** Jamie said the inverse of $f(x) = \sqrt{x - 9}$ is $f^{-1}(x) = (x + 9)^2$. Is Jamie correct? Explain.

59. **Analyze and Persevere** An electrician charges \$50 for a house visit plus \$40 per hour. Write a function for the cost $C$ of an electrician charging for $h$ hours. Find the inverse of the function. If the bill is \$150, how long did the electrician work?

TOPIC 6

# Exponential and Logarithmic Functions

## TOPIC ESSENTIAL QUESTION

**How do you use exponential and logarithmic functions to model situations and solve problems?**

## Topic Overview

**enVision® STEM Project:**
Analyze Elections

**6-1** Key Features of Exponential Functions
AR.5.4, AR.5.5, AR.5.7, F.1.1, F.1.7, MTR.1.1, MTR.2.1, MTR.7.1

**6-2** Exponential Models
AR.5.4, AR.5.5, AR.5.7, F.1.1, FL.3.1, FL.3.2, FL.3.4, DP.2.9, MTR.2.1, MTR.3.1, MTR.6.1

**Mathematical Modeling in 3 Acts:** The Crazy Conditioning
AR.5.4, AR.5.5, MTR.7.1

**6-3** Logarithms
NSO.1.6, AR.5.2, AR.5.7, F.3.7, MTR.2.1, MTR.4.1, MTR.5.1

**6-4** Logarithmic Functions
AR.5.7, AR.5.8, AR.5.9, F.1.7, F.2.2, F.2.3, F.2.5, F.3.7, MTR.1.1, MTR.5.1, MTR.7.1

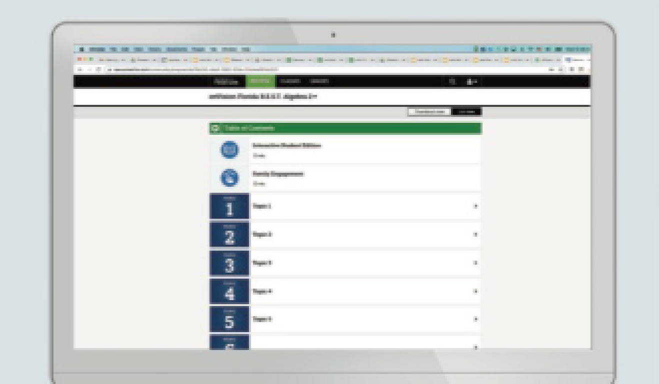
**6-5** Properties of Logarithms
NSO.1.6, NSO.1.7, AR.5.2, MTR.4.1, MTR.5.1, MTR.6.1

**6-6** Exponential and Logarithmic Equations
AR.5.2, MTR.3.1, MTR.4.1, MTR.7.1

**6-7** Geometric Sequences
AR.10.2, MTR.4.1, MTR.6.1, MTR.7.1

## Topic Vocabulary

- Change of Base Formula
- common logarithm
- compound interest
- continuously compounded interest
- decay factor
- exponential equation
- exponential function
- exponential decay function
- exponential growth function
- growth factor
- logarithm
- logarithmic equation
- logarithmic function
- natural base e
- natural logarithm



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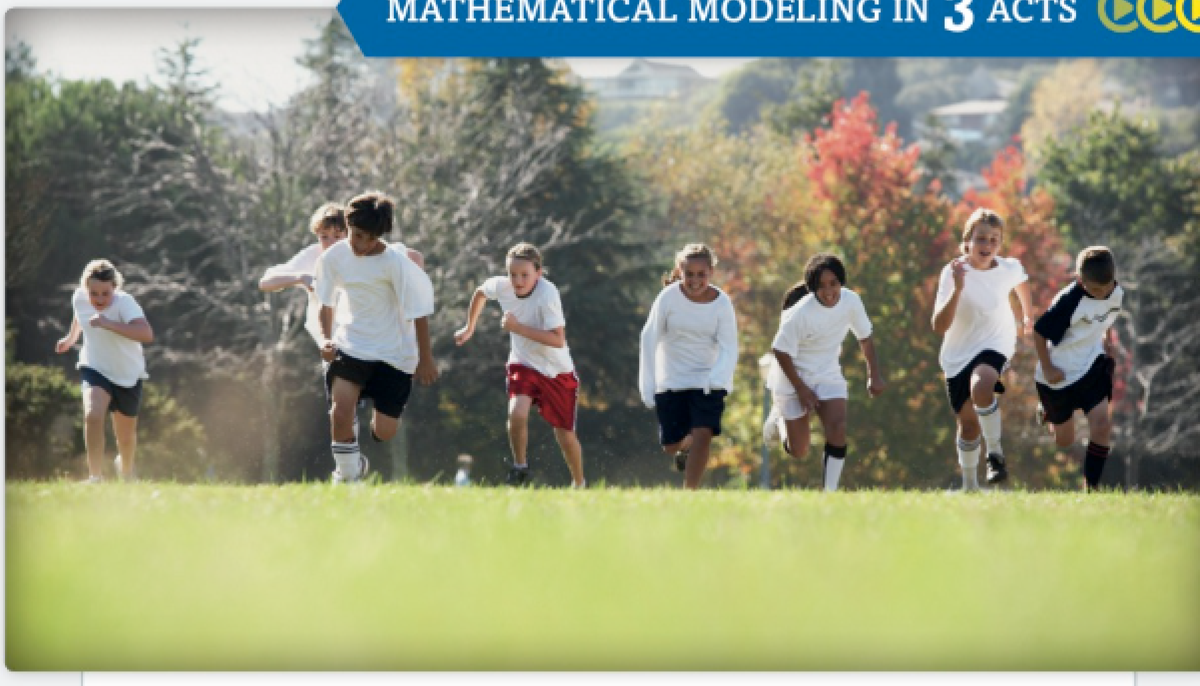
**ANIMATION** View and interact with real-world applications.

**PRACTICE** Practice what you've learned.



MATHEMATICAL MODELING IN 3 ACTS

## The Crazy Conditioning

Like all sports, soccer requires its players to be well trained. That is why players often have to run sprints in practice.

To make sprint drills more interesting, many coaches set up competitions. Coaches might split the players into teams and have them run relay races against each other. Or they might have the players sprint around cones and over barriers. What other ways would make doing sprints more fun? Think about this during the Mathematical Modeling in 3 Acts lesson.

**VIDEOS** Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

**ADAPTIVE PRACTICE** Practice that is *just right* and *just for you.*

**GLOSSARY** Read and listen to English and Spanish definitions.

**CONCEPT SUMMARY** Review key lesson content through multiple representations.

**ASSESSMENT** Show what you've learned.

**TUTORIALS** Get help from ***Virtual Nerd***, right when you need it.

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**DESMOS** Use Anytime and as embedded Interactives in Lesson content.

**QR CODES** Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

## Did You Know?

While you might expect the digits 1 through 9 to lead off the numbers in a data set with equal frequency, Benford's Law shows that they do not. **Benford's Law** states that, in real-world data, the leading digit **is 1 more than 30%** of the time, while the leading digit **is 9 less than 5%** of the time.

Data Sets That **Follow** Benford's Law


Bacterial growth | Expansion of $2^n$ | Price × Quantity

Data Sets That **Do Not Follow** Benford's Law


Price | Quantity | Zip Codes | Social Security numbers

**Bacteria** (plural of bacterium) exist in soil, water, plants, glaciers, hot springs, and the oceans. Bacteria **grow by duplicating themselves,** so a population grows by doubling. In a laboratory, a population can double at regular intervals. These intervals vary from about 12 minutes to as much as 24 hours.

## Your Task: Analyze Elections

You and your classmates will use Benford's law to analyze election results and determine which, if any, may be fraudulent.

# 6-1 Key Features of Exponential Functions

**I CAN…** recognize the key features of exponential functions.

## VOCABULARY

- decay factor
- exponential decay function
- exponential function
- exponential growth function
- growth factor

MA.912.AR.5.5–Given an expression or equation representing an exponential function, reveal the constant percent rate of change per unit interval using the properties of exponents. Interpret the constant percent rate of change in terms of a real-world context. Also AR.5.4, AR.5.7, F.1.1, F.1.7

MA.K12.MTR.1.1, MTR.2.1, MTR.7.1

## EXPLORE & REASON

**Margaret investigates three functions: $y = 3x$, $y = x^3$, and $y = 3^x$. She is interested in the differences and ratios between consecutive $y$-values. Here is the table she started for $y = 3x$.**

| Investigating $y = 3x$ | | | |
|---|---|---|---|
| x | y | Difference between y-values | Ratio between y-values |
| 1 | 3 | | |
| 2 | 6 | $6 - 3 = 3$ | $\frac{6}{3} = 2$ |
| 3 | 9 | $9 - 6 = 3$ | $\frac{9}{6} = 1.5$ |
| 4 | 12 | $12 - 9 = 3$ | $\frac{12}{9} \approx 1.33$ |

A. Create tables like Margaret's for all three functions and fill in more rows.

B. Which functions have a constant difference between consecutive $y$-values? Constant ratio?

C. Use Patterns and Structure Which of these three functions will have $y$-values that increase the fastest as $x$ increases? Why?

## ESSENTIAL QUESTION

**How do graphs and equations reveal key features of exponential growth and decay functions?**

## EXAMPLE 1 Identify Key Features of Exponential Functions

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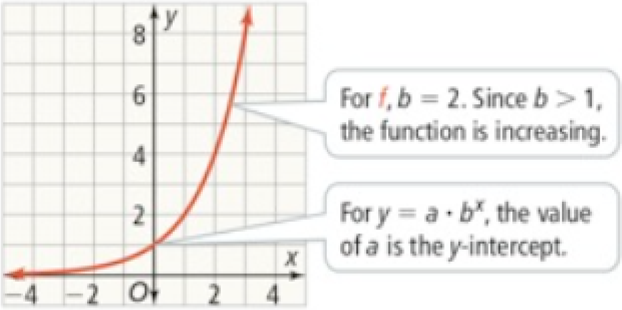
**What are the key features of each function? Include domain, range, intercepts, asymptotes, and end behavior.**

An **exponential function** is any function of the form $y = a \cdot b^x$ where $a$ and $b$ are constants with $a \neq 0$, and $b > 0$, $b \neq 1$.

A. $f(x) = 2^x$

**STUDY TIP**
Recall your investigation of the ratios of consecutive $y$-values in the Explore & Reason activity. That ratio, which was 3 for the function $y = 3^x$, is equal to the value of $b$ in the equation $y = a \cdot b^x$.

| Graphing $y = a{\cdot}b^x$ | |
|---|---|
| $x$ | $f(x) = 2^x$ |
| −2 | 0.25 |
| −1 | 0.5 |
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |




**Domain:** all real numbers
**Range:** $\{y \mid y > 0\}$
**$y$-intercept:** 1;

**Asymptote:** $x$-axis
**End Behavior:**
As $x \to -\infty$, $y \to 0$. As $x \to \infty$, $y \to \infty$.

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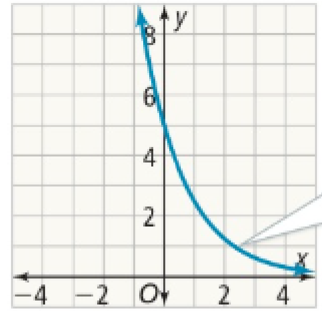
EXAMPLE 1 CONTINUED

**B.** $g(x) = 5\left(\frac{1}{2}\right)^x$

| Graphing $y = a \cdot b^x$ | |
|---|---|
| $x$ | $g(x) = 5\left(\frac{1}{2}\right)^x$ |
| −2 | 20 |
| −1 | 10 |
| 0 | 5 |
| 1 | 2.5 |
| 2 | 1.25 |




**Domain:** all real numbers
**Range:** $\{y \mid y > 0\}$
***y*-intercept:** 5;

**Asymptote:** $x$-axis
**End Behavior:** As $x \to -\infty$, $y \to \infty$.
As $x \to \infty$, $y \to 0$.

**Try It!** **1.** Graph $f(x) = 4(0.5)^x$. What are the domain, range, intercepts, asymptote, and the end behavior for this function?

## EXAMPLE 2 Graph Transformations of Exponential Functions, $f(x) = a \cdot b^{(x-h)} + k$

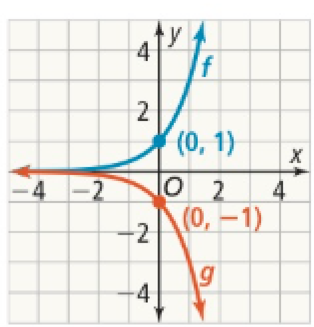
Powered By desmos

**Graph each function. Describe the graph in terms of transformations of the parent function $f(x) = 3^x$. How do the asymptote and intercept of the given function compare to the asymptote and intercept of the parent function?**

**COMMON ERROR**
You may confuse reflection across the axes. Recall that if the $y$-value is multiplied by −1 (as in this case, with $g(x) = -3^x$), the reflection is across the $x$-axis. Each $y$-value is replaced by its opposite.

**A.** $g(x) = -3^x$

When the sign of $a$ changes, the function is reflected across the $x$-axis.



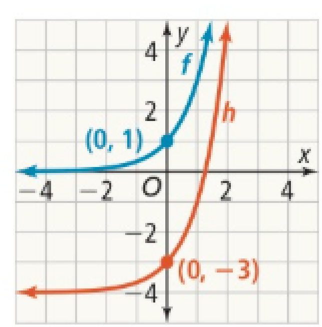
$g(x) = -3^x = -f(x)$

The intercept changes from $a$ to $-a$.

The asymptote of the function does not change. It is still the $x$-axis.

**B.** $h(x) = 3^x - 4$

When adding a constant $k$, the function shifts vertically by $k$ units.



$h(x) = 3^x - 4 = f(x) - 4$

The intercept changes from 1 to $1 + k$.

The asymptote of the function also changes. It is $y = k$ or $y = -4$.

**Try It!** **2.** How do the asymptote and intercept of the given function compare to the asymptote and intercept of the function $f(x) = 5^x$?

**a.** $g(x) = 5^{x+3}$ **b.** $h(x) = 5^{-x}$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Model with Exponential Functions

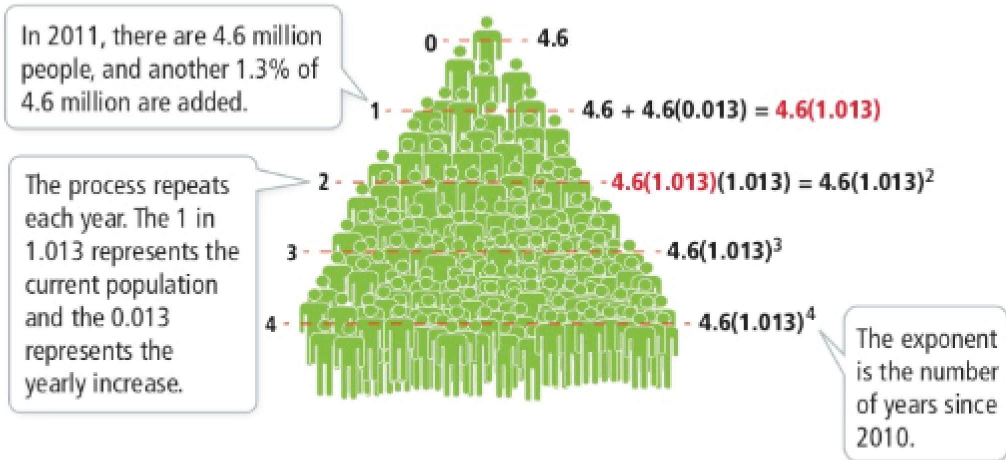
**The population of a large city was about 4.6 million in the year 2010 and grew at a rate of 1.3% for the next four years.**

**A. What exponential function models the population of the city over that 4-year period?**

Compute the population for the first few years to look for a pattern.

**USE PATTERNS AND STRUCTURE**
Exponential functions of the form $y = a \cdot b^x$ involve repeated multiplication by the factor $b$. To understand how to model population with an exponential function, look for repeated multiplication in your computations.

***t*** **(years since 2010)** ***p*** **(population in millions)**




The population can be modeled by the exponential function:

population after $t$ years | population in 2010 | growth factor | years since 2010

$$P = 4.6(1.013)^t$$

**B. If the population continues to grow at the same rate, what will the population be in 2040?**

To find the population in 2040, solve the equation for $t = 30$:

$P = 4.6(1.013)^{30} \approx 6.78.$

In 2040, the population will be about 6.78 million.

**Try It!** **3.** A factory purchased a 3D printer in 2012 for $15,000. The value of the printer is modeled by the function $f(x) = 15{,}000(0.93)^x$, where $x$ is the number of years since 2012.

**a.** What is the value of the printer after 3 years?

**b.** Does the printer lose more of its value in the first 3 years or in the second 3 years after it was purchased?

## CONCEPT Exponential Growth and Decay Models

**Exponential growth** and **exponential decay** functions model quantities that increase or decrease by a constant percent during each time period. Given an initial amount $a$ and the rate of increase or decrease $r$, the amount $A(t)$ after $t$ time periods is given by:

| Exponential Growth Model | Exponential Decay Model |
| --- | --- |
| $A(t) = a(1 + r)^t$ | $A(t) = a(1 - r)^t$ |
| $a > 0;\ b > 1;\ b = 1 + r$ | $a > 0;\ 0 < b < 1;\ b = 1 - r$ |

The **growth or decay factor** is equal to $b$, and is the ratio between two consecutive $y$-values. The *constant percent rate of change* is the value $r$ expressed as a percent.

## EXAMPLE 4 Interpret an Exponential Function

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**A car was purchased for \$24,000. The function $y = 24 \cdot 0.8^x$ can be used to model the value of the car (in thousands of dollars) $x$ years after it was purchased.**

**A. Does the function represent exponential growth or decay?**

$$y = 24 \cdot 0.8^x$$

$b = 0.8$, so $b < 1$ and the function represents exponential decay.

**B. What is the constant percent rate of change for this function? What does it mean?**

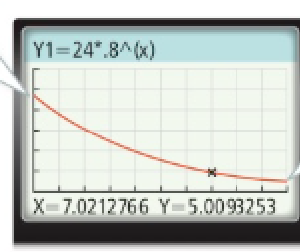
$$b = 1 - r$$
$$0.8 = 1 - r$$
$$r = 0.2$$

The rate of decay is 0.2, or 20%. This means that the value of the car decreases by 20% each year.

> **STUDY TIP**
> For a function of the form $y = a \cdot b^x$, if $b > 1$, the function is increasing. If $0 < b < 1$, the function is decreasing.

**C. Graph the function on a reasonable domain. What do the $y$-intercept and asymptote represent? When will the value of the car be about \$5,000?**

The intercept of 24 shows the car was bought for \$24,000.




The asymptote of $y = 0$ means the value will approach 0 after many years.

Find the value of $x$ when $y = 5$.

The graph (approximately) passes through the point (7, 5). This means that the value of the car will be about \$5,000 after 7 years.

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**Try It!** 4. Two-hundred twenty hawks were released into a region on January 2, 2016. The function $f(x) = 220(1.05)^x$ can be used to model the number of hawks in the region $x$ years after 2016.

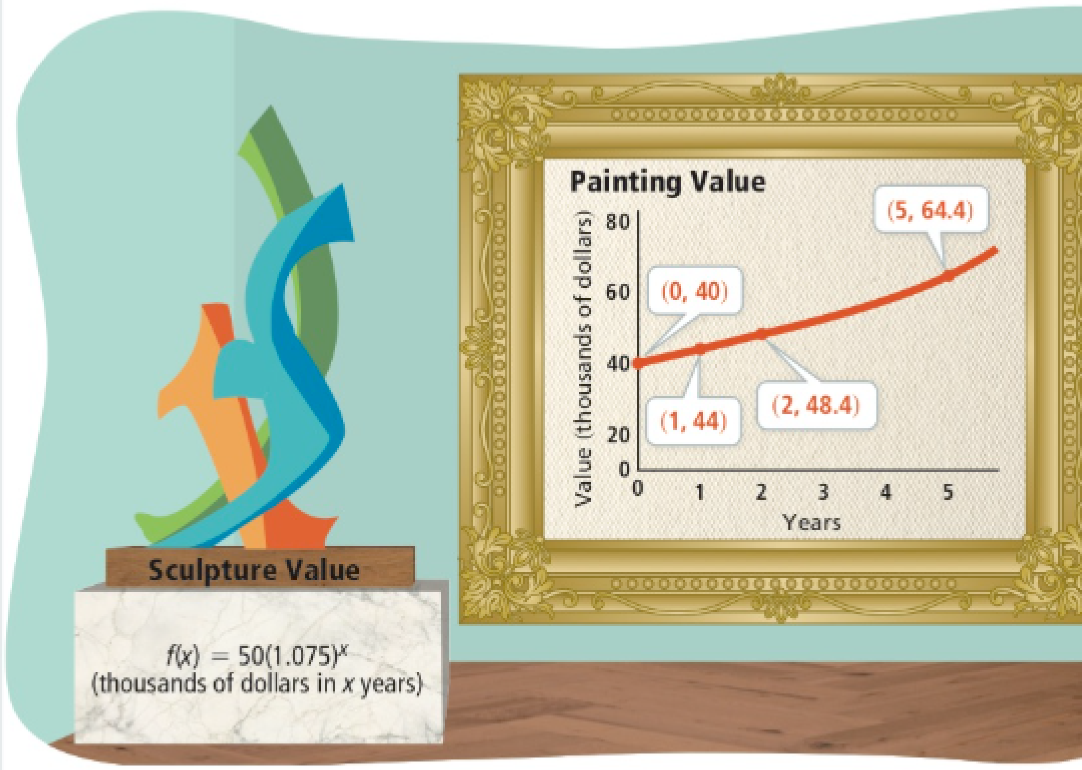
**a.** Is the population increasing or decreasing? Explain.

**b.** In what year will the number of hawks reach 280?

APPLICATION

## EXAMPLE 5 Compare Two Exponential Functions

**A museum purchased a painting and a sculpture in the same year. Their changing values are modeled as shown. Which art work's value is increasing more quickly?**




Identify key features for the sculpture.

$f(x) = 50(1.075)^x$

The initial purchase price is $50,000
The growth factor is 1.075

Find the growth factor, $r$, for the painting.

$r = \frac{44}{40} = 1.1$ and $r = \frac{48.4}{44} = 1.1$

$g(x) = 40(1.1)^x$

Verify for $x = 5$.

$g(5) = 40\,(1.1)^5 \approx 64.4$

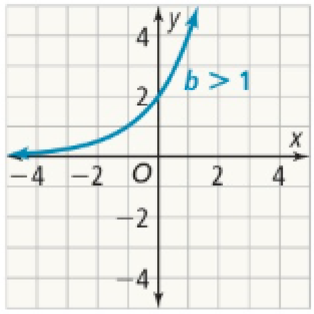
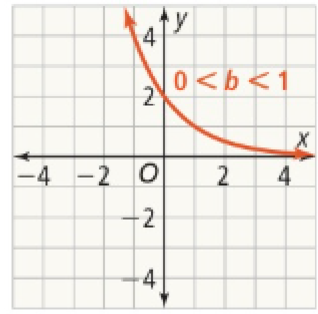
The growth factor for the painting is 1.10.

The painting's growth factor is greater than the sculpture's, so its value is increasing more rapidly.

**CHECK FOR REASONABLENESS**
The growth factor, $b$, is the ratio between two consecutive $y$-values. When the data is available it is good to check multiple places.

**Try It!** 5. In Example 5, in what year will the value of the painting become greater than value of sculpture?

## CONCEPT SUMMARY Key Features of Exponential Functions

| | Exponential Growth | Exponential Decay |
|---|---|---|
| **GRAPHS** |  Growth factor: $1 + r$ |  Decay factor: $1 - r$ |
| **EQUATIONS** | $y = a \cdot b^x$, for $b > 1$, $a \neq 0$ | $y = a \cdot b^x$, for $0 < b < 1$ |
| **KEY FEATURES** | Domain: All real numbers<br>Range: $\{y \mid y \geq 0\}$<br>Intercepts: $(0, a)$<br>Asymptote: $x$-axis | Domain: All real numbers<br>Range: $\{y \mid y \geq 0\}$<br>Intercepts: $(0, a)$<br>Asymptote: $x$-axis |
| **END BEHAVIOR** | As $x \to -\infty$, $y \to 0$<br>As $x \to \infty$, $y \to \infty$ | As $x \to -\infty$, $y \to \infty$<br>As $x \to \infty$, $y \to 0$ |
| **MODELS** | Growth: $A(t) = a(1 + r)^t$ | Decay: $A(t) = a(1 - r)^t$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do graphs and equations reveal key features of exponential growth and decay functions?

2. **Vocabulary** How do *exponential functions* differ from polynomial and rational functions?

3. **Error Analysis** Charles claimed the function $f(x) = \left(\frac{3}{2}\right)^x$ represents exponential decay. Explain the error Charles made.

4. **Analyze and Persevere** How are exponential growth functions similar to exponential decay functions? How are they different?

## Do You KNOW HOW?

5. Graph the function $f(x) = 4(3)^x$. Identify the domain, range, intercept, and asymptote, and describe the end behavior.

6. The exponential function $f(x) = 2500(0.4)^x$ models the amount of money in Zachary's savings account over the last 10 years. Is Zachary's account balance increasing or decreasing? Write the base in terms of the rate of growth or decay.

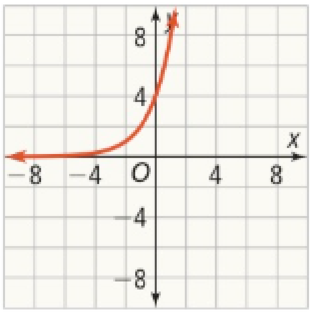
7. Describe how the graph of $g(x) = 4(0.5)^{x-3}$ compares to the graph of $f(x) = 4(0.5)^x$.

8. Two trucks were purchased by a landscaping company in 2016. Their values are modeled by the functions $f(x) = 35(0.85)^x$ and $g(x) = 46(0.75)^x$ where $x$ is the number of years since 2016. Which function models the truck that is worth the most after 5 years? Explain.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

9. **Use Patterns and Structure** What value of $a$ completes the equation $y = a \cdot 2^x$ for the exponential growth function shown below?




10. **Analyze and Persevere** Cindy found a collection of baseball cards in her attic worth \$8,000. The collection is estimated to increase in value by 1.5% per year. Write an exponential growth function and find the value of the collection after 7 years.

11. **Error Analysis** Describe and correct the error a student made in identifying the growth or decay factor for the function $y = 2.55(0.7)^x$.

> Step 1 The base of the function is 0.7, so it represents exponential decay.
> Step 2 The function in the form $y = a(1 - r)^x$ is $y = 2.55(1 - 0.7)^x$.
> Step 3 The decay factor is 0.3. ✗

12. **Represent and Connect** In 2000, the population of Pensacola was 56,255 and it decreased to 51,923 in 2010. If this population decrease were modeled by an exponential decay function, what value could represent the $y$-intercept? Explain your reasoning.

13. **Mathematical Connections** Describe how the graph of $g(x) = 6 \cdot 2^{x+1} - 4$ compares to the graph of $f(x) = 6 \cdot 2^x$.

### PRACTICE

**Identify the domain, range, intercept, and asymptote of each exponential function. Then describe the end behavior.** SEE EXAMPLE 1

14. $f(x) = 5 \cdot 3^x$

15. $f(x) = 0.75\left(\frac{2}{3}\right)^x$

16. $f(x) = 4\left(\frac{1}{2}\right)^x$

17. $f(x) = 7 \cdot 2^x$

**Determine whether each function represents exponential growth or decay. Write the base in terms of the rate of growth or decay, identify $r$, and interpret the rate of growth or decay.**
SEE EXAMPLES 3 AND 4

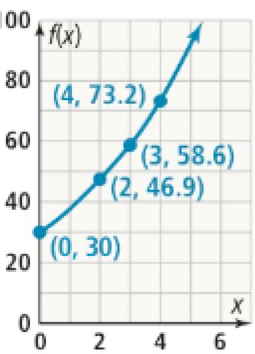
18. $y = 100 \cdot 2.5^x$

19. $f(x) = 10{,}200\left(\frac{3}{5}\right)^x$

20. $f(x) = 12{,}000\left(\frac{7}{10}\right)^x$

21. $y = 450 \cdot 2^x$

22. The function $f(x)$, shown in the graph, represents an exponential growth function. Find $r$ (the growth factor) for $f(x)$. If $g(x) = 25\,(1.4)^x$, which function is increasing faster? SEE EXAMPLE 5



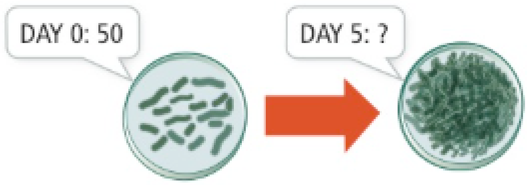

23. Write a function $g(x)$ that represents the exponential function $f(x) = 2^x$ after a vertical stretch of 6 and a reflection across the $x$-axis. Graph both functions. SEE EXAMPLE 2

24. The population of Port St. Joe, Florida, was 3,644 in 2000. It is expected to decrease by about 0.55% per year. Write an exponential decay function and use it to approximate the population in 2020. SEE EXAMPLE 4

# PRACTICE & PROBLEM SOLVING

## APPLY

25. **Apply Math Models** A colony of bacteria starts with 50 organisms and quadruples each day. Write an exponential function, $P(t)$, that represents the population of the bacteria after $t$ days. Then find the number of bacteria that will be in the colony after 5 days.




26. **Higher Order Thinking** The number of teams $y$ remaining in a single elimination tournament can be found using the exponential function $y = 128\left(\frac{1}{2}\right)^x$, where $x$ is the number of rounds played in the tournament.
    a. Determine whether the function represents exponential growth or decay. Explain.
    b. What does 128 represent in the function?
    c. What percent of the teams are eliminated after each round? Explain how you know.
    d. Graph the function. What is a reasonable domain and range for the function? Explain.

27. **Communicate and Justify** The function shown in the graph represents the number of lions in a region $x$ years from now, where the constant percent rate of change is 20%. The number of zebras in that same region $x$ years from now can be modeled by the function $f(x) = 300(0.95)^x$. A representative for a conservationist group claims there will be fewer lions than zebras within 2 years. Is the representative correct? Justify your answer.


## ASSESSMENT PRACTICE

28. The exponential function $g(x) = 3^{x-1} + 6$ is a transformation of the function $f(x) = 3^x$. Choose the statement that accurately describes how the graph of $g(x)$ compares to the graph of $f(x)$. F.2.2

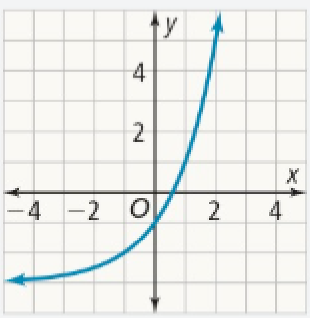
    Ⓐ $g(x)$ is translated 1 unit to the left and 6 units down.

    Ⓑ $g(x)$ is translated 1 unit to the left and 6 units up.

    Ⓒ $g(x)$ is translated 1 unit to the right and 6 units down.

    Ⓓ $g(x)$ is translated 1 unit to the right and 6 units up.

29. **SAT/ACT** Which of the functions defined below could be the one shown in this graph?




    Ⓐ $f(x) = 4(2)^{x-1} + 3$ Ⓒ $f(x) = 4(2)^{x-1} - 3$

    Ⓑ $f(x) = 4(2)^{x+1} + 3$ Ⓓ $f(x) = 4(2)^{x+1} - 3$

30. **Performance Task** A radioactive isotope of the element osmium Os-182 has a half-life of 21.5 hours. This means that if there are 100 grams of Os-182 in a sample, after 21.5 hours there will only be 50 grams of that isotope remaining.

    **Part A** Write an exponential decay function to model the amount of Os-182 in a sample over time. Use $A_0$ for the initial amount and $A$ for the amount after time $t$ in hours.

    **Part B** Use your model to predict how long it would take a sample containing 500 g of Os-182 to decay to the point where it contained only 5 g of Os-182.

# 6-2 Exponential Models

**I CAN…** write exponential models in different ways to solve problems.

### VOCABULARY

- compound interest formula
- continuously compounded interest formula
- natural base $e$

MA.912.FL.3.2–Solve real-world problems involving simple, compound and continuously compounded interest. Also AR.5.4, AR.5.5, AR.5.7, F.1.1, FL.3.1, FL.3.4, DP.2.9

MA.K12.MTR.2.1, MTR.3.1, MTR.6.1

## EXPLORE & REASON

**Juan is studying exponential growth of bacteria cultures. Each culture is carefully controlled to maintain a specific growth rate. Copy and complete the table to find the number of bacteria cells in each culture.**

| Culture | Initial Number of Bacteria | Growth Rate per Day | Time (days) | Final Number of Bacteria |
|---|---|---|---|---|
| A | 10,000 | 8% | 1 | |
| B | 10,000 | 4% | 2 | |
| C | 10,000 | 2% | 4 | |
| D | 10,000 | 1% | 8 | |

**A.** What is the relationship between the daily growth rate and the time in days for each culture?

**B. Generalize** Would you expect a culture with a growth rate of $\frac{1}{2}$% and a time of 16 days to have more or fewer cells than the others in the table? Explain.

## ESSENTIAL QUESTION

**How can you develop exponential models to represent and interpret situations?**

## EXAMPLE 1 Rewrite an Exponential Function to Identify a Rate

**In 2015, the population of a small town was 8,000. The population is increasing at a rate of 2.5% per year. Rewrite an exponential growth function to find the monthly growth rate.**

Write an exponential growth function using the annual rate to model the town's population $y$, in $t$ years after 2015.

initial population

annual growth rate

$$y = 8{,}000(1 + 0.025)^t$$

years after 2015

$$y = 8{,}000(1.025)^t$$

To identify the monthly growth rate, you need the exponent to be the number of months in $t$ years, or $12t$.

$$y = 8{,}000(1.025)^{\frac{12t}{12}}$$

Multiply the exponent by $\frac{12}{12}$ so that $12t$ represents the number of months.

$$y = 8{,}000\left(1.025^{\frac{1}{12}}\right)^{12t}$$

$$y \approx 8{,}000(1.00206)^{12t}$$

Applying the Power of a Power rule helps to reveal the monthly growth rate by producing an expression with the exponent $12t$.

The monthly growth rate is about $1.00206 - 1 = 0.00206$. The population is increasing about 0.206% per month.

### COMMON ERROR

Dividing the annual growth rate by 12 does not give the exact monthly growth rate. This Example shows how to find an expression for the exact monthly rate: $1.025^{\frac{1}{12}} - 1$.

**Try It!** **1.** The population in a small town is increasing annually by 1.8%. What is the quarterly rate of population increase?

## CONCEPT Compound Interest

When interest is paid monthly, the interest earned after the first month becomes part of the new principal for the second month, and so on. Interest is earned on interest already earned. This is compound interest.

The **compound interest formula** is an exponential model that is used to calculate the value of an investment when interest is compounded.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$P$ = the initial principal invested

$r$ = annual interest rate, written as a decimal

$n$ = number of compounding periods per year

$A$ = the value of the account after $t$ years

## EXAMPLE 2 Understand Compound Interest

**Tamira invests $5,000 in an account that pays 4% annual interest.**

**A. How much will there be in the account after 3 years if the interest is compounded annually, quarterly, or monthly?**

Use the Compound Interest formula to find the amount in Tamira's account after 3 years.

| | Compound Interest Formula | Amount After 3 Years ($) |
|---|---|---|
| **Annually** | $A = 5000\left(1 + \frac{0.04}{1}\right)^{1(3)}$ | 5,624.32 |
| **Quarterly** | $A = 5000\left(1 + \frac{0.04}{4}\right)^{4(3)}$ | 5,634.13 |
| **Monthly** | $A = 5000\left(1 + \frac{0.04}{12}\right)^{12(3)}$ | 5,636.36 |

**REPRESENT AND CONNECT**
The more frequently interest is added to the account, the earlier that interest generates more interest. This reasoning supports the trend shown in the table.

As the number of compounding periods increases, the amount in the account also increases.

**B. What is the annual growth rate for each possible account?**

Using the properties of exponents, you can transform the compound interest formula to find the annual growth rate over a single year.

| | Interest Formula | Annual Growth Rate |
|---|---|---|
| **Annually** | $A = 5000\left(1 + \frac{0.04}{1}\right)^{1t} = 5000(1.04)^{t}$ | 4% |
| **Quarterly** | $A = 5000\left(1 + \frac{0.04}{4}\right)^{4t} = 5000((1 + 0.01)^{4})^{t}$ $\approx 5000(1.0406)^{t}$ | 4.06% |
| **Monthly** | $A = 5000\left(1 + \frac{0.04}{12}\right)^{12t}$ $\approx 5000((1 + 0.003)^{12})^{t}$ $\approx 5000(1.0407)^{t}$ | 4.07% |

CONTINUED ON THE NEXT PAGE

**Try It!** 2. \$3,000 is invested in an account that earns 3% annual interest, compounded monthly.

a. What is the value of the account after 10 years? 100 years?

b. What is the effective annual interest rate of the account?

CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Understanding Continuously Compounded Interest

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**HAVE A GROWTH MINDSET**
How can you use mistakes as opportunities to learn and grow?

Consider an investment of \$1 in an account that pays a 100% annual interest rate for one year. The equation $A = 1\left(1 + \frac{1}{n}\right)^{n(1)} = \left(1 + \frac{1}{n}\right)^{n}$ gives the amount in the account after one year for the number of compounding periods $n$. Find the value of the account for the number of periods given in the table.

| Number of Periods, $n$ | Value of $\left(1 + \frac{1}{n}\right)^{n}$ |
|---|---|
| 1 | $\left(1 + \frac{1}{1}\right)^{1} = 2$ |
| 10 | $\left(1 + \frac{1}{10}\right)^{10} = 2.59374246$ |
| 100 | $\left(1 + \frac{1}{100}\right)^{100} = 2.704813829$ |
| 1000 | $\left(1 + \frac{1}{1{,}000}\right)^{1{,}000} = 2.716923932$ |
| 10000 | $\left(1 + \frac{1}{10{,}000}\right)^{10{,}000} = 2.718145927$ |
| 100000 | $\left(1 + \frac{1}{100{,}000}\right)^{100{,}000} = 2.718268237$ |

Notice that as n continues to increase, the value of the account remains very close to 2.718. This special number is called the *natural base*.

The **natural base $e$** is defined as the value that the expression $\left(1 + \frac{1}{x}\right)^{x}$ approaches as $x \to +\infty$. The number $e$ is an irrational number.

$$e = 2.718281828459\ldots$$

The number $e$ is the base in the **continuously compounded interest formula**.

$A = Pe^{rt}$
$P$ = the initial principal invested
$e$ = the natural base
$r$ = annual interest rate, written as a decimal
$A$ = the value of the account after $t$ years

**Try It!** 3. If you continued the table for $n = 1{,}000{,}000$, would the value in the account increase or decrease? How do you know?

APPLICATION

## EXAMPLE 4 Compare Simple, Compound, and Continuously Compounded Interest

**Regina considers investing $12,000 in 3 different accounts as shown.**


**A. Which investment will be worth the most money after 5 years?**

Use the simple, compound, and continuously compounded interest formulas.

**COMMUNICATE AND JUSTIFY**
The account at 5.1% compounded annually will never exceed the account at 5% compounded continuously. How can you use the properties of exponents to explain why?

| | Formula | Amount After 5 Years |
|---|---|---|
| 6% Simple Interest | $A = 12{,}000(1 + 0.06 \cdot 5)$ | \$15,600 |
| 5.1% Compounded Annually | $A = 12{,}000\left(1 + \frac{0.051}{1}\right)^{(1)(5)}$ | \$15,388 |
| 5% Continuously Compounded Interest | $A = 12{,}000e^{(0.05)(5)}$ | \$15,408 |

The 6% simple interest account is worth the most after 5 years.

**B. Which investment earns the most interest in the 5th year?**

Use the formulas again to find the amount after 4 years, and then calculate the amount of interest earned in year five.

| | Amount After 4 Years | Interest Earned in Year 5 |
|---|---|---|
| 6% Simple Interest | $12{,}000\,(1 + 0.06 \cdot 4) = 14{,}880$ | $15{,}600 - 14{,}880 = \$720$ |
| 5.1% Compounded Annually | $12{,}000\left(1 + \frac{0.051}{1}\right)^{(1)(4)} = 14{,}642$ | $15{,}388 - 14{,}642 = \$746$ |
| 5% Continuously Compounded Interest | $A = 12{,}000e^{(0.05)(4)} = 14{,}657$ | $15{,}408 - 14{,}657 = \$751$ |

Even though the account with simple interest has a higher annual percentage rate than either of the other investments, it grows the slowest of all 3 in the 5th year. The account at 5% compounded continuously earns the most in the 5th year, yet it is the one with the lowest interest rate.

The simple interest formula is a linear function, while the compound and continuously compounded interest formulas are exponential functions. Increasing exponential functions eventually overtake linear functions as time passes.

CONTINUED ON THE NEXT PAGE

**Try It!** **4.** Use the investments from Example 4 to answer the following questions.

**a.** Which investment is worth the most after 10 years?

**b.** Which investment earns the least interest in the 3rd year?

APPLICATION

## EXAMPLE 5 Use Two Points to Find an Exponential Model

**Tia knew that the number of e-mails she sent was growing exponentially. She generated a record of the number of e-mails she sent each year since 2009. What is an exponential model that describes the data?**


Write an exponential model in the form $y = a \cdot b^x$, with $y$ equal to the number of e-mails in hundreds and $x$ equal to the number of years since 2009. Use the data to find the values of the constants $a$ and $b$.

**COMMON ERROR**
Remember that the growth factor $(1 + r)$ is different from the growth rate $(r)$. In this example, the growth factor is 1.4 while the growth rate is 0.4, or 40%.

The growth factor for Tia's e-mails in the two consecutive years was $\frac{14}{10}$, or 1.4.

When data points have $x$-values that differ by 1, the growth factor, $b$, is the ratio of their $y$-values.

Use the value of $b$ and one of the data points to find the initial value, $a$.

$y = a \cdot b^x$ ........ Write an exponential growth equation.

$14 = a(1.4)^7$ ........ Substitute 1.4 for $b$, 7 for $x$, and 14 for $y$.

$\frac{14}{(1.4)^7} = a$ ........ Division Property of Equality

$1.33 \approx a$ ........ Simplify.

So, the function $y = 1.33(1.4)^x$ models the number of e-mails (in hundreds) Tia sends $x$ years after 2009.

**Try It!** **5.** A surveyor determined the value of an area of land over a period of several years since 1950. The land was worth \$31,000 in 1954 and \$35,000 in 1955. Use the data to determine an exponential model that describes the value of the land.

## CONCEPT SUMMARY Writing Exponential Models

| | General Exponential Model | Compound Interest | Continuously Compounded Interest |
|---|---|---|---|
| **ALGEBRA** | $y = a \cdot b^x$ | $A = P\left(1 + \frac{r}{n}\right)^{nt}$ | $A = Pe^{rt}$ |
| **NUMBERS** | A necklace costs \$250 and increases in value by 2% per year.<br>$a =$ initial amount \$250<br>$b =$ growth factor 1.02<br>$x =$ number of years<br>$y = 250(1.02)^x$ | A principal of \$3,000 is invested at 5% annual interest, compounded monthly, for 4 years.<br>$P = 3{,}000$<br>$r = 5\%$<br>$n = 12$ compounding periods per year<br>$t = 4$ years<br>$A = 3000\left(1 + \frac{0.05}{12}\right)^{(12)(4)}$ | A principal of \$3,000 is invested at 5% continuously compounded interest for 4 years.<br>$P = 3{,}000$<br>$r = 5\%$<br>$t = 4$ years<br>$A = 3000e^{(0.05)(4)}$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** Why do you develop exponential models to represent and interpret situations?

2. **Error Analysis** The exponential model $y = 5{,}000(1.05)^t$ represents the amount Yori earns in an account after $t$ years when \$5,000 is invested. Yori said the monthly interest rate of the exponential model is 5%. Explain Yori's error.

3. **Vocabulary** Explain the similarities and differences between compound interest and continuously compounded interest.

4. **Represent and Connect** Write a math story that would use the general exponential model to solve. Explain the meanings of the variables and parameters in your model.

## Do You KNOW HOW?

**The exponential function models the annual rate of increase. Find the monthly and quarterly rates.**

5. $f(t) = 2{,}000(1.03)^t$

6. $f(t) = 500(1.055)^t$

**Find the total amount of money in an account at the end of the given time period.**

7. compounded monthly, $P = \$2{,}000$, $r = 3\%$, $t = 5$ years

8. continuously compounded, $P = \$1{,}500$, $r = 1.5\%$, $t = 6$ years

**Write an exponential model given two points.**

9. (3, 55) and (4, 70)

10. (7, 12) and (8, 25)

11. Paul invests \$6,450 in an account that earns continuously compounded interest at an annual rate of 2.8%. What is the value of the account after 8 years?

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

12. **Error Analysis** Suppose \$6,500 is invested in an account that earns interest at a rate of 2% compounded quarterly for 10 years. Describe and correct the error a student made when finding the value of the account.

$$A = 6500\left(1 + \frac{0.02}{12}\right)^{12(10)}$$

$$A = 7937.80$$

13. **Choose Efficient Methods** The points (2, 54.61) and (4, 403.48) are points on the graph of an exponential model in the form $y = a \cdot e^x$.

    a. Explain how to write the exponential model, and then write the model.

    b. How can you use the exponential model to find the value of $y$ when $x = 8$?

14. **Apply Math Models** Enrique invests \$3225 in a bond that he holds for 15 years. The table shows the value of the bond for years 3 through 8. How much is the bond worth after 15 years?

| Time (yr) | Amount (\$) |
|---|---|
| 3 | 4040.04 |
| 4 | 4355.17 |
| 5 | 4694.87 |
| 6 | 5061.07 |
| 7 | 5455.83 |
| 8 | 5881.39 |

15. **Higher Order Thinking** A power model is a type of function in the form $y = a \cdot x^b$. Use the points (1, 4), (2, 8), (3, 16) and (4, 64) and a calculator to find an exponential model and a power model for the data. Then use each model to predict the value of $y$ when $x = 6$. Graph the points and models in the same window. What do you notice?

## PRACTICE

**Find the amount in the account for the given principal, interest rate, time, and compounding period.** SEE EXAMPLES 2 AND 4

16. $P = 800$, $r = 6\%$, $t = 9$ years; compounded quarterly

17. $P = 3{,}750$, $r = 3.5\%$, $t = 20$ years; compounded monthly

18. $P = 2{,}400$, $r = 5.25\%$, $t = 12$ years; compounded semi-annually

19. $P = 1{,}500$, $r = 4.5\%$, $t = 3$ years; compounded daily

20. $P = \$1{,}000$, $r = 2.8\%$, $t = 5$ years; compounded continuously

21. $P = \$16{,}000$, $r = 4\%$, $t = 25$ years; compounded continuously

**Write an exponential model given two points.**
SEE EXAMPLE 5

22. (9, 140) and (10, 250)

23. (6, 85) and (7, 92)

24. (10, 43) and (11, 67)

25. In 2012, the population of a small town was 3,560. The population is decreasing at a rate of 1.7% per year. How can you rewrite an exponential growth function to find the quarterly decay rate? SEE EXAMPLE 1

26. Talisha invests \$5000 in each of 3 investments: a city bond that pays 7.15% simple interest, a mutual fund that guarantees $6\frac{1}{2}$ % interest compounded semi-annually and a high-risk account with a return of 5.8% that compounds continuously. Which investment will be worth the most after 8 years, assuming each account continues without incident?

# PRACTICE & PROBLEM SOLVING

## APPLY

27. **Check for Reasonableness** Adam invests \$8,000 in an account that earns 1.25% interest, compounded quarterly for 20 years. On the same date, Jacinta invests \$8,000 in an account that earns continuous compounded interest at a rate of 1.25% for 20 years. Who do you predict will have more money in their account after 20 years? Explain your reasoning.


28. **Analyze and Persevere** A blogger found that the number of visits to her website increases 5.6% annually. The Web site had 80,000 visits this year. Write an exponential model to represent this situation. By what percent does the number of visits increase daily? Explain how you found the daily rate.

29. **Use Patterns Structure** Jae invested \$3,500 at a rate of 2.25% compounded continuously in 2010. How much will be in the account in 2025? How much interest will the account have earned by 2025?

30. **Apply Math Models** Ricardo invests \$20,000 in an account that pays 5% interest. How much will be there be in the account after 10 years if the interest is compounded annually, semi-annually, quarterly, and monthly? What is the annual growth rate for each possible account?


## ASSESSMENT PRACTICE

31. The table shows the account information of five investors. Which statement is true, assuming no withdrawals are made? AR.5.4

| Employee | $P$ | $r$ | $t$(years) | Compound |
|---|---|---|---|---|
| Anna | 4000 | 1.5% | 12 | Quarterly |
| Nick | 2500 | 3% | 8 | Monthly |
| Lori | 7200 | 5% | 15 | Annually |
| Tara | 2100 | 4.5% | 6 | Continuously |
| Steve | 3800 | 3.5% | 20 | Semi-annually |

Ⓐ After 12 years, Anna will have about \$4,788.33 in her account.

Ⓑ After 8 years, Nick will have about \$3,177.17 in his account.

Ⓒ After 15 years, Lori will have about \$15,218.67 in her account.

Ⓓ After 20 years, Steve will have about \$7,629.00 in his account.

32. **SAT/ACT** Rick invested money in a continuous compound account with an interest rate of 3%. How long will it take Rick's account to double?

Ⓐ about 2 years

Ⓑ about 10 years

Ⓒ about 23 years

Ⓓ about 46 years

Ⓔ about 67 years

33. **Performance Task** Cassie is financing a \$2,400 treadmill. She is going to use her credit card for the purchase. Her card charges 17.5% interest compounded monthly. She is not required to make minimum monthly payments.

**Part A** How much will Cassie pay in interest if she waits a full year before paying the full balance?

**Part B** How much additional interest will Cassie pay if she waits two full years before paying the full balance?

**Part C** If both answers represent a single year of interest, why is the answer in B greater than the answer in A?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.AR.5.4–Write an exponential function to represent a relationship between two quantities from a graph, a written description or a table of values within a mathematical or real-world context. Also AR.5.5 MA.K12.7.1

## The Crazy Conditioning

Like all sports, soccer requires its players to be well-trained. That is why players often have to run sprints in practice.

To make sprint drills more interesting, many coaches set up competitions. Coaches might split the players into teams and have them run relay races against each other. Or they might have the players sprint around cones and over barriers. What other ways would make doing sprints more fun? Think about this during this Mathematical Modeling in 3 Acts lesson.

### ACT 1 Identify the Problem

1. Write down the Main Question you will answer.
2. Make an initial conjecture that answers this Main Question.
3. Explain how you arrived at your conjecture.
4. Write a number that you know is too small.
5. Write a number that you know is too high.
6. What information do you need to know to answer the main question? How can you get it? How will you use that information?

### ACT 2 Develop a Model

1. Use the math that you have learned in this Topic to refine your conjecture.
2. Is your refined conjecture between the high and low estimates you came up with earlier?

### ACT 3 Interpret the Results

1. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

# 6-3 Logarithms

**I CAN…** evaluate and simplify logarithms.

**VOCABULARY**

- common logarithm
- logarithm
- logarithmic function
- natural logarithm

MA.912.AR.5.2–Solve one-variable equations involving logarithms or exponential expressions. Interpret solutions as viable in terms of the context and identify any extraneous solutions.
Also NSO.1.6, AR.5.7, F.3.7
MA.K12.MTR.2.1, MTR.4.1, MTR.5.1

## CRITIQUE & EXPLAIN

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**Earthquakes make seismic waves through the ground. The equation $y = 10^x$ relates the height, or amplitude, in microns, of a seismic wave, $y$, and the power, or magnitude, $x$, of the ground-shaking it can cause.**

| Magnitude, $x$ | Amplitude, $y$ |
|---|---|
| 2 | 100 |
| 3 | 1,000 |
| **?** | 5,500 |
| 4 | 10,000 |

Taylor and Chen used different methods to find the magnitude of the earthquake with amplitude 5,500.

Taylor

5,500 is halfway between 1,000 and 10,000.

3.5 is halfway between 3 and 4.

The magnitude is about 3.5.

Chen

$y = 10^x$

$10^3 = 1{,}000$
$10^4 = 10{,}000$
$10^{3.5} \approx 3{,}162$
$10^{3.7} \approx 5{,}012$
$10^{3.8} \approx 6{,}310$
$10^{3.74} \approx 5{,}500$

The magnitude is about 3.74.

**A.** What is the magnitude of an earthquake with amplitude 100,000? How do you know?

**B.** **Communicate and Justify** Critique Taylor's and Chen's work. Is each method valid? Could either method be improved?

**C.** Describe how to express the exact value of the desired magnitude.

**ESSENTIAL QUESTION** What are logarithms and how are they evaluated?

**CONCEPTUAL UNDERSTANDING**

## EXAMPLE 1 Understand Logarithms

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**Solve the equations $2x = 8$ and $2^x = 8$.**

You can use inverse operations to solve the first equation.

$$\frac{2x}{2} = \frac{8}{2}$$
$$x = 4$$

Division is the inverse of multiplication, so you can divide both sides by 2 to solve the equation.

The operation in $2^x = 8$ is exponentiation. To solve this equation, you need an inverse for exponentiation that answers the question, "To what exponent would you raise the base 2 to get 8?"

The inverse of applying an exponential function is applying a *logarithm* function. To solve the equation $2^x = 8$, you can write $\log_2 8 = x$. Solving this gives $\log_2 8 = 3$ because $2^3 = 8$.

This is read "logarithm base 2 of 8" or "log base 2 of 8."

**USE PATTERNS AND STRUCTURE**

Creating the notation $\log_2 x$ to represent the exponent to which you raise 2 to get $x$ is similar to creating the radical notation $\sqrt{x}$ to represent one number you can square to get $x$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

The **logarithm** base $b$ of $x$ is defined as follows.

$$\log_b x = y \text{ if and only if } b^y = x, \text{ for } b > 0, b \neq 1, \text{ and } x > 0.$$

The **logarithmic function** $y = \log_b x$ is the inverse of the exponential function $y = b^x$.

**Try It!** **1.** Write the logarithmic form of $y = 8^x$.

## CONCEPT Exponential and Logarithmic Forms

**Exponential form** shows that a base raised to an exponent equals the result.

$$a^b = c$$

**Logarithmic form** shows that the log of the result with the given base equals the exponent.

$$\log_a c = b$$

When written in logarithmic form, the number that was the result of the exponential equation is often called the argument.

## EXAMPLE 2 Convert Between Exponential and Logarithmic Forms

**STUDY TIP**
Do you remember writing *fact families* for related operations like addition and subtraction? Think of exponential and logarithmic forms as a *fact family* for the three numbers given.

**A. What is the logarithmic form of $3^4 = 81$?**

The base is 3, the exponent is 4, and the result is 81.

So, in logarithmic form,

$3^4 = 81 \rightarrow \log_3 81 = 4.$

The logarithmic form of $3^4 = 81$ is $\log_3 81 = 4$.

**B. What is the exponential form of $\log_{10} 1{,}000 = 3$?**

The base is 10, the exponent is 3, and the result (or argument) is 1,000.

So, in exponential form,

$\log_{10} 1{,}000 = 3 \rightarrow 10^3 = 1{,}000.$

The exponential form of $\log_{10} 1{,}000 = 3$ is $10^3 = 1{,}000$.

**Try It!** **2. a.** What is the logarithmic form of $7^3 = 343$?

**b.** What is the exponential form of $\log_4 16 = 2$?

## EXAMPLE 3 Evaluate Logarithms

**GENERALIZE**
The output of any exponential function of the form $y = b^x$, with $b > 0$, is always a positive number. Therefore, the input of a logarithmic function must also be a positive number.

**What is the value of each logarithmic expression?**

| | |
|---|---|
| **A.** $\log_5 125$<br>THINK: $5^? = 125$<br>Since $5^3 = 125$, $\log_5 125 = 3$. | **B.** $\log_{\frac{1}{4}} 16$<br>THINK: $\left(\frac{1}{4}\right)^? = 16$<br>Since $\left(\frac{1}{4}\right)^{-2} = 16$, $\log_{\frac{1}{4}} 16 = -2$. |
| **C.** $\log_3 0$<br>THINK: $3^? = 0$<br>There is no such power, so $\log_3 0$ is undefined. | **D.** $\log_2 2^8$<br>THINK: $2^? = 2^8$<br>Since $2^8 = 2^8$, $\log_2 2^8 = 8$. |

**Try It!** **3.** What is the value of each logarithmic expression?

**a.** $\log_3\left(\frac{1}{81}\right)$ **b.** $\log_7(-7)$ **c.** $\log_5 5^9$

## CONCEPT Common Logarithms and Natural Logarithms

The base 10 logarithm is called the **common logarithm** and is written as $\log x$ with the base of 10 implied.

The base $e$ logarithm is called the **natural logarithm** and is written as $\ln x$.

The expressions $\log_{10} x$ and $\log x$ mean the same thing, as do $\log_e x$ and $\ln x$.

## EXAMPLE 4 Evaluate Common and Natural Logarithms

**STUDY TIP**
Most calculators have keys for the common logarithm (LOG) and the natural logarithm (LN).

**What is the value of each logarithmic expression to the nearest ten-thousandth?**

**A.** $\log 900$

$\log 900 \approx 2.9542$ $\quad 10^{2.9542} \approx 900$

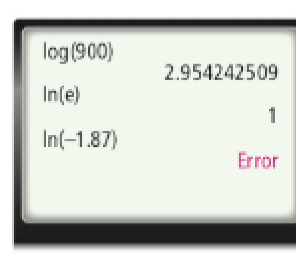
Check by writing the expression in exponential form and evaluating.

**B.** $\ln e$

$\ln e = 1$ $\quad e^1 = e$

**C.** $\ln(-1.87)$

$\ln(-1.87)$ $\quad e^? = -1.87$




There is no exponent to which $e$ can be raised in order to get a negative number, so $\ln(-1.87)$ is undefined.

**Try It!** **4.** What is the value of each logarithmic expression to the nearest ten-thousandth?

a. $\log 321$ b. $\ln 1{,}215$ c. $\log 0.17$

## EXAMPLE 5 Solve Equations With Logarithms

**What is the solution to each equation? Round to the nearest thousandth.**

**COMMON ERROR**
Remember that 10 is not a coefficient, but a base. You cannot divide both sides by 10 and then add 1 to solve for $x$.

**A.** $25 = 10^{x-1}$

$$25 = 10^{x-1}$$

$\log 25 = x - 1$ ········ Convert to logarithmic form.

$1 + \log 25 = x$ ········ Addition Property

$2.398 \approx x$ ········ Use calculator to evaluate.

**B.** $\ln(2x + 3) = 4$

$$\ln(2x + 3) = 4$$

$2x + 3 = e^4$ ········ Convert to exponential form.

$2x + 3 \approx 54.598$ ········ Use calculator to evaluate.

$2x \approx 51.598$ ········ Addition Property

$x \approx 25.799$ ········ Multiplication Property

**Try It!** **5.** Solve each equation. Round to the nearest thousandth.

**a.** $\log(3x - 2) = 2$ **b.** $e^{x+2} = 8$

**APPLICATION**

## EXAMPLE 6 Use Logarithms to Solve Problems

**The seismic energy, $x$, in joules can be estimated based on the magnitude, $m$, of an earthquake by the formula $x = 10^{1.5m+12}$. What is the magnitude of an earthquake with a seismic energy of $4.2 \times 10^{20}$ joules?**


**Formulate** Substitute $4.2 \times 10^{20}$ for $x$ in the formula.

$$4.2 \times 10^{20} = 10^{1.5m+12}$$

**Compute** Solve the equation for $m$.

$4.2 \times 10^{20} = 10^{1.5m+12}$ ········ Write the original equation.

$\log(4.2 \times 10^{20}) = 1.5m + 12$ ········ Write the equation in logarithmic form.

$20.6 \approx 1.5m + 12$ ········ Evaluate the logarithm.

$5.75 \approx m$ ········ Solve for $m$.

**Interpret** The magnitude of the earthquake is about 5.75.
Verify the answer: $10^{1.5(5.75)+12} \approx 4.2 \times 10^{20}$

**Try It!** **6.** What is the magnitude of an earthquake with a seismic energy of $1.8 \times 10^{23}$ joules?

## CONCEPT SUMMARY Logarithms

| | Exponential Form | | Logarithmic Form |
|---|---|---|---|
| ALGEBRA | $b^x = y$ | $\longleftrightarrow$ | $\log_b y = x$ |
| WORDS | The base raised to the exponent is equal to a result. | | The logarithm with a base b of the result (or argument) is equal to the exponent. |
| NUMBERS | $3^4 = 81$ | $\longleftrightarrow$ | $\log_3 81 = 4$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are logarithms and how are they evaluated?

2. **Error Analysis** Amir said the expression $\log_5(-25)$ simplifies to $-2$. Explain Amir's possible error.

3. **Vocabulary** Explain the difference between the common logarithm and the natural logarithm.

4. **Analyze and Persevere** How can logarithms help to solve an equation such as $10^t = 656$?

## Do You KNOW HOW?

**Write each given form in logarithmic form.**

**5.** $2^{-6} = \frac{1}{64}$ **6.** $e^4 \approx 54.6$

**Write each equation in exponential form.**

**7.** $\log 200 \approx 2.301$ **8.** $\ln 25 \approx 3.22$

**Evaluate the expression.**

**9.** $\log_4 64$

**10.** $\log \frac{1}{100}$

**11.** $\ln e^5$

**12.** Solve for $x$. $4e^x = 7$.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

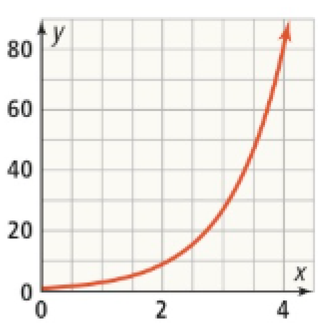
13. **Analyze and Persevere** If the LN button on your calculator were broken, how could you still use your calculator to find the value of the expression ln 65?

14. **Error Analysis** Describe and correct the error a student made in solving an exponential equation.

$$16e^t = 98$$
$$e^t = 6.125$$
$$6.125t = \ln e$$
$$t = \frac{\ln e}{6.125}$$

✗

15. **Higher Order Thinking** Use the graph of $y = 3^x$ to estimate the value of $\log_3 50$. Explain your reasoning.




16. **Generalize** For what values of $x$ is the expression $\log_4 x < 0$ true?

17. **Use Patterns and Structure** A student says that $\log_3\left(\frac{1}{27}\right)$ simplifies to $-3$. Is the student correct? Explain.

18. **Use Patterns and Structure** Why is the expression ln(1,000) not equal to 3?

### PRACTICE

**Write the inverse of each exponential function.**
SEE EXAMPLE 1

19. $y = 4^x$
20. $y = 10^x$
21. $y = 7^x$
22. $y = a^x$

**Write each equation in logarithmic form.**
SEE EXAMPLE 2

23. $3^8 = 6{,}561$
24. $e^{-3} \approx 0.0498$
25. $5^0 = 1$
26. $7^3 = 343$

**Write each equation in exponential form.**
SEE EXAMPLE 2

27. $\log\frac{1}{100} = -2$
28. $\log_8 64 = 2$
29. $\ln 148.41 \approx 5$
30. $\log_2 \frac{1}{32} = -5$

**Evaluate each logarithmic expression.** SEE EXAMPLE 3

31. $\log_5 \frac{1}{125}$
32. $\log_6(-216)$
33. $\log_3 3^4$
34. $\log_2 32$
35. $\log_9 729$
36. $\log_8 \frac{1}{64}$
37. $\log_7 0$
38. $\log_7 7^a$

**Use a calculator to evaluate each expression. Round to the nearest ten-thousandth.** SEE EXAMPLE 4

39. $\log 78.5$
40. $\log 0.24$
41. $\ln(-37)$
42. $\ln 41.5$
43. $\log 12$
44. $\ln 3$

**Solve each equation. Round answers to the nearest ten-thousandth.** SEE EXAMPLES 5 AND 6

45. $\log(7x + 6) = 3$
46. $2.75e^t = 38.6$
47. $\ln(3x - 1) = 2$
48. $10^{t+1} = 50$
49. $1.5e^t = 27$
50. $\log(x - 3) = -1$

51. How long does it take for \$250 to grow to \$600 at 4% annual percentage rate compounded continuously? Round to the nearest year.

## PRACTICE & PROBLEM SOLVING

### APPLY

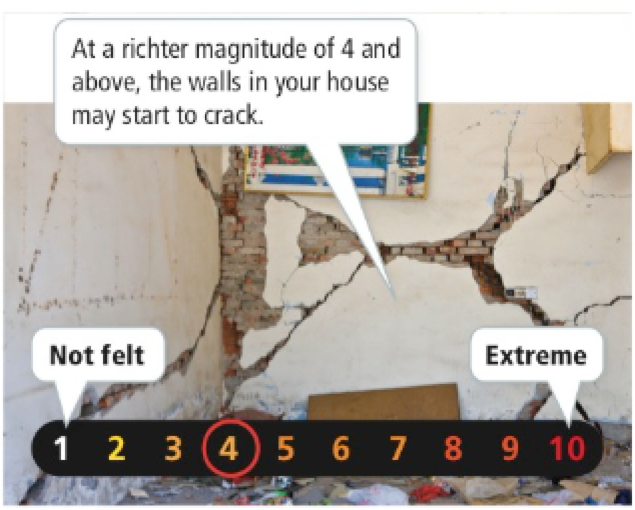
52. **Apply Math Models** Michael invests \$1,000 in an account that earns a 4.75% annual percentage rate compounded continuously. Peter invests \$1,200 in an account that earns a 4.25% annual percentage rate compounded continuously. Which person's account will grow to \$1,800 first?

53. **Analyze and Persevere** The Richter magnitude of an earthquake is $R = 0.67\log(0.37E) + 1.46$, where $E$ is the energy (in kilowatt-hours) released by the earthquake.

    a. What is the magnitude of an earthquake that releases 11,800,000,000 kilowatt-hours of energy? Round to the nearest tenth.

    b. How many kilowatt-hours of energy would an earthquake have to release in order to be an 8.2 on the Richter scale? Round to the nearest whole number.

    c. What number of kilowatt-hours of energy would an earthquake have to release in order for walls to crack? Round to the nearest whole number.




54. **Represent and Connect** The function $c(t) = 108e^{-0.08t} + 75$ calculates the temperature, in degrees Fahrenheit, of a cup of coffee that was handed out a drive-thru window $t$ minutes ago.

    a. What is the temperature of the coffee in the instant that it is handed out the window?

    b. After how many minutes is the coffee in the cup 98 degrees Fahrenheit? Round to the nearest whole minute.

### ASSESSMENT PRACTICE

55. Sandra invests \$500 in an account that earns a 2.5% annual percentage rate compounded continuously. How long will it take for her account to grow to \$700? AR.5.7

56. **SAT/ACT** In the equation $\log_3 a = b$, if $b$ is a whole number, which of the following CANNOT be a value for $a$?

    Ⓐ 1 Ⓑ 3 Ⓒ 6 Ⓓ 9 Ⓔ 81

57. **Performance Task** Money is deposited into two separate accounts. The interest in one account is compounded continuously. The money in the other account earns simple annual interest. Neither account has any money withdrawn in the first 6 years.

| Year | Account 1 Balance (\$) | Account 2 Balance (\$) |
|---|---|---|
| 0 | 400 | 500 |
| 1 | 433.31 | 575 |
| 2 | 469.40 | 650 |
| 3 | 508.50 | 725 |
| 4 | 550.85 | 800 |
| 5 | 596.72 | 875 |

**Part A** Write a function to calculate the amount of money in each account given $t$, the number of years since the account was opened. Describe the growth in each account.

**Part B** Will the amount of money in Account 1 ever exceed the amount of money in Account 2? Explain. If so, when will that occur?

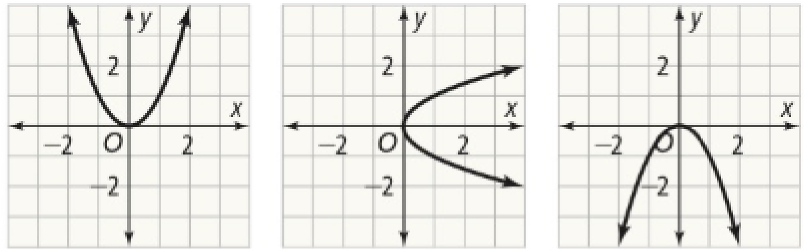
# 6-4 Logarithmic Functions

**I CAN…** graph logarithmic functions and find equations of the inverses of exponential and logarithmic functions.

MA.912.F.3.7–Represent the inverse of a function algebraically, graphically or in a table. Use composition of functions to verify that one function is the inverse of the other. Also AR.5.7, AR.5.8, AR.5.9, F.1.7, F.2.2, F.2.3, F.2.5
MA.K12.MTR.1.1, MTR.5.1, MTR.7.1

## EXPLORE & REASON

**Compare the graphs.**



**A.** Which two graphs represent inverse relations of each other? Explain.

**B.** **Use Patterns and Structure** What is the relationship between the domain and the range of the two inverse relations?

## ESSENTIAL QUESTION

**How is the relationship between logarithmic and exponential functions revealed in the key features of their graphs?**

## EXAMPLE 1 Identify Key Features of Logarithmic Functions

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**Graph $y = \log_2 x$. What are the domain, range, $x$-intercept, and asymptote? What is the end behavior of the graph?**

Create a table of values for $y = 2^x$.

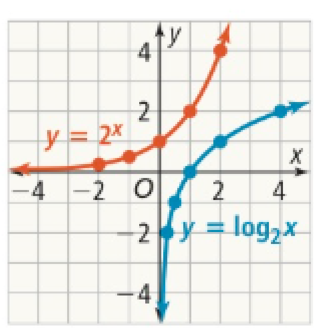
| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 0.25 | 0.5 | 1 | 2 | 4 |

$y = \log_2 x$ and $y = 2^x$ are inverse functions, so start by making a table of values for $y = 2^x$.

Interchange the corresponding $x$- and $y$-values. These ordered pairs represent $y = \log_2 x$.

| $x$ | 0.25 | 0.5 | 1 | 2 | 4 |
|---|---|---|---|---|---|
| $y$ | $-2$ | $-1$ | 0 | 1 | 2 |

Graph the ordered pairs.




domain: $\{x \mid x > 0\}$

The logarithmic function accepts only positive input values.

range: all real numbers

$x$-intercept: 1

asymptote: $y$-axis

The graph of the logarithmic function approaches $x = 0$ but does not touch it.

end behavior: As $x \to 0$, $y \to -\infty$.
As $x \to \infty$, $y \to \infty$.

**USE PATTERNS AND STRUCTURE**
The functions $y = \log_2 x$ and $y = 2^x$ are inverse functions. The graphs are reflections of each other across the line $y = x$.

**Try It!** **1.** Graph each function and identify the domain and range. List any intercepts or asymptotes. Describe the end behavior.

**a.** $y = \ln x$ **b.** $y = \log_{\frac{1}{2}} x$

## EXAMPLE 2 Graph Transformations of Logarithmic Functions, $f(x) = a\log_b(x - h) + k$

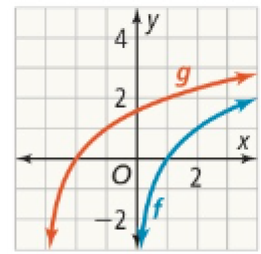
**Graph the function. How do the asymptote and *x*-intercept of the given function compare to those of the parent function?**

**$g(x) = \log_2(x + 3)$**

> **COMMON ERROR**
> With exponential functions, the value of *k* dictates the shift of the horizontal asymptote. With logarithmic functions, the asymptote is vertical. So the value of *h* determines the shift of the asymptote.

In *g*, the value of *h* is –3, so the logarithmic function is translated 3 units to the left.

$$f(x) = \log_2 x$$
$$g(x) = \log_2(x + 3) = f(x - (-3))$$




The vertical asymptote and the *x*-intercept each shift 3 units to the left.

**Try It!** **2.** Describe how each graph compares to the graph of $f(x) = \ln x$.

**a.** $g(x) = \ln x + 4$

**b.**

| $x$ | 0.1 | $e^{-1}$ | 1 | 2 | $e$ | $e^2$ |
|---|---|---|---|---|---|---|
| $h(x)$ | –11.51 | –5 | 0 | 3.47 | 5 | 10 |

**CONCEPTUAL UNDERSTANDING**

## EXAMPLE 3 Inverses of Exponential and Logarithmic Functions

**What is the equation of the inverse of the functions?**

**A.** $f(x) = 10^{x+1}$

> **GENERALIZE**
> $f(x)$ is a translation of the parent function $y = 10^x$ one unit left. $f^{-1}(x)$ is a translation of the parent function $y = \log x$ one unit down. Graphical translations of a function and its inverse are directly related, with horizontal and vertical effects switching places.

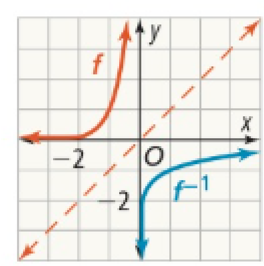
Write the function in $y = f(x)$ form and then interchange *x* and *y*.

$y = 10^{x+1}$ ............ Write the function in $y = f(x)$ form.

$x = 10^{y+1}$ ............ Interchange *x* and *y*.

$y + 1 = \log x$ ............ Write in log form.

$y = \log x - 1$ ............ Solve for *y*.




The equation of the inverse of $f(x) = 10^{x+1}$ is $f^{-1}(x) = \log x - 1$.

**B.** $g(x) = \log_7(x + 5)$

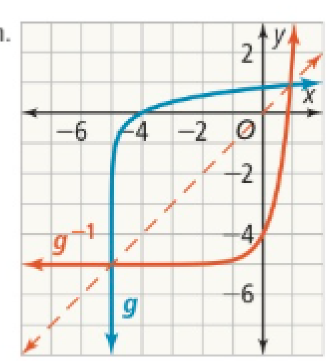
Write the function in $y = g(x)$ form and then interchange *x* and *y*.

$y = \log_7(x + 5)$ .... Write the function in $y = g(x)$ form.

$x = \log_7(y + 5)$ .... Interchange *x* and *y*.

$y + 5 = 7^x$ ............ Write in exponential form.

$y = 7^x - 5$ ............ Solve for *y*.




The equation of the inverse of $g(x) = \log_7(x + 5)$ is $g^{-1}(x) = 7^x - 5$.

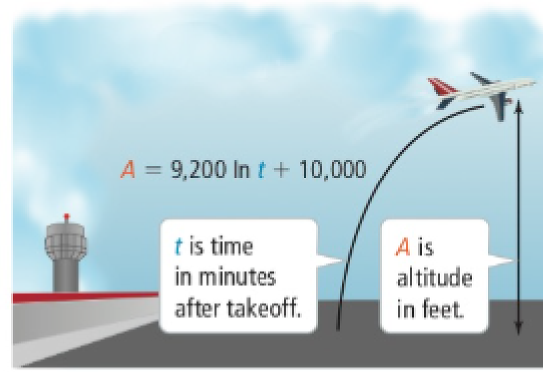
**Try It!** **3.** Find the inverse of each function.

**a.** $f(x) = 3^{x+2}$

**b.** $g(x) = \log_7 x - 2$

## EXAMPLE 4 Use a Logarithmic Function as a Model

**A logarithmic function can approximate the altitude of a plane over time.**




**A. What is the altitude of the plane 10 minutes after takeoff?**

Substitute 10 for $t$ in the equation of the function.

$$\begin{aligned} A &= 9{,}200 \ln 10 + 10{,}000 \\ &\approx 9{,}200(2.34) + 10{,}000 \\ &\approx 21{,}528 + 10{,}000 \\ &\approx 31{,}528 \end{aligned}$$

After 10 minutes, the plane's altitude is about 31,528 feet.

**B. How can you determine the time after takeoff when given the altitude of the plane? How long after takeoff will the plane reach an altitude of 35,000 feet?**

Rearrange the equation to isolate the variable $t$.

$$A = 9{,}200 \ln t + 10{,}000$$

$$A - 10{,}000 = 9{,}200 \ln t \quad \text{Subtract 10,000 from each side.}$$

$$\frac{A - 10{,}000}{9{,}200} = \ln t \quad \text{Divide each side by 9,200.}$$

$$t = e^{\left(\frac{A - 10{,}000}{9{,}200}\right)} \quad \text{Rewrite in exponential form.}$$

**STUDY TIP**
The original equation is a logarithmic function, while the rearranged equation is an exponential function. The exponential function is the inverse of the logarithmic function.

In the rearranged equation, time is a function of altitude, so it can be used to determine how long after takeoff the plane reaches a given altitude.

$$\begin{aligned} A &= e^{\left(\frac{35{,}000 - 10{,}000}{9{,}200}\right)} \\ &\approx 15.14 \end{aligned}$$

The plane will reach an altitude of 35,000 feet after about 15.14 minutes.

**C. What constraints are needed to make this logarithmic function a better model at the beginning of the flight?**

The logarithmic function has a vertical asymptote at $x = 0$, and on the domain interval (0, 0.337) the function is negative. The altitude of the plane is not correct during this interval.

The domain should be restricted to $x > 0.34$ seconds, with rounding, to eliminate the negative values of the logarithmic function.

**Try It!** **4.** Commercial airplanes usually fly no higher than 38,000 ft. How long will it take the plane to reach this altitude? How does a maximum cruising height affect the constraints of the model?

## EXAMPLE 5 Compose Exponential and Logarithmic Functions

**You are given the functions $f(x) = 2^{2x+3}$ and $g(x) = \log_2 x$. What is a simplified expression for the composed function $g \circ f$? What shape is its graph?**

$(g \circ f)(x) = \log_2(2^{2x+3})$ ········ Definition of function composition.

$2^{(g \circ f)(x)} = 2^{2x+3}$ ········ Rewrite in exponential form.

$(g \circ f)(x) = 2x + 3$ ········ Set exponents equal to each other.

The composed function $g \circ f$ is a linear function, so its graph is a line.

**Try It!** 5. Given $h(x) = 2^x$, what is a simplified expression for $h \circ g$? What shape is its graph?

## EXAMPLE 6 Use Logarithms and Regression to Find an Exponential Model

**A radioactive iodine isotope that decays over time is used for medical treatments. A hospital pharmacy must keep a back-up supply. When there is less than 40 mcg of the isotope, the pharmacy orders a new supply.**


| Day | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| Mass | 120.0 | 77.7 | 51.0 | 33.0 | 21.8 | 12.8 |

**A. Find a model for the amount of iodine isotope after $x$ days.**

Create a scatterplot.

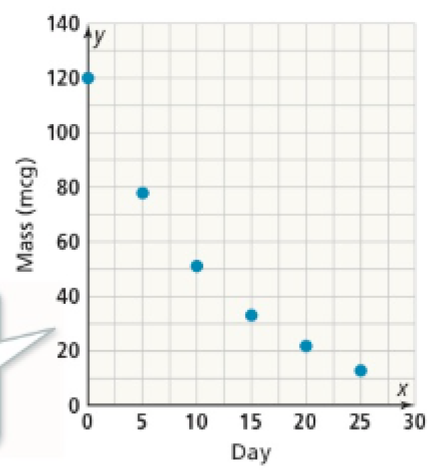
If the data can be modeled by an exponential function $y = a \cdot b^x$, taking the natural logarithm of the $y$-values will transform the data into a linear relationship.

$\ln(120.0) = 4.7875$

$\ln(77.7) = 4.3529$

and so on.

The data appear to be exponential, as they approach but do not cross the $x$-axis.

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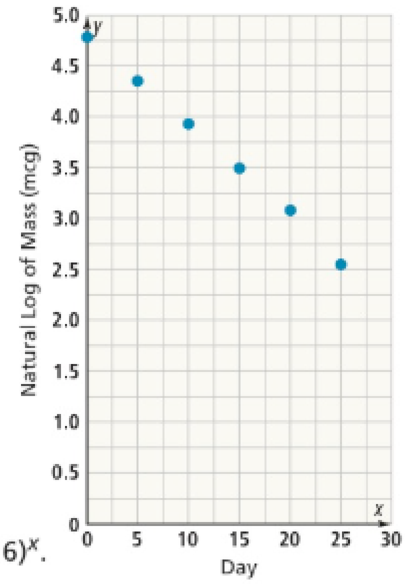
EXAMPLE 6 CONTINUED

| $x$ (days) | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $y$ (mcg) | 120.0 | 77.7 | 51.0 | 33.0 | 21.8 | 12.8 |
| $\ln y$ | 4.7875 | 4.3529 | 3.9318 | 3.4965 | 3.0819 | 2.5494 |

Graph the ordered pairs $(x, \ln(y))$.




The transformed data are approximately linear. Use linear regression on a calculator to find a line of fit for the transformed data.

$$\ln y \approx -0.0882x + 4.8028$$

You can use the line of fit to find the exponential model.

$$\ln y \approx -0.0882x + 4.8028$$
$$e^{\ln y} \approx e^{-0.0882x + 4.8028}$$
$$y \approx e^{4.8028} \cdot (e^{-0.0882})^x$$
$$y \approx 121.85 \cdot (0.916)^x$$

An exponential model is $y \approx 121.85 \cdot (0.916)^x$.

**APPLY MATH MODELS**
The initial value of the exponential model is e raised to the power of the $y$-intercept of the linear model. The decay factor of the exponential model is $e$ raised to the power of the slope.

**B. Should the pharmacy reorder a new supply before the 12th day?**

Use the model to find the amount of isotope on the 12th day.

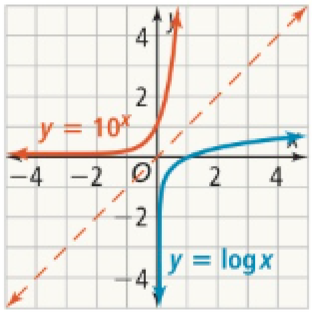
$$y \approx 121.85 \cdot (0.916)^{12}$$
$$y \approx 42.52$$

On the 12th day, there are approximately 42.52 mcg of iodine isotope. The hospital can wait until after the 12th day to reorder a supply.

**Try It!** **6.** The pharmacy hopes to keep a supply of at least 10 mcg of iodine isotope. Will the pharmacy have enough iodine on the 27th day if it has not received a new supply yet?

## CONCEPT SUMMARY Logarithmic Functions

**GRAPH**




The functions are inverses so their graphs are reflections of each other across the line with equation $y = x$.

| | | |
|---|---|---|
| **EQUATIONS** | $y = \log x$ | $y = 10^x$ |
| **KEY FEATURES** | Domain: $\{x \mid x > 0\}$<br>Range: all real numbers<br>$x$-intercept: 1<br>Asymptote: $y$-axis | Domain: all real numbers<br>Range: $\{y \mid y > 0\}$<br>$y$-intercept: 1<br>Asymptote: $x$-axis |
| **END BEHAVIOR** | As $x \to 0$, $y \to -\infty$<br>As $x \to \infty$, $y \to \infty$ | As $x \to -\infty$, $y \to 0$<br>As $x \to \infty$, $y \to \infty$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is the relationship between logarithmic and exponential functions revealed in the key features of their graphs?

2. **Error Analysis** Raynard claims the domain of the function $y = \log_3 x$ is all real numbers. Explain the error Raynard made.

3. **Communicate and Justify** How are the graphs of $f(x) = \log_5 x$ and $g(x) = -\log_5 x$ related?

4. **Use Patterns and Structure** Explain the necessary steps to find the inverse of $h(x) = \log_6(x + 4)$. Find the inverse.

## Do You KNOW HOW?

5. Graph the function $y = \log_4 x$ and identify the domain and range. List any intercepts or asymptotes. Describe the end behavior.

6. Write the equation for the function $g(x)$, which can be described as a vertical shift $1\frac{1}{2}$ units up from the function $f(x) = \ln x - 1$.

7. The function $y = 5 \ln(x + 1)$ gives $y$, the number of downloads, in hundreds, $x$ minutes after the release of a song. Find the equation of the inverse and interpret its meaning.


8. Sketch the functions represented by the tables. Identify which graph is the logarithmic function. Are the two functions inverses?

a.

| $x$ | $10^{-10}$ | 0.01 | 1 | 2 | 10 |
|---|---|---|---|---|---|
| $g(x)$ | −8 | 0 | 2 | 2.301 | 3 |

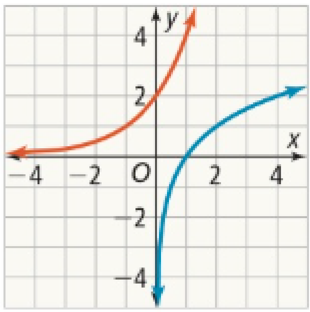
b.

| $x$ | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $h(x)$ | 0.001 | 0.01 | 0.1 | 1 | 10 | 100 |

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

9. **Use Patterns and Structure** Are the logarithmic and exponential functions shown inverses of each other? Explain.



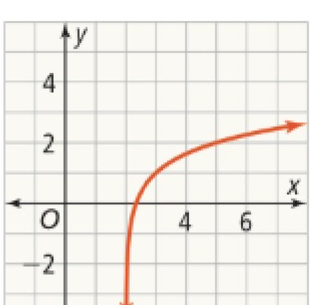

10. **Communicate and Justify** How is the graph of the logarithmic function $g(x) = \log_2 (x - 7)$ related to the graph of the function $f(x) = \log_2 x$? Explain your reasoning.

11. **Error Analysis** Describe and correct the error a student made in finding the inverse of the exponential function $f(x) = 5^{x-6} + 2$.

| | |
|---|---|
| $y = 5^{x-6} + 2$ | Write in $y = f(x)$ form. |
| $x = 5^{y-6} + 2$ | Interchange $x$ and $y$. |
| $x - 2 = 5^{y-6}$ | Subtract 2 from each side. |
| $\log_5 x - 2 = y - 6$ | Rewrite in logarithmic form. |
| $\log_5 x - 2 + 6 = y$ | Add 6 to each side. |
| $\log_5 x + 4 = y$ | Simplify. |
| $f^{-1}(x) = \log_5 x + 4$ | Rewrite as an inverse function. ✗ |

12. **Analyze and Persevere** The number of members $m$ who joined a new workout center $w$ weeks after opening is modeled by the equation $m = 1.6^{w+2}$, where $0 \leq w \leq 10$. Find the inverse of the function and explain what the inverse tells you.

13. **Use Patterns and Structure** The graph shows a transformation of the parent graph $f(x) = \log_3 x$. Write an equation for the graph.




## PRACTICE

**Graph each function and identify the domain and range. List any intercepts or asymptotes. Describe the end behavior.** SEE EXAMPLE 1

14. $y = \log_5 x$

15. $y = \log_8 x$

16. $y = \log_{\frac{3}{10}} x$

17. $y = \log_{0.1} x$

**Describe the graph in terms of transformations of the parent function $f(x) = \log_6 x$. Compare the asymptote and $x$-intercept of the given function to the parent function.** SEE EXAMPLE 2

18. $g(x) = \frac{1}{2} \log_6 x$

19. $g(x) = \log_6 (-x)$

20. Describe how the graph of $g(x) = -\ln(x + 0.5)$ is related to the graph of $f(x) = \ln x$. SEE EXAMPLE 2

**Find the equation of the inverse of each function.** SEE EXAMPLE 3

21. $f(x) = 5^{x-3}$

22. $f(x) = 6^{x+7}$

23. $f(x) = \ln (x + 3) - 1$

24. $f(x) = 4 \log_2 (x - 3) + 2$

25. The altitude $y$, in feet, of a plane $t$ minutes after takeoff is approximated by the function $y = 5{,}000 \ln(.05t) + 8{,}000$. Solve for $t$ in terms of $y$. SEE EXAMPLE 4

**For items 26–29, let $f(x) = e^x$, $g(x) = \ln x$, $h(x) = e^{-x^2}$, and $j(x) = \ln x^2$. Find a simplified expression for each composition of functions.** SEE EXAMPLE 5

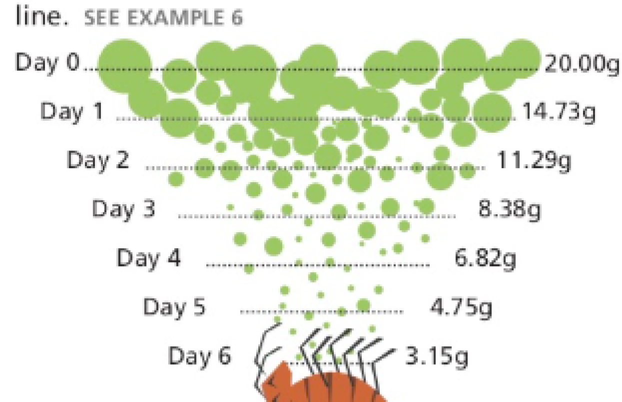
26. $f \circ g$

27. $g \circ f$

28. $g \circ h$

29. $f \circ j$

30. **Apply Math Models** A scientist is conducting an experiment with a pesticide. Use a calculator to find an exponential model for the data in the table. Use the model to determine how much pesticide remains after 180 days. Then transform the function so it graphs a straight line. SEE EXAMPLE 6



## PRACTICE & PROBLEM SOLVING

### APPLY

**31. Apply Math Models** The equation $r = 90 - 25 \log(t + 1)$ is to model a student's retention $r$ after taking a physics course where $r$ represents a student's test score (as a percent), and $t$ represents the number of months since taking the course.

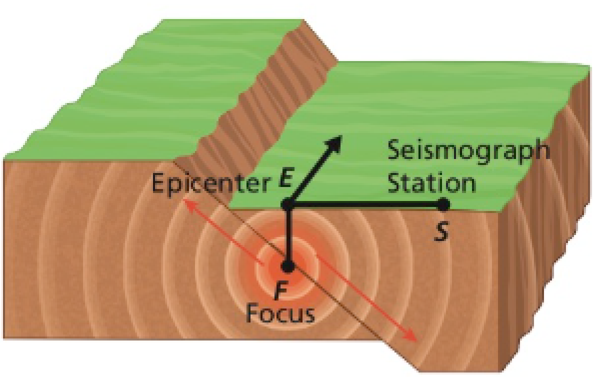
**a.** Make a table of values for ordered pairs that represent $r = 90 - 25 \log(t + 1)$, rounding to the nearest tenth. Then sketch the graph of the function on a coordinate plane through those ordered pairs. (You may use a graphing calculator to check.)

**b.** Find the equation of the inverse. Interpret the meaning of this function.

**32. Higher Order Thinking** As shown by the diagram, an earthquake occurs below Earth's surface at point $F$ (the focus). Point $E$, on the surface above the focus, is called the *epicenter*. A seismograph station at point $S$ records the waves of energy generated by the earthquake. The surface wave magnitude $M$ of the earthquake is given by this formula:

$$M = \log\left(\frac{A}{T}\right) + 1.66(\log D) + 3.3$$

In the formula, $A$ is the amplitude of the ground motion in micrometers, $T$ is the period in seconds, and $D$ is the measure of $\widehat{ES}$ in degrees.




**a.** Find surface wave magnitude of an earthquake with $A = 700$ micrometers, $T = 2$ and $D = 100°$.

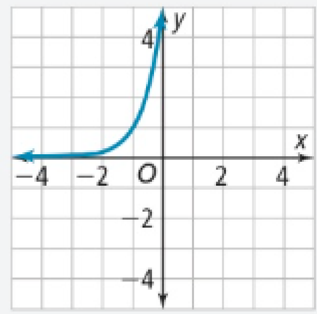
**b.** In the formula, $20° < D \leq 160°$. By how much can the size of arc $ES$ affect the surface wave magnitude? Explain.

### ASSESSMENT PRACTICE

**33.** Find the $x$-intercept of the logarithmic function $g(x) = \ln x - 1$. Then, graph the function.

AR.5.8

**34. SAT/ACT** The graph shows the exponential function $f(x) = 5^{x+1}$. Which of the following functions represents its inverse, $f^{-1}(x)$?




Ⓐ $f^{-1}(x) = 1 + \log_5 x$

Ⓑ $f^{-1}(x) = \log_5 x - 1$

Ⓒ $f^{-1}(x) = \log_5 (x - 1)$

Ⓓ $f^{-1}(x) = \log_5 (x + 1)$

**35. Performance Task** The logarithmic function $M(d) = 5 \log d + 2$ is used to find the limiting magnitude of a telescope, where $d$ represents the diameter of the lens of the telescope (mm) that is being used for the observation.

**Part A** Find the limiting magnitude of a telescope having a lens diameter of 40 mm.

**Part B** Find the equation of the inverse of this function.

**Part C** Interpret why astronomers may wish to use the inverse of this function. Justify your reasoning.

**Part D** Using the inverse function, find the diameter of the lens that has a limiting magnitude of 13.5. Check your answer with the table function of your graphing calculator.

# 6-5 Properties of Logarithms

**I CAN…** use properties of logarithms to rewrite expressions.

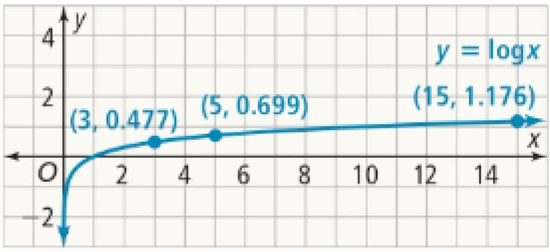
**VOCABULARY**

- Change of Base Formula

MA.912.NSO.1.6–Given a numerical logarithmic expression, evaluate and generate equivalent numerical expressions using the properties of logarithms or exponents. Also NSO.1.7, AR.5.2
MA.K12.MTR.4.1, MTR.5.1, MTR.6.1

## EXPLORE & REASON

**Look at the graph of $y = \log x$ and the ordered pairs shown.**




**A.** Complete the table shown.

| $x$ | 3 | 5 | 15 |
|---|---|---|---|
| $\log x$ | | | |

**B.** **Use Patterns and Structure** What is the relationship between the numbers 3, 5, and 15? What is the relationship between the logarithms of 3, 5, and 15?

**C.** What is your prediction for the value of log 45? log 75? Explain.

**ESSENTIAL QUESTION** How are the properties of logarithms used to simplify expressions and solve logarithmic equations?

## CONCEPT Properties of Logarithms

For positive numbers $b$, $m$, and $n$ with $b \neq 1$, the following properties hold.

$\log_b mn = \log_b m + \log_b n$ ········ Product Property of Logarithms

$\log_b \frac{m}{n} = \log_b m - \log_b n$ ········ Quotient Property of Logarithms

$\log_b m^n = n \log_b m$ ········ Power Property of Logarithms

## EXAMPLE 1 Prove a Property of Logarithms

**LEARN TOGETHER**
How do you listen actively as others share ideas?

**How can you prove the Product Property of Logarithms?**

Let $x = \log_b m$ and $y = \log_b n$. Then $b^x = m$ and $b^y = n$.

$b^x \cdot b^y = m \cdot n$ ········ Multiply the expressions $b^x$ and $b^y$.

$b^{x+y} = mn$ ········ Product Property of Exponents

$x + y = \log_b mn$ ········ Rewrite the equation in logarithmic form.

$\log_b m + \log_b n = \log_b mn$ ········ Substitute.

**Try It!** 1. Prove the Quotient Property of Logarithms.

## EXAMPLE 2 Expand Logarithmic Expressions

**ANALYZE AND PERSEVERE**
Expanding logarithmic expressions requires the use of a variety of properties. Look at the whole expression first to determine the sequence of properties that will be used. Can you determine an order of operations for Properties of Logarithms depending on your goal?

**How can you use the properties of logarithms to expand each expression?**

**A.** $\log_5(a^2b^7)$

$$\log_5(a^2b^7) = \log_5(a^2) + \log_5(b^7) \quad \text{Product Property of Logarithms}$$
$$= 2\log_5 a + 7\log_5 b \quad \text{Power Property of Logarithms}$$

**B.** $\ln\left(\frac{25}{3}\right)$

$$\ln\left(\frac{25}{3}\right) = \ln\left(\frac{5^2}{3}\right) \quad \text{Rewrite the numerator as a power.}$$
$$= \ln(5^2) - \ln 3 \quad \text{Quotient Property of Logarithms}$$
$$= 2\ln 5 - \ln 3 \quad \text{Power Property of Logarithms}$$

**Try It!** **2.** Use the properties of logarithms to expand each expression.

**a.** $\log_7\left(\frac{r^3t^4}{v}\right)$ **b.** $\ln\left(\frac{7}{225}\right)$

## EXAMPLE 3 Write Expressions as Single Logarithms

**STUDY TIP**
Recall that the Properties of Logarithms are each associated with a different operation. Addition signals the Product Property, subtraction signals the Quotient Property, and multiplication by a constant signals the Power Property.

**What is each expression written as a single logarithm?**

**A.** $4\log_4 m + 3\log_4 n - \log_4 p$

$$4\log_4 m + 3\log_4 n - \log_4 p$$
$$= \log_4(m^4) + \log_4(n^3) - \log_4 p \quad \text{Power Property of Logarithms}$$
$$= \log_4(m^4n^3) - \log_4 p \quad \text{Product Property of Logarithms}$$
$$= \log_4\left(\frac{m^4n^3}{p}\right) \quad \text{Quotient Property of Logarithms}$$

**B.** $3\ln 2 - 2\ln 5$

$$3\ln 2 - 2\ln 5 = \ln(2^3) - \ln(5^2) \quad \text{Power Property of Logarithms}$$
$$= \ln\left(\frac{2^3}{5^2}\right) \quad \text{Quotient Property of Logarithms}$$
$$= \ln\left(\frac{8}{25}\right) \quad \text{Simplify exponents}$$

**Try It!** **3.** Write each expression as a single logarithm.

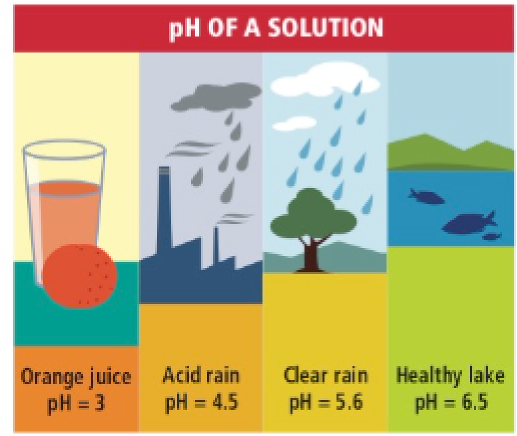
**a.** $5\log_2 c - 7\log_2 n$ **b.** $2\ln 7 + \ln 2$

APPLICATION

## EXAMPLE 4 Apply Properties of Logarithms

The pH of a solution is a measure of its concentration of hydrogen ions. This concentration (measured in moles per liter) is written $[H^+]$ and is given by the formula

$$\text{pH} = \log\frac{1}{[H^+]}.$$




**What is the concentration of hydrogen ions in the acid rainfall?**

$4.5 = \log\frac{1}{[H^+]}$ ........ Substitute 4.5 for pH.

$4.5 = \log 1 - \log[H^+]$ ........ Quotient Property

$-4.5 = \log[H^+]$ ........ Solve for $\log[H^+]$

$10^{-4.5} = H^+$ ........ Write in exponential form.

The concentration of hydrogen ions in the acid rainfall is $10^{-4.5} \approx 0.0000316$ moles per liter.

**COMMON ERROR**
The expression $\log\frac{1}{[H^+]}$ is equal to $\log 1 - \log[H^+]$, not $\log[1 - H^+]$.

**Try It!** 4. What is the concentration of hydrogen ions in a liter of orange juice?

CONCEPTUAL UNDERSTANDING

## EXAMPLE 5 Evaluate Logarithmic Expressions by Changing the Base

**How can you use base 10 logarithms to evaluate base 2 logarithms?**

To evaluate $\log_2 3$ with a calculator, you need to express $\log_2 3$ in terms of base 10 logarithms.

$\log_2 3 = \frac{\log_2 3 \cdot \log 2}{\log 2}$ ........ Multiply $\log_2 3$ by 1 in the form of $\frac{\log 2}{\log 2}$.

$= \frac{\log 2^{\log_2 3}}{\log 2}$ ........ Power Property of Logarithms.

$= \frac{\log 3}{\log 2}$ ........ Since $2^{\log_2 3} = 3$, the numerator simplifies to log 3.

Remember that exponents and logarithms are inverse operations, so they undo one another.

$= \frac{0.477}{0.301} \approx 1.585$ ........ Use a calculator to evaluate.

This illustrates the **Change of Base Formula**:

For positive numbers $m$, $b$, and $a$, with $b \neq 1$ and $a \neq 1$, $\log_b m = \frac{\log_a m}{\log_a b}$.

**STUDY TIP**
You can use other bases to solve the problem too. For example, $\frac{\ln 3}{\ln 2}$ is also equal to $\log_2 3$, using $e$ as the base of the logarithm in the Change of Base Formula.

**Try It!** 5. Estimate the value of each logarithm. Then use a calculator to find the value of each logarithm to the nearest thousandth.

a. $\log_2 7$ b. $\log_5 3$

## EXAMPLE 6 Use the Change of Base Formula

**What is the solution of the equation $2^x = 7$? Express the solution as a logarithm and then evaluate. Round to the nearest thousandth.**

$2^x = 7$ ........ Write the equation.

$x = \log_2 7$ ........ Rewrite in logarithmic form.

$x = \frac{\log 7}{\log 2}$ ........ Use the Change of Base Formula.

$x \approx 2.807$ ........ Use a calculator.

Check $2^{2.807} \approx 7$

**STUDY TIP**
The equation $2^x = 7$ can also be solved using natural logarithms.

**Try It!** 6. What is the solution to the equation $3^x = 15$? Express the solution as a logarithm, make an estimate, and then evaluate. Round to the nearest thousandth.

## CONCEPT SUMMARY Properties of Logarithms

| | Product Property | Quotient Property | Power Property | Change of Base |
|---|---|---|---|---|
| **ALGEBRA** | $\log_b(mn) = \log_b m + \log_b n$ | $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$ | $\log_b(m^n) = n \cdot \log_b m$ | $\log_b m = \frac{\log_a m}{\log_a b}$ |
| **WORDS** | The log of a product is the sum of the logs. | The log of a quotient is the difference of the logs. | The log of a number raised to a power is the power multiplied by the log of the number. | The log base $b$ of a number is equal to the log base $a$ of the number divided by the log base $a$ of $b$. |
| **NUMBERS** | $\log_2(20) = \log_2(4) + \log_2(5)$ | $\log_{10}\left(\frac{2}{3}\right) = \log_{10}2 - \log_{10}3$ | $\log_3(16) = 4 \cdot \log_3 2$ | $\log_5 7 = \frac{\log 7}{\log 5}$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the properties of logarithms used to simplify expressions and solve logarithmic equations?

2. **Vocabulary** While it is not necessary to change to base 10 when applying the Change of Base Formula, why is it common to do so?

3. **Error Analysis** Amanda claimed the expanded form of the expression $\log_4(c^2d^5)$ is $5\log_4 c + 5\log_4 d$. Explain the error Amanda made.

## Do You KNOW HOW?

4. Use the properties of logarithms to expand the expression $\log_6\left(\frac{49}{5}\right)$.

5. Use the properties of logarithms to write the expression $5\ln s + 6\ln t$ as a single logarithm.

6. Use the formula $\text{pH} = \log\frac{1}{[H^+]}$ to write an expression for the concentration of hydrogen ions, $[H^+]$, in a container of baking soda with a pH of 8.9.

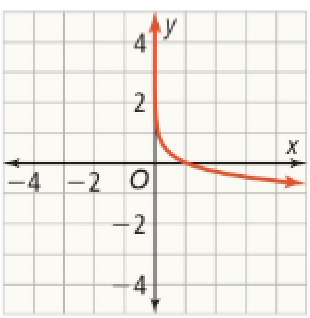
## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

7. **Use Patterns and Structure** Without applying the Change of Base Formula, explain how to use $\log_3 2 \approx 0.631$ and $\log_3 5 \approx 1.465$ to approximate $\log_3\left(\frac{2}{5}\right)$.

8. **Communicate and Justify** Explain what is meant by *expanding a logarithmic expression*. How are the processes of *expanding logarithmic expressions* and *writing logarithmic expressions as a single logarithm* related?

9. **Higher Order Thinking** The graph of $y = \log\left(\frac{1}{x}\right)$ and $y = -\log x$ are shown. Notice the graph is the same for both equations. Use properties of logarithms to explain why the graphs are the same.




10. **Check for Reasonableness** Emma used the Change of Base Formula to solve the equation $6^x = 72$ and found that $x = 2.387$. How can Emma check her solution?

11. **Error Analysis** Describe and correct the error a student made in writing the logarithmic expression in terms of a single logarithm.

$$\log_3 2 + \frac{1}{2}\log_3 y = \log_3 2y^2$$

12. **Error Analysis** A student wants to approximate $\log_2 9$ with her calculator. She enters the equivalent expression $\frac{\ln 2}{\ln 9}$, but the decimal value is not close to her estimate of 3. What happened?

$$\log_2 9 = \frac{\ln 2}{\ln 9}$$

### PRACTICE

13. Use the properties of exponents to prove the Power Property of Logarithms. SEE EXAMPLE 1

**Use the properties of logarithms to expand each expression.** SEE EXAMPLE 2

14. $\log_5\left(\frac{2}{3}\right)$

15. $\log_6(2m^5n^3)$

16. $\ln 2x^5$

17. $\log_2\left(\frac{x}{5y}\right)$

**Use the properties of logarithms to write each expression as a single logarithm.** SEE EXAMPLE 3

18. $9\ln x - 6\ln y$

19. $\log_5 6 + \frac{1}{2}\log_5 y$

20. $2\log 10 + 4\log(3x)$

21. $\frac{1}{3}\ln 27 - 3\ln(2y)$

22. $8\log_3 2 + 5\log_3 c + 7\log_3 d$

23. Use properties of logarithms to show that $\text{pH} = \log\frac{1}{[H^+]}$ can be written as $\text{pH} = -\log[\text{H}^+]$. SEE EXAMPLE 4

**Use the Change of Base Formula to evaluate each logarithm. Round to the nearest thousandth.** SEE EXAMPLE 5

24. $\log_4 9$

25. $\log_6 5$

26. $\ln 3$

27. $\log_2 7$

28. $\log_9 12$

29. $\ln 23$

**Use the Change of Base Formula to solve each equation for *x*. Give an exact solution as a logarithm and an approximate solution rounded to the nearest thousandth.** SEE EXAMPLE 6

30. $3^x = 4$

31. $5^x = 11$

32. $8^x = 10$

33. $2^x = 30$

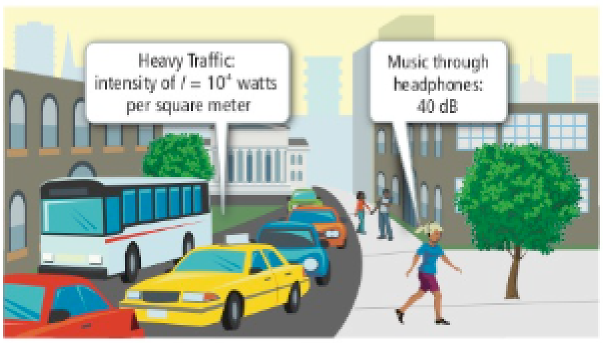
34. $7^x = 100$

35. $4^x = 55$

## PRACTICE & PROBLEM SOLVING

### APPLY

**36. Analyze and Persevere** The loudness of sound is measured in decibels. For a sound with intensity $I$ (in watts per square meter), its loudness $L(I)$ (in decibels) is modeled by the function $L(I) = 10 \log \frac{I}{I_0}$, where $I_0$ represents the intensity of a barely audible sound (approximately $10^{-12}$ watts per square meter).




**a.** Find the decibel level of the sound made by the heavy traffic.

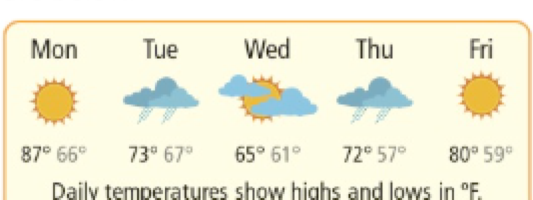
**b.** Find the intensity of the sound that is made by music playing at 40 decibels.

**c.** How many times as great is the intensity of the traffic than the intensity of the music?

**37. Apply Math Models** Miguel collected data on the attendance at an amusement park and the daily high temperature. He found that the model $A = 2\log t + \log 5$ approximated the attendance $A$, in thousands of people, at the amusement park, when the daily high temperature is $t$ degrees Fahrenheit.

**a.** Use properties of logarithms to simplify Miguel's formula.

**b.** The daily high temperatures for the week are below.




What is the expected attendance on Wednesday? Round to the nearest person.

### ASSESSMENT PRACTICE

**38.** Select all the expression equal to $\log_2 3$. NS0.1.6

☐ **A.** $\frac{\log 3}{\log 2}$

☐ **B.** $\frac{\log 2}{\log 3}$

☐ **C.** $\frac{\log_2 3}{\log_2 2}$

☐ **D.** $\frac{\log_3 2}{\log_3 3}$

☐ **E.** $\frac{\log_2 3}{\log_3 2}$

**39. SAT/ACT** Use the properties of logarithms to write the following expression in terms of a single logarithm.

$$2(\log_3 20 - \log_3 4) + 0.5 \log_3 4$$

Ⓐ $\log_3 4$

Ⓑ $\log_3 5$

Ⓒ $\log_3 25$

Ⓓ $\log_3 50$

**40. Performance Task** The magnitude, or intensity, of an earthquake is measured on the Richter scale. For an earthquake where the amplitude of its seismographic trace is $A$, its magnitude is modeled by the function:

$$R(A) = \log \frac{A}{A_0},$$

where $A_0$ represents the amplitude of the smallest detectable earthquake.

**Part A** An earthquake occurs with an amplitude 200 times greater than the amplitude of the smallest detectable earthquake, $A_0$. What is the magnitude of this earthquake on the Richter scale?

**Part B** Approximately how many times as great is the amplitude of an earthquake measuring 6.8 on the Richter scale than the amplitude of an earthquake measuring 5.9 on the Richter scale?

**Part C** Suppose the intensity of one earthquake is 150 times as great as that of another. How much greater is the magnitude of the more intense earthquake than the less intense earthquake?

# 6-6 Exponential and Logarithmic Equations

**I CAN…** solve exponential and logarithmic equations.

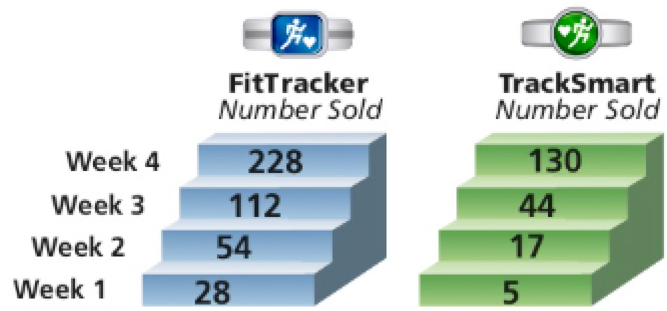
**VOCABULARY**

- exponential equation
- logarithmic equation

MA.912.AR.5.2–Solve one-variable equations involving logarithms or exponential expressions. Interpret solutions as viable in terms of the context and identify any extraneous solutions.
MA.K12.MTR.3.1, MTR.4.1, MTR.7.1

## MODEL & DISCUSS

**A store introduces two new models of fitness trackers to its product line. A glance at the data is enough to see that sales of both types of fitness trackers are increasing. Unfortunately, the store has limited space for the merchandise. The manager decides that the store will sell both models until sales of TrackSmart exceed those of FitTracker.**




**A.** **Apply Math Models** Find an equation of an exponential function that models the sales for each fitness tracker. Describe your method.

**B.** Based on the equations that you wrote, determine when the store will stop selling FitTracker.

## ESSENTIAL QUESTION

**How do properties of exponents and logarithms help you solve equations?**

**VOCABULARY**
An **exponential equation** is an equation that contains variables in the exponents.

### CONCEPT Property of Equality for Exponential Equations

**Symbols** Suppose $b > 0$ and $b \neq 1$, then $b^x = b^y$ if and only if $x = y$.

**Words** If two powers of the same base are equal, then their exponents are equal; if two exponents are equal, then the powers with the same base are equal.

## EXAMPLE 1 Solve Exponential Equations Using a Common Base

**What is the solution to $\left(\frac{1}{2}\right)^{x+7} = 4^{3x}$?**

$\left(\frac{1}{2}\right)^{x+7} = 4^{3x}$ ........ Write the original equation.

$(2^{-1})^{x+7} = (2^2)^{3x}$ ........ Rewrite each side with a common base.

$2^{-x-7} = 2^{6x}$ ........ Power of a Power Property

$-x - 7 = 6x$ ........ Property of Equality for Exponential Equations

$-7 = 7x$ ........ Add $x$ to each side.

$-1 = x$ ........ Divide each side by 7.

**Try It!** **1.** Solve each equation using a common base.

**a.** $25^{3x} = 125^{x+2}$ **b.** $0.001 = 10^{6x}$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Rewrite Exponential Equations Using Logarithms

**How can you rewrite the equation $17 = 4^x$ using logarithms?**

There is no common base for 17 and 4. Write each number as a power of 10.

$17 = 4^x$ ........ Write the original equation.

$10^{\log 17} = 10^{\log 4^x}$ ........ Write the equation using the powers of 10.

$\log 17 = \log 4^x$ ........ Property of Equality for Exponential Equations

Rewriting expressions using logarithms can help you solve many types of problems.

**Try It!** **2.** Rewrite the equation $5^x = 12$ using logarithms.

### CONCEPT Property of Equality for Logarithmic Equations

**Symbols** If $x > 0$, then $\log_b x = \log_b y$ if and only if $x = y$.

**Words** If two logarithms (exponents) of the same base are equal, then the quantities are equal; if two quantities are equal, and the bases are the same, then the logarithms (exponents) are equal.

## EXAMPLE 3 Solve Exponential Equations Using Logarithms

**What is the solution to $3^{x+1} = 5^x$?**

$3^{x+1} = 5^x$ ........ Write the original equation.

$\log(3^{x+1}) = \log(5^x)$ ........ Property of Equality for Logarithmic Equations

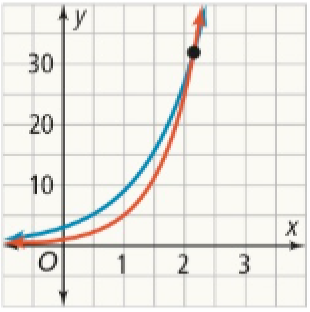
$(x + 1) \log 3 = x \log 5$ ........ Power Property of Logarithms

$x \log 3 + \log 3 = x \log 5$ ........ Use the Distributive Property.

$x(\log 3 - \log 5) = -\log 3$ ........ Move terms and factor out $x$.

$x = \dfrac{-\log 3}{\log 3 - \log 5}$ ........ Divide.

$x \approx 2.15$ ........ Evaluate.




**COMMON ERROR**
The entire quantity of $x + 1$ is the exponent, so it must be written as a quantity to be multiplied by the logarithmic expression.

**Check**

Substitute 2.15 into the equation:

$3^{x+1} = 3^{2.15+1} \approx 31.8$

$5^x = 5^{2.15} \approx 31.8$

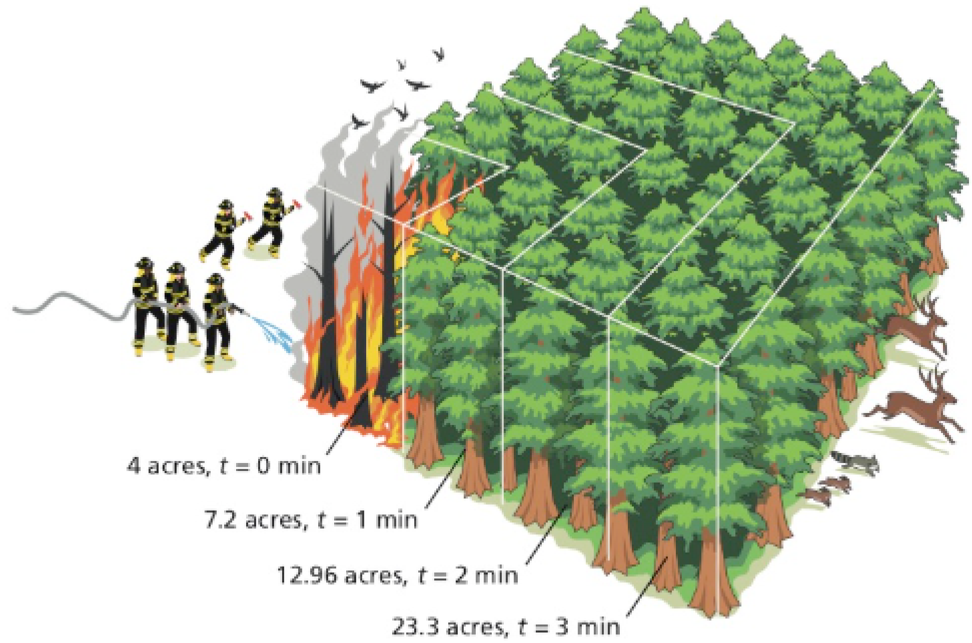
The point of intersection of the graphs is about (2.15, 31.8).

**Try It!** **3.** What is the solution to $2^{3x} = 7^{x+1}$?

APPLICATION

## EXAMPLE 4 Use an Exponential Model

**The diagram shows how a forest fire grows over time. The fire department can contain a 160-acre fire without needing additional resources. About how many minutes does it take for a fire to become too big for the fire department to contain without additional resources? Round to the nearest minute.**




**Formulate** Because the ratios of the number of acres from the diagram are all 1.8, the exponential growth model uses $b = 1.8$. The model is $160 = 4(1.8)^t$.

Use the model $y = ab^t$ where $y$ represents the number of acres, $a$ is the initial number of acres, $b$ is the growth rate of the fire, and $t$ is the number of minutes the fire has raged.

**Compute** Solve the equation for $t$.

$160 = 4(1.8)^t$ ........ Write the original equation.

$40 = (1.8)^t$ ........ Divide each side by 4.

$\log 40 = \log(1.8)^t$ ........ Property of Equality for Logarithmic Equations

$\log 40 = t \log 1.8$ ........ Power Property of Logarithms

$\dfrac{\log 40}{\log 1.8} = t$ ........ Isolate the variable $t$.

$6.276 \approx t$ ........ Evaluate.

**Interpret** Verify the answer by evaluating the expression $4(1.8)^{6.276}$.

$4(1.8)^{6.276} \approx 160.01$

The fire department has a little more than 6 minutes to contain the fire before they will require additional resources.

**Try It!** 4. About how many minutes does it take the fire to spread to cover 100 acres?

**VOCABULARY**
A **logarithmic equation** contains one or more logarithms of variable expressions.

## EXAMPLE 5 Solve Logarithmic Equations

**What is the solution to $\ln(x^2 - 16) = \ln(6x)$?**

$\ln(x^2 - 16) = \ln(6x)$ ........ Write the original equation.

$x^2 - 16 = 6x$ ........ Property of Equality for Logarithmic Equations

$x^2 - 6x - 16 = 0$ ........ Set quadratic equation equal to 0.

$(x - 8)(x + 2) = 0$ ........ Factor.

$x = 8$ or $-2$ ........ Apply the Zero Product Property.

**Check** Substitute each value into the original equation.

| $x = 8$ | $x = -2$ |
|---|---|
| $\ln(8^2 - 16) = \ln(6 \cdot 8)$ | $\ln((-2)^2 - 16) = \ln(6 \cdot (-2))$ |
| $\ln(48) = \ln(48)$ ✔ | $\ln(-12) = \ln(-12)$ ✘ |

Because logarithms are not defined for negative values, only $x = 8$ is a solution. The value $x = -2$ is an extraneous solution.

**Try It!** **5.** Solve each equation.

**a.** $\log_5(x^2 - 45) = \log_5(4x)$ **b.** $\ln(-4x - 1) = \ln(4x^2)$

## EXAMPLE 6 Solve Logarithmic and Exponential Equations by Graphing

Powered By desmos

**CHOOSE EFFICIENT METHODS**
When typing this equation into a calculator, it is helpful to write the equation using the Power Property of Logarithms rather than risking incorrect input of the exponent.

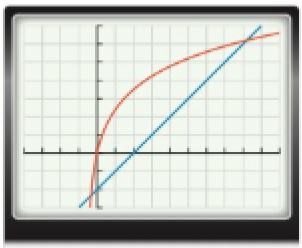
**What is the solution to $\log(2x + 1)^5 = x - 2$?**

Let $y_1 = 5\log(2x + 1)$ and $y_2 = x - 2$.

Graph both equations.

Use the INTERSECT feature to find the point(s) of intersection.

The points of intersection, to the nearest thousandth, are $(-0.329, -2.329)$ and $(8.204, 6.204)$.



x scale: 1 y scale: 1

**Check**

| | |
|---|---|
| $\log(2(-0.329) + 1)^5 = -0.329 - 2$ | $\log(2(8.204) + 1)^5 = 8.204 - 2$ |
| $\log(0.342)^5 = -2.329$ | $\log(17.408)^5 = 6.204$ |
| $-2.329 = -2.329$ ✔ | $6.204 = 6.204$ ✔ |

The solutions are $x \approx -0.329$ and $x \approx 8.204$.

**Try It!** **6.** Solve each equation by graphing. Round to the nearest thousandth.

**a.** $3(2)^{x+2} - 1 = 3 - x$ **b.** $\ln(3x - 1) = x - 5$

## CONCEPT SUMMARY Exponential and Logarithmic Equations

| | Property of Equality for Exponential Equations | | Property of Equality for Logarithmic Equations | |
|---|---|---|---|---|
| **ALGEBRA** | If b is a positive number other than 1, $b^x = b^y$ if and only if $x = y$. | | If b is a positive number other than 1, $\log_b x = \log_b y$ if and only if $x = y$. | |
| **WORDS** | If two powers of the same base are equal, then their exponents are equal. | If two exponents are equal, then the powers with the same base are equal. | If two logarithms of the same base are equal, then the arguments are equal. | If two arguments are equal and the bases are the same, then the logarithms are equal. |
| **NUMBERS** | If $2^x = 2^4$, then $x = 4$. | If $x = 4$, then $2^x = 2^4$. | If $\log_3 x = \log_3 8$, then $x = 8$. | If $x = 8$, then $\log_3 x = \log_3 8$. |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do properties of exponents and logarithms help you solve equations?

2. **Vocabulary** Jordan claims that $x^2 + 3 = 12$ is an exponential equation. Is Jordan correct? Explain your thinking.

3. **Communicate and Justify** How can properties of logarithms help to solve an equation such as $\log_6 (8x - 2)^3 = 12$?

## Do You KNOW HOW?

**Solve. Round to the nearest hundredth, if necessary. List any extraneous solutions.**

4. $16^{3x} = 256^{x+1}$

5. $6^{x+2} = 4^x$

6. $\log_5 (x^2 - 44) = \log_5 (7x)$

7. $\log_2 (3x - 2) = 4$

8. $4^{2x} = 9^{x-1}$

9. A rabbit farm had 200 rabbits in 2015. The number of rabbits increases by 30% every year. How many rabbits are on the farm in 2031?

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

10. **Use Patterns and Structure** Would you use the natural log or the common log when solving the equation $10^{x+2} = 78$? Is it possible to use either the natural log or common log? Explain.

11. **Analyze and Persevere** Explain why logarithms are necessary to solve the equation $3^{x+2} = 8$, but are not necessary to solve the equation $3^{x+2} = 27^{4x}$.

12. **Communicate and Justify** Tristen solved the equation $\log_3 (x + 1) - \log_3 (x - 6) = \log 3 (2x + 2)$. Justify each step of solving the equation in Tristen's work. Are both numbers solutions to the equation? Explain.

$$\log_3(x + 1) - \log_3(x - 6) = \log_3(2x + 2)$$
$$\log_3(x + 1) = \log_3(2x + 2) + \log_3(x - 6)$$
$$\log_3(x + 1) = \log_3(2x + 2)(x - 6)$$
$$(x + 1) = (2x + 2)(x - 6)$$
$$x + 1 = 2x^2 - 10x - 12$$
$$0 = 2x^2 - 11x - 13$$
$$x = 6.5 \text{ or } x = -1$$

13. **Error Analysis** The number of milligrams of medicine in a person's system after $t$ hours is given by the function $A = 20e^{-0.40t}$. Thomas sets $A = 0$ to find the number of hours it takes for all of the medicine to be removed from a person's system. What mistake did Thomas make? Explain.

14. **Mathematical Connections** Explain the importance of the Power Property of Logarithms when solving exponential equations.

15. **Error Analysis** Find the student error in the solution of the logarithmic equation.

$$\log (x + 3) + \log x = 1$$
$$\log x (x + 3) = 1$$
$$x(x + 3) = 10^1$$
$$x^2 + 3x - 10 = 0$$
$$(x - 2)(x + 5) = 0$$
$$x = 2, -5$$

✗

### PRACTICE

**Find all solutions of the equation. Round answers to the nearest ten-thousandth.** SEE EXAMPLE 1

16. $3^{2-3x} = 3^{5x-6}$
17. $7^{3x} = 54$
18. $25^{x^2} = 125^{x+3}$
19. $4^{3x-1} = \left(\frac{1}{2}\right)^{x+5}$
20. $4^{2x+1} = 4^{3x-5}$
21. $6^{x-2} = 216$

**Find all solutions of the equation. Round answers to the nearest ten-thousandth.** SEE EXAMPLES 2 AND 3

22. $2^{3x-2} = 5$
23. $4 + 5^{6-x} = 15$
24. $6^{3x+1} = 9^x$
25. $-3 = \left(\frac{1}{2}\right)^x - 12$
26. $3^{2x-3} = 4^x$
27. $4^{x+2} = 8^{x-1}$

28. Dale has \$1,000 to invest. He has a goal to have \$2,500 in this investment in 10 years. At what annual rate compounded continuously will Dale reach his goal? Round to the nearest hundredth. SEE EXAMPLE 4

**Find all solutions of the equation. Round answers to the nearest thousandth.** SEE EXAMPLE 5

29. $\log_2 (4x + 5) = \log_2 x^2$
30. $2\ln (3x - 2) = \ln (5x + 6)$
31. $\log_4 (x^2 - 2x) = \log_4 (3x + 8)$
32. $\ln (5x - 2) = \ln (x - 1)$
33. $\ln (2x^2 + 5x) = \ln (2x + 7)$
34. $2\log (x + 1) = \log (x + 1)$
35. $\log_2 x + \log_2 (x - 3) = 2$
36. $\log_2 (3x - 2) = \log_2 (x - 1) + 4$
37. $\log_6 (x^2 - 2x) = \log_6 (2x - 3) + \log_6 (x + 1)$

**Solve by graphing. Round answers to the nearest thousandth.** SEE EXAMPLE 6

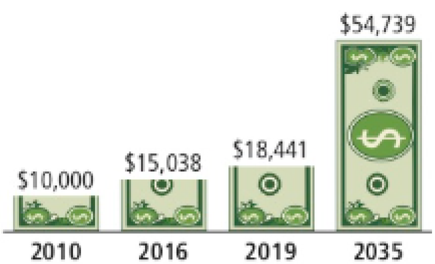
38. $\log(5x - 3)^2 = x - 4$
39. $\ln(2x) = 3x - 5$
40. $\log(4x) = x + \log x$

## PRACTICE & PROBLEM SOLVING

### APPLY

**41.** Apply Math Models The population of a city is modeled by the function $P = 250{,}000e^{0.013t}$, where $t$ is the number of years since 2000. In what year, to the nearest year, will the population reach 450,000?

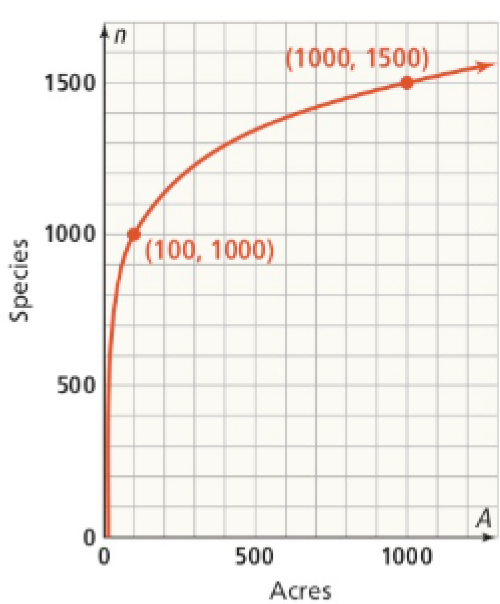
**42.** Use Patterns and Structure Felix invested \$10,000 into a retirement account in 2010. He then projected the amount of money that would be in the account for several years assuming that interest would compound continuously at an annual rate. Later, when he looked back the data, he could not recall the annual rate that he used for the projections. Use the data below to determine the annual rate.




**43.** Higher Order Thinking A biologist is using the logarithmic model $n = k\log(A)$ to determine the number of a species $n$, that can live on a land mass of area $A$. The constant $k$ varies according to the species.

a. Use the graph to determine the constant $k$ for the species that the scientist is studying.

b. Determine the land mass in acres that is needed to support 3,000 of the species.




### ASSESSMENT PRACTICE

**44.** A scientist has a sample of 125 bacteria cells. The number of cells doubles each week. After how many weeks will there be 16,000 bacteria cells? AR.5.2

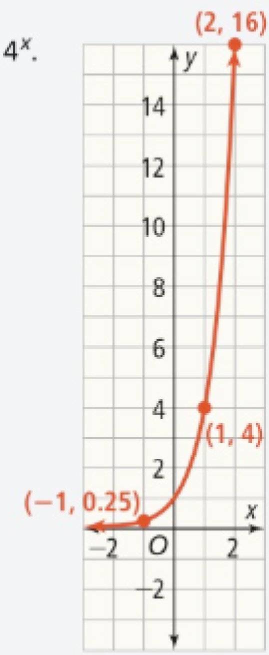
**45.** SAT/ACT The graph shows the function $y = 4^x$. Determine when the function shown in the graph is greater than the function $y = 2^{3x-1}$.

Ⓐ $x > 1$

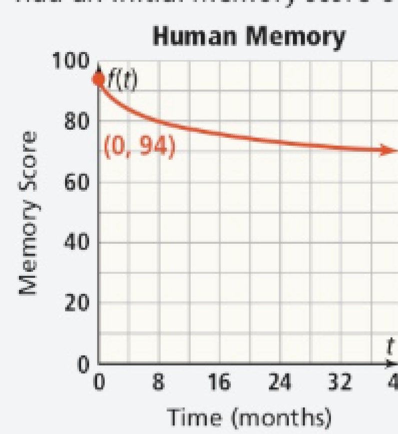
Ⓑ $x < 1$

Ⓒ $x > -1$

Ⓓ $x < -1$




**46.** Performance Task A professor conducted an experiment to find the relationship between time and memory. The professor determined the model $f(t) = t_0 - 15\log(t + 1.1)$ gives the memory score after $t$ months when a student had an initial memory score of $t_0$.




**Part A** Write a model for a student with the given initial memory score.

**Part B** After about how many years will the student have a memory score of 65?

# 6-7 Geometric Sequences

**I CAN…** identify, write, and use geometric sequences

**VOCABULARY**

- common ratio
- geometric sequence

MA.912.AR.10.2–Given a mathematical or real-world context, write and solve problems involving geometric sequences.
MA.K12.MTR.4.1, MTR.6.1, MTR.7.1

## EXPLORE & REASON

**A store offered customers two plans for getting bonus points:**


**A.** What expression represents the number of points received each day for Plan A?

**B.** What expression represents the number of points received each day for Plan B?

**C.** **Communicate and Justify** On the 7th day, which plan would offer the most bonus points? Explain.

## ESSENTIAL QUESTION

**How can you represent and use geometric sequences?**

## EXAMPLE 1 Identify Geometric Sequences

**A. Is the sequence shown in the table a geometric sequence? If so, write a recursive definition for the sequence.**

A **geometric sequence** is a sequence with a constant ratio between consecutive terms. This ratio is called the **common ratio**, $r$.

Notice the relationship between the terms in this sequence:

| Term Number ($n$) | Term ($a_n$) |
|---|---|
| 1 | 4 |
| 2 | 12 |
| 3 | 36 |
| 4 | 108 |
| 5 | 324 |

x3
x3

Each term is 3 times the preceding term.

This sequence is a geometric sequence, since for $n = 1$, $a_1 = 4$, $r = 3$, and $a_2 = a_1 \cdot 3$

$a_3 = a_2 \cdot 3$

$a_n = a_{n-1} \cdot 3$

First term

Term number

The recursive definition for this sequence is $a_n = \begin{cases} 4, & n = 1 \\ 3a_{n-1}, & n > 1. \end{cases}$

common ratio

The preceding term $a_{n-1}$ multiplied by the common ratio is the next term in the sequence $a_n$.

The general recursive defintion for a geometric sequence is $a_n = \begin{cases} a_1, & n = 1 \\ a_{n-1} \cdot r, & n > 1 \end{cases}$

CONTINUED ON THE NEXT PAGE

**STUDY TIP**
Notice that $r$ serves in a similiar capacity as $d$ in an arithmetic sequence. Where $d$ is added to each preceding term, $r$ is multiplied by each preceding term.

EXAMPLE 1 CONTINUED

**B. Is the sequence 12, 9.6, 7.68, 6.144, ... a geometric sequence? If so, write the recursive definition for the sequence.**

Find the ratio between consecutive terms.

$\frac{9.60}{12} = 0.80$ $\quad \frac{7.68}{9.6} = 0.80$ $\quad \frac{6.144}{7.68} = 0.80$

12 $\quad$ 9.60 $\quad$ 7.68 $\quad$ 6.144

The ratio between the consecutive terms is constant. This is a geometric sequence with $a_1 = 12$ and $r = 0.8$.

The recursive definition for the sequence is $a_n = \begin{cases} 12, & n = 1 \\ 0.8\, a_{n-1}, & n > 1 \end{cases}$

**Try It!** **1.** Is the sequence a geometric sequence? If so, write a recursive definition for the sequence.

**a.** 1.22, 1.45, 1.68, 1.91, ...

**b.** −1.5, 0.75, −0.375, 0.1875, ...

## EXAMPLE 2 Translate Between Recursive and Explicit Definitions

**A. Given the recursive definition $a_n = \begin{cases} 5, & n = 1 \\ \frac{1}{2} a_{n-1}, & n > 1 \end{cases}$ what is the explicit definition for the geometric sequence?**

**VOCABULARY**
Recall that an explicit definition allows you to find any term in the sequence without knowing the previous term.

The first term is 5, and the common ratio is $\frac{1}{2}$.

The recursive definition is $a_n = \frac{1}{2}a_{n-1}$. Use this definition to find a pattern:

$a_1 = 5$ ........ Identify the first term.

$a_2 = 5\left(\frac{1}{2}\right)$ ........ 1 common ratio multiplied to the first term

$a_3 = a_2 \cdot \left(\frac{1}{2}\right) = (5)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = (5)\left(\frac{1}{2}\right)^2$ ........ 2 common ratios multiplied to the first term

$a_4 = a_3 \cdot \left(\frac{1}{2}\right) = (5)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = (5)\left(\frac{1}{2}\right)^3$ ........ 3 common ratios multiplied to the first term

The pattern reveals the explicit definition for the sequence: $a_n = (5)\left(\frac{1}{2}\right)^{n-1}$.

The general explicit definition for any geometric sequence is: $a_n = a_1 r^{n-1}$.

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EXAMPLE 2 CONTINUED

**B. Given the explicit definition $a_n = 3(2)^{n-1}$, what is the recursive definition for the geometric sequence?**

From the explicit definition, $a_1 = 3$ and $r = 2$.

The recursive definition for the sequence is $a_n = \begin{cases} 3, & n = 1 \\ 2a_{n-1}, & n > 1 \end{cases}$.

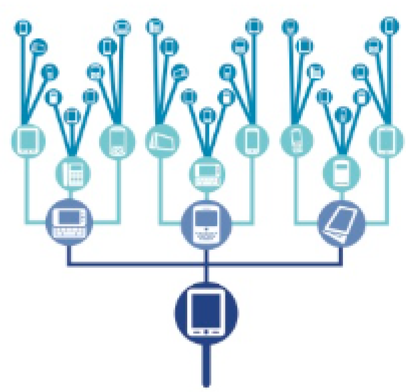
**Try It!** **2. a.** Given the recursive definition $a_n = \begin{cases} 12, & n = 1 \\ \frac{1}{3}a_{n-1}, & n > 1 \end{cases}$ what is the explicit definition for the sequence?

**b.** Given the explicit definition $a_n = 6(1.2)^{n-1}$, what is the recursive definition?

APPLICATION

## EXAMPLE 3 Solve Problems With Geometric Sequences

**A phone tree is when one person calls a certain number of people, then those people each call the same number of people, and so on. In the fifth round of calls, 243 people were called.**



**A. Write an explicit definition to find the number of people called in each round.**

GENERALIZE
Notice that the equation for an explicit definition is in the same form as the equation for an exponential function.

So $a_1 = 3$ and $a_5 = 243$. Use the explicit definition to find $r$.

$a_n = a_1 r^{n-1}$ Write the general explicit formula.

$243 = 3r^{5-1}$ Substitute 243 for $a_5$, 3 for $a_1$, and 5 for $n$.

$81 = r^4$ Isolate the power.

$3 = r$ Solve. Disregard the negative solution.

The explicit definition is $a_n = 3(3)^{n-1}$.

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EXAMPLE 3 CONTINUED

**B. How many people were called in the eighth round of the phone tree?**

Use the explicit definition and solve for $n = 8$.

$$a_r = a_1 r^{n-1}$$
$$a_8 = 3(3)^{8-1}$$
Substitute 8 for $n$ and simplify.
$$a_8 = 3(3)^7$$
$$a_8 = 6{,}561$$

On the eighth round, 6,561 people were called.

**Try It!** **3.** A geometric sequence can be used to describe the growth of bacteria in an experiment. On the first day of the experiment there were 9 bacteria in a Petri dish. On the 10th day, there are $3^{20}$ bacteria in the dish. How many bacteria were in the dish on the 7th day of the experiment?

## EXAMPLE 4 Number of Terms in a Finite Geometric Sequence

**How many terms are in the finite geometric sequence 200, 300, 450, …, 7688.7?**

**COMMON ERROR**
Be careful to calculate the common ratio as a term divided by the previous term, not the next term.

Since the sequence is geometric, you can find that $r = 1.5$. Use the explicit definition to find $n$, the number of terms.

$$7{,}688.7 = 200(1.5)^{n-1}$$
$$38.4435 = (1.5)^{n-1}$$
Substitute the last term in the sequence into the explicit definition to determine $n$.
$$\log 38.4435 = \log (1.5)^{n-1}$$
$$\log 38.4435 = (n - 1)\log (1.5)$$
$$\frac{\log 38.4435}{\log 1.5} = n - 1$$
$$n = \frac{\log 38.4435}{\log 1.5} + 1$$
$$n \approx 10$$

There are 10 terms in the finite geometric sequence.

**Try It!** **4.** How many terms are in the finite geometric sequence? 3, 6, 12, …, 768

## CONCEPT SUMMARY Geometric Sequences

In a geometric sequence, the ratio defined by a term divided by the previous term is a constant, $r$. Alternately, any term in a geometric sequence multiplied by $r$ gives the next term.

The sequence 1, 5, 25, 125, 625, … is a geometric sequence, since $r = 5$.

| WORDS | ALGEBRA | EXAMPLE |
|---|---|---|
| Each term in the sequence is $r$ times the previous term. | The recursive definition for a geometric sequence is $a_n = \begin{cases} a_1, & n = 1 \\ a_{n-1} \cdot r, & n > 1 \end{cases}$ | $a_n = \begin{cases} 1 & n = 1 \\ 5a_{n-1}, & n > 1 \end{cases}$ |
| The fourth term in a sequence is the first term multiplied by three common ratios. | The explicit definition is $a_n = a_1 r^{n-1}$ | $a_n = 1(5)^{n-1}$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you represent and use geometric sequences?

2. **Error Analysis** Denzel claims the sequence 0, 7, 49, 343, … is a geometric sequence and the next number is 2,401. What error did he make?

3. **Vocabulary** Describe the similarities and differences between a common difference and a common ratio.

4. **Use Patterns and Structure** What happens to the terms of a sequence if $a_1$ is positive and $r > 1$? What happens if $0 < r < 1$? Explain.

## Do You KNOW HOW?

**Find the common ratio and the next three terms of each geometric sequence.**

5. 2, −4, 8, −16, …

6. −64, −16, −4, −1, …

7. 0.8, 2.4, 7.2, 21.6, …

8. 2, −10, 50, −250, …

9. 100, 50, 25, 12.5, …

10. In a video game, players earn 10 points for finishing the first level and twice as many points for each additional level. How many points does a player earn for finishing the fifth level?

## PRACTICE & PROBLEM SOLVING

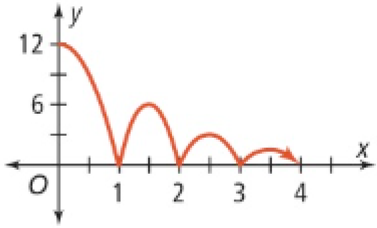
### UNDERSTAND

11. **Check for Reasonableness** True or False: If the first two terms of a geometric sequence are positive, then the third term is positive. Explain your reasoning.

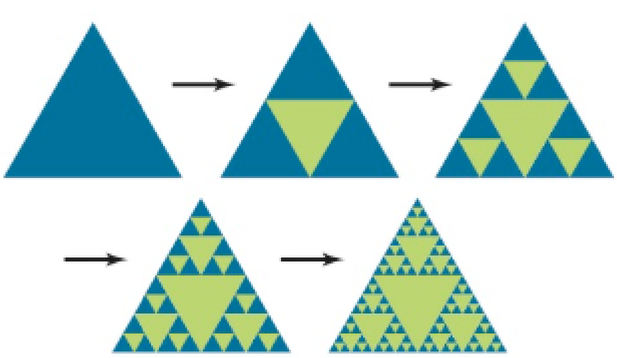
12. **Error Analysis** The first term of a geometric sequence is 4 and grows exponentially by a factor of 3. Murphy writes out the terms and says that the sum of the 4th and 5th terms is 1,296. Explain Murphy's error and correct it.

13. **Communicate and Justify** Write a geometric sequence with at least four terms and describe it using both an explicit and recursive definitions. How can you confirm that your sequence is geometric?

14. **Higher Order Thinking** Adam drops a ball from a height of 12 feet. Each bounce is 50% as high as the previous bounce. What is the vertical distance the ball travels between the 3rd and 4th times it hits the ground?




15. **Apply Math Models** The Sierpinski Triangle is a fractal made by cutting an equilateral triangle into four congruent pieces and removing the center piece, leaving three smaller triangles. The process is repeated on each triangle, creating more triangles that are even smaller. Continuing this pattern, how many triangles would there be after the tenth step in the process?



### PRACTICE

**Is the sequence geometric? If so, write a recursive definition for the sequence.** SEE EXAMPLE 1

16. 1, −3, 9, −27, …

17. 3, −15, 75, −375, …

18. 4, 5, 6, 7, …

19. 24, 8, $\frac{8}{3}$, $\frac{8}{9}$, …

20. 2, 4, 6, 8, …

21. 10, 40, 160, 640, …

**Translate between the recursive and explicit definitions for each sequence.** SEE EXAMPLE 2

22. $a_n = 1{,}024\left(\frac{1}{2}\right)^{n-1}$

23. $a_n = \begin{cases} 2, & n = 1 \\ -2a^{n-1}, & n > 1 \end{cases}$

24. $a_n = 35(2)^{n-1}$

25. $a_n = -6(-3)^{n-1}$

26. $a_n = \begin{cases} 1, & n = 1 \\ \frac{2}{3}a_{n-1}, & n > 1 \end{cases}$

27. In an experiment, the number of bacteria present each day form a geometric sequence. On the first day, there were 100 bacteria. On the eighth day, there were 12,800 bacteria. How many bacteria were there on the fourth day? SEE EXAMPLE 3

**Write the first 6 terms of each sequence.**
SEE EXAMPLE 4

28. $a_n = 4(2)^{n-1}$

29. $a_n = 6(2)^{n-1}$

30. $a_n = -4(3)^{n-1}$

31. $a_n = (-4)^{n-1}$

**Write an explicit definition for each geometric sequence. Then find what term the last number of the sequence is.** SEE EXAMPLE 4

32. 8, 16, 32, …, 1,024

33. −7, −42, −252, …, −54,432

34. $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{20}$, …, $\frac{1}{80}$

35. 4, −12, 36, …, 2,916

## PRACTICE & PROBLEM SOLVING

### APPLY

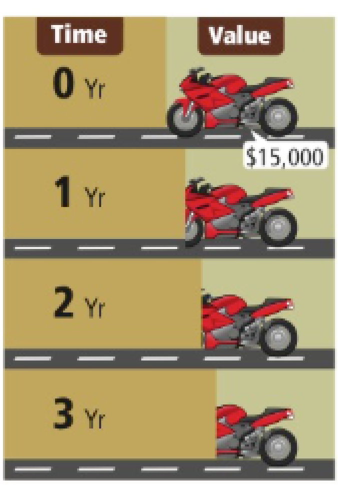
36. **Apply Math Models** Kelley opens a bank account to save for a new car. Her initial deposit is \$250, and she plans to deposit 10% more each month. In how many months will Kelley be depositing at least \$1,000 per month?

37. **Analyze and Persevere** Henry just started his own cleaning business. He is using word-of-mouth from his current clients to promote his business. He currently has seven clients.

    a. Five of his clients really like Henry's work and each told two friends the following month. This group each told two friends the following month, and so on for a total of five months. Assuming no one heard twice, how many people heard of a positive experience with Henry's cleaning business in the fifth month?

    b. The two unhappy clients each told five people the following month. This group each told five people, and so on, for five months. Assuming no one heard twice, how many people heard of a negative experience with Henry's cleaning business in the fifth month?

38. **Apply Math Models** Ricardo bought a motorcycle for \$15,000. The value depreciates 15% at the start of every year. What is the value of the motorcycle after three years?




### ASSESSMENT PRACTICE

39. A school has a phone tree in which 1 person calls 3 people, and then those people each call 3 people, and so on. How many people are called in the 8th round of calls? AR.10.2

40. **SAT/ACT** What is the value of the 11th term in the following geometric sequence?

$$\frac{1}{27}, \frac{1}{9}, \frac{1}{3}, \ldots$$

Ⓐ $3^4$
Ⓑ $3^5$
Ⓒ $3^6$
Ⓓ $3^7$
Ⓔ $3^8$

41. **Performance Task** An avid collector wants to purchase a signed basketball from a particular playoff game. To account for the increasing value of the basketball, he plans to put away 4% more money each year, in a safe at his home, to save up for the basketball. In the sixth year, he puts \$580 in the safe and realizes that he has exactly enough money to purchase the basketball.


**Part A** How much money did the collector put into the safe the first year?

**Part B** To the nearest dollar, how much did the collector pay for the signed playoff basketball? Create a table of the amount the collector put away each month and the total amount in the safe.

TOPIC 6

# Topic Review

## TOPIC ESSENTIAL QUESTION

1. How do you use exponential and logarithmic functions to model situations and solve problems?

## Vocabulary Review

**Choose the correct term to complete each sentence.**

2. A(n) __________ has base $e$.
3. A(n) __________ has the form $f(x) = a \cdot b^x$.
4. In an exponential function, when $0 < b < 1$, $b$ is a(n) __________.
5. The __________ is helpful for evaluating logarithms with a base other than 10 or $e$.
6. A(n) __________ has base 10.
7. The inverse of an exponential function is a(n) __________.

- decay factor
- exponential function
- logarithmic function
- growth factor
- common logarithm
- natural logarithm
- Change of Base Formula

## Concepts & Skills Review

### LESSON 6-1 Key Features of Exponential Functions

#### Quick Review

An **exponential function** has the form $f(x) = a \cdot b^x$. When $a > 0$ and $b > 1$, the function is an **exponential growth function**. When $a > 0$ and $0 < b < 1$, the function is an **exponential decay function**.

#### Example

**Paul invests \$4,000 in an account that pays 2.5% interest annually. How much money will be in the account after 5 years?**

Write and use the exponential growth function model.

$A(t) = a(1 + r)^n$

$A(5) = 4{,}000(1 + 0.025)^5$

$A(5) = 4{,}000(1.025)^5$

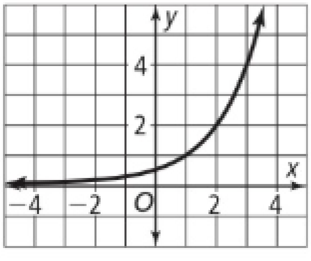
$A(5) = 4{,}525.63$

There will be about \$4,525.63 in Paul's account after 5 years.

#### Practice & Problem Solving

**Identify the domain, range, intercept, and asymptote of each exponential function. Then describe the end behavior.**

8. $f(x) = 400 \cdot \left(\frac{1}{2}\right)^x$
9. $f(x) = 2 \cdot (3)^x$
10. **Represent and Connect** Seth invests \$1,400 at 1.8% annual compound interest for 6 years. How much will Seth have at the end of the sixth year?
11. **Apply Math Models** Bailey buys a car for \$25,000. The car depreciates in value 18% per year. How much will the car be worth after 3 years?
12. Identify the domain, range, intercept, and asymptote.



## LESSON 6-2 Exponential Models

### Quick Review

Interest may be compounded over different time periods, such as quarterly, monthly, or daily. The formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ is used to calculate the amount of money available after it has been invested for an amount of time. Interest may also be compounded continuously. The formula $A = Pe^{rt}$ is used to calculate the amount of money available in an account in which the interest is compounded continuously.

### Example

**Jenny invests $2,500 in an account that pays 2.4% interest annually. The interest is compounded quarterly. How much will Jenny have in the account after 6 years?**

Use the formula $\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{nt}}$.

$A = 2{,}500\left(1 + \frac{0.024}{4}\right)^{4(6)}$ Substitute for $A$, $P$, $n$, and $r$.

$A = 2{,}500(1.006)^{24}$ Simplify.

$A = 2{,}885.97$ Use a calculator.

Jenny will have about $2,885.97.

### Practice & Problem Solving

**Find the total amount of money in the account after the given amount of time.**

**13.** Compounded quarterly, $P = \$12{,}000$, $r = 3.6\%$, $t = 4$ years

**14.** Compounded monthly, $P = \$5{,}000$, $r = 2.4\%$, $t = 8$ years

**15.** Continuously compounded, $P = \$7{,}500$, $r = 1.6\%$, $t = 10$ years

**Write an exponential model given two points.**

**16.** (12, 256) and (13, 302)

**17.** (3, 54) and (4, 74)

**18.** **Apply Math Models** Jason's parents invested some money for Jason's education when Jason was born. The table shows how the account has grown.

| Number of Years | Amount ($) |
|---|---|
| 1 | 2,250 |
| 3 | 2,525 |
| 6 | 3,480 |
| 7 | 4,400 |
| 9 | 6,000 |
| 13 | 9,250 |

Predict how much will be in the account after 18 years.

## LESSON 6-3 Logarithms

### Quick Review

A logarithm is an exponent. Common logarithms have base 10 and natural logarithms have base *e*. Exponential expressions can be rewritten in logarithmic form, and logarithmic expressions can be converted to exponential form.

$5^3 = 125$ can be rewritten as $\log_5 125 = 3$.

$\log 100 = 2$ can be rewritten as $10^2 = 100$.

### Example

Evaluate $\log_2 \frac{1}{8}$.

$\log_2 \frac{1}{8} = x$ ⋯⋯ Write an equation.

$2^x = \frac{1}{8}$ ⋯⋯ Rewrite the equation in exponential form.

$2^x = 2^{-3}$ ⋯⋯ Rewrite the equation with a common base.

$x = -3$ ⋯⋯ Since the two expressions have a common base, the exponents are equal.

### Practice & Problem Solving

**Use Patterns and Structure** **If an equation is given in exponential form, write the logarithmic form. If an equation is given in logarithmic form, write the exponential form.**

**19.** $4^3 = 64$

**20.** $10^2 = 100$

**21.** $\log_6 216 = 3$

**22.** $\ln 20 = x$

**Evaluate each logarithmic expression.**

**23.** $\log_8 \frac{1}{64}$

**24.** $\log_3 81$

**Choose Efficient Methods** **Evaluate each logarithmic expression using a calculator. Round answers to the nearest thousandth.**

**25.** $\log 628$

**26.** $\ln 0.55$

**Evaluate each logarithmic expression.**

**27.** $\log_5 5^9$

**28.** $7^{\log_7 49}$

## LESSON 6-4 Logarithmic Functions

### Quick Review

A logarithmic function is the inverse of an exponential function.

### Example

**Find the inverse of $f(x) = 10^{x-2}$. Identify any intercepts or asymptotes.**

$y = 10^{x-2}$ ⋯⋯ Write in $y = f(x)$ form.

$x = 10^{y-2}$ ⋯⋯ Interchange $x$ and $y$.

$y - 2 = \log x$ ⋯⋯ Write in log form.

$y = \log x + 2$ ⋯⋯ Solve for $y$.

The equation of the inverse is $f^{-1}(x) = \log x + 2$. It has an $x$-intercept at $x = \frac{1}{100}$ and a vertical asymptote at the $y$-axis.

### Practice & Problem Solving

**Use Patterns and Structure** **Graph each function and identify the domain and range. List any intercepts or asymptotes. Describe the end behavior.**

**29.** $f(x) = \log_4 x$

**30.** $f(x) = \ln(x - 2)$

**Use Patterns and Structure** **Find the equation of the inverse of each function.**

**31.** $f(x) = 8^{x-2}$

**32.** $f(x) = \frac{5^{x-2}}{8}$

## LESSON 6-5 Properties of Logarithms

### Quick Review

Properties of logarithms can be used to either expand a single logarithmic expression into individual logarithms or condense several logarithmic expressions into a single logarithm.

The Change of Base Formula can be used to find logarithms of numbers with bases other than 10 or $e$.

### Example

**Use the properties of logarithms to expand the expression $\log_6 \frac{x^3 y^5}{z}$.**

$\log_6 \frac{x^3 y^5}{z}$

$= \log_6 x^3y^5 - \log_6 z$ ········ Quotient Property of Logarithms

$= \log_6 x^3 + \log_6 y^5 - \log_6 z$ ···· Product Property of Logarithms

$= 3\log_6 x + 5\log_6 y - \log_6 z$ ···· Power Property of Logarithms

### Practice & Problem Solving

**Use Patterns and Structure** **Use the properties of logarithms to write each as a single logarithm.**

**33.** $3\log r - 2\log s + \log t$

**34.** $2\ln 3 + 4\ln 2 - \ln 36$

**Evaluate each logarithm using a calculator.**

**35.** $\log_4 12$

**36.** $\log_7 70$

**Analyze and Persevere** **Solve each equation for $x$. Give an exact solution written as a logarithm and use the Change of Base Formula to provide an approximated solution rounded to the nearest thousandth.**

**37.** $5^x = 200$

**38.** $7^x = 486$

## LESSON 6-6 Exponential and Logarithmic Equations

### Quick Review

You can solve exponential equations using the Property of Equality for Logarithmic Equations. You can solve a logarithmic equation by combining the logarithmic terms into one logarithm and then converting to exponential form.

### Example

**Solve $7^{2x} = 10^{x+1}$.**

$7^{2x} = 10^{x+1}$

$\log 7^{2x} = \log 10^{x+1}$ ········ Apply the Property of Equality for Logarithmic Equations.

$2x \log 7 = (x + 1) \log 10$ ··· Power Property of Logarithms

$2x \log 7 = x + 1$ ········ Since $\log 10 = 1$

$2x \log 7 - x = 1$ ········ Subtract $x$ from each side.

$x(2 \log 7 - 1) = 1$ ········ Factor out $x$.

$x = \frac{1}{2 \log 7 - 1}$ ········ Divide each side by $2 \log 7 - 1$.

$x \approx 1.449$ ········ Use a calculator.

### Practice & Problem Solving

**Find all solutions of the equation. Round answers to the nearest ten-thousandth.**

**39.** $2^{5x+1} = 8^{x-1}$

**40.** $9^{2x+3} = 27^{x+2}$

**41.** $3^{x-2} = 5^{x-1}$

**42.** $7^{x+1} = 12^{x-1}$

**Find all solutions of the equation.**

**43.** $\log_5 (3x - 2)^4 = 8$

**44.** $\ln(x^2 - 32) = \ln(4x)$

**45.** $\log_6(2x - 1) = 2 - \log_6 x$

**46.** **Apply Math Models** Geri has \$1,500 to invest. She has a goal to have \$3,000 in this investment in 10 years. At what annual rate, compounded continuously, will Geri reach her goal? Round the answer to the nearest tenth.

## LESSON 6-7 Geometric Sequences

### Quick Review

A geometric sequence is defined by a common ratio between consecutive terms. It can be defined explicitly or recursively.

### Example

**A geometric sequence is defined by**

$$a_n = \begin{cases} \frac{1}{9}, & n = 1 \\ 3a_{n-1}, & n > 1 \end{cases}.$$

**What is the tenth term of this sequence?**

$a_n = \frac{1}{9}(3)^{n-1}$ ········· Write the explicit definition.

$a_{10} = \frac{1}{9}(3)^9 = 2187$ ········· Find the 10th term.

### Practice & Problem Solving

**Determine whether or not each sequence is geometric.**

**47.** 2, 4, 6, 8, 10, 12, …

**48.** 2, 4, 8, 16, 32, 64, …

**Convert between recursive and explicit forms.**

**49.** $a_n = \begin{cases} \frac{1}{8}, & n = 1 \\ \frac{3}{2}a_{n-1}, & n > 1 \end{cases}$

**50.** $a_n = -2(5)^{n-1}$

**Write the first 5 terms of each sequence.**

**51.** $a_n = 6(2)^{n-1}$

**52.** $a_n = 81\left(\frac{1}{3}\right)^{n-1}$

**53. Analyze and Persevere** The half-life of carbon-14 is 5,730 years. This is the amount of time it takes for half of a sample to decay. From a sample of 24 grams of carbon 14, how long will it take until only 3 grams of the sample remains?

TOPIC

7

# Matrices

## TOPIC ESSENTIAL QUESTION

How can you use matrices to help you solve problems?

## Topic Overview

**enVision™ STEM Project:**
Analyzing Traffic With Markov Chains

**7-1** Operations With Matrices
NSO.4.1, NSO.4.3, MTR.2.1, MTR.3.1, MTR.5.1

**7-2** Matrix Multiplication
NSO.4.1, NSO.4.3, MTR.1.1, MTR.2.1, MTR.6.1

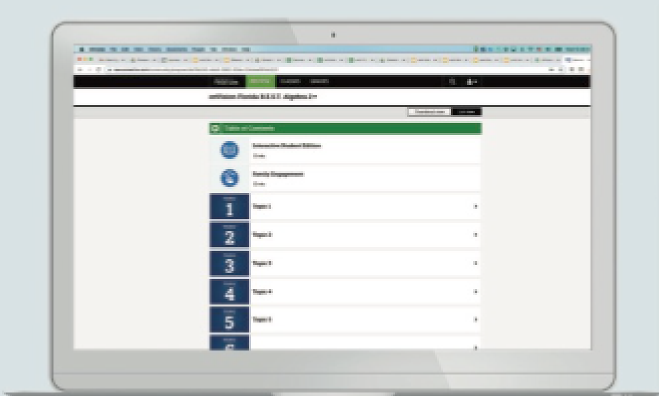
**7-3** Inverses and Determinants
NSO.4.4, MTR.1.1, MTR.3.1, MTR.4.1

**7-4** Inverse Matrices and Systems of Equations
NSO.4.2, NSO.4.4, MTR.4.1, MTR.5.1, MTR.7.1

**Mathematical Modeling in 3 Acts:**
The Big Burger
NSO.4.2, NSO.4.4, MTR.7.1

## Topic Vocabulary

- coefficient matrix
- component form
- constant matrix
- determinant of a $2 \times 2$ matrix
- equal matrices
- identity matrix
- inverse matrix
- matrix, matrices
- scalar
- scalar multiplication
- square matrix
- variable matrix
- zero matrix



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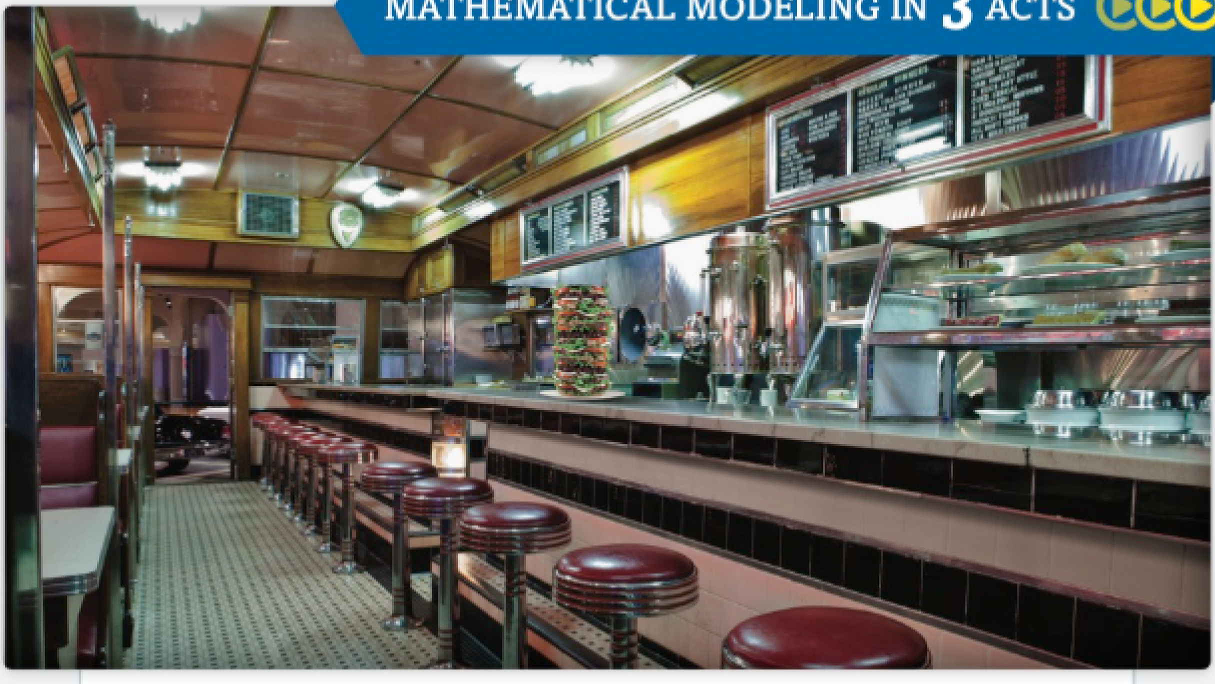
**ANIMATION** View and interact with real-world applications.

**PRACTICE** Practice what you've learned.

## MATHEMATICAL MODELING IN 3 ACTS



### The Big Burger

For many people, hamburgers are a hallmark of American food. Nearly every restaurant, from fast food chains, to diners, to fine dining establishments, offers some kind of hamburger on their menu. Some restaurants offer various types of burgers: beef, turkey, and veggie burgers are all quite popular. You can also often choose extras to add to your burger: double patties of beef, cheese, pickles, onions, lettuce, tomatoes . . . The options are endless! Think about this during the Mathematical Modeling in 3 Acts lesson.

**VIDEOS** Watch clips to support *Mathematical Modeling in 3 Acts Lessons* and **enVision**® ***STEM Projects***.

**ADAPTIVE PRACTICE** Practice that is *just right* and *just for you*.

**GLOSSARY** Read and listen to English and Spanish definitions.

**CONCEPT SUMMARY** Review key lesson content through multiple representations.

**ASSESSMENT** Show what you've learned.

**TUTORIALS** Get help from ***Virtual Nerd***, right when you need it.

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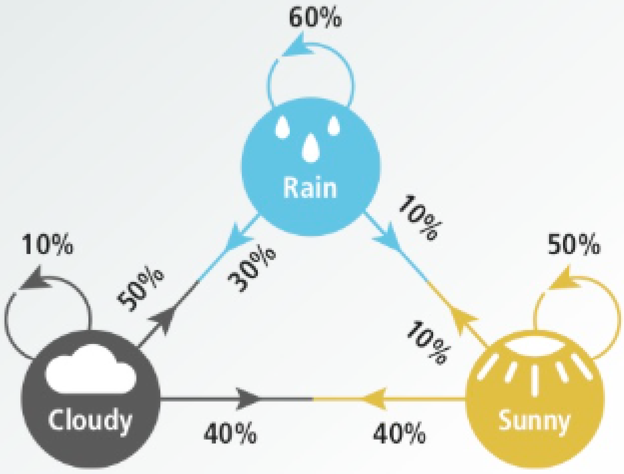
## Did You Know?

A **Markov chain** is a mathematical system that explores the probabilities of changing from one state to another. A diagram can be used to represent a Markov chain, such as **weather conditions.**




Markov chains and probability matrices are used in many fields, including **weather analysis, the outcomes of sporting events**, and **predictive texting.**

A **large matrix** can represent all the connections between roads in a city. Powers of this matrix can be used to **estimate traffic patterns,** including the effects of closing or expanding a road.

## Your Task: Analyzing Traffic With Markov Chains

You and your classmates will create a matrix to represent a map, then test the effects of road closures or expansion.

# 7-1
## Operations With Matrices

**I CAN…** interpret the parts of a matrix and use matrices for addition, subtraction, and scalar multiplication.

**VOCABULARY**
- equal matrices
- scalar
- scalar multiplication
- zero matrix

MA.912.NSO.4.1-Given a mathematical or real-world context, represent and manipulate data using matrices. Also NSO.4.3
MA.K12.MTR.2.1, MTR.3.1, MTR.5.1

## MODEL & DISCUSS

**This screen shows the number of Small, Medium, Large, and Extra Large limited-edition silkscreen shirts on sale at an online store.**


A. Construct a table to summarize the inventory that is on sale.

B. At the end of the day, the store has sold this many of each T-shirt from the sale items: red: 4 S, 6 M, 3 L, 5 XL; blue: 2 S, 8 M, 4 L, 0 XL. Make two new tables, one showing the merchandise sold and one showing the inventory that is left.

C. **Use Patterns and Structure** What relationships did you use in creating the two tables in Part B?

## ESSENTIAL QUESTION

**How can you interpret matrices and operate with matrices?**

## EXAMPLE 1 Represent Data With a Matrix

**A. What could the data values in the matrix represent?**

$$\begin{bmatrix} 5 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix}$$

A **matrix** is a rectangular array of values. Matrices help organize information. For example, if the columns of this matrix represent sizes (Small, Medium, Large) and the rows represent clothing items (Shirts, Sweaters), then "4" can represent 4 large sweaters.

$$\begin{array}{l} \\ \text{Shirts} \\ \text{Sweaters} \end{array} \begin{array}{c} \begin{array}{ccc} \text{S} & \text{M} & \text{L} \end{array} \\ \begin{bmatrix} 5 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix} \end{array}$$

**COMMUNICATE AND JUSTIFY**
The dimensions of a matrix are given in the form of row by column, often written as $r \times c$. What are the dimensions of this matrix?

**B. How can you refer to an entire matrix or to the elements of the matrix?**

$$A = \begin{bmatrix} 7 & 5 \\ -2 & 0 \\ 4 & 8 \end{bmatrix}$$

A capital letter is used to refer to the entire matrix. This matrix is referred to as Matrix $A$, or just $A$.

A specific element can be referred to using subscripts to indicate the row and column where the element is located. In general, $a_{ij}$ indicates the element in row $i$ and column $j$.

$$\begin{array}{c} \\ g \\ h \\ i \end{array} \begin{array}{c} \begin{array}{cc} j & k \end{array} \\ \begin{bmatrix} a_{gj} & a_{gk} \\ a_{hj} & a_{hk} \\ a_{ij} & a_{ik} \end{bmatrix} \end{array}$$

The subscript lists the row number, then the column number. In a general matrix, $a_{32}$ refers to the element in row 3, column 2, which is the number 8 in matrix $A$.

CONTINUED ON THE NEXT PAGE

**LEARN TOGETHER**
How can you provide constructive feedback while being aware of the feelings and reactions of others?

EXAMPLE 1 CONTINUED

**C. What are the values of the variables in the matrix equation?**

$$\begin{bmatrix} 12 & 0 \\ 2x & 10 \end{bmatrix} = \begin{bmatrix} 12 & y \\ 14 & 10 \end{bmatrix}$$

**Equal matrices** have the same dimensions, and corresponding elements are equal.

$a_{11}$, $12 = 12$. $a_{12}$, $0 = y$, so $y = 0$.

$a_{21}$, $2x = 14$, so $x = 7$. $a_{22}$, $10 = 10$.

The matrices are equal for $y = 0$ and $x = 7$.

**Try It!** **1.** In matrix $C$, the entries are the numbers of students on a committee. Column 1 lists girls, column 2 lists boys, row 1 lists sophomores, and row 2 lists juniors. Find $a_{12}$, $a_{21}$, and $a_{22}$, and tell what each number represents.

$$C = \begin{bmatrix} 7 & 5 \\ 8 & 10 \end{bmatrix}$$

APPLICATION

## EXAMPLE 2 Apply Scalar Multiplication

**Cool Threads, a clothing store, uses a matrix $C$ to represent the prices of women's clothes.**

$$C = \begin{bmatrix} 320 & 210 & 160 \\ 240 & 110 & 65 \end{bmatrix}$$

**The columns represent the brands Vintage, Casual, and Distressed, and the rows represent jeans and jackets. If the sales tax rate is 5%, how can you use a matrix operation to find the amount of sales tax on each item?**

Formulate ◀ If the sales tax rate is 5%, you can multiply each element in the matrix by 0.05 to get $S$, the matrix of sales tax amounts.

**Scalar multiplication** is the multiplication of each element in a matrix by a single real number, called a **scalar**.

Compute ◀

$$S = 0.05 \cdot C$$

$$= 0.05 \cdot \begin{bmatrix} 320 & 210 & 160 \\ 240 & 110 & 65 \end{bmatrix}$$

Scalar multiplication does not change the dimensions or the meanings of the rows and columns of the original matrix.

$$= \begin{bmatrix} (320)(0.05) & (210)(0.05) & (160)(0.05) \\ (240)(0.05) & (110)(0.05) & (65)(0.05) \end{bmatrix} = \begin{bmatrix} 16 & 10.50 & 8 \\ 12 & 5.50 & 3.25 \end{bmatrix}$$

Interpret ◀ The new matrix represents the sales tax amounts for each item of clothing.

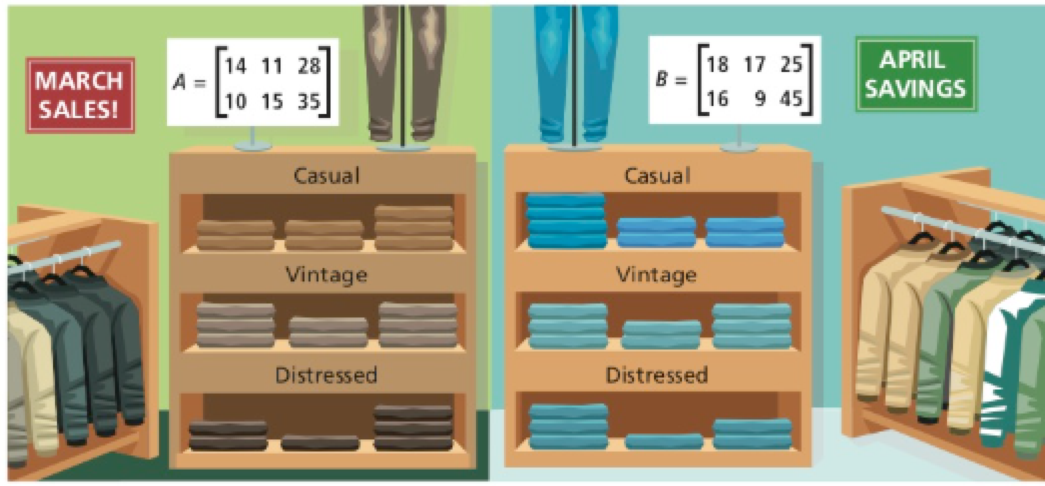
**Try It!** **2.** In this matrix $C$, the rows represent prices for shirts and khakis. The columns have the same meaning as in Example 2. If the sales tax rate is 6%, use scalar multiplication to find the sales tax for each item.

$$C = \begin{bmatrix} 75 & 40 & 25 \\ 100 & 60 & 30 \end{bmatrix}$$

APPLICATION

## EXAMPLE 3 Add and Subtract Matrices

**Cool Threads uses matrices to keep track of monthly sales. In matrices *A* and *B*, Row 1 represents the sale of jeans and Row 2 represents the sale of jackets, with the columns representing Vintage, Casual, and Distressed brands, respectively, during two consecutive months.**




**A. What is the sales total for the two months?**

The sum $A + B$ represents the totals sold during the two months. Add corresponding elements of the two matrices.

The entry "25" means that 25 pairs of distressed jeans were sold in the second month.

$$A + B = \begin{bmatrix} 14 & 11 & 28 \\ 10 & 15 & 35 \end{bmatrix} + \begin{bmatrix} 18 & 17 & 25 \\ 16 & 9 & 45 \end{bmatrix} = \begin{bmatrix} 14+18 & 11+17 & 28+25 \\ 10+16 & 15+9 & 35+45 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 28 & 53 \\ 26 & 24 & 80 \end{bmatrix}$$

The entry "26" means that in these two months, 26 Vintage jackets were sold.

**STUDY TIP**
Adding matrices is similar to adding complex numbers. When finding $(a + bi) + (c + di)$, you combine $a + c$ and $bi + di$, just like you add the corresponding parts of the matrices.

**B. What is the difference in the number of items sold each month?**

The difference $A - B$ represents the differences in the monthly sales. Subtract corresponding elements of the two matrices.

$$A - B = \begin{bmatrix} 14 & 11 & 28 \\ 10 & 15 & 35 \end{bmatrix} - \begin{bmatrix} 18 & 17 & 25 \\ 16 & 9 & 45 \end{bmatrix} = \begin{bmatrix} 14-18 & 11-17 & 28-25 \\ 10-16 & 15-9 & 35-45 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -6 & 3 \\ -6 & 6 & -10 \end{bmatrix}$$

In the matrix $A - B$, a positive number means more of this type were sold in the first month. A negative number means that more were sold in the second month.

**Try It!** 3. Consider matrices $M$ and $N$.

$$M = \begin{bmatrix} -3 & 5 \\ 2 & 0 \end{bmatrix}, N = \begin{bmatrix} 6 & 5 \\ -8 & 0.2 \end{bmatrix}$$

a. What are matrices $M + N$ and $N + M$?

b. What are matrices $M - N$ and $N - M$?

CONCEPTUAL UNDERSTANDING

## EXAMPLE 4 Understand Matrix Addition and Subtraction

**Consider the matrices below. How can you add and subtract these matrices?**

$$A = \begin{bmatrix} 5 & -7 & 3 \\ 4 & 8 & -2 \end{bmatrix}, B = \begin{bmatrix} 6 & 5 \\ -2 & 0 \\ 3 & -4 \end{bmatrix}, C = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 15 & -9 \end{bmatrix}, D = \begin{bmatrix} -5 & 7 & -3 \\ -4 & -8 & 2 \end{bmatrix}$$

**A. Which matrices can be combined using addition or subtraction?**

Adding (or subtracting) matrices involves adding (or subtracting) corresponding pairs of elements. To have corresponding pairs of elements, the matrices must have the same dimensions.

**COMMON ERROR**
The dimensions of a matrix, $r \times c$, always give the number of rows first and number of columns second.

The matrices $A$, $C$, and $D$ are $2 \times 3$ matrices, so they can be added or subtracted in any order, but matrix $B$ is a $3 \times 2$ matrix so it cannot be combined with $A$, $C$, or $D$ using addition or subtraction.

**B. How can you interpret a matrix of zeros?**

$$A + D = \begin{bmatrix} 5 & -7 & 3 \\ 4 & 8 & -2 \end{bmatrix} + \begin{bmatrix} -5 & 7 & -3 \\ -4 & -8 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the zero $2 \times 3$ matrix. A zero $3 \times 2$ matrix would also have all zeros, but it would have 3 rows and 2 columns.

Every element in $A + D$ is zero. A matrix with all zeros is a **zero matrix.** When the sum of two matrices is a zero matrix, the matrices are additive inverses.

**C. What is the additive inverse of matrix $B$?**

Solve the equation $B + X = 0$, so that $X$ is the additive inverse of $B$, $X = -B$.

$$\begin{bmatrix} 6 & 5 \\ -2 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -6 & -5 \\ 2 & 0 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Think about what elements of Matrix $X$ will make every element of the sum matrix equal to 0.

$$X = \begin{bmatrix} -6 & -5 \\ 2 & 0 \\ -3 & -4 \end{bmatrix}$$

The additive inverse of a matrix is the scalar multiple of $-1$ times the matrix.

A zero matrix of the same dimensions of any matrix is the additive identity matrix. Adding the zero matrix will return the original matrix.

**D. How does $A + C$ relate to $C + A$?**

$$A + C = \begin{bmatrix} 5 & -7 & 3 \\ 4 & 8 & -2 \end{bmatrix} + \begin{bmatrix} 12 & 0 & 0 \\ 0 & 15 & -9 \end{bmatrix} = \begin{bmatrix} 17 & -7 & 3 \\ 4 & 23 & -11 \end{bmatrix}$$

$$C + A = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 15 & -9 \end{bmatrix} + \begin{bmatrix} 5 & -7 & 3 \\ 4 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 17 & -7 & 3 \\ 4 & 23 & -11 \end{bmatrix}$$

In this example, $A + C = C + A$. Matrix addition is real number addition of corresponding elements. Because real number addition is commutative, a Commutative Property of Addition holds for matrices.

**Try It!**

**4.** Consider the matrices below.

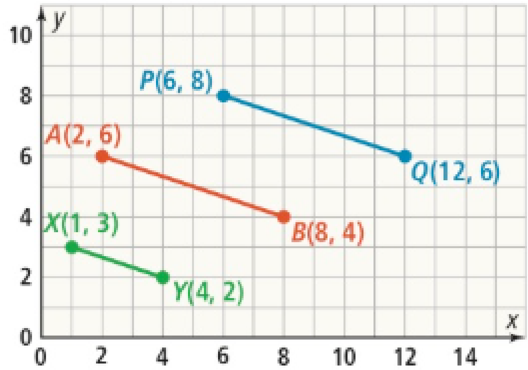
$$P = \begin{bmatrix} 5 & 2 & -3 \\ 7 & 0 & -5 \end{bmatrix}, Q = \begin{bmatrix} 2 & -2 \\ 5 & -5 \\ -7 & 7 \end{bmatrix}, R = \begin{bmatrix} 6 & 0.5 \\ -3 & 0 \\ -2 & -2 \end{bmatrix}$$

**a.** Find $R - Q$. What other matrix sums or differences can be calculated?

**b.** Find the additive inverses of $P$, $Q$, and $R$.

## EXAMPLE 5 Use Matrices to Translate and Dilate Figures

**In this diagram $\overline{AB}$ is translated 4 units right and 2 units up to $\overline{PQ}$. Also, $\overline{XY}$ is a dilation of $\overline{AB}$ by a factor of $\frac{1}{2}$, centered at the origin.**




**How can you use matrices to show a translation and a dilation?**

> **GENERALIZE**
> The coordinates of the vertices of an $n$-sided polygon can be represented as a $2 \times n$ matrix, with 2 rows and $n$ columns.

**A. Show the translation of $\overline{AB}$ to $\overline{PQ}$ using matrices.**

A matrix with one column can represent an ordered pair. A matrix with two columns can represent two ordered pairs, where the first row gives the $x$-coordinates and the second row gives the $y$-coordinates.

Point $A$: $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ endpoints of $\overline{AB}$: $\begin{bmatrix} 2 & 8 \\ 6 & 4 \end{bmatrix}$

A translation can be represented as a matrix. The matrix for $\overline{AB}$ has two columns, so a matrix for a translation of $\overline{AB}$ also has two columns.

Since the translation results in the $x$-coordinates increasing by 4 and the $y$-coordinates increasing by 2, add the matrix $\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$ to the matrix representing $\overline{AB}$.

To show the translation of $\overline{AB}$, use matrix addition:

$$\underbrace{\begin{bmatrix} 2 & 8 \\ 6 & 4 \end{bmatrix}}_{\text{preimage}} + \underbrace{\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}}_{\text{translation}} = \underbrace{\begin{bmatrix} 6 & 12 \\ 8 & 6 \end{bmatrix}}_{\text{image}}$$

The sum represents the endpoints $P(6, 8)$ and $Q(12, 6)$.

**B. Show the dilation of $\overline{AB}$ by a scale factor of $\frac{1}{2}$.**

Use scalar multiplication:

$$\underbrace{\frac{1}{2}}_{\text{scalar}} \cdot \underbrace{\begin{bmatrix} 2 & 8 \\ 6 & 4 \end{bmatrix}}_{\text{preimage}} = \underbrace{\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}}_{\text{image}}$$

This method can be used only for a dilation centered at the origin.

**Try It!** 5. A segment has endpoints $M(8, -7)$ and $N(1, 2)$.

a. Use matrices to represent a translation of $\overline{MN}$ to $\overline{RS}$ by 6 units left and 3 units down. What are the coordinates of $R$ and $S$?

b. Use matrices to represent a dilation of $\overline{MN}$ to $\overline{DE}$ by a scale factor of 3, centered at the origin. What are the coordinates of $D$ and $E$?

## CONCEPT SUMMARY Operating With and Interpreting Matrices

**DIMENSIONS** The dimensions of a matrix are stated as the number of rows ($r$) by the number of columns ($c$).

$$A = \begin{bmatrix} t & u & v \\ x & y & z \end{bmatrix}$$

The dimensions of this matrix are $2 \times 3$, because it has 2 rows and 3 columns. $a_{13} = v$, because $v$ is in the 1st row and 3rd column.

**OPERATIONS** To multiply a matrix by a scalar, multiply each element in the matrix by the scalar.

$$k \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} kw & kx \\ ky & kz \end{bmatrix}$$

To add matrices, add the corresponding elements.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

To subtract matrices, subtract the corresponding elements.

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} - \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} p-w & q-x \\ r-y & s-z \end{bmatrix}$$

**DILATIONS** A matrix can represent the transformation of a figure such as a dilation, centered at the origin, of a triangle.

$$3\begin{bmatrix} 5 & -3 & 0 \\ 6 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 15 & -9 & 0 \\ 18 & 15 & -6 \end{bmatrix}$$

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you interpret matrices and operate with matrices?

2. **Error Analysis** Tonya says the matrix subtraction would produce a zero matrix. Explain her error.
$$\begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

3. **Communicate and Justify** Explain how you know if two matrices can be added. Then explain how to add them.

4. **Vocabulary** What are equal matrices? Give an example of equal matrices.

## Do You KNOW HOW?

**Identify the element for each matrix.**

5. $\begin{bmatrix} 4 & 1 & 0 \\ 7 & 3 & 5 \end{bmatrix}$; $a_{23}$

6. $\begin{bmatrix} -6 \\ 2 \end{bmatrix}$; $a_{11}$

**Given $A = \begin{bmatrix} 3 & -2 \\ 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 7 \\ -4 & 12 \end{bmatrix}$, calculate each of the following.**

7. $A + B$

8. $B - A$

9. $4A$

10. $A - B$

11. The endpoints of $\overline{AB}$ are represented by the matrix $\begin{bmatrix} 3 & 7 \\ 1 & 5 \end{bmatrix}$.
Find the image of the segment after a dilation, centered at the origin, by a scale factor of 2.

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

12. **Represent and Connect** Explain how you would solve for each variable. Then find the value of each variable. $\begin{bmatrix} a & b-3 \\ c & d+5 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & 4 \end{bmatrix}$

13. **Analyze and Persevere** Find the sum of $A = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}$ and the additive inverse of $P = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$

14. **Error Analysis** Describe and correct the error a student made in translating the points $A(1, -3)$, $B(2, 1)$ and $C(-3, -2)$ 3 units left and 1 unit up.

Original points $\begin{pmatrix} 1 & 2 & -3 \\ -3 & 1 & -2 \end{pmatrix}$

Translation matrix $\begin{pmatrix} -3 & -3 & -3 \\ -1 & -1 & -1 \end{pmatrix}$

Answer matrix $\begin{pmatrix} -2 & -1 & -6 \\ -4 & 0 & -3 \end{pmatrix}$ ✗

15. **Communicate and Justify** Suppose $A$ and $B$ are two matrices with the same dimensions. Explain how to find $A + B$, $A - B$, and matrix $C$ such that $A + C$ is the zero matrix.

16. **Higher Order Thinking** Explain why $A = \begin{bmatrix} 0.5 \\ 4 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{2} \\ 1+3 \end{bmatrix}$ have the same additive inverse.

17. **Mathematical Connections** For the set of real numbers, if the sum of two numbers is the additive identity element, then the two numbers are additive inverses of each other. How does this property relate to matrix addition?

18. **Choose Efficient Methods** The coordinates of the vertices of a square are represented in a matrix. The matrix is then multiplied by the scalar 3. How does the area of the new square compare to the area of the original square?

19. $A + X = A$. What must $X$ be?

## PRACTICE

20. In matrix $D$, the entries are the number of students playing volleyball at a high school. Column 1 lists boys, column 2 lists girls, row 1 lists juniors, and row 2 lists seniors. Find $d_{22}$, $d_{12}$, and $d_{11}$, and tell what each number represents.

$D = \begin{bmatrix} 4 & 5 \\ 7 & 6 \end{bmatrix}$ SEE EXAMPLE 1

21. In the price matrix $P$, the rows represent prices for sweatshirts and sweatpants. The columns represent the color scheme of the items: white, red, and tie-dye. If the sales tax rate is 7%, find the sales tax of each item.

$P = \begin{bmatrix} 30 & 40 & 50 \\ 25 & 35 & 55 \end{bmatrix}$ SEE EXAMPLE 2

**Given the following matrices, calculate the sum or difference. If not possible, state so.** SEE EXAMPLE 3

$X = \begin{bmatrix} 7 & 2 & 1 \\ 4 & -3 & 6 \end{bmatrix}$, $Y = \begin{bmatrix} -2 & 4 \\ 3 & 8 \end{bmatrix}$, and $Z = \begin{bmatrix} 0 & 3 & 7 \\ 1 & -2 & 6 \end{bmatrix}$,

22. $X + Y$

23. $Z - X$

24. $X + Z$

25. $X - Z$

**Find the additive inverse of each matrix.**
SEE EXAMPLE 4

26. $Q = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

27. $R = \begin{bmatrix} 2 & 0 \\ 6 & -5 \\ -4 & 11 \end{bmatrix}$

28. $S = \begin{bmatrix} 4 & -7 & -8 & 9 \end{bmatrix}$

29. $T = \begin{bmatrix} 9 & -1 \\ 4 & 10 \\ 3 & -7 \end{bmatrix}$

**A segment has endpoints $E$ (5, −1) and $F$ (6,11).**
SEE EXAMPLE 5

30. Use matrices to represent a translation of $\overline{EF}$ to $\overline{YZ}$ by 5 units right and 1 unit down. What are the coordinates of $Y$ and $Z$?

31. Use matrices to represent a dilation of $\overline{EF}$ to $\overline{UV}$ by a scale factor of 4, centered at the origin. What are the coordinates of $U$ and $V$?

## PRACTICE & PROBLEM SOLVING

### APPLY

**32. Represent and Connect** Using a 10 × 10 grid, create a battleship game board with 5 ships placed. Write a matrix *B* for your battleship board. Use a 1 for a space a ship is placed and a 0 for a space no ship exists.

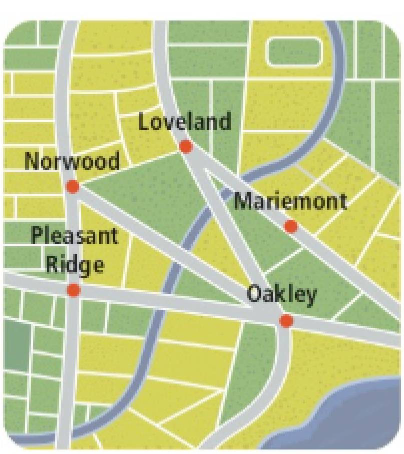
**33. Apply Math Models** The table shows some of the men's running records in seconds in 2020.

| Distance (meters) | World record | American record | Olympic record |
|---|---|---|---|
| 100 | 9.58 | 9.69 | 9.63 |
| 200 | 19.19 | 19.32 | 19.30 |
| 400 | 43.03 | 43.18 | 43.03 |
| 1,500 | 206 | 209.3 | 212.07 |

**a.** Write a matrix that represents the difference between the Olympic and World records for each race distance expressed as a column matrix.

**b.** If all of the records in the table are expressed in seconds and are represented by a matrix *B*, what matrix expression could be used to convert all data to minutes?

**34. Use Patterns and Structure** A matrix can be used to represent which towns are connected by a single road to each other on a map. Use a 1 to represent two towns connected to each other and a 0 to represent two towns not connected to each other. Use a 0 to show that the indicated row and column both represent the same town. Create a matrix *C* to represent this situation.




### ASSESSMENT PRACTICE

**35.** Use these matrices to complete the statements.

$$A = \begin{bmatrix} 0 & 9 & 6 \\ 1 & 2 & 4 \\ 7 & -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -7 & -2 \\ 0 & 5 & 8 \\ -3 & 1 & 1 \end{bmatrix}$$

In matrix *A*, the value of $a_{31}$ is ______ the value of $a_{12}$. In matrix *B*, the value of $b_{31}$ is ______ the value of $b_{12}$. NSO.4.1

Ⓐ less than; less than

Ⓑ less than; greater than

Ⓒ greater than; less than

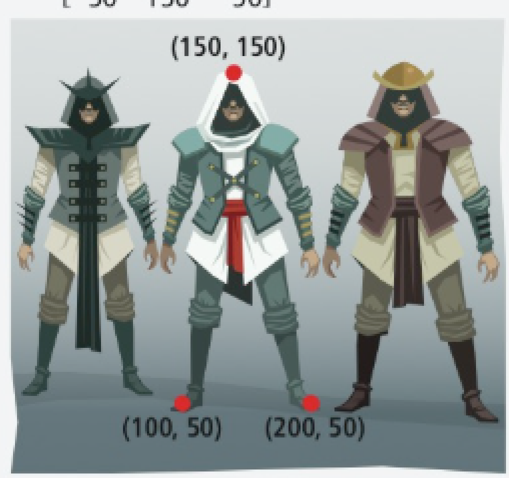
Ⓓ greater than; greater than

**36. SAT/ACT** If $5\begin{bmatrix} a \\ b \end{bmatrix} = 14\begin{bmatrix} 20 \\ 12 \end{bmatrix}$, then what is the value of $a + b$?

Ⓐ 29 Ⓑ $\frac{148}{5}$ Ⓒ $\frac{448}{5}$ Ⓓ $\frac{191}{4}$ Ⓔ $\frac{41}{5}$

**37. Performance Task** A computer animator uses a screen that is 1,000 pixels wide and 800 pixels tall. The animator uses matrix columns to represent three locator points on an avatar. The top row represents the horizontal coordinate of each point, and the bottom row represents the vertical coordinate. Let *P* represent the initial postion of the avatar.

$$P = \begin{bmatrix} 100 & 150 & 200 \\ 50 & 150 & 50 \end{bmatrix}$$




**Part A** The animator wants the avatar to move up at a rate of 100 pixels per second. Use addition of matrices to show the position of the avatar after 2 seconds and after 5 seconds.

**Part B** The animator wants the avatar to move right at a rate of 50 pixels per second. Use addition of matrices to show the position of the avatar after 3 seconds and after 8 seconds.

**Part C** How could the animator use scalar multiplication and matrix addition to show how the avatar moves across the screen?

# 7-2 Matrix Multiplication

I CAN… find the product of matrices or explain why the product does not exist.

**VOCABULARY**
- identity matrix
- square matrix

MA.912.NSO.4.3-Solve mathematical and real-world problems involving addition, subtraction and multiplication of matrices. Also NSO.4.1

MA.K12.MTR.1.1, MTR.2.1, MTR.6.1

## EXPLORE & REASON

**Two stores, Quick Repair and TechRite, buy and sell pre-owned phones, tablets, and computers. The matrices below represent their average revenue *R*, purchase costs *C*, and repair expenses *E* for each item:**

| | | | | |
|---|---|---|---|---|
| R | Quick Repair | $150 | $100 | $400 |
| | TechRite | $200 | $250 | $500 |
| C | Quick Repair | $100 | $50 | $200 |
| | TechRite | $125 | $75 | $300 |
| E | Quick Repair | $25 | $20 | $50 |
| | TechRite | $10 | $50 | $50 |

A. Would it make sense to find the sum and/or difference of any two of the three matrices? Explain.

B. **Analyze and Persevere** Quick Repair and TechRite both need to estimate their total purchase and repair costs. They each predict that they will need to purchase 100 phones, 100 tablets, and 100 computers, and that they will need to repair 50% of them. Explain what you would do to find the total costs.

**ESSENTIAL QUESTION** What does it mean to multiply a matrix by another matrix?

CONCEPTUAL UNDERSTANDING

## EXAMPLE 1 Understand Matrix Multiplication

**A teacher assigns final grades based on a weighted system. Students are graded on unit assessments, a semester project, and the final exam, each with a different weight. The matrices given below represent the weights for each kind of work and the grades for two students Oscar and Reagan.**

**Weighting** **Grades**

$$W = \begin{array}{ccc} \text{unit} & \text{project} & \text{final} \\ \end{array} \begin{bmatrix} 0.50 & 0.30 & 0.20 \end{bmatrix} \qquad G = \begin{array}{r} \\ \text{unit} \\ \text{project} \\ \text{final} \end{array} \begin{array}{c} \begin{array}{cc} \text{O} & \text{R} \end{array} \\ \begin{bmatrix} 90 & 80 \\ 95 & 70 \\ 75 & 85 \end{bmatrix} \end{array}$$

**What are the final grades for each student?**

REPRESENT AND CONNECT
Grades that are weighted aren't simply averaged but, instead, are multiplied by their weight, then added.

The final grade is the sum of the weighted averages.

Final grade = 0.5(unit grade) + 0.3(project grade) + 0.2(final exam grade)

This equation shows that computing the final grade for each student is like multiplying the elements of row 1 of *W* (weights) by the corresponding elements of each student's column and finding the sum of the products. This method is similar to multiplying two matrices.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

To find each student's final grade, multiply a $1 \times 3$ matrix by a $3 \times 1$ matrix. The product of two matrices is a new matrix. The elements of the new matrix are the sums of the products of the corresponding row and column elements as shown below.

**Oscar's Final Grade**

Weight Matrix • Grade Matrix

$$= [0.50\ 0.30\ 0.20] \begin{bmatrix} 90 \\ 95 \\ 75 \end{bmatrix}$$

$$= 0.50(90) + 0.30(95) + 0.20(75)$$
$$= [88.3]$$

**Reagan's Final Grade**

Weight Matrix • Grade Matrix

$$= [0.50\ 0.30\ 0.20] \begin{bmatrix} 80 \\ 70 \\ 85 \end{bmatrix}$$

$$= 0.50(80) + 0.30(70) + 0.20(85)$$
$$= [78]$$

**COMMON ERROR**
Scalar multiplication is often confused with matrix multiplication. Remember that matrix multiplication considers the product of two matrices while scalar multiplication considers the product of a matrix and a number.

So, Oscar's final grade is 88.5, and Reagan's final grade is 78.

These can be represented in a matrix $F$. Each element in matrix $F$ is the sum of the products of the corresponding row elements in $W$, multiplied by the corresponding column elements in $G$.

$$F = WG = [0.50\ \ 0.30\ \ 0.20] \begin{bmatrix} 90 & 80 \\ 95 & 70 \\ 75 & 85 \end{bmatrix} = [88.5\ \ 78]$$

Since you are multiplying elements of the rows of $W$ by elements of the columns of $G$, it is important that the number of elements in each row of $W$ is equal to the number of elements in each column of $G$.

In general, the product matrix $QR$ has the same number of rows, $r_Q$, as $Q$, and the same number of columns, $c_R$, as $R$.

$$\underset{r_Q \times c_Q}{Q} \quad \bullet \quad \underset{r_R \times c_R}{R} \quad = \quad \underset{r_Q \times c_R}{QR}$$

If the number of columns in the first matrix does not match the number of rows in the second matrix ($c_Q \neq r_R$) then the product matrix does not exist.

Since 88.5 results from row 1 of $W$ by the element in column 1 of $G$, it is element $a_{11}$ of $F$. Similarly, 78 results from multiplying the element in multiplying the element in row 1 of $W$ by the element in column 2 of $G$, so it is element $a_{12}$of $F$. Then $F$ will be a $1 \times 2$ matrix.

The matrix that shows the final average for each student is $F = \begin{array}{cc} \text{O} & \text{R} \\ [88.5 & 78]. \end{array}$

**Try It!** **1. a.** Explain why a product matrix $QR$ does not exist when the columns of matirx $Q$ does not match the rows of Matrix $R$, $c_Q \neq r_R$.

**b.** How could you organize the weighting and grade information differently so that Oscar's and Reagan's final grades are given by GW?

## EXAMPLE 2 Examine Multiplication of Square Matrices

**Below are square matrices $A$ and $B$. Determine if the given equations are true for $A$ and $B$. What conclusions can we make about the Commutative and Distributive Properties for multiplying square matrices?**

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -5 \\ 0 & 4 \end{bmatrix}$$

The element $AB_{12} = 14$ is found by multiplying the elements in the first row of $A$ by the elements of the second column of $B$.

**A. Is $AB = BA$?**

Find the product on each side of the equation.

$$AB = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -2(-1) + 1(0) & -2(-5) + 1(4) \\ -1(-1) + 0(0) & -1(-5) + 0(4) \end{bmatrix} = \begin{bmatrix} 2 & 14 \\ 1 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & -5 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1(-2) + (-5)(-1) & -1(1) + (-5)(0) \\ 0(-2) + 4(-1) & 0(1) + 4(0) \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ -4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 14 \\ 1 & 5 \end{bmatrix} \neq \begin{bmatrix} 7 & -1 \\ -4 & 0 \end{bmatrix}$$

A **square matrix** is a matrix that has the same number of rows as columns. Since $AB \neq BA$ above, the Commutative Property does NOT hold for all square matrices.

**CHECK FOR REASONABLENESS**
Explain why the Commutative Property does not hold for matrix multiplication.

**B. Is $A(A + B) = AA + AB$?**

What is the product on each side of the equation?

$$A(A+B) = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 0 & 4 \end{bmatrix}\right) = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2(-3) + 1(-1) & -2(-4) + 1(4) \\ -1(-3) + 0(-1) & -1(-4) + 0(4) \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 3 & 4 \end{bmatrix}$$

The product of two square matrices with the same dimensions will always exist, because the number of rows in the first equals the number of columns in the second.

$$AA + AB = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -5 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2(-2) + 1(-1) & -2(1) + 1(0) \\ -1(-2) + 0(-1) & -1(1) + 0(0) \end{bmatrix} + \begin{bmatrix} -2(-1) + 1(0) & -2(-5) + 1(4) \\ -1(-1) + 0(0) & -1(-5) + 0(4) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 14 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 3 & 4 \end{bmatrix}$$

Since $A(A + B) = AA + AB$, this case suggests the Distributive Property holds.

Proving the equation for a specific example is not sufficient to prove that the equation is true for all matrices.

**Try It!** 2. Determine whether each equation is true for the following matrices.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & -2 \end{bmatrix},\ B = \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix},\ C = \begin{bmatrix} 6 & 2 \\ 4 & 8 \end{bmatrix}$$

**a.** $(AB)C = A(BC)$ **b.** $(A + B)C = AC + BC$

## EXAMPLE 3 Understand Identity Matrices

**HAVE A GROWTH MINDSET**
What other strategies can you try when you get stuck?

**What is the product of the $3 \times 3$ matrices**

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & -3 & 2 \\ -4 & 5 & -6 \\ 9 & -7 & 8 \end{bmatrix}?$$

Multiply the matrices, and compare $IA$ and $AI$.

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ -4 & 5 & -6 \\ 9 & -7 & 8 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(1) + 0(-4) + 0(9) & 1(-3) + 0(5) + 0(-7) & 1(2) + 0(-6) + 0(8) \\ 0(1) + 1(-4) + 0(9) & 0(-3) + 1(5) + 0(-7) & 0(2) + 1(-6) + 0(8) \\ 0(1) + 0(-4) + 1(9) & 0(-3) + 0(5) + 1(-7) & 0(2) + 0(-6) + 1(8) \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & -3 & 2 \\ -4 & 5 & -6 \\ 9 & -7 & 8 \end{bmatrix}$$

When you multiply $3 \times 3$ matrices, use the same process as with $2 \times 2$ matrices.

$$AI = \begin{bmatrix} 1 & -3 & 2 \\ -4 & 5 & -6 \\ 9 & -7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AI = \begin{bmatrix} 1(1) + (-4)(0) + 9(0) & -3(1) + 5(0) + (-7)(0) & 2(1) + (-6)(0) + 8(0) \\ 1(0) + (-4)(1) + 9(0) & -3(0) + 5(1) + (-7)(0) & 2(0) + (-6)(1) + 8(0) \\ 1(0) + (-4)(0) + 9(1) & -3(0) + 5(0) + (-7)(1) & 2(0) + (-6)(0) + 8(1) \end{bmatrix}$$

$$AI = \begin{bmatrix} 1 & -3 & 2 \\ -4 & 5 & -6 \\ 9 & -7 & 8 \end{bmatrix}$$

**STUDY TIP**
When multiplying a matrix by 1, as in $1A = A$, you are using scalar multiplication. The identity for multiplication of matrices must be a matrix where $AI = A$ and $IA = A$.

So the product matrices $IA$ and $AI$ are both equal to $A$.

$$A = AI = IA = \begin{bmatrix} 1 & -3 & 2 \\ -4 & 5 & -6 \\ 9 & -7 & 8 \end{bmatrix}$$

Although the identity matrix commutes with any matrix, matrix multiplication is not commutative in general.

The matrix $I$ is an **identity matrix**, because $M = MI = IM$ for every $3 \times 3$ matrix M. Identity matrices are always square matrices with a 1 as each element of the main diagonal and zeros for all other elements.

**Try It!** **3. a.** What is the product of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

**b.** What is the product of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$?

## CONCEPT SUMMARY Products of Matrices

**MATRIX MULTIPLICATION** The product of two matrices is a new matrix. The elements of the new matrix are the sums of the products of the corresponding row and column elements.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

**THE IDENTITY MATRIX** For the $n \times n$ matrix $A$, the multiplicative identity matrix $I$ is an $n \times n$ square matrix with 1s on the main diagonal and 0s for all other elements: $AI = IA = A$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What does it mean to multiply a matrix by another matrix?

2. **Use Patterns and Structure** Would it be possible to multiply $A$ and $B$ if $A$ is a $3 \times 5$ matrix and $B$ is a $4 \times 5$ matrix? Explain your reasoning.

3. **Vocabulary** Explain why a matrix with ones on the main diagonal and zeros for all the other elements is called the *identity matrix*.

4. **Communicate and Justify** A student thought that the product of $A$, a $1 \times 5$ matrix, and $B$, a $5 \times 1$ matrix, should have five elements in the answer. Is the student correct? If not, how many elements will there be?

## Do You KNOW HOW?

Let $A = \begin{bmatrix} 3 & 0 \\ -1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}$.

5. Find $AB$ and $BA$ to demonstrate that matrix multiplication is not commutative. Show your work.

**Find each product.**

6. $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 8 & 2 \end{bmatrix}$

8. The coordinates of the vertices of a triangle are $A(-2, 3)$, $B(1, 1)$, and $C(2, -1)$. The coordinates are multiplied by a transformation matrix, $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Find the coordinates of the image of the triangle after the transformation. $T$ needs to be on the left in the matrix multiplication.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

9. **Generalize** Suppose square matrices $A$ and $B$ have dimensions $n \times n$, where $n$ is a positive integer greater than or equal to 2. What are the dimensions of their product $A \times B$?

10. **Use Patterns and Structure** If you wanted to find a product of the two matrices shown below, explain why it is necessary to write them in this order.

$$\begin{bmatrix} 10 & 15 & 12 \\ 7 & 11 & 20 \end{bmatrix} \begin{bmatrix} 50 \\ 14 \\ 38 \end{bmatrix}$$

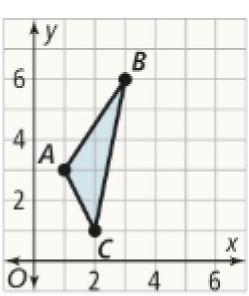
11. **Error Analysis** Describe and correct the error a student made in mulitiplying matrix $A$ by matrix $B$.

$$\begin{array}{cc} A & B \\ \begin{pmatrix} 6 & 2 \\ -3 & 5 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 4 & -2 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 6 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ -12 & -10 \end{pmatrix}$$

✗

12. **Higher Order Thinking** The triangle shown is transformed using two matrices, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, in that order. The transformation matrix needs to be on the left in matrix multiplication. Be sure to set up any matrix with the correct dimensions for multiplication.




a. What transformation occurs as a result of multiplication by matrix $A$?

b. What transformation occurs as a result of multiplication by matrix $B$?

### PRACTICE

13. A math teacher assigns final grades based on a weighted system. Matrix $W$ represents the weights of each type of assignment, and matrix $G$ represents the grades for two students, Jacob and Lucy. Use matrix multiplication to find matrix $F$ that represents the final class grades for these two students. SEE EXAMPLE 1

$$W = \begin{array}{ccc} \text{hw} & \text{tests} & \text{exam} \\ [0.20 & 0.50 & 0.30] \end{array}$$

$$G = \begin{array}{c} \text{hw} \\ \text{tests} \\ \text{exam} \end{array} \begin{array}{c} \begin{array}{cc} \text{Jacob} & \text{Lucy} \end{array} \\ \begin{bmatrix} 95 & 85 \\ 80 & 90 \\ 75 & 85 \end{bmatrix} \end{array}$$

**Determine whether each equation is true for the following matrices.** SEE EXAMPLE 2

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix},\ B = \begin{bmatrix} -4 & 0 \\ -1 & 8 \end{bmatrix},\ C = \begin{bmatrix} 5 & 1 \\ 7 & -2 \end{bmatrix}$$

14. $(A + B)C = AC + BC$

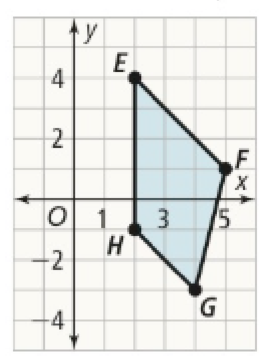
15. $A(BC) = (AB)C$

16. Find $IQ$, if

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -3 & 2 \\ -4 & 5 & -6 \\ 9 & -7 & 8 \end{bmatrix}.$$

SEE EXAMPLE 3

17. Create a $2 \times 4$ matrix $A$ to represent the coordinates of quadrilateral $EFGH$.




a. Multiply matrix $A$ by $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

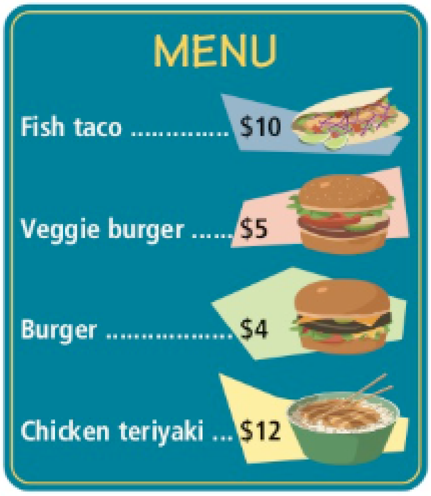
b. Graph the quadrilateral represented by the resulting matrix, and describe the movement of the quadrilateral in the coordinate plane.

## PRACTICE & PROBLEM SOLVING

### APPLY

**18. Represent and Connect** The following matrix represents the inventory of the three snack bars at a state park.

$$\begin{array}{l} \\ \text{Snack Bar A} \\ \text{Snack Bar B} \\ \text{Snack Bar C} \end{array}\begin{array}{c} \begin{array}{cccc} \text{fish taco} & \text{veggie burger} & \text{burger} & \text{chicken teriyaki} \end{array} \\ \begin{bmatrix} 20 & 15 & 7 & 11 \\ 22 & 18 & 6 & 8 \\ 15 & 19 & 10 & 5 \end{bmatrix} \end{array}$$




Use matrix multiplication to find the total value of the inventory for each snack bar.

**19. Apply Math Models** Raul owns and operates two souvenir stands. At his baseball park stand, sweatshirts cost \$45 and T-shirts cost \$20. At his football stadium stand, sweatshirts cost \$50 and T-shirts cost \$15. Today Raul sold 20 sweatshirts and 25 T-shirts at each stand. Use matrix multiplication to find the total amount in daily sales at each souvenir stand.

**20. Choose Efficient Methods** A drama teacher assigns final grades in her class based on the weighted system shown below. The matrix $G$ represents the grades for Kiyo and his two friends, Rachel and Leo.

$$G = \begin{array}{l} \\ \text{tests} \\ \text{projects} \\ \text{part} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Kiyo} & \text{Rachel} & \text{Leo} \end{array} \\ \begin{bmatrix} 90 & 83 & 78 \\ 94 & 88 & 96 \\ 98 & 94 & 89 \end{bmatrix} \end{array}$$

| Drama Syllabus |
|---|
| Tests 45% |
| Projects 30% |
| Participation 25% |

**a.** Write matrix $W$ as a $1 \times 3$ matrix to represent the weighted grading system.

**b.** Perform matrix multiplication to find the final grades for each of the three students.

### ASSESSMENT PRACTICE

**21.** Find the product of the two matrices. NS0.4.3

$$\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

**22. SAT/ACT** Select the undefined matrix product.

Ⓐ $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -2 & 0 & -4 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

**23. Performance Task** Paula has a candle-making business. The candles come in four different types. The cost of making each type of candle is \$0.50, \$1, \$5, and \$7, in order of size. Paula's candle sales for her first three years of business are shown in the table below.

| | Tea \$1 | Floating \$2 | Jar \$12 | Pillar \$15 |
|---|---|---|---|---|
| Year 1 | 20 | 15 | 40 | 30 |
| Year 2 | 25 | 20 | 50 | 35 |
| Year 3 | 15 | 20 | 60 | 45 |

**Part A** Write matrix $C$ as a $4 \times 1$ matrix to represent the cost of making each type of candle, write matrix $P$ as a $4 \times 1$ matrix to represent the selling price of each candle, and write matrix $S$ as a $3 \times 4$ matrix to represent Paula's candle sales for the first three years.

**Part B** Use matrix subtraction to find a matrix, $X$, that represents the amount of profit that Paula makes per candle.

**Part C** Use matrix multiplication to find the product of matrices $S$ and $X$. Explain what the elements of this product represent.

# 7-3 Inverses and Determinants

**I CAN…** find and use the inverse of a matrix.

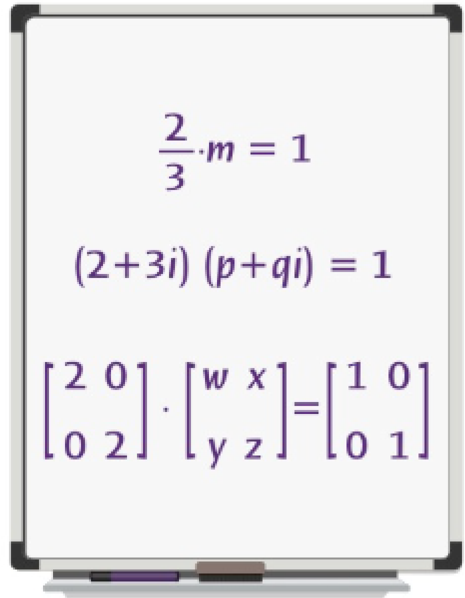
**VOCABULARY**

- determinant of a $2 \times 2$ matrix
- inverse matrix

MA.912.NSO.4.4-Solve mathematical and real-world problems using the inverse and determinant of matrices.
MA.K12.MTR.1.1, MTR.3.1, MTR.4.1

## EXPLORE & REASON

A teacher writes these three equations on the board.




A. Carolina notices that the solution to the first equation is given by $\frac{3}{2}$, and she hypothesizes that

$p + qi = \frac{1}{2+3i}$ and $\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \dfrac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}$.

Is Carolina correct?

B. **Use Patterns and Structure** What do the methods for solving these equations have in common?

## ESSENTIAL QUESTION

How do you find and use an inverse matrix?

**CONCEPTUAL UNDERSTANDING**

## EXAMPLE 1 Explore Inverses of $2 \times 2$ Matrices

**A. What is the inverse matrix of $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$?**

An **inverse matrix** of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that the product of the two matrices is the identity matrix. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Use Patterns and Structure**
For a matrix $A$ and its inverse matrix $B$, $AB = BA = I$, where $I$ is the identity matrix.

**Step 1** Write the equation for the inverse.

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Step 2** Multiply the matrices on the left side of the equation.

$$\begin{bmatrix} 2w + y & 2x + z \\ 3w + 0y & 3x + 0z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Step 3** Solve the equations that result from the matrix multiplication. Set each element of the first matrix equal to the corresponding element in the second matrix. This results in a system of four equations:

$$2w + y = 1 \qquad 2x + z = 0$$

$$3w + 0y = 0 \qquad 3x + 0z = 1$$

Solve the equations that involve only one variable first.

$$3w + 0y = 0 \qquad 3x + 0z = 1$$

$$3w = 0 \qquad 3x = 1$$

$$w = 0 \qquad x = \frac{1}{3}$$

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Substituting $w$ and $x$ into the first two equations yields the following.

$$2w + y = 1 \qquad 2x + z = 0$$
$$2(0) + y = 1 \qquad 2\left(\frac{1}{3}\right) + z = 0$$
$$y = 1 \qquad z = -\frac{2}{3}$$

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix}$$

**Step 4** Verify the solution:

$$\begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0(2) + \frac{1}{3}(3) & 0(1) + \frac{1}{3}(0) \\ 1(2) - \frac{2}{3}(3) & 1(1) - \frac{2}{3}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The inverse matrix of $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix}$.

**B. What is the inverse matrix of $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$?**

Find a matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$, so that $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Multiplying the matrices on the left of the equation yields

$$\begin{bmatrix} w - y & x - z \\ -w + y & -x + z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Create a system of four equations from the equal matrices.

$$w - y = 1 \qquad x - z = 0$$
$$-w + y = 0 \qquad -x + z = 1$$

Combining these equations to solve for $x$ and $z$ leads to the contradiction $0 = 1$.

**USE PATTERNS AND STRUCTURE**
Do all real numbers have a multiplicative inverse?

To solve, combine two equations that contain the same variables.

This system has no solution. Some matrices *do not have* inverses.

The matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ does not have an inverse.

**Try It!** **1.** What is the inverse matrix of $\begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix}$?

## CONCEPT The Inverse of a 2 × 2 Matrix

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant of a 2 × 2 matrix $A$**, denoted det $A$, is the value $ad - bc$.

The inverse of $A$ is denoted $A^{-1}$ and exists if and only if det $A \neq 0$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A$ has an inverse, then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

## EXAMPLE 2 Find Inverses of Square Matrices

**A. What is the inverse of $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$?**

> **VOCABULARY**
> A *determinant* is a number associated with a square matrix that *determines* whether or not the matrix has an inverse.

The matrix is a square matrix, $2 \times 2$. The inverse exists because $\det A = ad - bc = 4(2) - 1(-1) = 9$.

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{4}{9} \end{bmatrix}$$

The inverse of $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ is $\begin{bmatrix} \frac{2}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{4}{9} \end{bmatrix}$.

Check your work by using your calculator to multiply $A$ by the matrix you found for $A^{-1}$. If your answer is correct, the product will be the identity matrix.

**B. What is the inverse of $B = \begin{bmatrix} 6 & 8 \\ 3 & 4 \end{bmatrix}$?**

The matrix is a square matrix, $2 \times 2$.
Since $\det B = ad - bc = 6(4) - 8(3) = 0$, the inverse matrix does not exist.

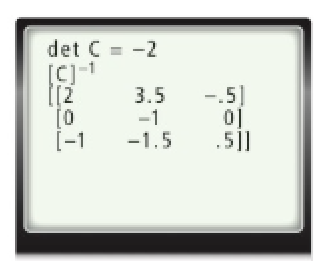
**C. What is the inverse of $C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 4 \end{bmatrix}$?**

The matrix is a square matrix, $3 \times 3$. The formulas that we have shown you for finding the determinant and inverse matrix apply only to matrices that are $2 \times 2$. Use a calculator to find the determinant and inverse of a $3 \times 3$ matrix.

Since $\det C = -2$, the inverse exists.

The calculator shows that the inverse matrix is

$$C^{-1} = \begin{bmatrix} 2 & \frac{7}{2} & -\frac{1}{2} \\ 0 & -1 & 0 \\ -1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$




**Try It!** **2.** Does each given matrix have an inverse? If so, find it.

**a.** $P = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}$ **b.** $Q = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$ **c.** $R = \begin{bmatrix} 5 & 1 & -1 \\ 2 & 0 & 5 \\ 1 & 0 & 2 \end{bmatrix}$

## EXAMPLE 3 Use a Matrix Inverse

**Use matrices to encode and decode the message WE COME IN PEACE.**

Matrix multiplication can be used to encode and decode messages.

**Step 1** Convert the message into a string of numbers.

Assign every letter in the alphabet a number, from $A = 1$ to $Z = 26$. Let 27 represent a space between words.

Write the message, and translate each letter into its corresponding number.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| W | E | * | C | O | M | E | * | I | N | * | P | E | A | C | E |
| 23 | 5 | 27 | 3 | 15 | 13 | 5 | 27 | 9 | 14 | 27 | 16 | 5 | 1 | 3 | 5 |

**Step 2** Choose a matrix that has an inverse to encode the message, such as

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -3 & -1 \\ 2 & -1 & 1 \end{bmatrix}.$$

The encoding matrix is a $3 \times 3$ matrix, so break the message into three-digit strings.

$$\begin{bmatrix} 23 \\ 5 \\ 27 \end{bmatrix}, \begin{bmatrix} 3 \\ 15 \\ 13 \end{bmatrix}, \begin{bmatrix} 5 \\ 27 \\ 9 \end{bmatrix}, \begin{bmatrix} 14 \\ 27 \\ 16 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 27 \\ 27 \end{bmatrix}$$

Insert spaces at the end to complete the string.

Create a matrix from these strings, and multiply by the encoding matrix.

**CHOOSE EFFICIENT METHODS**
Use technology to perform the matrix multiplications and to find the inverse matrix.

**Encoding Matrix** **Matrix representation of message**

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & -3 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 23 & 3 & 5 & 14 & 5 & 5 \\ 5 & 15 & 27 & 27 & 1 & 27 \\ 27 & 13 & 9 & 16 & 3 & 27 \end{bmatrix} =$$

$$\begin{bmatrix} 137 & 75 & 91 & 130 & 21 & 145 \\ -19 & -55 & -85 & -83 & -1 & -103 \\ 68 & 4 & -8 & 17 & 12 & 10 \end{bmatrix}$$

**Step 3** Multiply by the inverse of $A$ to decode the message.

The inverse matrix is $A^{-1} = \begin{bmatrix} -4 & -5 & 7 \\ -3 & -4 & 5 \\ 5 & 6 & -8 \end{bmatrix}$.

Try decoding the matrix by making a chart of letters to convert the matrix product back into the message.

$$\begin{bmatrix} -4 & -5 & 7 \\ -3 & -4 & 5 \\ 5 & 6 & -8 \end{bmatrix} \begin{bmatrix} 137 & 75 & 91 & 130 & 21 & 145 \\ -19 & -55 & -85 & -83 & -1 & -103 \\ 68 & 4 & -8 & 17 & 12 & 10 \end{bmatrix} =$$

$$\begin{bmatrix} 23 & 3 & 5 & 14 & 5 & 5 \\ 5 & 15 & 27 & 27 & 1 & 27 \\ 27 & 13 & 9 & 16 & 3 & 27 \end{bmatrix}$$

Now, this matrix can be reorganized into a string of numbers that represents the letter in the message.

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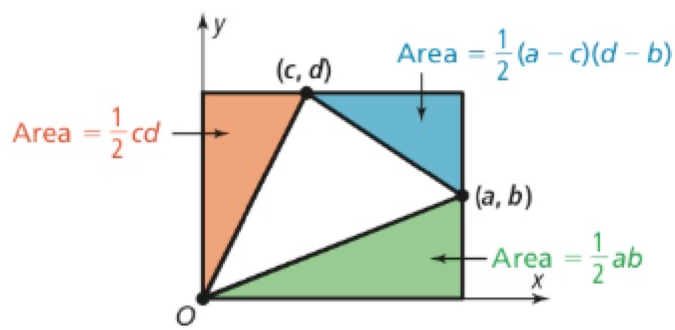
**Try It!** **3.** The matrix $\begin{bmatrix} -3 & -5 & 11 & 6 \\ 130 & 105 & 106 & 65 \\ 323 & 269 & 205 & 128 \end{bmatrix}$ was encoded using the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 5 & 3 & 0 \\ 4 & 3 & 8 \end{bmatrix}$. What is the message?

## EXAMPLE 4 Use Determinants to Find the Area of a Triangle

**A. How can you use determinants to find the area of a triangle?**

Translate the triangle so that one vertex is at the origin. The area of the triangle can be found by subtracting the areas of the surrounding triangles from the area of the rectangle.




The area of the rectangle is $ad$, so the area of the triangle is

$$\begin{aligned} \text{Area} &= \left|ad - \tfrac{1}{2}(a - c)(d - b) - \tfrac{1}{2}ab - \tfrac{1}{2}cd\right| \\ &= \tfrac{1}{2}|2ad - ad + ab + cd - cb - ab - cd| \\ &= \tfrac{1}{2}|ad - cb| \end{aligned}$$

This is the same as half the determinant of the matrix with the coordinates of the two points:

$$\text{Area} = \frac{1}{2}|ad - cb| = \frac{1}{2}\left|\det \begin{bmatrix} a & c \\ b & d \end{bmatrix}\right|$$

**REPRESENT AND CONNECT**
Determinants can be used for more than determining the existence of an inverse matrix. You can also use them to find the areas of triangles in the coordinate plane.

**B. What is the area of the triangle with vertices at (3, 2), (6, 6), and (2, 4)?**

Translate the triangle so the first vertex is at the origin:

$$\begin{bmatrix} 3 & 6 & 2 \\ 2 & 6 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -3 & -3 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 4 & 2 \end{bmatrix}$$

Write the matrix representing the other two vertices:

$$T = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$$

Then area of the triangle $= \frac{1}{2}|\det T| = \frac{1}{2}|(3)(2) - (-1)(4)| = \frac{1}{2}|10| = 5$

The area of the triangle is 5 square units.

**Try It!** **4.** Find the area of each triangle.

**a.** Vertices (0, 0), (–2, 10), and (–1, –5).

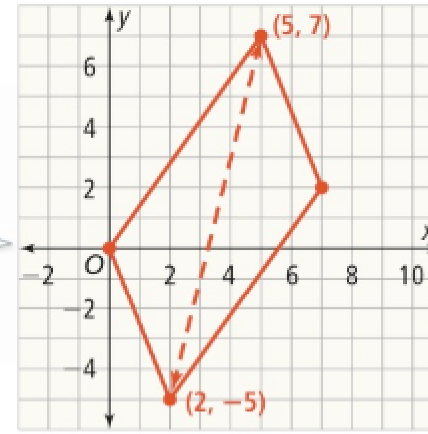
**b.** Vertices (–1, 6), (7, 10), and (6, 3).

APPLICATION

## EXAMPLE 5 Use a Determinant to Find the Area of a Parallelogram

**A. What is the area of a parallelogram, *P*, with one vertex at the origin and defined by the defined by the points (0, 0), (5, 7), (7, 2), and (2, −5)?**

Use the endpoints of the diagonal that does not include the origin. The determinant of the matrix that contains the origin as a point is zero. Either diagonal divides the parallelogram into two equal triangles.




The area of the parallelogram is twice the area of the triangle with a vertex at the origin, so the area of $P = |\det T|$.

Using the matrix $T = \begin{bmatrix} 5 & 2 \\ 7 & -5 \end{bmatrix}$, the area of the parallelogram is $|\det T|$.

$|\det T| = |5(-5) - 7(2)| = 39$

The area of the parallelogram is 39 square units.

**COMMON ERROR**

The sign of the determinant changes based on the order of points in your matrix. Remember to use absolute value when calculating area.

**B. A parallelogram has vertices at the origin, (−2, 1) and (1, *a*), and has an area of 5 square units. What are the possible value(s) of *a*?**

$T = \begin{bmatrix} -2 & 1 \\ 1 & a \end{bmatrix}$ ······ Write the matrix defined by the two diagonal endpoints.

$|-2a - 1(1)| = 5$ ······ Find the absolute value of the determinant, and set it equal to the area of the parallelogram.

$|-2a - 1| = 5$ ······ Simplify.

$-2a - 1 = 5$ or $-2a - 1 = -5$ ······ Rewrite using definition of absolute value.

$-2a = 6$ or $-2a = -4$ ······ Solve.

$a = -3$ or $a = 2$

Check the solutions.

$$\left|\det\begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}\right| = 5 \qquad \left|\det\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\right| = 5$$

$$|(-2)(-3) - (1)(1)| = 5 \qquad |(-2)(2) - (1)(1)| = 5$$

$$5 = 5\ ✓ \qquad 5 = 5\ ✓$$

The possible values for $a$ are −3 and 2.

**Try It!** **5.** Find the area of the parallelogram defined by the points (0, 0), (3, 8), (4, 12) and (1, 4).

## CONCEPT SUMMARY Matrix Inverses and Determinants

**DETERMINANT** The determinant of a $2 \times 2$ matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is denoted det $A$ and is equal to $ad - bc$.

If $A$ is an $n \times n$ matrix with $n \geq 3$, use technology to calculate the determinant.

**INVERSE** The multiplicative inverse of a square matrix $A$, denoted $A^{-1}$, exists if and only if det $A \neq 0$. It is the unique matrix such that:

- $A \cdot A^{-1} = I$
- $A^{-1} \cdot A = I.$

For a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse matrix is $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If $A$ is an $n \times n$ matrix with $n \geq 3$, use technology to calculate the inverse matrix.

**APPLICATIONS** The area of the parallelogram defined by the origin, $(a, b)$, $(c, d)$, and $(a + c, b + d)$ is given by $|\det P|$ when

$P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

For a parallelogram with vertices (0, 0), (4, –3), (10, 4), and (6, 7),

$P = \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}$; $\det P = |4(7) - (-3)(6)| = |28 - (-18)| = 46$

The area of the parallelogram is 46 square units.

The area of the triangle defined by the origin and points $(a, b)$, and $(c, d)$ is given by $\frac{1}{2}|\det T|$ when

$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

For a triangle with points: (0, 0), (2, 3) and (6, –2),

$T = \begin{bmatrix} 2 & 6 \\ 3 & -2 \end{bmatrix}$

$\text{Area}_{\Delta} = \frac{1}{2}|2(-2) - 3(6)| = \frac{1}{2}|-4 - 18| = \frac{1}{2}(22) = 11$

The area of the triangle is 11 square units.

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do you find and use an inverse matrix?

2. **Vocabulary** What is the determinant of a $2 \times 2$ matrix?

3. **Error Analysis** Enrique says the matrix $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ has an inverse matrix. Explain his error.

4. **Communicate and Justify** Explain how to use the determinant of a matrix to find the area of a triangle.

## Do You KNOW HOW?

**Find the inverse of each matrix, if it exists.**

5. $\begin{bmatrix} -2 & -4 \\ 2 & 3 \end{bmatrix}$

6. $\begin{bmatrix} -1 & 3 \\ -3 & 9 \end{bmatrix}$

7. $\begin{bmatrix} -3 & -2 & 1 \\ 5 & 4 & -3 \\ 6 & -4 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 0 & -4 \\ 0 & 6 & 3 \\ -1 & 1 & 3 \end{bmatrix}$

9. **Analyze and Persevere** What is the area of a triangle determined by the vertices at (0, 0), (2, 3), and (6, –1)?

10. What is the area of a parallelogram with vertices (0, 0), (5, 2), (4, –8), and (–1, –10)?

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

**11.** **Use Patterns and Structure** Write a $2 \times 2$ matrix that does not have an inverse. Explain how you can tell that it does not have an inverse.

**12.** **Generalize** Can a $2 \times 3$ matrix have an inverse? Explain.

**13.** **Error Analysis** Leah wants to find the area of a parallelogram with points at the origin, $(-4, -10)$, $(-1, -4)$, and $(3, 6)$. Explain and correct Leah's error in finding the area of the parallelogram.

Let $T = \begin{bmatrix} 3 & -4 \\ 6 & -10 \end{bmatrix}$.

$A = \frac{1}{2}|\det T|$

$A = \frac{1}{2}|-6| = 3$

The area of the parallelogram is 3 square units. ✗

**14.** **Represent and Connect** Are $\begin{bmatrix} 8 & 4 \\ 4 & -2 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{16} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{4} \end{bmatrix}$ inverses? Explain how you know.

**15.** **Higher Order Thinking** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find values of $a$, $b$, $c$, and $d$ such that $A$ is the same matrix as its inverse.
(*Hint:* There are four distinct possible values.)

**16.** **Generalize** Monisha said that to find det $B$, where $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, you can use the expression $ad - bc$ or the expression $bc - ad$. Is Monisha correct? Explain.

**17.** **Analyze and Persevere** Matrix $A$ does not have an inverse. Find the value of $b$ and explain how you know that this value for $b$ is correct.

$A = \begin{bmatrix} -1 & b \\ 3 & 6 \end{bmatrix}$

### PRACTICE

**Find the inverse of each matrix.** SEE EXAMPLE 1

**18.** $\begin{bmatrix} 10 & 2 \\ -5 & -3 \end{bmatrix}$

**19.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

**Does each given matrix have an inverse? If so, find it.** SEE EXAMPLE 2

**20.** $P = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

**21.** $R = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$

**22.** $Q = \begin{bmatrix} -6 & -9 \\ -4 & -6 \end{bmatrix}$

**23.** $S = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

**24.** The matrix $\begin{bmatrix} 30 & 15 & 106 & 63 & 33 & 121 \\ 18 & 120 & 80 & 102 & 102 & 164 \\ 101 & 24 & 154 & 43 & 111 & 162 \end{bmatrix}$

was encoded using the matrix $A = \begin{bmatrix} -1 & 3 & 2 \\ 4 & 6 & -2 \\ 0 & 1 & 5 \end{bmatrix}$.

What is the message? SEE EXAMPLE 3

**25.** The matrix $\begin{bmatrix} 49 & 145 & 173 & 124 & 76 & 215 \\ 18 & 50 & 62 & 46 & 30 & 78 \end{bmatrix}$

was encoded using the matrix $\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$.

What is the secret word?

**Find the area of the triangle defined by the given points.** SEE EXAMPLE 4

**26.** $(0, 0)$, $(2, 10)$, $(-1, 5)$

**27.** $(2, 1)$, $(11, -2)$, $(-4, 0)$

**What is the area of a parallelogram defined by the given points?** SEE EXAMPLE 5

**28.** $(0, 0)$, $(-2, 12)$, $(-6, 28)$, and $(-4, 16)$

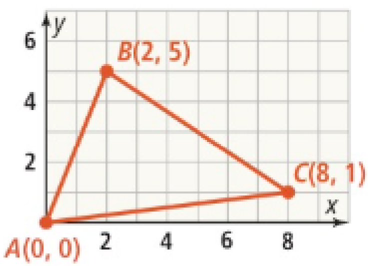
**29.** $(-5, 3)$, $(2, 2)$, $(7, -1)$, and $(0, 0)$

# PRACTICE & PROBLEM SOLVING

## APPLY

30. **Analyze and Persevere** A job title was hidden in the matrix
$$\begin{bmatrix} -34 & -29 & -35 & -23 & -19 & -92 \\ 123 & 93 & 114 & 219 & 66 & 153 \\ 97 & 26 & -137 & -83 & 11 & 16 \end{bmatrix}$$
using the encoding matrix $A = \begin{bmatrix} -4 & -2 & 1 \\ 0 & 3 & 6 \\ 5 & -8 & 2 \end{bmatrix}$.
Only those who discovered the title could apply. What job title was advertised?

31. **Apply Math Models** The coordinate plane shows the location of a triangular park, where each unit on the grid represents 10 ft.

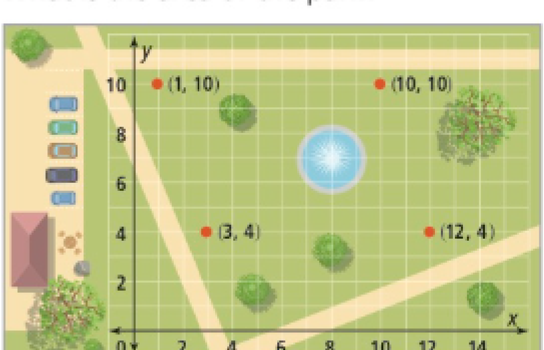



a. Use the vectors to write a matrix that represents the coordinates of the vertices of the triangular park.

b. What is the area of the triangular park?

c. A five-pound bag of grass seed covers about 300 square feet and costs \$17.98. How much will it cost to cover the park with grass seed? Explain.

32. **Analyze and Persevere** A city planner uses a coordinate plane to plan out a new neighborhood. Each grid square represents 4,000 square feet. A park, in the shape of a parallelogram, is to be built so that one vertex of the parallelogram is located at (1, 10) on the planner's coordinate plane. Using this as an initial point, the other points that determine the parallelogram are (10, 10), (12, 4), and (3, 4). What is the area of the park?




## ASSESSMENT PRACTICE

33. Does the matrix have an inverse? Write *Yes* or *No*. NS0.4.4

a. $\begin{bmatrix} -5 & 10 \\ -2 & -3 \end{bmatrix}$ b. $\begin{bmatrix} 15 & 2 \\ -12 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 9 & -6 \\ 12 & 8 \end{bmatrix}$ d. $\begin{bmatrix} -3 & -6 \\ -6 & -12 \end{bmatrix}$

34. **SAT/ACT** The area of a triangle defined by the points (0, 0), (2, $y$), and (4, 7) is 11 square units. What are the possible values of $y$?

Ⓐ 2 and 9 Ⓑ −2 and −9 Ⓒ −9 and 2 Ⓓ −2 and 9

35. **Performance Task** A credit card company encodes its issued credit card numbers when transmitting them electronically so that a customer's number is more secure. The company uses a $2 \times 8$ matrix to represent the 16 digits in the card number. The first column represents the first two digits, and so on.


**Part A** The matrix
$$\begin{bmatrix} 450 & 450 & 280 & 30 & 10 & 330 & 100 & 370 \\ 945 & 945 & 580 & 75 & 25 & 685 & 210 & 765 \end{bmatrix}$$
was encoded using the matrix $\begin{bmatrix} 10 & 40 \\ 25 & 80 \end{bmatrix}$.
What was the customer's credit card number?

**Part B** Create your own 16-digit credit card number and an encoding matrix, and encode your card number. Trade encoded card number matrices and the encoding matrix you used with a partner, and decode each other's numbers.

**Part C** The head of the company decides that a $2 \times 2$ encoding matrix is not secure enough and wants to institute a policy of using $3 \times 3$ encoding matrices. How will this affect the encoding process?

# 7-4

## Inverse Matrices and Systems of Equations

**I CAN…** use matrices to represent and solve systems of equations.

**VOCABULARY**

- constant matrix
- variable matrix

MA.912.NSO.4.4-Solve mathematical and real-world problems using the inverse and determinant of matrices. Also NSO.4.2

MA.K12.MTR.4.1, MTR.5.1, MTR.7.1

### CRITIQUE & EXPLAIN

**Let $A$, $B$, and $C$ be $2 \times 2$ matrices such that $B = C$. Recall that usually $AB \neq BA$.**

**A.** Is the product defined for every choice of two of the three matrices? What are the dimensions of the products?

**B.** Consider multiplying both sides of $B = C$ by matrix $A$. Which multiplications will always keep the equality?

$$AB = AC \qquad AB = CA \qquad BA = CA$$

**C.** Generalize Summarize your observations.

### ESSENTIAL QUESTION

**How can an inverse matrix be used to simplify the process of solving a system of linear equations?**

### EXAMPLE 1 Solve a Matrix Equation

**How can you solve the matrix equation $\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$?**

Let $A$ represent the matrix $\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$, and let $A^{-1}$ represent the inverse of the matrix. The goal is to isolate the variable matrix to determine the values of $x$ and $y$.

$$A \cdot X = B \qquad \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot B \qquad \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$I \cdot X = A^{-1} \cdot B \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$X = A^{-1} \cdot B \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

**GENERALIZE**
Multiplying by an inverse matrix is like multiplying by the reciprocal of a real number.

So the solution to the equation is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

You can check your work by multiplying A by the solution matrix to be sure the product is B.

**Try It!** 1. Solve the matrix equation $A \cdot X = B$ for

$$A = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -1 & 0 \\ -1 & -4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix}.$$

## EXAMPLE 2 Write a System of Linear Equations as a Matrix Equation

**How can each system of linear equations be represented as a matrix equation?**

**A.** $\begin{cases} 3x - 5y = 7 \\ 2x + y = 4 \end{cases}$

$$\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

The coefficient of $y$ in the second equation is 1.

The **coefficient matrix** has rows which contain the coefficients from a single equation. Each column contains all coefficients of a single variable.

The **variable matrix** has one column that represents all the variables in the system of equations.

Use the constant values from the *right-hand side* of the equations to make the **constant matrix**.

**USE PATTERNS AND STRUCTURE**
Multiply the matrices on the left-hand side and use properties of equivalent matrices to recover the original system of linear equations from the matrix equation.

**B.** $\begin{cases} x - 2y + 4z = 9 \\ 2x - 8z = -24 \\ 3x + y - 2z = -1 \end{cases}$

Be sure to include a zero coefficient for any missing variables.

$$\begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & -8 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -24 \\ -1 \end{bmatrix}$$

**Try It!** **2.** Express each system of linear equations as a matrix equation.

**a.** $10x - 9y = 1$
$7x + 6y = 13$

**b.** $4x + 2y - z = 14$
$2x - 3y + 5z = 20$
$3x - 6y = 8$

**CONCEPTUAL UNDERSTANDING**

## EXAMPLE 3 Solve a System of Linear Equations Using an Inverse Matrix

**How can you use matrix inverses to solve a system of linear equations?**

**A.** If possible, use inverse matrices to solve $\begin{cases} -3x + z = -9 \\ -x + 2z = 2 \\ 2x - y = 10 \end{cases}$

**Step 1** Express the system of linear equations as a matrix equation.

$$\begin{cases} -3x + z = -9 \\ -x + 2z = 2 \\ 2x - y = 10 \end{cases} \longrightarrow \begin{bmatrix} -3 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \\ 10 \end{bmatrix}$$

**Step 2** Find the inverse of the $3 \times 3$ coefficient matrix using technology.

The inverse of $\begin{bmatrix} -3 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & 0 \end{bmatrix}$ is $\begin{bmatrix} -0.4 & 0.2 & 0 \\ -0.8 & 0.4 & -1 \\ -0.2 & 0.6 & 0 \end{bmatrix}$.

**STUDY TIP**
Notice the original matrix has to be invertible for this method to work.

CONTINUED ON THE NEXT PAGE

EXAMPLE 3 CONTINUED

**Step 3** Multiply each side of the matrix equation by the inverse matrix.

$$\begin{bmatrix} -0.4 & 0.2 & 0 \\ -0.8 & 0.4 & -1 \\ -0.2 & 0.6 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.4 & 0.2 & 0 \\ -0.8 & 0.4 & -1 \\ -0.2 & 0.6 & 0 \end{bmatrix} \cdot \begin{bmatrix} -9 \\ 2 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.4(-9) + 0.2(2) + 0(10) \\ -0.8(-9) + 0.4(2) - 1(10) \\ -0.2(-9) + 0.6(2) + 0(10) \end{bmatrix}$$

Multiplying a matrix by its inverse results in an identity matrix.

Multiplying the inverse matrix by the constant matrix gives the solution matrix.

**Step 4** Multiply by the identity matrix, and write your solution.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$$

The solution to the original system of equations is (4, –2, 3).

**B.** If possible, use inverse matrices to solve $\begin{cases} \frac{1}{4}x - \frac{5}{8}y = \frac{1}{2} \\ -2x + 5y = -4 \end{cases}$

Express the system of linear equations as a matrix equation, $CX = D$.

$$\begin{bmatrix} \frac{1}{4} & -\frac{5}{8} \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -4 \end{bmatrix}$$

**COMMON ERROR**

The matrix is not invertible, so this system of equations has no unique solution. Graphically, the two equations either represent parallel lines or the same line.

To use the method from part (a), you need an invertible matrix. Find the determinant of the coefficient matrix $C$ to see if it has an inverse.

$$\det(C) = \frac{1}{4}(5) - \left(-\frac{5}{8}\right)(-2) = \frac{5}{4} - \frac{5}{4} = 0$$

The determinant of the coefficient matrix is 0, so this matrix does not have an inverse. You will have to solve this system using a different method, such as substitution or elimination.

$$8\left(\frac{1}{4}x - \frac{5}{8}y = \frac{1}{2}\right) \longrightarrow 2x - 5y = 4$$

$$\begin{array}{r} 2x - 5y = 4 \\ +(-2x + 5y = -4) \\ \hline 0 + 0 = 0 \end{array}$$

Using elimination results in the equation $0 = 0$, so the system of linear equations has infinitely many solutions. If elimination had resulted in a contradiction like $0 = 5$, then the system of equations would not have a solution.

While it is not possible to solve this system using an inverse matrix, there are infinitely many solutions to the system.

**Try It!** **3.** Solve the following systems of linear equations using inverse matrices, if possible.

**a.** $\begin{cases} 3x + 4y = 8 \\ \frac{3}{2}x + 2y = 5 \end{cases}$

**b.** $\begin{cases} x + 2y - 4z = 4 \\ x - 2y + 2z = -10 \\ -x - y + z = 4 \end{cases}$

APPLICATION

## EXAMPLE 4 Solve a Real-World System of Equations With an Inverse

**A company makes men's and women's sneakers. Last week, the company spent \$340 on labor and \$420 on materials. How many sneakers of each type did the company produce?**


**Formulate** To find a solution using an inverse matrix, first define the variables. Then write a system of equations to model the situation. Finally express the system as a matrix equation.

$w$ = the number of pairs of women's sneakers produced

$m$ = the number of pairs of men's sneakers produced

$$10w + 14m = 340$$

The first equation represents the *labor* costs.

$$12w + 18m = 420$$

The second equation represents the *materials* costs.

$$\begin{bmatrix} 10 & 14 \\ 12 & 18 \end{bmatrix} \begin{bmatrix} w \\ m \end{bmatrix} = \begin{bmatrix} 340 \\ 420 \end{bmatrix}$$

**Compute** Now solve the system of equations by multiplying by an inverse matrix.

Use technology to find that the inverse of $\begin{bmatrix} 10 & 14 \\ 12 & 18 \end{bmatrix}$ is $\begin{bmatrix} \frac{3}{2} & -\frac{7}{6} \\ -1 & \frac{5}{6} \end{bmatrix}$.

Multiply each side of the matrix equation by the inverse matrix.

$$\begin{bmatrix} \frac{3}{2} & -\frac{7}{6} \\ -1 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 10 & 14 \\ 12 & 18 \end{bmatrix} \begin{bmatrix} w \\ m \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{7}{6} \\ -1 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 340 \\ 420 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ m \end{bmatrix} = \begin{bmatrix} \frac{3}{2}(340) + \left(-\frac{7}{6}\right)(420) \\ -1(340) + \left(\frac{5}{6}\right)(420) \end{bmatrix}$$

Recall that a matrix multiplied by its inverse gives the identity matrix.

$$\begin{bmatrix} w \\ m \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

**Interpret** The solution is (20, 10). So the company made 20 pairs of women's sneakers and 10 pairs of men's sneakers.

**Try It!** 4. For a three-week period, the same company budgets \$860 for labor and \$1,080 for materials. How many pairs of men's and women's sneakers can they make in three weeks?

## CONCEPT SUMMARY Solving Systems of Linear Equations With Matrices

**LINEAR SYSTEMS**

- Express the linear system of equations as a matrix equation.

$$\begin{cases} x + 2y + 3z = 4 \\ 3x - 2y + z = -1 \\ x + y - 2z = 6 \end{cases} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}$$

- For a $2 \times 2$ matrix, find the determinant of the coefficient matrix, $A$. If det $A \neq 0$, solve the matrix equation using the inverse of the coefficient matrix. If det $A = 0$, solve the original system with an alternative method. Use substitution or elimination.
- For an $n \times n$ matrix, where $n \geq 3$, use technology to find the inverse of the matrix. Then multiply each side of the equation by the inverse matrix to find the solution.

**MATRIX EQUATIONS**

$$\begin{aligned} AX &= B \\ A^{-1} \cdot AX &= A^{-1} \cdot B \\ I \cdot X &= A^{-1} \cdot B \\ X &= A^{-1} \cdot B \end{aligned}$$

Multiply by the inverse matrix ($A^{-1}$) on the left to isolate the variable matrix.

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can an inverse matrix be used to simplify the process of solving a system of linear equations?

2. **Error Analysis** Corey says the matrix equation $\begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ 22 \end{bmatrix}$ represents the system of linear equations $\begin{cases} 3x + 2y = 8 \\ -y + 4z = 13. \\ 2x + 6z = 22 \end{cases}$ Explain Corey's error.

3. **Vocabulary** How do you determine the coefficient matrix for a particular system of linear equations?

4. **Communicate and Justify** Explain how to solve a system of linear equations using an inverse matrix.

## Do You KNOW HOW?

**Express the system of linear equations as a matrix equation.**

5. $\begin{cases} 5x + 3y = -21 \\ 2x - 4y = -24 \end{cases}$

6. $\begin{cases} 6x - 8y + 2z = -46 \\ -x + 5y + 3z = 29 \\ 9x - 4z = -35 \end{cases}$

7. Given the matrix equation $A \cdot X = B$ for $A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & -2 & 3 \\ -4 & -6 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -5 \\ -5 \end{bmatrix}$, find $A^{-1}$. Then use $A^{-1}$ to solve the matrix equation for $X$.

8. Write an equation that shows what your next step would be in solving this matrix equation for $x$, $y$, and $z$.

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & -13 & 9 \\ -4 & 12 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 2 \end{bmatrix}$$

# PRACTICE & PROBLEM SOLVING

## UNDERSTAND

9. **Communicate and Justify** Explain why the equation $X = BA^{-1}$ cannot be used to solve this matrix equation in the form $AX = B$.

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & -13 & 9 \\ -4 & 12 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 2 \end{bmatrix}$$

10. **Analyze and Persevere** Give an advantage of solving a system of linear equations using matrices instead of using the substitution or elimination method. (Assume you use technology to find the inverse matrix.)

11. **Error Analysis** Describe and correct the error a student made in solving the matrix equation $\begin{bmatrix} 5 & 9 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -26 \\ 16 \end{bmatrix}$.

$$\begin{bmatrix} 5 & 9 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -26 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} -26 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ -100 \end{bmatrix} \quad \times$$

12. **Use Patterns and Structure** Assume $a = c$ and $b = d$ for the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$. What is the relationship between $e$ and $f$ when there are infinitely many solutions? What is the relationship between $e$ and $f$ when there are no solutions?

13. **Higher Order Thinking** Write a system of linear equations in four variables, $w$, $x$, $y$, and $z$ that has integer solutions. Then write a matrix equation to represent your system of equations. Finally, solve the matrix equation using technology to verify the integer solutions. Hint: work backwards from a solution.

## PRACTICE

**Solve the matrix equation $A \cdot X = B$ for the given matrices.** SEE EXAMPLE 1

14. $A = \begin{bmatrix} 8 & -7 \\ -6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 11 \\ -12 \end{bmatrix}$

15. $A = \begin{bmatrix} 2 & 8 & 4 \\ 1 & -1 & -3 \\ -3 & 2 & -9 \end{bmatrix}$ and $B = \begin{bmatrix} 26 \\ -2 \\ 37 \end{bmatrix}$

**Express the system of linear equations as a matrix equation.** SEE EXAMPLE 2

16. $8x + y = -1$
$-12x - 2y = 6$

17. $2x + 3y + 7z = 4$
$10x + 8y - 2z = -12$
$6x - y = 30$

**Solve the following systems of linear equations using inverse matrices, if possible.** SEE EXAMPLE 3

18. $-x + 2y = 8$
$-3x + 6y = -12$

19. $9x + 2y + 3z = 1$
$-8x - 3y - 4z = 1$
$12x + y - 2z = -17$

20. $-3x + 4y = -4$
$\frac{1}{2}x - 3y = -11$

21. $2x + \frac{2}{3}y + z = -8$
$x + 2y - \frac{1}{3}z = 6$
$-\frac{1}{2}x + 3y - 2z = 22$

22. Katrina makes bracelets and necklaces. Last week, she made 5 bracelets and 2 necklaces. This week, she made 3 bracelets and 5 necklaces. On average, how many hours does it take Katrina to make one bracelet? On average, how many hours does it take Katrina to make one necklace? SEE EXAMPLE 4

## PRACTICE & PROBLEM SOLVING

### APPLY

**23.** **Use Patterns and Structure** Luke had some quarters and dimes in his pocket. The quarters and dimes are worth \$2.55. He has 3 times as many quarters as dimes.

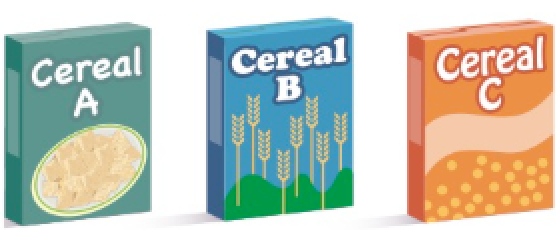
**a.** Write a matrix equation to find the number of quarters, $x$, and dimes, $y$, Luke has.

**b.** How many quarters and dimes does Luke have?

**24.** **Analyze and Persevere** Malia is training for a triathlon. The table shows the number of hours she swam, biked, and ran and the total distance traveled on three different days. Write and solve a matrix equation to find Malia's average speed while swimming, biking, and running.

| DAY | | | | Total Distance (mi.) |
|---|---|---|---|---|
| 1 | $\frac{1}{3}$ | 1 | 2 | 32 |
| 2 | $\frac{2}{3}$ | $\frac{4}{5}$ | 1 | 22 |
| 3 | 1 | 2 | $\frac{1}{2}$ | 37 |

**25.** **Apply Math Models** Steve wants to mix three different types of cereal to create a mixture with 3,400 calories, 90 grams of protein, and 90 grams of fiber. The boxes of cereal show the number of calories, grams of protein, and grams of fiber in one serving of cereal A, B, and C. Write a matrix equation to represent this situation. How many servings of each type of cereal does Steve need to include in the mixture?




Calories: 300
Protein: 11g
Fiber: 8g

Calories: 300
Protein: 7g
Fiber:6g

Calories: 320
Protein: 8g
Fiber: 10g

### ASSESSMENT PRACTICE

**26.** Which matrix equation has a unique solution? NSO.4.4

Ⓐ $\begin{bmatrix} 6 & 2 \\ 9 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 18 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 2 & 9 \\ -3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -46 \\ 36 \end{bmatrix}$

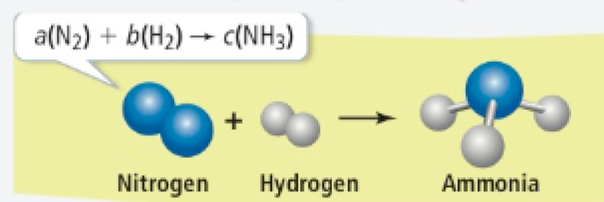
Ⓒ $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

Ⓓ $\begin{bmatrix} -4 & 2 \\ -14 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \end{bmatrix}$

**27.** **SAT/ACT** The coordinates $(x, y)$ of a point in a plane are the solution of the matrix equation $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$. In what quadrant is the point located?

Ⓐ I  Ⓑ II  Ⓒ III  Ⓓ IV

**28.** **Performance Task** Nitrogen ($N_2$) and hydrogen ($H_2$) can react to form ammonia ($NH_3$). To write an equation for the reaction, the number of molecules of each element that are combined must equal the number of molecules of each element in the result. You can use matrices to figure out the coefficients that will balance the reaction. $a(N_2) + b(H_2) \rightarrow c(NH_3)$

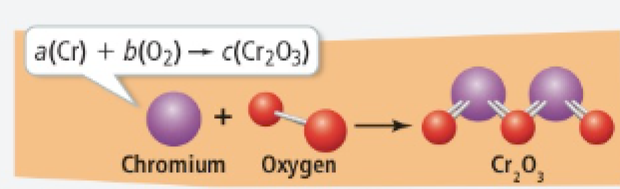



**Part A** Let $c = 1$. Write a system of equations in $a$ and $b$ for this reaction. Make one equation to represent N and one equation to represent H, and think of the reaction arrow as an equal sign.

**Part B** Rewrite your system of equations as a matrix equation, and solve for $a$ and $b$.

**Part C** Substitute the coefficients into the reaction equation. Multiply the equation by the least common denominator of all fractions so that all coefficients are whole numbers. Check that it is balanced.

**Part D** Use the same process to balance the reaction $a(Cr) + b(O_2) \rightarrow c(Cr_2O_3)$.



MATHEMATICAL MODELING IN 3 ACTS

MA.912.NSO.4.2-Given a mathematical or real-world context, represent and solve a system of two- or three-variable linear equations using matrices.
Also NSO.4.4
MA.K12.MTR.7.1

## The Big Burger

For many people, hamburgers are a hallmark of American food. Nearly every restaurant, from fast food chains, to diners, to fine dining establishments, offers some kind of hamburger on their menu.

Some restaurants offer various types of burgers: beef, turkey, and veggie burgers are all quite popular. You can also often choose extras to add to your burger: double patties of beef, cheese, pickles, onions, lettuce, tomatoes . . . The options are endless! Think about this during the Mathematical Modeling in 3 Acts lesson.

### ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

### ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

### ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC
7

# Topic Review

## TOPIC ESSENTIAL QUESTION

1. How can you use matrices to help you solve problems?

## Vocabulary Review

**Choose the correct term to complete each sentence.**

2. The ________ has one column that represents all the variables in the system of equations.
3. A/an ________ is a square matrix with ones on the main diagonal and zeros for all other elements.
4. ________ means the multiplication of each element of a matrix by a single real number.
5. The product of a matrix and its ________ is the identity matrix.
6. The ________ has one column that contains the constants from the right-hand side of the system of equations.
7. A(n) ________ is a matrix that has the same number of rows as columns.

- constant matrix
- identity matrix
- inverse matrix
- scalar multiplication
- square matrix
- variable matrix
- zero matrix

## Concepts & Skills Review

**LESSON 7-1** Operations With Matrices

### Quick Review

To multiply a matrix by a scalar, multiply each element in the matrix by the scalar.

To add (or subtract) matrices, add (or subtract) the corresponding elements.

### Example

**Add matrices $A$ and $B$.**

$A = \begin{bmatrix} 9 & 2 & 11 \\ -3 & 5 & 6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & -1 & 7 \\ 8 & 12 & 0 \end{bmatrix}$

Add corresponding elements of the two matrices.

$$A + B = \begin{bmatrix} 9 & 2 & 11 \\ -3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 7 \\ 8 & 12 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9+4 & 2+(-1) & 11+7 \\ -3+8 & 5+12 & 6+0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 1 & 18 \\ 5 & 17 & 6 \end{bmatrix}$$

### Practice & Problem Solving

**Given matrices $C = \begin{bmatrix} 9 & -5 \\ 3 & 6 \end{bmatrix}$ and $D = \begin{bmatrix} -7 & 1 \\ 8 & 2 \end{bmatrix}$, calculate each of the following.**

8. $D - C$
9. $5D$
10. A segment has endpoints $A(5, -3)$ and $B(2, 4)$. Use matrices to represent a translation of $\overline{AB}$ to $\overline{YZ}$ by 3 units right and 7 units down. What are the coordinates of $Y$ and $Z$?
11. **Communicate and Justify** Suppose $N$ is a $3 \times 3$ matrix. Explain how to find matrix $P$ so that $N + P$ is the zero matrix.
12. **Analyze and Persevere** A seminar has 6 women and 8 men register early. Then 18 women and 12 men register in class. Use matrix addition to find the total number of men and women in the seminar.

## LESSON 7-2 Matrix Multiplication

### Quick Review

The product of two matrices is a new matrix with the sums of the products of corresponding row and column elements.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

For an $n \times n$ matrix $A$, the multiplicative **identity matrix** $I$ is an $n \times n$ square matrix with 1s on the main diagonal and 0s for all other elements: $AI = IA = A$.

### Example

**Multiply matrices $A$ and $B$.**

$$A = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$$

Find the sums of products of corresponding row and column elements.

$$AB = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(-1) + (-2)(0) & (3)(6) + (-2)(5) \\ (1)(-1) + (-4)(0) & (1)(6) + (-4)(5) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 8 \\ -1 & -14 \end{bmatrix}$$

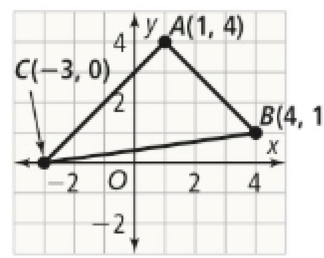
### Practice & Problem Solving

**Given matrices $A = \begin{bmatrix} 4 & -3 \\ 0 & 9 \end{bmatrix}$, $B = \begin{bmatrix} -7 & 8 \\ -5 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 6 & -1 \\ 2 & -2 \end{bmatrix}$, find each of the following.**

**13.** $AB$ **14.** $AC$ **15.** $BC$

**16.** $BA$ **17.** $CA$ **18.** $CB$

**19.** Represent the coordinates of the triangle as a matrix. Then multiply by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to find the coordinates of the image of triangle $ABC$ after a reflection across the $x$-axis.




**20. Generalize** Explain how to determine whether two matrices can be multiplied.

**21. Analyze and Persevere** At Store X, Television A costs \$800 and Television B costs \$500. At Store Y, Television A costs \$750 and Television B costs \$550. Last month, each store sold 25 of Television A and 20 of Television B. Write and solve a matrix equation to find the total amount in sales at each store.

## LESSON 7-3 Inverses and Determinants

### Quick Review

The determinant of a **2 × 2 matrix** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted det $A$ and is equal to $ad - bc$.

The **inverse matrix** is $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

### Example

**Find the inverse of matrix** $A = \begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix}$.

$\det A = ad - bc = (4)(6) - (2)(8) = 24 - 16 = 8$

Because the determinant does not equal 0, there is an inverse.

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 6 & -8 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

### Practice & Problem Solving

**Find the determinant of each matrix.**

**22.** $\begin{bmatrix} 12 & -6 \\ 8 & -3 \end{bmatrix}$ **23.** $\begin{bmatrix} 14 & -3 \\ 2 & 0 \end{bmatrix}$

**Does each given matrix have an inverse? If so, find it.**

**24.** $A = \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}$ **25.** $B = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -4 & 0 \\ -1 & -3 & 5 \end{bmatrix}$

**26. Error Analysis** Carla said the inverse of matrix $A = \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$.

Describe and correct Carla's error.

**27. Use Patterns and Structure** Find the area of the triangle defined by (−3, 5), (5, 11), and (−1, 9).

## LESSON 7-4 Inverse Matrices and Systems of Equations

### Quick Review

Matrices can be used to solve systems of equations.

The coefficient matrix has rows which contain the coefficients from a single equation. Each column contains all coefficients of a single variable.

The **variable matrix** has one column that represents all the variables in the system of equations.

The constant values from the right-hand side of the equations are used to make the **constant matrix**.

$$\begin{matrix} ax + by + cz = k \\ dx + ey + fz = m \\ gx + hy + jz = n \end{matrix} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ m \\ n \end{bmatrix}$$

### Example

**Solve the system of equations using matrices.** $\begin{cases} 3x + 6y = 0 \\ -2x + 3y = -7 \end{cases}$

$$\begin{matrix} 3x + 6y = 0 \\ -2x + 3y = -7 \end{matrix} \Rightarrow \begin{bmatrix} 3 & 6 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ -2 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 & 6 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -2 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{2}{21} & \frac{1}{7} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

### Practice & Problem Solving

**Solve the following systems of equations using inverse matrices.**

**28.** $\begin{cases} 2x + 4y = 4 \\ -3x - 7y = -4 \end{cases}$ **29.** $\begin{cases} -2x + 3y + 3z = 6 \\ 6x - 8y - 2z = -4 \\ 2x - 2y - 3z = -13 \end{cases}$

**30. Use Patterns and Structure** Explain how to write a system of equations given the matrix equation $\begin{bmatrix} 4 & 9 & 1 \\ 8 & -2 & 0 \\ -7 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 6 \end{bmatrix}$.

**31. Use Patterns and Structure** Two students visited the school store to buy supplies for the school year. One student purchased 8 folders and 6 notebooks for a total price of \$38. The other student purchased 2 folders and 9 notebooks for a total of \$47. If each folder is the same price and each notebook is the same price, how much does each folder and each notebook cost?

TOPIC

# 8

# Probability

## TOPIC ESSENTIAL QUESTION

How can you find the probability of events and combinations of events?

## Topic Overview


**enVision® STEM Project:**
Simulate Weather Conditions

**8-1** Probability Events
DP.4.1, DP.4.2, MTR.5.1, MTR.6.1, MTR.7.1

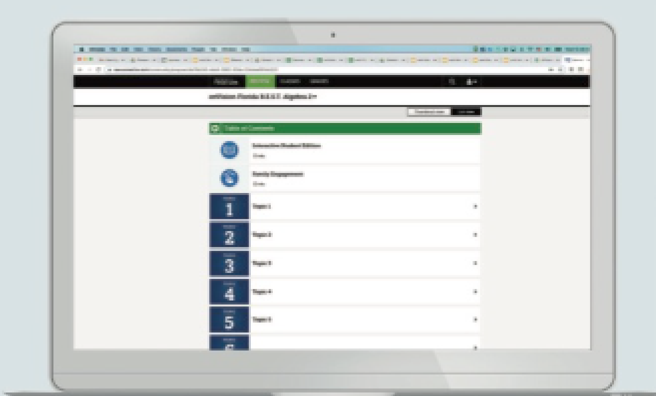
**8-2** Conditional Probability
DP.4.3, DP.4.4, MTR.1.1, MTR.4.1, MTR.5.1

**Mathematical Modeling in 3 Acts:**
Place Your Guess
DP.4.1, MTR.7.1

**8-3** Permutations and Combinations
DP.4.9, DP.4.10, AR.1.11, MTR.1.1, MTR.2.1, MTR.3.1


## Topic Vocabulary

- combination
- complement
- conditional probability
- dependent events
- factorial
- Fundamental Counting Principle
- independent events
- intersection
- mutually exclusive
- permutation
- union



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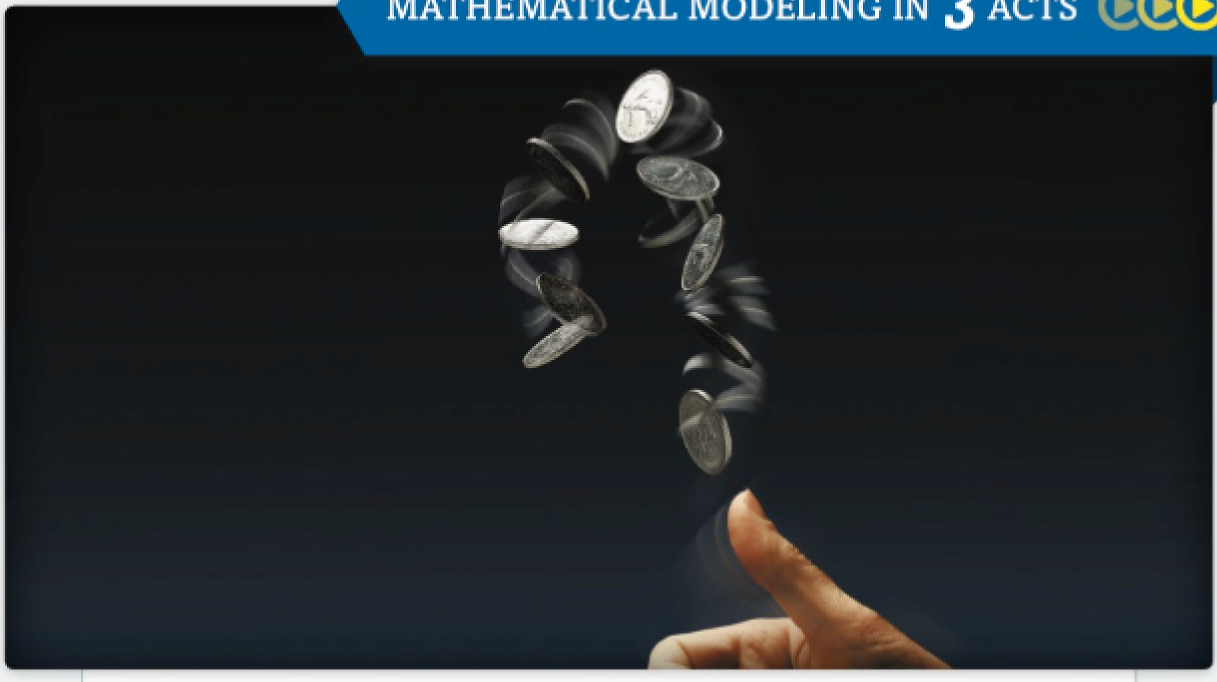
**FAMILY ENGAGEMENT** Involve family in your learning.

**ACTIVITIES** Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its.***

**ANIMATION** View and interact with real-world applications.

**PRACTICE** Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



## Place Your Guess

A coin toss is a popular way to decide between two options or settle a dispute. The coin toss is popular because it is a simple and unbiased way of deciding. Assuming the coin being tossed is a fair coin, both parties have an equally likely chance of winning.

What other methods could you use to decide between two choices fairly? Think about this during the Mathematical Modeling in 3 Acts lesson.

**VIDEOS** Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

**ADAPTIVE PRACTICE** Practice that is *just right* and *just for you.*

**GLOSSARY** Read and listen to English and Spanish definitions.

**CONCEPT SUMMARY** Review key lesson content through multiple representations.

**ASSESSMENT** Show what you've learned.

**TUTORIALS** Get help from ***Virtual Nerd***, right when you need it.

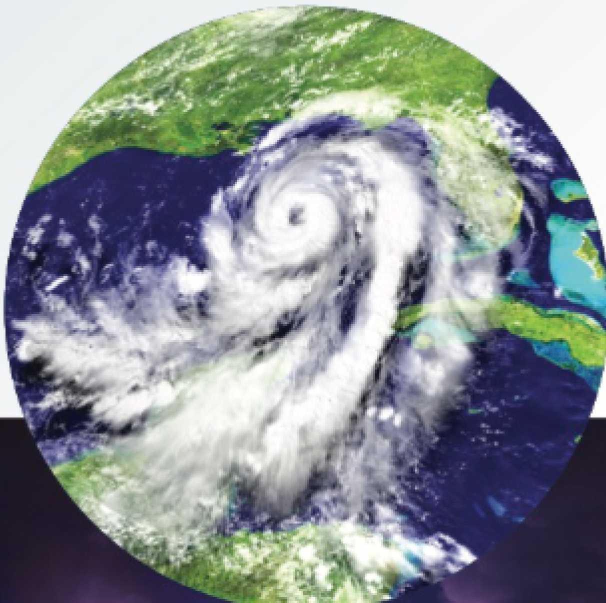
**MATH TOOLS** Explore math with digital tools and manipulatives.

**DESMOS** Use Anytime and as embedded Interactives in Lesson content.

**QR CODES** Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

## Did You Know?

**Meteorologists** use past climate data for a particular location and date as well as **weather models** to make weather predictions. Some regions in the U.S. are more predictable than others.



The **greatest temperature change** in a one-day period occurred in Loma, Montana, in 1972. The temperature rose an incredible **103 degrees**, from –54 to 49 °F, in 24 hours.

**Weather events** can surprise experts, and can vary greatly even within a few miles.

**Climate** is the long-term average of weather conditions. So the difference between weather and climate is a measure of time.

## Your Task: Simulate Weather Conditions

You and your classmates will research climate data for a specific location for one month. You'll use probability to simulate a plausible set of weather conditions for each day of February, including temperature and precipitation, and whether the precipitation will be rain or snow.

# 8-1 Probability Events

**I CAN…** use relationships among events to find probabilities.

**VOCABULARY**

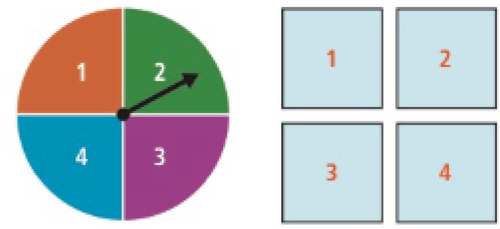
- complement
- independent events
- intersection
- mutually exclusive
- union

MA.912.DP.4.1–Describe events as subsets of a sample space using characteristics, or categories, of the outcomes, or as unions, intersections or complements of other events. Also DP.4.2

MTR.5.1, MTR.6.1, MTR.7.1

## EXPLORE & REASON

**Allie spins the spinner and draws one card without looking. She gets a 3 on the spinner and the 3 card. Then she sets the card aside, spins again, and draws another card.**




**A.** Is it possible for Allie to get a 3 on her second spin? On her second card? Explain.

**B.** **Communicate and Justify** How does getting the 3 card on her first draw affect the probability of getting the 2 card on her second draw? Explain.

## ESSENTIAL QUESTION

**How does describing events as mutually exclusive or independent affect how you find probabilities?**

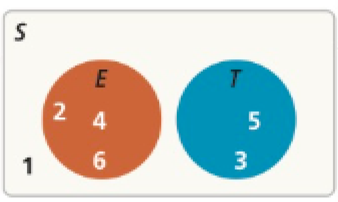
## EXAMPLE 1 Find Probabilities of Mutually Exclusive Events

**You roll a standard number cube once. Let $E$ represent the event "roll an even number." Let $T$ represent the event "roll a 3 or 5."**

**A. What is the probability that you roll an even number or roll a 3 or 5?**

Show the outcomes of events $E$ and $T$ as subsets of the sample space $S$.

Events $E$ and $T$ are **mutually exclusive** because there is no outcome in both sets.




There are 5 outcomes that are even numbers or a 3 or 5. {2, 3, 4, 5, 6} There are a total of 6 possible outcomes in the sample space.

$$P(E \text{ or } T) = \frac{\text{number of favorable outcomes}}{\text{number of total possible outcomes}}$$

$$= \frac{3+2}{6}$$

There are 3 outcomes in event $E$ and 2 outcomes in event $T$.

$$= \frac{3}{6} + \frac{2}{6}$$

This is equivalent to $P(E) + P(T)$.

$$= \frac{5}{6}$$

The probability of rolling an even number or rolling a 3 or a 5 is $\frac{5}{6}$.

**B. You roll a standard number cube once. What is the probability that you roll an even number and a 3 or 5?**

$$P(E \text{ and } T) = \frac{\text{number of favorable outcomes}}{\text{number of total possible outcomes}}$$

$$= \frac{0}{6}$$

$$= 0$$

Because events $E$ and $T$ are mutually exclusive, there are no outcomes that are in both sets.

The probability of rolling an even number and rolling a 3 or a 5 is 0.

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**STUDY TIP**

Notice the roles of *or* and *and* when working with probability of mutually exclusive events.

**GENERALIZE**
What is the probability of any pair of mutually exclusive events? Explain.

EXAMPLE 1 CONTINUED

**C. You roll a standard number cube once. What is the probability that you do *not* roll an even number?**

$P(\text{not } E) = \frac{3}{6}$ or $\frac{1}{2}$ — There are 3 outcomes that are not even numbers.

The probability of not rolling an even number is $\frac{1}{2}$.

**Try It!** **1.** A box contains 100 balls. Thirty of the balls are purple and 10 are orange. If you select one of the balls at random, what is the probability of each of the following events?

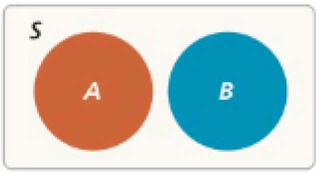
**a.** The ball is purple or orange.

**b.** The ball is not purple and not orange.

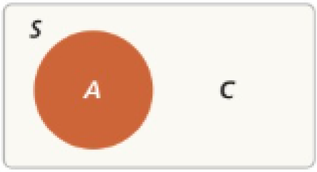
## CONCEPT Probabilities of Mutually Exclusive Events

The **union** of two events (A or B) is the set of outcomes in either A or B. The **intersection** of two events (A and B) is the set of outcomes in both A and B. If *A* and *B* are mutually exclusive events, there are no outcomes in their intersection. Therefore,

- $P(A \text{ or } B) = P(A) + P(B)$
- $P(A \text{ and } B) = 0$




The **complement** of an event is the set of all outcomes in a sample space that are not included in the event.



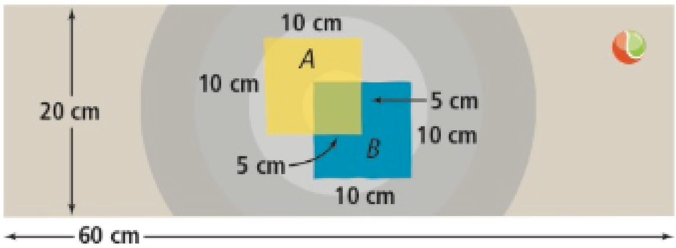

If *C* is the event that *A* does not occur, then

$P(C) = 1 - P(A)$.

APPLICATION

## EXAMPLE 2 Find the Probabilities of Non-Mutually Exclusive Events

**A student-made target includes two overlapping squares. Assume that a sticky ball thrown at the target is equally likely to land anywhere on the target. What is the probability that the ball lands inside one or both of the squares?**

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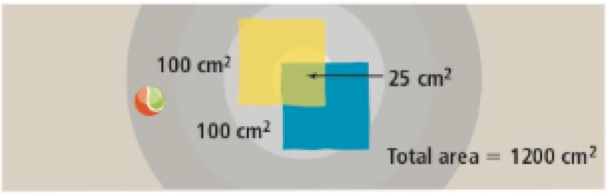
USE PATTERNS AND STRUCTURE

What other method could you use to compute $P(A \text{ and } B)$?

EXAMPLE 2 CONTINUED

**Step 1** Find the areas of the board, the squares, and their overlapping area.




**Step 2** Find the probabilities.

$$P(A \text{ or } B) = \frac{100}{1{,}200} + \frac{100}{1{,}200} - \frac{25}{1{,}200}$$
$$= \frac{175}{1{,}200} = \frac{7}{48}$$

Subtract the probability of the overlapping area because it was included twice, once for each large square.

The probability that the ball will land inside one or both squares is $\frac{7}{48}$, or about 15%.

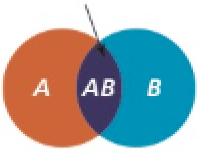
**Try It!** 2. The student changed square A to a rectangle 10 cm by 12 cm with an area overlap of square B of 5 cm by 7 cm. What is the probability that the ball lands inside the union of A's and B's areas?

## CONCEPT Probabilities of Non-Mutually Exclusive Events

If $A$ and $B$ are not mutually exclusive events, there are outcomes in their intersection which must not be counted twice. Therefore,

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Events $A$ and $B$




CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Identify Independent Events

GENERALIZE

The number of green marbles, $n$, and the number of violet marbles, m, are mutually exclusive events. We use addition to find the total number of ways we draw any marble from the sample space, $n + m$. This basic idea of counting is used again and again in finding probability which by one definition is the number of considered results divided by the total number of results.

**A jar contains 12 green marbles and 8 violet marbles.**

**A. A marble is chosen at random from the jar and replaced. Another marble is chosen at random from the jar. Does the color of the first marble chosen affect the possible outcomes for the second marble chosen?**

Determine the probabilities for each choice to decide whether the first marble chosen affects the possibilities for the second marble.

| First Choice-Sample Space |
|---|
| 12 Green |
| 8 Violet |

$P(G) = \frac{12}{20} = \frac{3}{5}$

$P(V) = \frac{8}{20} = \frac{2}{5}$

| Second Choice-Sample Space |
|---|
| 12 Green |
| 8 Violet |

$P(G) = \frac{12}{20} = \frac{3}{5}$

$P(V) = \frac{8}{20} = \frac{2}{5}$

The probabilities are the same.

The color of the first marble chosen does not affect the possible outcomes for the second marble chosen.

Two events are **independent events** if and only if the occurrence of one event does not affect the probability of a second event.

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EXAMPLE 3 CONTINUED

**B. A marble is chosen at random from the jar and not replaced. Another marble is chosen at random from the jar. Does the color of the first marble chosen affect the possible outcomes for the second marble chosen?**

Determine whether the events are independent.

| First Choice-Sample Space |
|---|
| 12 Green |
| 8 Violet |

**COMMON ERROR**
When two or more items are selected from the same set, you must determine whether the first item(s) is replaced before the next item is selected. Then find the probabilities.

$P(G) = \frac{12}{20} = \frac{3}{5}$ $\qquad$ $P(V) = \frac{8}{20} = \frac{2}{5}$

Assume a green marble was chosen first.

| Second Choice-Sample Space |
|---|
| 11 Green |
| 8 Violet |

$P(G) = \frac{11}{19}$ $\qquad$ $P(V) = \frac{8}{19}$

Assume a violet marble is chosen first.

| Second Choice-Sample Space |
|---|
| 12 Green |
| 7 Violet |

$P(G) = \frac{12}{19}$ $\qquad$ $P(V) = \frac{7}{19}$

When the first marble is not replaced in the jar, the color of the first marble chosen does affect the possible outcomes for the second marble chosen. These events are not independent.

**Try It!** **3.** In each scenario, there are 10 cards in a box, 5 black and 5 red. Two cards are selected from the box, one at a time. Does the scenario define an independent event? Explain.

**a.** A card is chosen at random and then replaced. Another card is chosen.

**b.** A card is chosen at random and *not* replaced. Another card is chosen.

## CONCEPT Probability of Independent Events

If $A$ and $B$ are independent events, then $P(A \text{ and } B) = P(A) \cdot P(B)$.

If $P(A \text{ and } B) = P(A) \cdot P(B)$, then $A$ and $B$ are independent events.

Example: There are 12 possible results from the independent events rolling a number cube and tossing a coin.

$P(4) = \frac{1}{6}$ $\qquad$ $P(H) = \frac{1}{2}$

$P(4 \text{ and } H) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$

APPLICATION

## EXAMPLE 4 Find Probabilities of Independent Events

**Carmen cannot decide which shirt to wear today, so she chooses one at random.**

**The probability of rain today is 40%, or $\frac{2}{5}$.**

**A. What is the probability that Carmen chooses a yellow shirt and it does not rain today?**

Let $Y$ represent the event "yellow shirt." Let $N$ represent "no rain."

$Y$ and $N$ are independent because Alex's choice of shirt does not affect the weather, and the weather does not affect Alex's choice of shirt. Use the formula to find $P(Y \text{ and } N)$.

**STUDY TIP**
You can write probabilities as fractions, decimals, or percents. Choose the most convenient form for your purposes.

**Step 1** Find $P(N)$ and $P(Y)$.

$P(N) = 1 - \frac{2}{5} = \frac{3}{5}$.

Subtract the probability that it will rain, $\frac{2}{5}$, from 1 to find the probability that it will *not* rain.

$P(Y) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{2}{4} \text{ or } \frac{1}{2}$

**Step 2** Apply the formula and multiply the probabilities of $Y$ and $N$.

$P(Y \text{ and } N) = P(Y) \cdot P(N) = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} = 30\%$

Use the rule for the probability of independent events.

The probability that Carmen chooses a yellow shirt and it does not rain is 30%.

**B. What is the probability that Carmen chooses a yellow shirt and it does not rain today or that Carmen chooses a green shirt and it rains today?**

Let $G$ represent "green shirt." Let $R$ represent "rain."

The events "$Y$ and $N$" and "$G$ and $R$" are mutually exclusive because no outcomes are in both events.

**Step 1** Find $P(G \text{ and } R)$

$P(G \text{ and } R) = P(G) \cdot P(R) = \frac{1}{4} \cdot \frac{2}{5} = \frac{2}{20} = \frac{1}{10} = 10\%$

**Step 2** Find $P((Y \text{ and } N) \text{ or } (G \text{ and } R))$

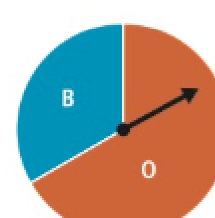
$P((Y \text{ and } N) \text{ or } (G \text{ and } R)) = P(Y \text{ and } N) + P(G \text{ and } R)$

$= 30\% + 10\% = 40\%$

Add to find the probability that either event will occur.

The probability that Carmen chooses a yellow shirt and it does not rain or that Carmen chooses a green shirt and it rains is 40%.

**Try It!** 4. You spin the spinner two times. Assume that the probability of Blue each spin is $\frac{1}{3}$ and the probability of Orange each spin is $\frac{2}{3}$. What is the probability of getting the same color both times? Explain.

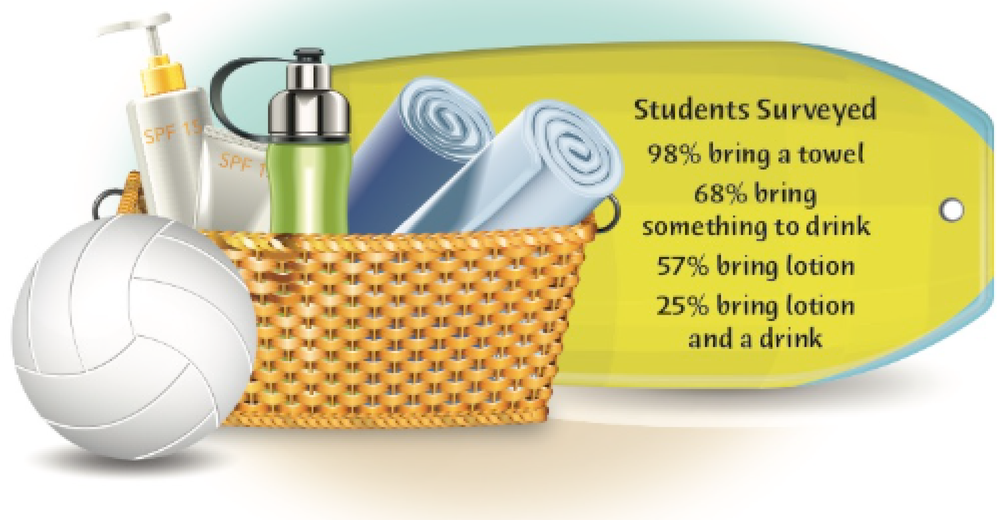


APPLICATION

**HAVE A GROWTH MINDSET**
In what ways do you give your best effort and persist?

## EXAMPLE 5 Use Probabilities to Determine Independence

**Students going to the beach for a few hours were surveyed about what they would bring with them. If a student is chosen at random, are the events of a student bringing lotion and of a student bringing something to drink to the beach independent events?**




If they are independent events, then the product of the two probabilities will be approximately equal to the probability of a student bringing both to the beach. Let A be the event of bringing lotion and let B be the event of bringing a drink.

$P(A \text{ and } B) = P(A) \cdot P(B)$

$0.25 \neq 0.57(0.68) \approx 0.39$

So the events of a student bringing lotion and of a student bringing something to drink are not independent events.

**Try It!** **5.** Consider a deck of cards. Explain the independence of events in drawing a club and drawing a seven.

## CONCEPT SUMMARY Probability Events

| | Mutually Exclusive Events | Independent Events |
|---|---|---|
| **WORDS** | $A$ and $B$ are mutually exclusive because no outcome is in both $A$ and $B$. | $D$ and $M$ are independent because the occurrence of one does not affect the probability of the other. |
| **ALGEBRA** | If $A$ and $B$ are mutually exclusive events, then $P(A \text{ or } B) = P(A) + P(B)$.<br>If $C$ is the event that $A$ does not occur, then $P(C) = 1 - P(A)$. | If $D$ and $M$ are independent events, then $P(D \text{ and } M) = P(D) \cdot P(M)$.<br>If $P(D \text{ and } M) = P(D) \cdot P(M)$, then $D$ and $M$ are independent events. |
| **EXAMPLES** | Experiment: spin the spinner. | Experiment: spin the spinner and roll a number cube |



Event $A$: number less than 3

Event $B$: number greater than 5

$P(A \text{ or } B) = P(A) + P(B) = \frac{2}{6} + \frac{1}{6} = \frac{1}{2}$

Event $D$: odd number on spinner

Event $M$: number greater than 4 on number cube

$P(D \text{ and } M) = P(D) \cdot P(M) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does describing events as independent or mutually exclusive affect how you find probabilities?

2. **Check for Reasonableness** Two marbles are chosen, one at a time, from a bag containing 6 marbles: 4 red marbles and 2 green marbles. Suppose the first marble chosen is green. Is the probability that the second marble will be red greater if the first marble is returned to the bag or if it is not returned to the bag? Explain.

3. **Error Analysis** The probability that Deshawn plays basketball (event $B$) after school is 20%. The probability that he talks to friends (event $T$) after school is 45%. He says that *the probability of the union of B and T is 65%*. Explain Deshawn's error.

4. **Vocabulary** What is the difference between mutually exclusive events and independent events?

## Do You KNOW HOW?

5. A bag contains 40 marbles. Eight are green and 2 are blue. You select one marble at random. What is the probability of each event?
   a. The marble is green or blue.
   b. The marble is not green and not blue.

6. A robot at a carnival booth randomly tosses a dart at a square target with 8 inch sides and circle with a 3 inch radius in the middle. To the nearest whole percent, what is the probability that the dart will land in the circle?

**For Exercises 7 and 8, assume that you roll a standard number cube two times.**

7. What is the probability of rolling an even number on the first roll and a number less than 3 on the second roll?

8. What is the probability of rolling an odd number on the first roll and a number greater than 3 on the second roll?

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

9. **Communicate and Justify** Let $S$ be a sample space for an experiment in which all outcomes are both equally likely and mutually exclusive. What can you conclude about the sum of the probabilities for all of the outcomes? Give an example.

10. **Error Analysis** At Lincoln High School, 6 students are members of both the Chess Club and the Math Club. There are 20 students in the Math Club, 12 students in the Chess Club, and 400 students in the entire school.

    Danielle calculated the probability that a student chosen at random belongs to the Chess Club or the Math Club. Explain her error.

    Event $C$: Student is in Chess Club
    Event $M$: Student is in Math Club

    $$P(C \text{ or } M) = P(C) + P(M)$$
    $$= \frac{12}{400} + \frac{20}{400}$$
    $$= \frac{32}{400} = 0.08$$

11. **Higher Order Thinking** Murphy's math teacher sometimes wears scarves to class. Murphy has been documenting the relationship between his teacher wearing a scarf and when the class has a math quiz. The probabilities are as follows:

    - $P$(wearing a scarf) = 10%
    - $P$(math quiz) = 15%
    - $P$(wearing a scarf and math quiz) = 5%

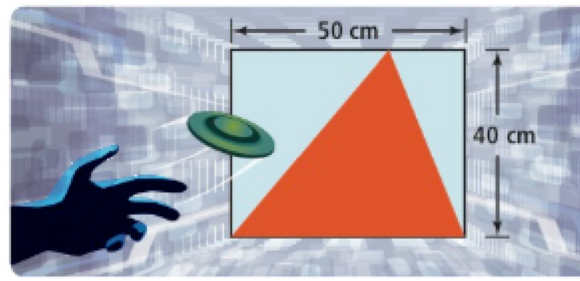
    Are the events "the teacher is wearing a scarf" and "there will be a quiz" independent events? Explain.

**Check for Reasonableness A card is drawn from a box containing 5 cards, each showing a different number from 1 to 5. Consider the events "even number," "odd number," "less than 3," and "greater than 3." Determine whether each pair of events is mutually exclusive.**

12. $< 3$, $> 3$
13. even, $> 3$
14. odd, $> 3$
15. odd, even

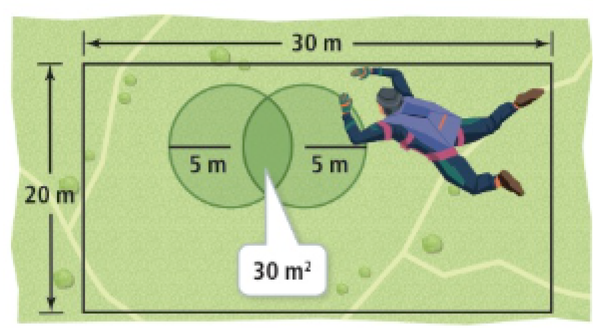
### PRACTICE

16. Hana is playing a virtual reality game in which she must toss a disc to strike the largest triangular section of the board. If the disc is equally likely to strike anywhere on the board, what is the probability that she will succeed? Explain. SEE EXAMPLE 1




**In a class of 25 students, 8 students have heights less than 65 inches and 10 students have heights of 69 inches or more. For Exercises 17–19, find the probabilities described.** SEE EXAMPLE 1

17. $P$(less than 65 inches or greater than 69 inches)
18. $P$(greater than or equal to 65 inches)
19. $P$(greater than or equal to 65 inches and less than or equal to 69 inches)
20. A skydiver is equally likely to land at any point on a rectangular field. Two overlapping circular targets of radius 5 meters are marked on the field. To the nearest percent, what is the probability that the sky diver will land in the union of the circles? In the intersection? SEE EXAMPLE 2




21. Two marbles are chosen at random, one at a time from a box that contains 7 marbles: 5 red and 2 green. SEE EXAMPLES 3, 4, and 5

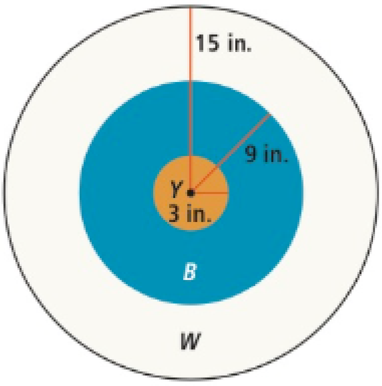
    a. Find the probability of drawing 2 red marbles when the first marble is replaced before the second marble is chosen.

    b. Determine whether the situation described is independent.

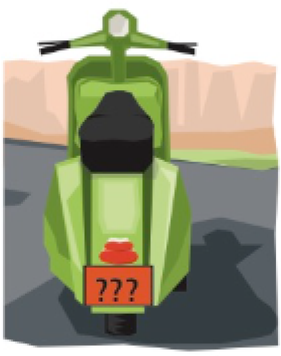
# PRACTICE & PROBLEM SOLVING

## APPLY

**22. Mathematical Connections** For a science fair project, Paige wants to test whether ants prefer certain colors. She releases ants on the colored surface shown. If the ants distribute themselves randomly on the surface without color preference across the entire surface, what is the probability that any given ant will be within the blue circle, but not within the yellow circle? Round to the nearest whole percent.




**23. Use Patterns and Structure** A city issues 3-digit license plates for motorized scooters. The digits 0–9 are chosen at random by a computer program. What is the probability that a license plate issued meets each set of criteria?




**a.** no digit is repeated

**b.** all 3 digits are the same

**c.** the 3-digit number formed is even

**d.** the first two digits are the same and the third digit is different

**24. Apply Math Models** During a football game, a kicker is called in twice to kick a field goal from the 30 yard line. Suppose that for each attempt, the probability that he will make the field goal is 0.8.

**a.** What is the probability that he will make both field goals?

**b.** What is the probability that he will make neither field goal?

## ASSESSMENT PRACTICE

**25.** The probability of events $A$ and $B$ both occurring is 15%. The probability of event $A$ or $B$ occurring is 60%. The probability of $B$ occurring is 50%. What is the probability of $A$ occurring? DP.4.1

**26. SAT/ACT** A robot spins the spinner shown twice. Assume that the outcomes 1, 2, 3, and 4 are equally likely for each spin. What is the probability that the sum of the two outcomes will be 6?

Ⓐ $\frac{1}{16}$

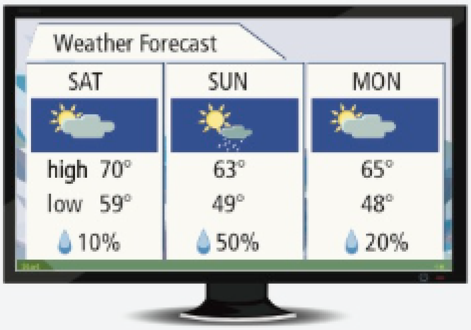
Ⓑ $\frac{1}{8}$

Ⓒ $\frac{3}{16}$

Ⓓ $\frac{1}{4}$

Ⓔ $\frac{3}{4}$


**27. Performance Task** Marta is packing to visit a friend in another city for a long weekend. She looks at the weather forecast shown below to find the chance of rain. Assume that whether it rains on each day is independent of whether it rains on any other day.




**Part A** What is the probability that it will not rain on any of the three days to the nearest percent?

**Part B** What is the probability that it will rain at least one of the three days to the nearest percent?

**Part C** Do you think Marta should pack an umbrella? Explain.

# 8-2 Conditional Probability

**I CAN…** find the probability of an event given that another event has occurred.

**VOCABULARY**
- conditional probability
- dependent events

MA.912.DP.4.3–Calculate the conditional probability of two events and interpret the result in terms of its context. **Also DP.4.4**
MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

## EXPLORE & REASON

**At Central High School, 85% of all senior girls attended and 65% of all senior boys attended the Spring Dance. Of all attendees, 20% won a prize.**

**A.** Assuming that the number of girls is about equal to the number of boys, estimate the probability that a randomly selected senior won a prize at the dance. Explain.

**B.** **Communicate and Justify** If you knew whether the selected student was a boy or a girl, would your estimate change? Explain.

## ESSENTIAL QUESTION

**How are conditional probability and independence related in real-world experiments?**

## EXAMPLE 1 Understand Conditional Probability

**A student committee is being formed to decide how after-school activities will be funded. The committee members are selected at random from current club members. The frequency table shows the current club membership data.**

**Monday Club Memberships by Grade**

| | Drama | Science | Art | Total |
|---|---|---|---|---|
| Sophomore | 3 | 9 | 24 | 36 |
| Junior | 6 | 18 | 16 | 40 |
| Senior | 8 | 13 | 18 | 39 |
| Total | 17 | 40 | 58 | 115 |

**What is the probability that a member of the art club selected at random is a junior?**

**One Method** Use the frequency table to find the probability that the student chosen is a junior given that the student is a member of the art club.

The probability that an event $B$ occurs given that another event $A$ has occurs is called a **conditional probability** and is written as $P(B \mid A)$.

$$P(\text{junior} \mid \text{member of the art club}) = \frac{\text{number of juniors in art club}}{\text{total number of art club members}}$$

$$= \frac{16}{58} = \frac{8}{29}$$

**Another Method** Use the formula for conditional probability.

For any two events $A$ and $B$, with $P(A) \neq 0$, $P(B \mid A) = \dfrac{P(B \text{ and } A)}{P(A)}$.

$$P(\text{junior} \mid \text{art}) = \frac{P(\text{junior and art})}{P(\text{art})} = \frac{\frac{16}{115}}{\frac{58}{115}} = \frac{8}{29}$$

Of the 115 Monday club members and 58 art club members, 16 are juniors and in the art club.

The probability that the student chosen is a junior member from the art club is $\frac{8}{29}$.

**COMMON ERROR**
Avoid confusing $P(A \mid B)$ with $P(B \mid A)$. In the first case, event $B$ is assumed, but in the second case event $A$ is assumed.

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EXAMPLE 1 CONTINUED

**Try It!** **1. a.** What is the probability that a member of the drama club is a sophomore, $P(\text{sophomore} \mid \text{drama})$?

**b.** What is the probability that a sophomore is member of the drama club, $P(\text{drama} \mid \text{sophomore})$? Is $P(\text{sophomore} \mid \text{drama})$ the same as $P(\text{drama} \mid \text{sophomore})$? Explain.

## CONCEPT Conditional Probability and Independent Events

Let $A$ and $B$ be events with $P(A) \neq 0$ and $P(B) \neq 0$.

| | |
|---|---|
| If events $A$ and $B$ are independent, then the conditional probability of $B$ given $A$ equals the probability of $B$ and the conditional probability of $A$ given $B$ equals the probability of $A$. | If events $A$ and $B$ are independent, then $P(B \mid A) = P(B)$ and $P(A \mid B) = P(A)$. |
| If the conditional probability of $B$ given $A$ equals the probability of $B$ and the conditional probability of $A$ given $B$ equals the probability of $A$, then events $A$ and $B$ are independent. | If $P(B \mid A) = P(B)$ and $P(A \mid B) = P(A)$, then events $A$ and $B$ are independent. |

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Use the Test for Independence

**The table below shows the vehicles in a parking garage one afternoon. A vehicle in the garage will be selected at random. Let $B$ represent "the vehicle is black" and $V$ represent "the vehicle is a van." Are the events $B$ and $V$ independent or dependent?**

**STUDY TIP**
When looking at a table of probabilities, consider outcomes that are impossible or guaranteed to occur. For example, it is impossible to select a red van.

| | Car | Van | Pickup | Totals |
|---|---|---|---|---|
| **Red** | 5 | 0 | 2 | 7 |
| **White** | 0 | 0 | 2 | 2 |
| **Black** | 6 | 3 | 4 | 13 |
| **Totals** | 11 | 3 | 8 | 22 |

**One Method**

Since $P(B) = \frac{13}{22} \neq 0$, $P(V) = \frac{3}{22}$, and $P(V \text{ and } B) = \frac{3}{22}$,

$$P(V \mid B) = \frac{P(V \text{ and } B)}{P(B)} = \frac{\frac{3}{22}}{\frac{13}{22}} = \frac{3}{13}$$

Since $P(V \mid B) \neq P(V)$, $B$ and $V$ are not independent events, they are **dependent events**.

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EXAMPLE 2 CONTINUED

**Another Method**

Since $P(V) = \frac{3}{22} \neq 0$, $P(B) = \frac{13}{22}$, and $P(B \text{ and } V) = P(V \text{ and } B)$,

$$P(B \mid V) = \frac{P(B \text{ and } V)}{P(V)}$$
$$= \frac{\frac{3}{22}}{\frac{3}{22}} = 1$$

A probability of 1, or 100%, indicates an outcome that is certain. Given that a van is selected, it must be black.

Since $P(B \mid V) \neq P(B)$, the events $B$ and $V$ are dependent events.

**Try It!** 2. Let $R$ represent "selecting a red vehicle" and $C$ represent "selecting a car." Are the events $R$ and $C$ independent or dependent? Explain.

## EXAMPLE 3 Apply the Conditional Probability Formula

**A band's marketing agent conducted a survey to determine how many high school fans the band has. What is the probability that a surveyed student plans to attend the band's concert and is a fan of the group?**

**Concert Survey Results**

Students who plan to attend concert

- 70% of students plan to attend,
- 80% of students who plan to attend are fans of the band.

Students who do not plan to attend

- 30% of students do not plan to attend,
- 25% are fans of the band.

Use the conditional probability formula to find the combined probability.

Rewrite $P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$ as $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$.

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

$$P(\text{attend and fan}) = P(\text{attend}) \cdot P(\text{fan} \mid \text{attend})$$

Event $A$ is "attend." Event $B$ is "fan."

$$= 0.7 \cdot 0.8$$
$$= 0.56, \text{ or } 56\%$$

Substitute 0.7 for $P(\text{attend})$ and 0.8 for $P(\text{fan} \mid \text{attend})$.

**USE PATTERNS AND STRUCTURE**
Why might $P(\text{fan} \mid \text{attend})$ not equal $P(\text{attend} \mid \text{fan})$ in this situation?

The probability that a surveyed student plans to attend the concert and is a fan of the group is 0.56, or 56%.

**Try It!** 3. What is the probability that a surveyed student plans to attend but is not a fan of the group?

APPLICATION

## EXAMPLE 4 Use Conditional Probability to Make a Decision

**A marketer is looking at mobile phone statistics to help plan an online advertising campaign. She wants to find which product is most likely to be purchased after a related search.**

**Mobile Phone Search and Buying Behavior**

| Product | Search(S) | Search & Buy (S and B) |
|---|---|---|
| W | 46% | 16% |
| X | 32% | 14% |
| Y | 35% | 12% |
| Z | 40% | 15% |

Find the probability a mobile phone customer buys, given that they performed a related search. Use the formula $P(B \mid S) = \frac{P(S \text{ and } B)}{P(S)}$.

| Product | $P(B \mid S)$ |
|---|---|
| W | $\frac{0.16}{0.46} \approx 0.348$ or about 34.8% |
| X | $\frac{0.14}{0.32} = 0.4375$ or 43.75% |
| Y | $\frac{0.12}{0.35} \approx 0.343$ or 34.3% |
| Z | $\frac{0.15}{0.40} = 0.375$ or 37.5% |

**ANALYZE AND PERSEVERE** Product W has the highest $P(S \text{ and } B)$ of 16% but not the highest $P(B \mid S)$. Can you explain why?

Product B has the highest probability of being purchased given that a related search was performed. So product B is probably a good choice for the online advertising campaign.

**Try It!** **4.** The marketer also has data from desktop computers. Which product is most likely to be purchased after a related search?

**Computer Search and Buying Behavior (% of computer-based site visitors)**

| Product | Search | Search & Buy |
|---|---|---|
| J | 35% | 10% |
| K | 28% | 9% |
| L | 26% | 8% |
| M | 24% | 5% |

## CONCEPT SUMMARY Conditional Probability

| | Conditional Probability Formula | Conditional Probability and Independent Events |
|---|---|---|
| **WORDS** | The probability that an event $B$ occurs given that another event $A$ occurs is called a **conditional probability**. | Events $A$ and $B$ are independent events if and only if the conditional probability of $A$ given $B$ is the same as the probability of $A$, and the conditional probability of $B$ given $A$ is the same as the probability of $B$. |
| **ALGEBRA** | For any two events $A$ and $B$, with $P(A) \neq 0$, $P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$ | For any events $A$ and $B$ with $P(A) \neq 0$ and $P(B) \neq 0$, $A$ and $B$ are independent if and only if $P(B \mid A) = P(B)$ and $P(A \mid B) = P(A)$. |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are conditional probability and independence related in real-world experiments?

2. **Vocabulary** How is the sample space for $P(B \mid A)$ different from the sample space for $P(B)$?

3. **Vocabulary** Why does the definition of $P(B \mid A)$ have the condition that $P(A) \neq 0$?

4. **Use Patterns and Structure** Why is $P(A) \cdot P(B \mid A) = P(B) \cdot P(A \mid B)$?

5. **Error Analysis** Taylor knows that $P(\text{red}) = 0.8$, $P(\text{blue}) = 0.2$, and $P(\text{red and blue}) = 0.05$. Explain Taylor's error.

$$P(B|R) = \frac{0.05}{0.2} = 0.25$$

6. **Communicate and Justify** At a sports camp, a coach wants to find the probability that a soccer player is a local camper. Because 40% of the students in the camp are local, the coach reasons that the probability is 0.4. Is his conclusion justified? Explain.

## Do You KNOW HOW?

7. Let $P(A) = \frac{3}{4}$, $P(B) = \frac{2}{3}$, and $P(A \text{ and } B) = \frac{1}{6}$. Find each probability.

   **a.** What is $P(B \mid A)$?

   **b.** What is $P(A \mid B)$?

8. Students randomly generate two digits from 0 to 9 to create a number between 0 and 99. Are the events "first digit 5" and "second digit 6" independent or dependent in each case? What is $P(56)$ in each experiment?

   **a.** The digits may not be repeated.

   **b.** The digits may be repeated.

9. Suppose that you select one card at random from the set of 6 cards below.

| | | | | | |
|---|---|---|---|---|---|
| W2 | B3 | W4 | B3 | B2 | W3 |

Let $B$ represent the event "select a blue card" and $T$ represent the event "select a card with a 3." Are $B$ and $T$ independent events? Explain your reasoning.

# PRACTICE & PROBLEM SOLVING

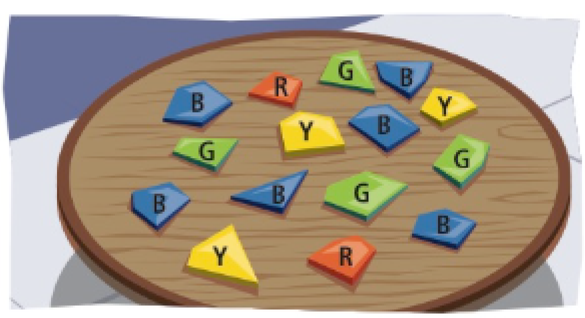
## UNDERSTAND

**10.** **Mathematical Connections** How can the formula $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$ be simplified to find the probability of $A$ and $B$ when the events are independent? Explain.

**11.** **Error Analysis** From a bag containing 3 red marbles and 7 blue marbles, 2 marbles are selected without replacement. Esteban calculated the probability that two red marbles are selected. Explain Esteban's error.

$$\begin{aligned} P(\text{red}) &= 0.3 \\ P(\text{red and red}) &= P(\text{red}) \cdot P(\text{red}) \\ &= 0.3 \cdot 0.3 \\ &= 0.09 \end{aligned}$$

**12.** **Generalize** Kiyo is creating a table using mosaic tiles chosen and placed randomly. She is picking tiles without looking. How does $P(\text{yellow second} \mid \text{blue first})$ compare to $P(\text{yellow second} \mid \text{yellow first})$ if the tiles are selected without replacement? How do they compare if the tiles are selected and returned to the pile because Kiyo wants a different color?




**13.** **Use Patterns and Structure** At a fundraiser, a participant is asked to guess what is inside an unlabeled can for a possible prize. If there are two crates of cans to choose from, each having a mixture of vegetables and soup, what is the probability that the first participant will select a vegetable can from the left crate given each situation?

**a.** The left crate has 2 cans of vegetables and 8 cans of soup, and the right crate has 7 cans of vegetables and 3 cans of soup.

**b.** The left crate has 8 cans of vegetables and 2 cans of soup, and the right crate has 5 cans of vegetables and 5 cans of soup.

## PRACTICE

**For Exercises 14–18, use the data in the table to find the probability of each event.** SEE EXAMPLE 1

**Technology Class Enrollment by Year**

| | Sophomore | Junior |
|---|---|---|
| Robotics | 16 | 24 |
| Game Design | 18 | 22 |

**14.** $P(\text{Junior} \mid \text{Robotics})$

**15.** $P(\text{Robotics} \mid \text{Junior})$

**16.** $P(\text{Game Design} \mid \text{Sophomore})$

**17.** $P(\text{Sophomore} \mid \text{Game Design})$

**18.** Are year and technology class enrollment dependent or independent events? Explain. SEE EXAMPLE 2

**19.** At a high school, 40% of the students play an instrument. Of those students, 20% are freshmen. Of the students who do not play an instrument, 30% are freshmen. What is the probability that a student selected at random is a freshman who plays an instrument? SEE EXAMPLE 3

**In a study of an experimental medication, patients were randomly assigned to take either the medication or a placebo.**

**Effectiveness of New Medication As Compared to a Placebo**

| | Medication | Placebo |
|---|---|---|
| Health Improved | 53 | 47 |
| Health Did Not Improve | 65 | 35 |

**20.** What is the probability that a patient taking the medication showed improvement? Round to the nearest whole percent. SEE EXAMPLE 1

**21.** Are taking the medication and having improved health independent or dependent events? SEE EXAMPLE 2

**22.** Based on the data in the table, would you recommend that the medication be made available to doctors? Explain. SEE EXAMPLE 4

## PRACTICE & PROBLEM SOLVING

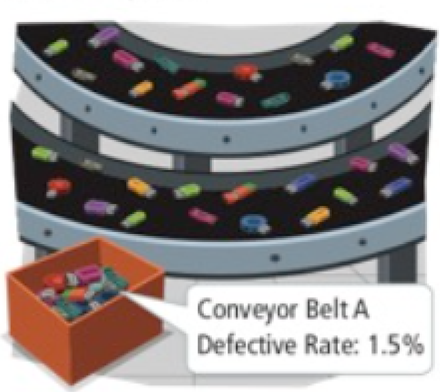
### APPLY

23. **Analyze and Persevere** In a recreation center with 1,500 members, 200 are high school students. Of the members, 300 regularly swim. The 45 students of the high school swim team are all members and practice at the pool every week. What is the probability that a high school member selected at random is on the swim team?

24. **Use Patterns and Structure** At the school fair, 5% of students will win a prize. A winner has an equally likely chance to win each prize type shown. What is the probability that a student at the fair will win a comic book? Explain.


25. **Analyze and Persevere** A box contains 50 batteries, of which 10 are dead and 5 are weak. Suppose you select batteries at random from the box and set them aside for recycling if they are dead or weak. If the first battery you select is dead and the second one is weak, what is the probability that the next battery you select will be weak?

26. **Higher Order Thinking** An inspector at a factory has determined that 1% of the flash drives produced by the plant are defective. If assembly line A produces 20% of all the flash drives, what is the probability that a defective flash drive chosen at random is from conveyor belt A? Explain.




### ASSESSMENT PRACTICE

27. Select all pairs of independent events. DP.4.4

☐ **A.** A student selected at random has a backpack. A student selected at random has brown hair.

☐ **B.** Events $A$ and $B$, where $P(B \mid A) = \frac{1}{3}$, $P(A) = \frac{3}{5}$ and $P(B) = \frac{5}{9}$

☐ **C.** A student selected at random is a junior. A student selected at random is a freshman.

☐ **D.** Events $A$ and $B$, where $P(A) = 0.30$, $P(B) = 0.25$ and $P(A \text{ and } B) = 0.075$

☐ **E.** Events $A$ and $B$, where $P(A) = 0.40$, $P(B) = 0.3$ and $P(A \text{ and } B) = 0.012$

28. **SAT/ACT** The table shows student participation in the newspaper and yearbook by year. A student on the newspaper staff is selected at random to attend a symposium. What is the probability that the selected student is a senior?

**Journalism Club Members**

| | Junior | Senior |
|---|---|---|
| Newspaper | 16 | 9 |
| Yearbook | 8 | 17 |

Ⓐ $\frac{9}{50}$

Ⓑ $\frac{9}{26}$

Ⓒ $\frac{9}{25}$

Ⓓ $\frac{9}{17}$

Ⓔ $\frac{9}{16}$

29. **Performance Task** In a survey of 50 male and 50 female high school students, 60 students said they exercise daily. Of those students, 32 were female.

**Part A** Use the data to make a two-way frequency table.

**Part B** What is the probability that a surveyed student who exercises daily is female? What is the probability that a surveyed student who exercises regularly is male?

**Part C** Based on the survey, what can you conclude about the relationship between exercise and gender? Explain.

# MATHEMATICAL MODELING IN 3 ACTS

MA.912.DP.4.1–Describe events as subsets of a sample space using characteristics, or categories, of the outcomes, or as unions, intersections or complements of other events. Also DP.4.9
MA.K12.MTR.7.1

## Place Your Guess

A coin toss is a popular way to decide between two options or settle a dispute. The coin toss is popular because it is a simple and unbiased way of deciding. Assuming the coin being tossed is a fair coin, both parties have an equally likely chance of winning.

What other methods could you use to decide between two choices fairly? Think about this during the Mathematical Modeling in 3 Acts lesson.

### ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

### ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

### ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

# 8-3 Permutations and Combinations

**I CAN…** use permutations and combinations to find the number of outcomes in a probability experiment.

## VOCABULARY

- combination
- factorial
- Fundamental Counting Principle
- permutation

MA.912.DP.4.10–Given a mathematical or real-world situation, calculate the appropriate permutation or combination. Also DP.4.9, AR.1.11
MA.K12.MTR.1.1, MTR.2.1, MTR.3.1

## EXPLORE & REASON

**Holly, Tia, Kenji, and Nate are eligible to be officers of the Honor Society. Two of the four students will be chosen at random as president and vice-president. The table summarizes the possible outcomes.**

**Honor Society Officers**

| | | Vice-President | | | |
|---|---|---|---|---|---|
| | | Holly | Tia | Kenji | Nate |
| President | Holly | – | HT | HK | HN |
| | Tia | TH | – | TK | TN |
| | Kenji | KH | KT | – | KN |
| | Nate | NH | NT | NK | – |

A. Holly wants to be an officer with her best friend Tia. How many outcomes make up this event?

B. How many outcomes show Holly as president and Tia as vice-president?

C. **Generalize** How many outcomes have only one of them as an officer? Explain.

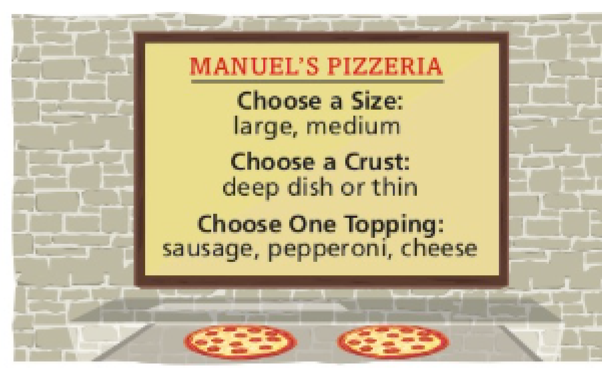
## ESSENTIAL QUESTION

**How are permutations and combinations useful when finding probabilities?**

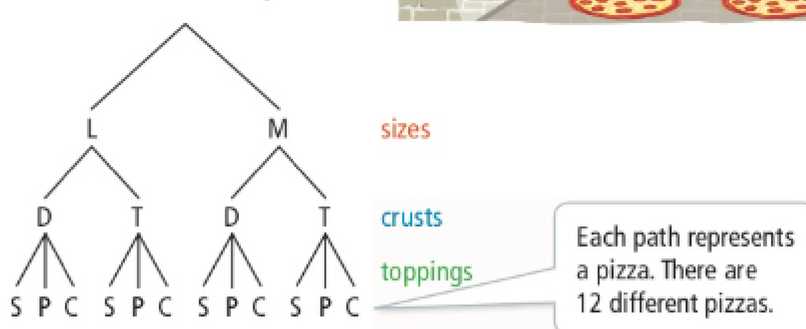
**COMMON ERROR**
When you compare a tree diagram to the Fundamental Counting Principle, remember to count the total number of paths from the beginning to the end of the tree diagram, not the number of branches in each section.

## EXAMPLE 1 Use the Fundamental Counting Principle

**Manuel wants to advertise the number of one-topping pizzas he offers to his customers. How many different one-topping pizzas are available at Manuel's Pizzeria?**




Make a tree diagram to find the number of pizzas.




sizes × crusts × toppings = number of pizzas

$2 \times 2 \times 3 = 12$

This example illustrates the **Fundamental Counting Principle**. If there are $m$ ways to make the first selection and $n$ ways to make the second selection, then there are $m \times n$ ways to make the two selections. If a third selection, $p$, is added, then there are $m \times n \times p$ to make all three selections, and so on.

CONTINUED ON THE NEXT PAGE

**Try It!** 1. The car that Ms. Garcia is buying comes with a choice of 3 trim lines (standard, sport, or luxury), 2 types of transmission (automatic or manual), and 8 colors. How many different option packages does Ms. Garcia have to choose from? Explain.

CONCEPTUAL UNDERSTANDING

## EXAMPLE 2 Find the Number of Permutations

**A. Gabriela is making a playlist with her 3 favorite songs. How many possible orders are there for the songs?**

**Method 1** Use an organized list.

Let A, B, and C represent the 3 songs. There are 6 different possible orders for the songs.

| | |
|---|---|
| ABC | ACB |
| BAC | BCA |
| CAB | CBA |

**Method 2** Use the Fundamental Counting Principle.

$3 \cdot 2 \cdot 1$

There are 3 choices for the first song, 2 choices for the second song, and 1 choice for the third song.

The **factorial** of a positive integer $n$ is the product of all positive integers less than or equal to $n$. It is written $n!$ and is read as *n factorial*. By definition, 0! equals 1.

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

The number of different possible orders for the songs is 3!

$$3! = 3 \cdot 2 \cdot 1 = 6$$

There are 6 different possible orders for the 3 songs.

**B. Gabriela wants to make another playlist using 5 of the 8 songs from her favorite artist's latest album to orchestrate the mood during a slide show. How many playlists are possible?**

**Method 1** Use the Fundamental Counting Principle.

There are 8 choices for the first song, 7 choices for the second song, and so on.

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6{,}720$$

There are 6,720 possible playlists with 5 of the 8 songs.

**Method 2** Use factorials.

To count the number of ways to order 5 songs from A, B, C, D, E, F, G, and H, consider the list ABCDE. The diagram shows all the possible ways that sequence appears among the 8! ways to list all songs.

For any sequence of 5 songs, there are $(8-5)! = 3!$ ways that sequence appears as the first 5 songs when listing all 8! lists. So divide 8! by 3! to find the number of 5-song playlists.

| | |
|---|---|
| ABCDEFGH | ABCDEFHG |
| ABCDEGFH | ABCDEGHF |
| ABCDEHFG | ABCDEHGF |

$$\frac{8!}{3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 6{,}720$$

There are 6,720 possible playlists with 5 of the 8 songs.

USE PATTERNS AND STRUCTURE
How is the Fundamental Counting Principle related to the number of permutations?

CONTINUED ON THE NEXT PAGE

**Try It!** **2.** How many possible playlists are there for each situation?

**a.** Gabriela's 4 favorite songs **b.** 5 of the 10 most popular songs

## CONCEPT Permutations

A **permutation** is an arrangement of some or all objects of a set in a specific order.

The number of permutations of $n$ items arranged $r$ items at a time is

$${}_{n}P_{r} = \frac{n!}{(n-r)!} \text{ for } 0 \leq r \leq n.$$

CONCEPTUAL UNDERSTANDING

## EXAMPLE 3 Find the Number of Combinations

**LEARN TOGETHER**
What are ways to stay positive and work toward goals?

**Marisol is planning to be a counselor at summer camp. She can choose 3 activities for her session. How many different combinations of 3 activities are possible?**


Use the formula to write an expression for the number of permutations of 10 choices taken 3 at a time.

$${}_{10}P_{3} = \frac{10!}{(10-3)!} = 720$$

However, in this situation, the order of the 3 chosen activities does not matter, so you must adjust the formula.

A **combination** is a set of objects with no specific order.

3! Permutations 1 Combination

| ABC | ACB |
|---|---|
| BAC | BCA |
| CAB | CBA |

→ {A, B, C}

A group of 3 items can be arranged in 3! ways, so you must divide the number of permutations, ${}_{10}P_{3}$, by the number of arrangements of the chosen items, 3!.

**CHOOSE EFFICIENT METHODS**
Most scientific and graphing calculators can calculate permutations (${}_{n}P_{r}$) and combinations (${}_{n}C_{r}$).

The notation ${}_{n}C_{r}$ indicates the number of combinations of $n$ items chosen $r$ items at a time.

$$\begin{aligned} {}_{10}C_{3} &= \frac{{}_{10}P_{3}}{3!} = \frac{10!}{3!(10-3)!} \\ &= \frac{720}{6} \\ &= 120 \end{aligned}$$

${}_{10}C_{3}$ denotes the number of combinations of 10 items taken 3 at a time.

There are 120 different combinations of activities that Marisol can choose.

**Try It!** **3.** How many ways can a camper choose 5 activities from the 10 available activities at the summer camp?

## CONCEPT Combinations

A **combination** is a set of objects with no specific order.

The number of combinations of $n$ items chosen $r$ at a time is

$${}_nC_r = \frac{n!}{r!(n-r)!} \text{ for } 0 \le r \le n.$$

${}_nC_r$ is sometimes written as $\binom{n}{r}$.

## EXAMPLE 4 Combinations and the Binomial Theorem

**A. Find the relationships between the Binomial Theorem, Pascal's Triangle and combinations.**

Earlier, to find the coefficients for the expansion $(a+b)^n$, Pascal's Triangle was the tool used, but this is awkward for large values of $n$.

Pascal's Triangle

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | 1 | | | | | |
| | | | | 1 | | 1 | | | | |
| | | | 1 | | 2 | | 1 | | | |
| | | 1 | | 3 | | 3 | | 1 | | |
| | 1 | | 4 | | 6 | | 4 | | 1 | |
| 1 | | 5 | | 10 | | 10 | | 5 | | 1 |

The Binomial Theorem is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$\sum$ is the upper case Greek letter sigma. It stands for the summation of $n+1$ terms starting with the value $k$ equal to 0 and counting up to $n$. Each term is the product of ${}_nC_k$ multiplied by $a^{n-k}$ and $b^k$.

The exponents always sum to $n$.

$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

Coefficients agree with Pascal's Triangle.

$$(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The Binomial Theorem is used to expand the terms of a binomial raised to a power. Using combinations to find coefficients of the expansion along with technology prevents the awkwardness of a larger Pascal's Triangle.

**USE PATTERNS AND STRUCTURES**
When the binomial uses the difference of terms, think of adding the opposite $(a+(-b))^n$ to get the correct sign for the expansion.

**B. Find the sixth and eleventh terms of $(x^2-3y)^{12}$.**

The value of $k$ is one less than the term number.

The sixth term is $\binom{12}{5}(x^2)^7(-3y)^5 = 792x^{14}(-243y^5) = -192{,}456x^{14}y^5$.

The eleventh term is $\binom{12}{10}(x^2)^2(-3y)^{10} = 66x^4(59{,}049y^{10}) = 3{,}897{,}234x^4y^{10}$.

**Try It!** **4. a.** Using factorials, show $\binom{6}{2} = \binom{6}{4} = 15$.

**b.** Find the eighth term of $(2d^3 - y^5)^9$.

APPLICATION

## EXAMPLE 5 Use Permutations and Combinations to Find Probabilities

**A teacher chooses 5 students at random from the names shown to work together on a group project. What is the probability that the 5 students' names begin with a consonant?**


Formulate ◀ Determine if the problem is about permutations or combinations.

Since the order in which the students are chosen does not matter, use combinations to find the numbers of possible outcomes and desirable outcomes to calculate the probability.

Compute ◀ **Step 1** Find the total number of possible outcomes.

$${}_{18}C_5 = \frac{18!}{5!(18-5)!} = 8{,}568$$

There are 8,568 ways the teacher could choose 5 students.

**Step 2** Find the number of possible outcomes in which all the names begin with a consonant and none of the names begin with a vowel.

$${}_{13}C_5 = \frac{13!}{5!(13-5)!} = 1{,}287 \qquad {}_{5}C_0 = \frac{5!}{0!(5-0)!} = 1$$

Choose 5 out of 13 names.

Choose 0 out of 5 names.

Use the Fundamental Counting Principle. Multiply the number of possible outcomes for the two subsets to find the total number of outcomes.

$${}_{13}C_5 \bullet {}_{5}C_0 = 1{,}287 \bullet 1 = 1{,}287$$

There are 1,287 outcomes with all the names beginning with consonants.

**Step 3** Find the probability.

$$P(\text{all consonants}) = \frac{\text{number of outcomes with all consonants}}{\text{total number of possible outcomes}}$$

$$= \frac{1{,}287}{8{,}568} \approx 0.15$$

Interpret ◀ The probability that all 5 names begin with a consonant is about 0.15, or 15%.

**Try It!** **5.** Using the data from Example 5, what is the probability that the 5 students' names end with a vowel?

## CONCEPT SUMMARY Permutations and Combinations

| | Permutation | Combination |
|---|---|---|
| **WORDS** | An arrangement of items in which the order of the items is important<br><br>${}_nP_r$ represents the number of permutations of $n$ items arranged $r$ at a time. | An arrangement of items in which the order of the items is not important<br><br>${}_nC_r$ represents the number of combinations of $n$ items chosen $r$ at a time. |
| **ALGEBRA** | ${}_nP_r = \frac{n!}{(n-r)!}$ for $0 \le r \le n$ | ${}_nC_r = \frac{n!}{r!(n-r)!}$ for $0 \le r \le n$ |
| **NUMBERS** | The number of permutations of 6 items taken 3 at a time is<br>${}_6P_3 = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$ | The number of combinations of 6 items taken 3 at a time is<br>${}_6C_3 = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)} = 20$ |

## Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are permutations and combinations useful when finding probabilities?

2. **Use Patterns and Structure** How is the formula for a combination related to the formula for a permutation?

3. **Vocabulary** Why is it important to distinguish between a *permutation* and a *combination* when counting items?

4. **Generalize** How is ${}_9C_2$ related to ${}_9C_7$? Explain. How can you generalize this observation for any values of $n$ and $r$?

5. **Error Analysis** Explain Beth's error.

$$\frac{{}_3P_3}{{}_5P_3} = \frac{3!}{\frac{5!}{(5-3)!}} = \frac{3!}{5!2!} = \frac{1}{40}$$

✗

6. **Communicate and Justify** A company wants to form a committee of 4 people from its 12 employees. How can you use a combination to find the probability that the 4 people newest to the company will be selected?

## Do You KNOW HOW?

**Do the possible arrangements represent permutations or combinations?**

7. Jennifer will invite 3 of her 10 friends to a concert.

8. Jennifer must decide how she and her 3 friends will sit at the concert.

**Find the number of permutations.**

9. How many ways can 12 runners in a race finish first, second, and third?

**Find the number of combinations.**

10. In how many ways can 11 contestants for an award be narrowed down to 3 finalists?

11. How many different teams of 4 people can be chosen from a group of 8 people?

**Students will be chosen at random for school spirit awards. There are 6 athletes and 8 non-athletes who are eligible for 2 possible prizes. What is each probability?**

12. $P$(both prizes are awarded to athletes)

13. $P$(both prizes are awarded to non-athletes)

14. $P$(no prize is awarded to an athlete)

15. $P$(no prize is awarded to a non-athlete)

16. Explain how Exercises 12 and 13 are similar to Exercises 14 and 15.

## PRACTICE & PROBLEM SOLVING

### UNDERSTAND

17. **Represent and Connect** Dwayne bought a new bike lock, and the lock came with instructions to choose 3 out of 30 numbers on a circular dial to keep his bike secure. The numbers cannot be repeated. How many possible arrangements can Dwayne choose for his lock? Do the arrangements represent permutations or combinations? Explain.

18. **Communicate and Justify** Sage volunteers to read and play with sick children in a hospital. She selects some erasers at random from a bag to use as prizes. There are 8 alien erasers and 10 flying saucer erasers.

    a. How many groups of 6 erasers can be formed from the 18 erasers? Explain.

    b. In how many ways can 3 aliens be selected? Explain.

    c. In how many ways can 3 aliens and 3 flying saucers be selected? Explain.

    d. What is the probability that 3 aliens and 3 flying saucers will be selected? Explain.

19. **Error Analysis** There are 6 tiles numbered 1 to 6 in a box. Two tiles are selected at random without replacement to form a 2-digit number. Jeffrey found the probability that the number selected is 16. Explain his error.

    The number of ways to select 1 and 6 is given by ${}_6C_2 = 15$

    $$P(16) = \frac{1}{{}_6C_2} = \frac{1}{15} \quad ✗$$

20. **Mathematical Connections** How many lines are determined by the points, $P$, $Q$, $R$, and $S$? Explain.


21. **Higher Order Thinking** There are 11! different ways for a group of people to sit around a circular table. How many people are in the group? Explain.

### PRACTICE

**For Exercises 22–27, state if the possible arrangements represent permutations or combinations, then state the number of possible arrangements.** SEE EXAMPLES 1, 2, AND 3

22. A student chooses at random 4 books from a reading list of 11 books.

23. At the end of a season, 10 soccer teams are ranked by the state.

24. A committee of 5 people is being selected from a group of 9 to choose the food for a sport's banquet.

25. A class president, secretary, and treasurer are chosen from 12 students running for office.

26. A food truck has a lunch special on tacos. Customers choose a shell, three toppings, and two sides for one price.


27. There are 4 comedians and 5 musicians performing in a variety show. The order in which the performers are chosen is random. SEE EXAMPLE 5

    a. What is the probability that the first 3 performers are comedians?

    b. What is the probability that the first two performers are a comedian followed by a musician?

28. A jewelry maker chooses three beads at random from a bag with 10 beads labeled A, B, C, D, E, F, G, H, I, and J. SEE EXAMPLES 2, 3, AND 5

    a. How can you use permutations or combinations to find *P*(selected beads spell the initials DEB)? What is the probability?

    b. How can you use permutations or combinations to find *P*(selected beads are all vowels)? What is the probability?

29. Find the second term of the expansion of $(2xy^3 + 4z)^8$. SEE EXAMPLE 4

## PRACTICE & PROBLEM SOLVING

### APPLY

**30. Analyze and Persevere** Amaya's wallet contains three \$1 bills, two \$5 bills, and three \$10 bills. If she pulls 2 bills without looking, what is the probability that she draws a \$1-bill and a \$10-bill? Explain.

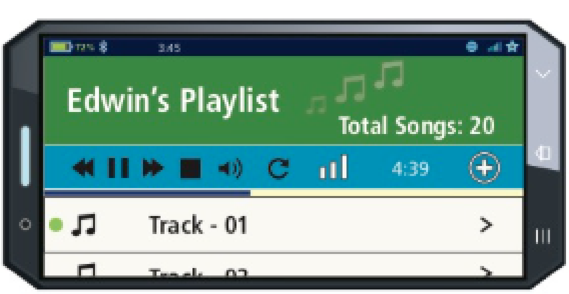
**31. Apply Math Models** Raul's favorite restaurant is running a prize game. Five of each of the winning tickets shown are available, and a customer must collect three winning tickets to receive the prize. What is the probability Raul will receive the prize for the baseball cap with his first 3 tickets?


**32. Use Patterns and Structure** Smart Phones, Inc. chooses a 5-digit security code at random from the digits 0–9.

**a.** Suppose the digits cannot be repeated. What is the probability that the security code is 30429? Explain.

**b.** Suppose the digits can be repeated. What is the probability that the security code is 30429? Explain.

**33. Analyze and Persevere** Edwin randomly plays 6 different songs from his playlist.




**a.** What is the probability that Edwin hears his 6 favorite songs?

**b.** What is the probability that he hears the songs in order from his most favorite to his sixth most favorite?

### ASSESSMENT PRACTICE

**34.** In a deck of 52 cards, 13 are red, 13 are blue, 13 are yellow, and 13 are green. Write an expression in terms of combinations that gives the probability of drawing 5 cards from the deck at random and ending up with 5 red cards. What is the probability? Round to the nearest ten-thousandth. DP.4.10

**35. SAT/ACT** Fifteen students enter a Safety Week poster contest in which prizes will be awarded for first through fourth place. In how many ways could the prizes be given out?

Ⓐ 4
Ⓑ 60
Ⓒ 1,365
Ⓓ 32,760
Ⓔ 50,625

**36. Performance Task** Use the word shown on the tiles below to find each probability.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| S | U | R | F | B | O | A | R | D |

**Part A** Two tiles are chosen at random without replacement. Use conditional probability to find the probability that both letters are vowels. Then find the probability using permutations or combinations. Explain.

**Part B** Four of the tiles are chosen at random and placed in the order in which they are drawn. Use conditional probability to find the probability the tiles spell the word SURF. Then find the probability using permutations or combinations. Explain.

TOPIC 8

# Topic Review

## TOPIC ESSENTIAL QUESTION

1. How can you find the probability of events and combinations of events?

## Vocabulary Review

**Choose the correct term to complete each sentence.**

2. An arrangement of items in a specific order is called a(n) ________.
3. Two events are ________ if there is no outcome that is in both events.
4. Two events are ________ if the occurrence of one event affects the probability of the other event.
5. The ________ of an event is the set of all outcomes in the sample space that are not included in the event.

- combination
- complement
- conditional probability
- dependent events
- independent events
- mutually exclusive
- permutation

## Concepts & Skills Review

**LESSON 8-1** Probability Events

### Quick Review

Two events are **mutually exclusive** if and only if there is no outcome that lies in the sample space of both events. Two events are **independent events** if and only if the outcome of one event does not affect the probability of a second event.

### Example

**Let $A$ represent the event "even number," or $A = \{2, 4, 6, 8\}$.**

**Let $B$ represent the event "odd number," or $B = \{1, 3, 5, 7\}$.**

**Let $C$ represent the event "divisible by 3," or $C = \{3, 6\}$.**


Are $A$ and $B$ mutually exclusive? Explain.
Yes; all of their elements are different.

Are $A$ and $C$ mutually exclusive? Explain.
No; they both have a 3 in their sample space.

### Practice & Problem Solving

**Ten craft sticks lettered A through J are in a coffee cup. Consider the events "consonant," "vowel," "letter before D in the alphabet," "letter after A in the alphabet," and "letter after E in the alphabet." State whether each pair of events is mutually exclusive.**

6. vowel, letter before D
7. letter before D, letter after E
8. letter after A, letter before D
9. **Communicate and Justify** Edward is rolling a number cube to decide on the new combination for his bicycle lock. If he only has one number to go, find the probability of each event. Use what you know about mutually exclusive events to explain your reasoning.
   a. Edward rolls a number that is both even and less than 2. Explain.
   b. Edward rolls a number that is even or less than 2. Explain.

## LESSON 8-2 Conditional Probability

### Quick Review

For any two events $A$ and $B$, with $P(A) \neq 0$, $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$. Events $A$ and $B$ are independent if and only if $P(B \mid A) = P(B)$.

### Example

**The table shows the number of students on different teams by grade. One of these students is selected at random for an interview. Are selecting a sophomore and selecting a member of the track team independent events?**

**Team Enrollment by Year**

| | Sophomore | Junior |
|---|---|---|
| Cross Country | 9 | 6 |
| Track | 12 | 23 |

$P(\text{Sophomore and Track}) = 0.24$

$P(\text{Sophomore}) = 0.42$ $\quad P(\text{Track} \mid \text{Sophomore}) \approx 0.57$

$P(\text{Soph and Track}) \neq P(\text{Soph}) \cdot P(\text{Track} \mid \text{Soph})$ because $0.24 \neq 0.42 \cdot 0.57$

No, selecting a sophomore and selecting a member of the track team are dependent events.

### Practice & Problem Solving

**Use the table in the Example for Exercises 10–14. All students are selected at random.**

10. $P(\text{Junior})$
11. $P(\text{Cross Country})$
12. $P(\text{Junior} \mid \text{Cross Country})$
13. $P(\text{Cross Country} \mid \text{Junior})$
14. Are selecting a junior and selecting a cross country runner dependent or independent events?
15. **Error Analysis** One card is selected at random from five cards numbered 1–5. A student says that drawing an even number and drawing a prime number are dependent events because $P(\text{prime} \mid \text{even}) = 0.5$ and $P(\text{even}) = 0.4$. Describe and correct the error the student made.
16. **Use Patterns and Structure** A person entered in a raffle has a 3% chance of winning a prize. A prize winner has a 25% chance of winning two theater tickets. What is the probability that a person entered in the raffle will win the theater tickets?

## LESSON 8-3 Permutations and Combinations

### Quick Review

The number of permutations of $n$ items taken $r$ at a time is ${}_nP_r = \frac{n!}{(n-r)!}$ for $0 \leq r \leq n$.

The number of combinations of $n$ items chosen $r$ at a time is ${}_nC_r = \frac{n!}{r!(n-r)!}$ for $0 \leq r \leq n$.

### Example

**A bag contains 4 blue tiles and 4 yellow tiles. Three tiles are drawn from the bag at random without replacement. What is the probability all three tiles are blue?**

Use combinations since order does not matter. Select 3 blue from 4 blue tiles ${}_4C_3$, or 4, ways. Select 3 tiles from 8 total tiles ${}_8C_3$, or 56, ways.

$P(\text{3 blue}) = \frac{4 \cdot 1}{56} = \frac{1}{14} \approx 0.07 \approx 7\%$

### Practice & Problem Solving

**Identify whether the situation is a combination or a permutation. Then find the number.**

17. How many ways can a team choose a captain and a substitute captain from 8 players?
18. How many ways can 3 numbers be selected from the digits 0–9 to set a lock code if the digits cannot be repeated?
19. **Error Analysis** A student computed ${}_5C_2$, and said that it is equal to 20. Describe and correct the error the student made.
20. **Use Patterns and Structure** A permutation and a combination must always evaluate to a natural number. Explain why.

# Visual Glossary

## English / Spanish

### A

**Arithmetic sequence** An arithmetic sequence is a number sequence formed by adding a fixed number to each previous term to find the next term. The fixed number is called the common difference.

**Secuencia aritmética** Una secuencia aritmética es una secuencia de números que se forma al sumar un número fijo a cada término para hallar el término que le sigue. El número fijo se denomina diferencia común.

**Example** The arithmetic sequence 1, 5, 9, 13, . . . has a common difference of 4.

**Asymptote** An asymptote is a line that a graph approaches. Asymptotes guide the end behavior of a function.

**Asíntota** Una asíntota es una recta a la cual se acerca una gráfica. Las asíntotas actúan como guías del comportamiento extremo de una función.

**Example** The function $y = \frac{x+2}{x-2}$ has $x = 2$ as a vertical asymptote and $y = 1$ as a horizontal asymptote.

### B

**Binomial Theorem** For every positive integer $n$, $(a + b)^n = P_0a^n + P_1a^{n-1}b + P_2\,a^{n-2}b^2 + \ldots + P_{n-1}\,ab^{n-1} + P_n\,b^n$ where $P_0, P_1, \ldots, P_n$ are the numbers in the row of Pascal's Triangle that has $n$ as its second number.

**Teorema binomial** Para cada número entero positivo $n$, $(a + b)^n = P_0a^n + P_1a^{n-1}\,b + P_2a^{n-2}\,b^2 + \ldots + P_{n-1}ab^{n-1} + P_nb^n$, donde $P_0, P_1, \ldots, P_n$ son los números de la fila del Triángulo de Pascal cuyo segundo número es $n$.

**Example**
$$(x + 1)^3 = {}_3C_0(x)^3 + {}_3C_1(x)^2(1)^1 + {}_3C_2(x)^1(1)^2 + {}_3C_3(1)^3 = x^3 + 3x^2 + 3x + 1$$

### C

**Change of Base Formula** This formula allows logarithms with a base other than 10 and $e$ to be evaluated. $\log_b m = \frac{\log_a m}{\log_a b}$, where $m$, $b$, and $a$ are positive numbers, and $b \neq 1$ and $a \neq 1$.

**Fórmula de cambio de base** Esta fórmula permite evaluar logaritmos con base distinta de 10 y $e$. $\log_b m = \frac{\log_a m}{\log_a b}$, donde $m$, $b$, y $a$ son números positivos y $b \neq 1$ y $a \neq 1$.

**Example** $\log_3 8 = \frac{\log 8}{\log 3} \approx 1.8928$

**Coefficient matrix** A coefficient matrix is a matrix that shows the coefficients of the variables in a system of equations.

**Matriz de coeficientes** Una matriz de coeficientes es una matriz que muestra los coeficientes de las variables en un sistema de ecuaciones.

**Example**
$$\begin{cases} x + 2y = 5 \\ 3x + 5y = 14 \end{cases}$$
coefficient matrix $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

**Combination** Any unordered selection of $r$ objects from a set of $n$ objects is a combination. The number of combinations of $n$ objects taken $r$ at a time is ${}_nC_r = \frac{n!}{r!(n-r)!}$ for $0 \leq r \leq n$.

**Combinación** Cualquier selección no ordenada de $r$ objetos tomados de un conjunto de $n$ objetos es una combinación. El número de combinaciones de $n$ objetos, cuando se toman $r$ objetos cada vez, ${}_nC_r = \frac{n!}{r!(n-r)!}$ para $0 \leq r \leq n$.

**Example** The number of combinations of seven items taken four at a time is ${}_7C_4 = \frac{7!}{4!(7-4)!} = 35$. There are 35 ways to choose four items from seven items without regard to order.

## English | Spanish

**Common difference** A common difference is the difference between consecutive terms of an arithmetic sequence.

**Diferencia común** La diferencia común es la diferencia entre los términos consecutivos de una progresión aritmética.

**Example** The arithmetic sequence 1, 5, 9, 13, . . . has a common difference of 4.

**Common logarithm** A common logarithm is a logarithm that uses base 10. You can write the common logarithm $\log_{10} y$ as $\log y$.

**Logaritmo común** El logaritmo común es un logaritmo de base 10. El logaritmo común $\log_{10} y$ se expresa como $\log y$.

**Example** $\log 1 = 0$
$\log 10 = 1$
$\log 50 = 1.698970004\ldots$

**Common ratio** A common ratio is the ratio of consecutive terms of a geometric sequence.

**Razón común** Una razón común es la razón de términos consecutivos en una secuencia geométrica.

**Example** The geometric sequence 2.5, 5, 10, 20, . . . has a common ratio of 2.

**Complement of an event** All possible outcomes that are not in the event.
$P(\text{complement of event}) = 1 - P(\text{event})$

**Complemento de un suceso** Todos los resultados posibles que no se dan en el suceso.
$P(\text{complemento de un suceso}) = 1 - P(\text{suceso})$

**Example** The complement of rolling a 1 or a 2 on a standard number cube is rolling a 3, 4, 5, or 6.

**Completing the square** Completing the square is the process of adding $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to form a perfect-square trinomial.

**Completar el cuadrado** Completar un cuadrado es el proceso mediante el cual se suma $\left(\frac{b}{2}\right)^2$ a $x^2 + bx$ para formar un trinomio cuadrado perfecto.

**Example** $x^2 - 12x + \blacksquare$
$x^2 - 12x + \left(\frac{-12}{2}\right)^2$
$x^2 - 12x + 36$

**Complex conjugates** Complex numbers with equivalent real parts and opposite imaginary parts are complex conjugates.

**Conjugados complejos** Los números complejos con partes reales equivalentes y partes imaginarias opuestas son conjugados complejos.

**Example** The complex numbers $2 - 3i$ and $2 + 3i$ are complex conjugates.

**Complex number** Complex numbers are numbers that can be written in the form $a + bi$, where $a$ and $b$ are real numbers and $i$ is the square root of $-1$.

**Número complejo** Los números complejos son los números que se pueden escribir como $a + bi$, donde $a$ y $b$ son números reales y donde $i$ es la raíz cuadrada de $-1$.

**Example** $6 + i$
$7$
$2i$

## English | Spanish

**Composite function** A composite function is a combination of two functions such that the output from the first function becomes the input for the second function.

**Función compuesta** Una función compuesta es la combinación de dos funciones. La cantidad de salida de la primera función es la cantidad de entrada de la segunda función.

**Example** $f(x) = 2x + 1,\ g(x) = x^2 - 1$
$(g \circ f)\ (5) = g(f(5)) = g(2(5) + 1)$
$= g(11)$
$= 11^2 - 1 = 120$

**Composition of functions** A composition of functions is the operation that forms composite functions.

**Composición de funciones** Una composición de funciones es la operación que forma funciones compuestas.

**Example** $f \circ g(x) = f(g(x))$
$g \circ f(x) = g(f(x))$

**Compound event** A compound event is an event that consists of two or more events linked by the word *and* or the word *or*.

**Suceso compuesto** Un suceso compuesto es un suceso que consiste en dos o más sucesos unidos por medio de la palabra *y* o la palabra *o*.

**Example** Rolling a 5 on a standard number cube and then rolling a 4 is a compound event.

**Compound fraction** A compound fraction is a fraction that has one or more fractions in the numerator and/or the denominator.

**Fracción compuesta** Una fracción compuesta está en forma de fracción y tiene una o más fracciones en el numerador o el denominador.

**Example** $\dfrac{\frac{x}{4} + \frac{x+2}{2}}{\frac{x-1}{3}}$

**Compound inequality** A combination of two or more inequalities is a compound inequality.

**Desigualdad compuesta** Una combinación de dos o más desigualdades es una desigualdad compuesta.

**Example** $-1 < x$ and $x \leq 3$
$x < -1$ or $x \geq 3$

**Compound interest** Interest that is paid on both the principal and the interest that has already been paid is compound interest.

**Interés compuesto** El interés calculado tanto sobre el capital como sobre los intereses ya pagados es el interés compuesto.

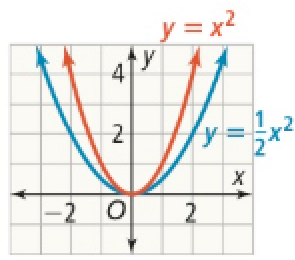
**Compound interest formula** This formula is an exponential model that is used to calculate the value of an investment when interest is compounded.

**Fórmula de interés compuesto** Esta fórmula es un modelo exponencial que se usa para calcular el valor de una inversión cuando el interés es compuesto.

**Compression** A compression is a transformation that decreases the distance between the points of a graph and a given line by the same factor.

**Compresión** La compresión es una transformación que reduce por el mismo factor la distancia entre los puntos de una gráfica y una recta dada.

**Example**




The graph of $y = \frac{1}{2}x^2$ is a compression of the graph of $y = x^2$.

## English | Spanish

**Conditional probability** A conditional probability is the probability that an event $B$ will occur given that another event $A$ has already occurred. The notation $P(B|A)$ is read "the probability of event $B$, given event $A$." For any two events $A$ and $B$ in the sample space, $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$.

**Probabilidad condicional** Una probabilidad condicional es la probabilidad de que ocurra un suceso $B$ cuando ya haya ocurrido otro suceso $A$. La notación $P(B|A)$ se lee "la probabilidad del suceso $B$, dado el suceso $A$". Para dos sucesos cualesquiera $A$ y $B$ en el espacio muestral $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$.

**Example** $= \frac{P(\text{departs and arrives on time})}{P(\text{departs on time})}$
$= \frac{0.75}{0.83}$
$\approx 0.9$

**Constant matrix** When representing a system of equations with a matrix equation, the matrix containing the constants of the system is the constant matrix.

**Matriz de constantes** Al representar un sistema de ecuaciones con una ecuación matricial, la matriz que contiene las constantes del sistema es la matriz de constantes.

**Example** $x + 2y = 5$
$3x + 5y = 14$
constant matrix $\begin{bmatrix} 5 \\ 14 \end{bmatrix}$

**Constant of variation** The constant of variation is the ratio of the two variables in a direct variation and the product of the two variables in an inverse variation.

**Constante de variación** La constante de variación es la razón de dos variables en una variación directa y el producto de las dos variables en una variación inversa.

**Example** In $y = 3.5x$, the constant of variation $k$ is 3.5. In $xy = 5$, the constant of variation $k$ is 5.

**Continuously compounded interest formula** This formula is a model for interest that has an infinitely small compounding period. The number $e$ is the base in the formula $A = Pe^{rt}$.

**Fórmula de interés compuesto continuo** Esta fórmula es un modelo para calcular el interés que tiene un período de capitalización muy reducido. El número e es la base de la fórmula $A = Pe^{rt}$.

**Example** Suppose that $P = \$1200$, $r = 0.05$, and $t = 3$. Then
$A = 1200e^{0.05 \cdot 3}$
$= 1200(2.718 \ldots)^{0.15}$
$\approx 1394.20$

## D

**Decay factor** In an exponential function of the form $y = ab^x$, $b$ is the decay factor if $0 < b < 1$.

**Factor de decaimiento** En la función exponencial de la forma $y = ab^x$, $b$ es el factor de decaimiento si $0 < b < 1$.

**Example** In the equation $y = 0.3^x$, 0.3 is the decay factor.

**Degree of a polynomial** The degree of a polynomial is the greatest degree among its monomial terms.

**Grado de un polinomio** El grado de un polinomio es el grado mayor entre los términos de monomios.

**Example** $P(x) = x^6 + 2x^3 - 3$ has degree 6

**Dependent events** When the outcome of one event affects the probability of a second event, the events are dependent events.

**Sucesos dependientes** Dos sucesos son dependientes si el resultado de un suceso afecta la probabilidad del otro.

**Example** You have a bag with marbles of different colors. If you pick a marble from the bag and pick another without replacing the first, the events are dependent events.

## English | Spanish

**Determinant of a 2 × 2 matrix** For the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the value $ad - bc$ is the determinant.

**Determinante de una matriz de 2 × 2** Para la matriz $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, el valor $ad - bc$ es el determinante.

**Example** The determinant of $\begin{bmatrix} 3 & -2 \\ 5 & 6 \end{bmatrix}$ is $3(6) - 5(-2) = 28$.

**Dimensions of a matrix** The dimensions of a matrix are determined by the number of rows and columns in a matrix.

**Dimensiones de una matriz** La dimensión de una matriz está determinada por la cantidad de filas y columnas de la matriz.

**Example** The matrix $\begin{bmatrix} 3 & 5 & 4 \\ -2 & 1 & 6 \end{bmatrix}$ has 2 rows and 3 columns; the dimensions are $2 \times 3$.

**Discriminant** The discriminant of a quadratic equation in the form $ax^2 + bx + c = 0$ is the value of the expression $b^2 - 4ac$. The value of the discriminant determines the number of solutions of the equation.

**Discriminante** El discriminante de una ecuación cuadrática en la forma $ax^2 + bx + c = 0$ es el valor de la expresión $b^2 - 4ac$. El valor del discriminante determina el número de soluciones de la ecuación.

**Example** $3x^2 - 6x + 1$

$$\begin{aligned} \text{discriminant} &= (-6)^2 - 4(3)(1) \\ &= 36 - 12 = 24 \end{aligned}$$

## E

**End behavior** End behavior of the graph of a function describes the directions of the graph as you move to the left and to the right, away from the origin.

**Comportamiento extremo** El comportamiento extremo de la gráfica de una función describe las direcciones de la gráfica al moverse a la izquierda y a la derecha, apartándose del origen.

**Equal matrices** Equal matrices are matrices with the same dimensions and equal corresponding elements.

**Matrices equivalentes** Dos matrices son equivalentes si y sólo si tienen las mismas dimensiones y sus elementos correspondientes son iguales.

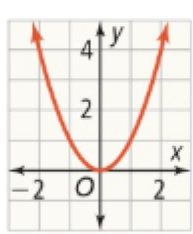
**Example** Matrices $A$ and $B$ are equal.

$$A = \begin{bmatrix} 2 & 6 \\ \frac{9}{3} & 1 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{6}{3} & 6 \\ 3 & \frac{-13}{-13} \end{bmatrix}$$

**Even function** A function that is symmetric about the $y$-axis is an even function; $f(x) = f(-x)$ for all $x$-values.

**Función par** Una función que es simétrica respecto del eje $y$ es una función par; $f(x) = f(-x)$ para todos los valores de $x$.

**Example** $f(x) = x^2$ is an even function since $f(-x) = (-x)^2 = x^2$.




**Event** Any group of outcomes in a situation involving probability.

**Suceso** En la probabilidad, cualquier grupo de resultados.

**Example** When rolling a number cube, there are six possible outcomes. Rolling an even number is an event with three possible outcomes, 2, 4, and 6.

| English | Spanish |
|---|---|
| | |

**Explicit definition** An explicit definition allows any term in a sequence to be found without knowing the previous term.

**Definición explícita** Una definición explícita permite hallar cualquier término de una progresión aunque no se conozca el término anterior.

**Example** The explicit definition $a_n = 3 + 4(n - 1)$ allows the 7th term to be calculated directly: $a_7 = 3 + 4(7 - 1) = 27$.

**Exponential decay function** Exponential decay is modeled by a function of the form $y = ab^x$ with $a > 0$ and $0 < b < 1$.

**Función de decaimiento** El decaimiento exponencial se expresa con una función $y = ab^x$ donde $a > 0$ y $0 < b < 1$.

**Exponential equation** An exponential equation contains the form $b^{cx}$, with the exponent including a variable.

**Ecuación exponencial** Una ecuación exponencial tiene la forma $b^{cx}$, y su exponente incluye una variable.

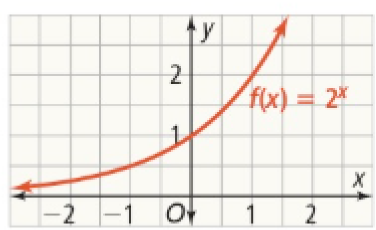
**Example**

$$\begin{aligned} 5^{2x} &= 270 \\ \log 5^{2x} &= \log 270 \\ 2x \log 5 &= \log 270 \\ 2x &= \frac{\log 270}{\log 5} \\ 2x &\approx 3.4785 \\ x &\approx 1.7392 \end{aligned}$$

**Exponential function** An exponential function is any function of the form $f(x) = ab^x$ where $a$ and $b$ are constants with $a \neq 0$, $b > 0$, and $b \neq 1$.

**Función exponencial** Una función exponencial es cualquier función de la forma $f(x) = ab^x$ donde $a$ y $b$ son constantes con $a \neq 0$, $b > 0$ y $b \neq 1$.

**Example**




**Exponential growth function** Exponential growth is modeled by a function of the form $y = ab^x$ with $a > 0$ and $b > 1$.

**Función de crecimiento exponencial** El crecimiento exponencial se expresa con una función de la forma $y = ab^x$ donde $a > 0$ y $b > 1$.

**Extraneous solution** An extraneous solution is a solution of an equation derived from an original equation, but it is not a solution of the original equation.

**Solución extraña** Una solución extraña es una solución de una ecuación derivada de una ecuación dada, pero que no satisface la ecuación dada.

**Example**

$$\begin{aligned} \sqrt{x - 3} &= x - 5 \\ x - 3 &= x^2 - 10x + 25 \\ 0 &= x^2 - 11x + 28 \\ 0 &= (x - 4)(x - 7) \\ x &= 4 \text{ or } 7 \end{aligned}$$

The number 7 is a solution, but 4 is not, since $\sqrt{4 - 3} \neq 4 - 5$.

## F

**Factorial** The factorial of a positive integer $n$ is the product of all positive integers less than or equal to $n$ and written $n!$

**Factorial** El factorial de un número entero positivo $n$ es el producto de todos los números positivos menores que o iguales a $n$, y se escribe $n!$

**Example** $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

| English | Spanish |
|---|---|

**Fundamental Counting Principle** If there are $m$ ways to make the first selection and $n$ ways to make the second selection, then there are $m \cdot n$ ways to make the two selections.

**Principio fundamental de Conteo** Si hay $m$ maneras de hacer la primera selección y $n$ maneras de hacer la segunda selección, quiere decir que hay $m \cdot n$ maneras de hacer las dos selecciones.

**Example** For 5 shirts and 8 pairs of shorts, the number of possible outfits is $5 \cdot 8 = 40$.

## G

**Geometric sequence** A geometric sequence is a sequence with a constant ratio between consecutive terms.

**Secuencia geométrica** Una secuencia geométrica es una secuencia con una razón constante entre términos consecutivos.

**Example** The geometric sequence 2.5, 5, 10, 20, 40 . . . , has a common ratio of 2.

**Greatest common factor** The greatest common factor (GCF) of an expression is the common factor of each term of the expression that has the greatest coefficient and the greatest exponent.

**Máximo factor común** El máximo factor común de una expresión es el factor común de cada término de la expresión que tiene el mayor coeficiente y el mayor exponente.

**Example** The GCF of $4x^2 + 20x - 12$ is 4.

**Growth factor** In an exponential function of the form $y = ab^x$, $b$ is the growth factor if $b > 1$.

**Factor de incremento** En una función exponencial de la forma $y = ab^x$, $b$ es el factor de incremento si $b > 1$.

**Example** In the exponential equation $y = 2^x$, 2 is the growth factor.

## I

**Identity** An identity is an equation between two polynomial expressions in which one side can be transformed into the other side using defined rules of calculation.

**Identidad** Una ecuación entre dos expresiones polinomiales para la cual un lado se puede transformar en el otro lado usando reglas de cálculo definidas.

**Identity matrix** An identity matrix is a matrix $I$ such that $IB = BI = B$; it is a square matrix with 1s along the main diagonal and 0 for all of the other elements.

**Matriz de identidad** Una matriz de identidad es una matriz $I$ tal que $IB = BI = B$; una matriz cuadrada con unos por la diagonal principal y ceros en los demás elementos.

**Example**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$$

**Imaginary number** An imaginary number is any number of the form $bi$, where $b$ is a nonzero real number and $i$ is the square root of $-1$.

**Número imaginario** Un número imaginario es cualquier número de la forma $bi$, donde $b$ es un número real distinto de cero y donde $i$ es la raíz cuadrada de $-1$.

**Example** $7i$
$i$

**Imaginary unit** The imaginary unit $i$ is the complex number whose square is $-1$.

**Unidad imaginaria** La unidad imaginaria $i$ es el número complejo cuyo cuadrado es $-1$.

**Inconsistent system** A system of equations that has no solution is an inconsistent system.

**istema incompatible** Un sistema incompatible es un sistema de ecuaciones para el cual no hay solución.

**Example** $\begin{cases} y = 2x + 3 \\ -2x + y = 1 \end{cases}$ is a system of parallel lines, so it has no solution. It is an inconsistent system.

| English | Spanish |
| --- | --- |

**Independent events** When the outcome of one event does not affect the probability of a second event, the two events are independent.

**Sucesos independientes** Cuando el resultado de un suceso no altera la probabilidad de otro, los dos sucesos son independientes.

**Example** The results of two rolls of a number cube are independent. Getting a 5 on the first roll does not change the probability of getting a 5 on the second roll.

**Index** With a radical sign, the index indicates the degree of the root.

**Índice** Con un signo de radical, el índice indica el grado de la raíz.

**Example** index 2 index 3 index 4

$\sqrt{16}$ $\sqrt[3]{16}$ $\sqrt[4]{16}$

**Interval notation** Interval notation represents a set of real numbers with a pair of values that are its left (minimum) and right (maximum) boundaries.

**Notación de intervalo** La notación de intervalo representa un conjunto de números reales con un par de valores que son sus límites a la izquierda (mínimo) y a la derecha (máximo).

**Example** The interval $(2, 7]$ represents the inequality $2 < x \leq 7$.

**Inverse function** If function $f$ pairs a value $b$ with $a$, then its inverse, denoted $f^{-1}$, pairs the value $a$ with $b$. If $f^{-1}$ is also a function, then $f$ and $f^{-1}$ are inverse functions.

**Funcion inversa** Si la función $f$ empareja un valor $b$ con $a$, entonces su inversa, cuya notación es $f^{-1}$, empareja el valor $a$ con $b$. Si $f^{-1}$ también es una función, entonces $f$ y $f^{-1}$ son funciones inversas.

**Example** If $f(x) = x + 3$, then $f^{-1}(x) = x - 3$.

**Inverse matrix** An inverse matrix is a matrix such that its product with another matrix yields the identity matrix.

**Matriz inversa** Una matriz inversa es una matriz tal que su producto con otra matriz da la matriz de identidad.

**Example** The inverse matrix of $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix}$

since $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

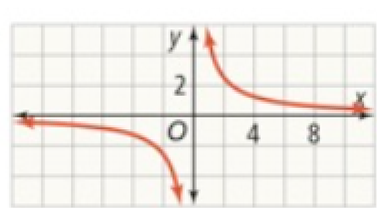
**Inverse relation** An inverse relation is formed when the roles of the independent and dependent variables are reversed.

**Relación inversa** Una relación inversa se forma cuando se invierten los roles de las variables independientes y dependientes.

**Inverse variation** An inverse variation is a relation represented by an equation of the form $xy = k$, $y = \frac{k}{x}$, or $x = \frac{k}{y}$, where $k \neq 0$.

**Variación inversa** Una variación inversa es una relación representada por la ecuación $xy = k$, $y = \frac{k}{x}$, ó $x = \frac{k}{y}$, donde $k \neq 0$.

**Example**




$xy = 5$, or $y = \frac{5}{x}$

# English / Spanish

## L

**Leading coefficient** In a polynomial, the non-zero term that is multiplied by the greatest power of $x$ is the leading coefficient.

**Coeficiente principal** En un polinomio, el término distinto de cero que se multiplica por la potencia de $x$ mayor es el coeficiente principal.

**Example** In the expression $4x^5 + 3x^2 - x - 6$, 4 is the leading coefficient.

**Like radicals** Like radicals are radical expressions that have the same index and the same radicand.

**Radicales semejantes** Los radicales semejantes son expresiones radicales que tienen el mismo índice y el mismo radicando.

**Example** $4\sqrt[3]{7}$ and $\sqrt[3]{7}$ are like radicals.

**Logarithm** For $b > 0$, $b \neq 1$, and $x > 0$, the logarithm base $b$ of a positive number $x$ is defined as follows: $\log_b x = y$, if and only if $x = b^y$.

**Logaritmo** Para $b > 0$, $b \neq 1$ y $x > 0$, la base del logaritmo $b$ de un número positivo $x$ se define como $\log_b x = y$, si y sólo si $x = b^y$.

**Example** $\log_2 8 = 3$
$\log_{10} 100 = \log 100 = 2$
$\log_5 5^7 = 7$

**Logarithmic equation** A logarithmic equation is an equation that includes a logarithm involving a variable.

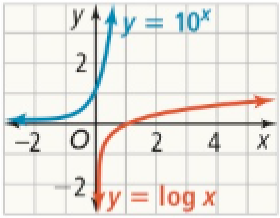
**Ecuación logarítmica** Una ecuación logarítmica es una ecuación que incluye un logaritmo con una variable.

**Example** $\log_3 x = 4$

**Logarithmic function** A logarithmic function is the inverse of an exponential function.

**Función logarítmica** Una función logarítmica es la inversa de una función exponencial.

**Example**




## M

**Matrix** A matrix is a rectangular array of numbers written within brackets.

**Matriz** Una matriz es un conjunto de números encerrados en corchetes y dispuestos en forma de rectángulo.

**Example** $A = \begin{bmatrix} 1 & -2 & 0 & 10 \\ 9 & 7 & -3 & 8 \\ 2 & -10 & 1 & -6 \end{bmatrix}$

The number 2 is the element in the third row and first column. $A$ is a $3 \times 4$ matrix.

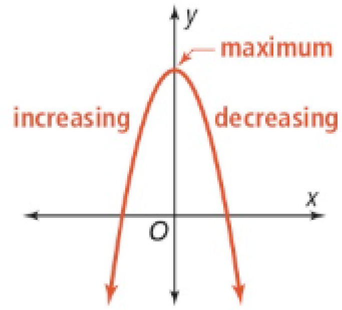
## English

## Spanish

**Maximum** The maximum of a function is the greatest value that the function attains in its domain. It is the $y$-coordinate of the highest point on the graph of the function.

**Máximo** El valor máximo de una función es el mayor valor que la función alcanza en su dominio. Es la coordenada $y$ del punto más alto de la gráfica de la función.

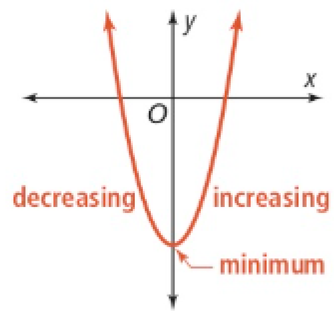
**Example**




**Minimum** The minimum of a function is the least value that the function attains in its domain. It is the $y$-coordinate of the lowest point on the graph of the function.

**Mínimo** El valor mínimo de una función es el menor valor que la función alcanza en su dominio. Es la coordenada y del punto más bajo de la gráfica de la función.

**Example**




**Monomial** A monomial is either a real number, a variable, or a product of real numbers and variables with whole number exponents.

**Monomio** Un monomio es un número real, una variable o un producto de números reales y variables cuyos exponentes son números enteros.

**Example** $1, x, 2z, 4ab^2$

**Multiplicity** The multiplicity of a zero of a polynomial function is the number of times the related linear factor is repeated in the factored form of the polynomial.

**Multiplicidad** La multiplicidad de un cero de una función polinomial es el número de veces que el factor lineal relacionado se repite en la forma factorizada del polinomio.

**Example** The zeros of the function $P(x) = 2x(x-3)^2(x+1)$ are 0, 3, and −1. Since $(x-3)$ occurs twice as a factor, the zero 3 has multiplicity 2.

**Mutually exclusive events** When two events cannot happen at the same time, the events are mutually exclusive. If $A$ and $B$ are mutually exclusive events, then $P(A \text{ or } B) = P(A) + P(B)$.

**Sucesos mutuamente excluyentes** Cuando dos sucesos no pueden ocurrir al mismo tiempo, son mutuamente excluyentes. Si $A$ y $B$ son sucesos mutuamente excluyentes, entonces $P(A \text{ o } B) = P(A) + P(B)$.

**Example** Rolling an even number $E$ and rolling a multiple of five $M$ on a standard number cube are mutually exclusive events.

$$\begin{aligned} P(E \text{ or } M) &= P(E) + P(M) \\ &= \frac{3}{6} + \frac{1}{6} \\ &= \frac{4}{6} \\ &= \frac{2}{3} \end{aligned}$$

| English | Spanish |
|---|---|

## N

**nth root** For any real numbers a and $b$, and any positive integer $n$, if $a^n = b$, then a is an $n$th root of $b$.

**raíz n-ésima** Para todos los números reales $a$ y $b$, y todo número entero positivo $n$, si $a^n = b$, entonces $a$ es la $n$-ésima raíz de $b$.

**Example** $\sqrt[5]{32} = 2$ because $2^5 = 32$.
$\sqrt[4]{81} = 3$ because $3^4 = 81$.

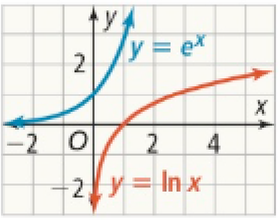
**Natural base e** The value that the expression $(1 + \frac{1}{x})^x$ approaches as $x \to \infty$. The value is approximately 2.7818282 . . .

**Base natural e** El valor al que se acerca la expresión $(1 + \frac{1}{x})^x$ a medida que $x \to \infty$. El valor es aproximadamente igual a 2.7818282 . . .

**Natural logarithmic function** A natural logarithmic function is a logarithmic function with base e. The natural logarithmic function $y = \ln x$ is $y = \log_e x$. It is the inverse of $y = e^x$.

**Función logarítmica natural** Una función logarítmica natural es una función logarítmica con base e. La función logarítmica natural $y = \ln x$ es $y = \log_e x$. Ésta es la función inversa de $y = e^x$.

**Example**




$\ln e^3 = 3$
$\ln 10 \approx 2.3026$
$\ln 36 \approx 3.5835$

## O

**Odd function** An odd function is a function that is symmetric about the origin; $f(-x) = -f(x)$ for all $x$-values.

**Función impar** Una función impar es una función que es simétrica respecto del origen; $f(-x) = -f(x)$ para todos los valores de $x$.

**Example** $f(x) = x^3$ is an odd function since $f(-x) = (-x)^3 = -x^3$.

**Outcome** An outcome is the result of a single trial in a probability experiment.

**Resultado** Un resultado es que se obtiene al hacer una sola prueba en un experimento de probabilidad.

**Example** The outcomes of rolling a number cube are 1, 2, 3, 4, 5, and 6.

**Overlapping events** Overlapping events are events that have at least one common outcome. If $A$ and $B$ are overlapping events, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

**Sucesos traslapados** Sucesos traslapados son sucesos que tienen por lo menos un resultado en común. Si $A$ y $B$ son sucesos traslapados, entonces $P(A \text{ ó } B) = P(A) + P(B) - P(A \text{ y } B)$.

**Example** Rolling a multiple of 3 and rolling an odd number on a number cube are overlapping events.

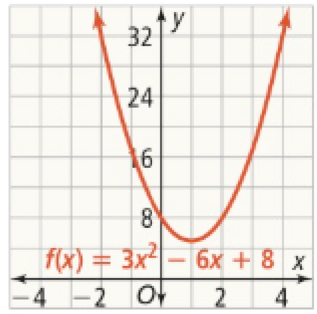
# English | Spanish

## P

**Parabola** A parabola is the graph of a quadratic function.

**Parábola** La parábola es la gráfica de una función cuadrática.

**Example**




**Pascal's Triangle** Pascal's Triangle is a triangular array of numbers in which the first and last number in each row is 1. Each of the other numbers in the row is the sum of the two numbers above it.

**Triángulo de Pascal** El Triángulo de Pascal es una distribución triangular de números en la cual el primer número y el último número son 1. Cada uno de los otros números en la fila es la suma de los dos números de encima.

**Example** **Pascal's Triangle**

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

**Permutation** A permutation is an arrangement of some or all of a set of objects in a specific order. You can use the notation ${}_nP_r$ to express the number of permutations, where $n$ equals the number of objects available and $r$ equals the number of selections to make.

**Permutación** Una permutación es una disposición de algunos o de todos los objetos de un conjunto en un orden determinado. El número de permutaciones se puede expresar con la notación ${}_nP_r$, donde $n$ es igual al número total de objetos y $r$ es igual al número de selecciones que han de hacerse.

**Example** How many ways can you arrange 5 objects 3 at a time?

$${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 60$$

There are 60 ways to arrange 5 objects 3 at a time.

**Piecewise-defined function** A piecewise-defined function has different rules for different parts of its domain.

**Función definida por partes** Una función de fragmentos tiene reglas diferentes para diferentes partes de su dominio.

**Polynomial** A polynomial is a monomial or the sum or difference of two or more monomials.

**Polinomio** Un polinomio es un monomio o la suma o la diferencia de dos o más monomios.

**Example** $3x^3 + 4x^2 - 2x + 5$
$8x$
$x^2 + 4x + 2$

**Polynomial function** A polynomial function is a function whose rule is a polynomial.

**Función polinomial** Una función polinomial es una función cuya regla es un polinomio.

**Example** $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is a polynomial function, where $n$ is a nonnegative integer and the coefficients $a_n, \ldots, a_0$ are real numbers.

# English

# Spanish

**Probability** Probability is the likelihood that an event will occur (written formally as $P$(event)).

**Probabilidad** Probabilidad es la posibilidad de que un suceso ocurra, escrita formalmente $P$(suceso).

**Example** You have 4 red marbles and 3 white marbles. The probability that you select one red marble, and then, without replacing it, randomly select another red marble is $P(\text{red}) = \frac{4}{7} \cdot \frac{3}{6} = \frac{2}{7}$.

## Q

**Quadratic Formula** The Quadratic Formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for $ax^2 + bx + c = 0$ and $a \neq 0$.

**Fórmula cuadrática** La fórmula cuadrática es $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ para $ax^2 + bx + c = 0$ y $a \neq 0$.

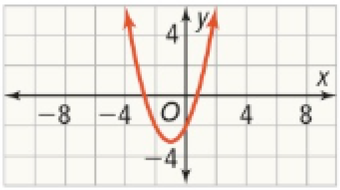
**Example** If $-x^2 + 3x + 2 = 0$, then

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(-1)(2)}}{2(-1)}$$
$$= \frac{-3 \pm \sqrt{17}}{-2}$$

**Quadratic function** A quadratic function is a function that you can write in the form $f(x) = ax^2 + bx + c$ with $a \neq 0$.

**Función cuadrática** Una función cuadrática es una función que puedes escribir como $f(x) = ax^2 + bx + c$ con $a \neq 0$.

**Example**




$f(x) = x^2 + 2x - 2$

**Quadratic regression** Quadratic regression is a method used to find the quadratic function that best fits a set of data.

**Regresión cuadrática** La regresión cuadrática es un método que se usa para hallar la función cuadrática que más se acerque a un conjunto de datos.

## R

**Radical function** A radical function is a function that can be written in the form $f(x) = a\sqrt[n]{x - h} + k$, where $a \neq 0$. For even values of $n$, the domain of a radical function is the real numbers $x \geq h$.

**Función radical** Una función radical es una función quepuede expresarse como $f(x) = a\sqrt[n]{x - h} + k$, donde $a \neq 0$. Para $n$ par, el dominio de la función radical son los números reales tales que $x \geq h$.

**Example** $f(x) = \sqrt{x - 2}$

**Radical symbol** The symbol denoting a root is a radical symbol.

**Símbolo de radical** El símbolo que expresa una raíz es un símbolo de radical.

**Example** $\sqrt{\phantom{x}}$

**Radicand** The number under a radical sign is the radicand.

**Radicando** La expresión que aparece debajo del signo radical es el radicando.

**Example** The radicand in $3\sqrt[4]{7}$ is 7.

**Rational equation** A rational equation is an equation that contains a rational expression.

**Ecuación racional** Una ecuación racional es una ecuación que contiene una expresión racional.

**Rational expression** A rational expression is the quotient of two polynomials.

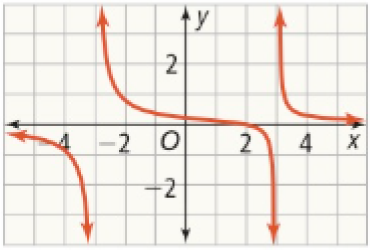
**Expresión racional** Una expresión racional es el cociente de dos polinomios.

## English | Spanish

**Rational function** A rational function $f(x)$ can be written as $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions. The domain of a rational function is all real numbers except those for which $Q(x) = 0$.

**Función racional** Una función racional $f(x)$ se puede expresar como $f(x) = \frac{P(x)}{Q(x)}$, donde $P(x)$ y $Q(x)$ son funciones de polinomios. El dominio de una función racional son todos los números reales excepto aquéllos para los cuales $Q(x) = 0$.

**Example**




The function $y = \frac{x-2}{x^2-9}$ is a rational function with three branches separated by asymptotes $x = -3$ and $x = 3$.

---

**Reciprocal function** The reciprocal function maps every non-zero real number to its reciprocal.

**Función recíproca** La función recíproca establece una correspondencia entre cada número real distinto de cero y su recíproco.

**Example** The function $f(x) = \frac{1}{x}$ is the reciprocal function.

---

**Recursive definition** A recursive definition of a sequence is a rule in which each term is defined by operations on the previous term.

**Definición recursiva** Una definición recursiva de una sucesión es una regla en la cual cada término se define por operaciones efectuadas en el término anterior.

**Example** Let $a_n = 2.5a_{n-1} + 3a_{n-2}$.
If $a_5 = 3$ and $a_4 = 7.5$, then
$a_6 = 2.5(3) + 3(7.5) = 30$.

---

**Reduced radical form** Reduced radical form is the form of an expression for which all $n$th roots of perfect $n$th powers in the radicand have been simplified and no radicals remain in the denominator.

**Forma radical reducida** La forma radical reducida es la forma de una expresión en la cual todas las raíces enésimas de potencias enésimas perfectas en el radicando se han simplificado y no quedan radicales en el denominador.

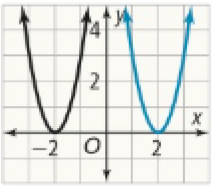
**Example** The reduced radical form of $\sqrt{50x^7} = 5x^3\sqrt{2x}$.

---

**Reflection** A reflection flips the graph of a function across a line, such as the $x$- or $y$-axis. Each point on the graph of the reflected function is the same distance from the line of reflection as is the corresponding point on the graph of the original function.

**Reflexión** Una reflexión voltea la gráfica de una función sobre una línea, como el eje de las $x$ o el eje de las $y$. Cada punto de la gráfica de la función reflejada está a la misma distancia del eje de reflexión que el punto correspondiente en la gráfica de la función original.

**Example**



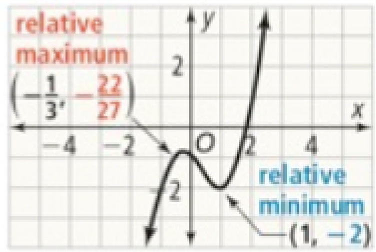

reflection across $y$-axis

| English | Spanish |
|---|---|

**Relative maximum (minimum)** A relative maximum (minimum) is the value of the function at an up-to-down (down-to-up) turning point.

**Máximo (mínimo) relativo** El máximo (mínimo) relativo es el valor de la función en un punto de giro de arriba hacia abajo (de abajo hacia arriba).

**Example**




**Remainder Theorem** If you divide a polynomial $P(x)$ of degree $n > 1$ by $x - a$, then the remainder is $P(a)$.

**Teorema del residuo** Si divides un polinomio $P(x)$ con un grado $n > 1$ por $x - a$, el residuo es $P(a)$.

**Example** If $P(x) = x^3 - 4x^2 + x + 6$ is divided by $x - 3$, then the remainder is $P(3) = 3^3 - 4(3)^2 + 3 + 6 = 0$ (which means that $x - 3$ is a factor of $P(x)$).

## S

**Sample space** A sample space is the set of all possible outcomes of a situation or experiment.

**Espacio muestral** Un espacio muestral es el conjunto de todos los resultados posibles de un suceso.

**Example** When you roll a standard number cube, the sample space is {1, 2, 3, 4, 5, 6}.

**Scalar** A real number factor in a special product, such as the 3 in the vector product $3\vec{v}$, is a scalar.

**Escalar** Un factor que es un número real en un producto especial, como el 3 en el producto vectorial $3\vec{v}$, es un escalar.

**Scalar multiplication** Scalar multiplication is an operation that multiplies a matrix $A$ by a scalar $c$. To find the resulting matrix $cA$, multiply each element of $A$ by $c$.

**Multiplicación escalar** La multiplicación escalar es la que multiplica una matriz $A$ por un número escalar $c$. Para hallar la matriz $cA$ resultante, multiplica cada elemento de $A$ por $c$.

**Example** $2.5\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2.5(1) & 2.5(0) \\ 2.5(-2) & 2.5(3) \end{bmatrix}$

$= \begin{bmatrix} 2.5 & 0 \\ -5 & 7.5 \end{bmatrix}$

**Sequence** A sequence is an ordered list of numbers that often forms a pattern.

**Progresión** Una progresión es una sucesión de números que suelen formar un patrón.

**Example** 1, 4, 7, 10, . . .

**Set-builder notation** Set-builder notation uses a verbal description or an inequality to describe the numbers in a set.

**Notación conjuntista** La notación conjuntista usa una descripción verbal o una desigualdad para describir los números.

**Example** $\{x \mid x$ is a real number$\}$
$\{x \mid x > 3\}$

**Simplest form of a radical expression** A radical expression with index $n$ is in simplest form if there are no radicals in any denominator, no denominators in any radical, and any radicand has no $n$th power factors.

**Mínima expresión de una expresión radical** Una expresión radical con índice $n$ está en su mínima expresión si no tiene radicales en ningún denominador ni denominadores en ningún radical y los radicandos no tienen factores de potencia.

| English | Spanish |
| --- | --- |

**Simplified form of a rational expression** A rational expression is in simplified form if its numerator and denominator are polynomials that have no common divisor other than 1.

**Forma simplificada de una expresión racional** Una expresión racional se encuentra en su mínima expresión si su numerador y su denominador son polinomios que no tienen otro divisor aparte de 1.

**Example** $\frac{x^2 - 7x + 12}{x^2 - 9} = \frac{(x-4)(x-3)}{(x+3)(x-3)} = \frac{x-4}{x+3}$, where $x \neq -3$

**Solution of a system of linear equations** A solution of a system of linear equations is a set of values for the variables that makes all the equations true.

**Solución de un sistema de ecuaciones lineales** Una solución de un sistema de ecuaciones lineales es un conjunto de valores para las variables que hace que todas las ecuaciones sean verdaderas.

**Square matrix** A square matrix is a matrix with the same number of columns as rows.

**Matriz cuadrada** Una matriz cuadrada es la que tiene la misma cantidad de columnas y filas.

**Example** Matrix $A$ is a square matrix.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

**Standard form of a polynomial function** The standard form of a polynomial function arranges the terms by degree in descending numerical order. A polynomial function, $P(x)$, in standard form is $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $n$ is a nonnegative integer and $a_n, \ldots, a_0$ are real numbers.

**Forma normal de una función polinomial** La forma normal de una función polinomial organiza los términos por grado en orden numérico descendiente. Una función polinomial, $P(x)$, en forma normal es $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, donde n es un número entero no negativo y $a_n, \ldots, a_0$ son números reales.

**Example** $2x^3 - 5x^2 - 2x + 5$

**Standard form of a quadratic function** The standard form of a quadratic function is $f(x) = ax^2 + bx + c$ with $a \neq 0$.

**Forma normal de una función cuadrática** La forma normal de una función cuadrática es $f(x) = ax^2 + bx + c$ con $a \neq 0$.

**Example** $f(x) = 2x^2 + 5x + 2$

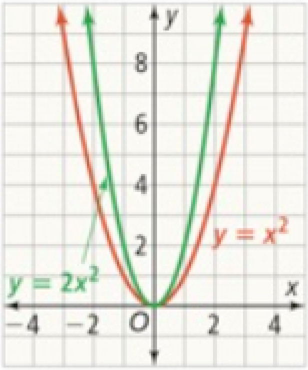
**Step function** A step function pairs every number in an interval with a single value. The graph of a step function can look like the steps of a staircase.

**Función escalón** Una función escalón empareja cada número de un intervalo con un solo valor. La gráfica de una función escalón se puede parecer a los peldaños de una escalera.

**Stretch** A stretch is a transformation that increases the distance between the points of a graph and a given line by the same factor.

**Estiramiento** Un estiramiento es una transformación que aumenta por el mismo factor la distancia entre los puntos de una gráfica y una recta dada.

**Example**



# English

# Spanish

**Synthetic division** Synthetic division is a process for dividing a polynomial by a linear expression $x - a$.

**División sintética** La división sintética es un proceso para dividir un polinomio por una expresión lineal $x - a$.

**Example**

$$\begin{array}{r|rrrrr} -3 & 2 & 5 & 0 & -2 & -8 \\ & & -6 & 3 & -9 & 33 \\ \hline & 2 & -1 & 3 & -11 & 25 \end{array}$$

Divide $2x^4 + 5x^3 - 2x - 8$ by $x + 3$. $2x^4 + 5x^3 - 2x - 8$ divided by $x + 3$ gives $2x^3 - x^2 + 3x - 11$ as quotient and 25 as remainder.

**System of linear equations** A system of equations is a set of two or more equations using the same variables.

**Sistema de ecuaciones lineales** Un sistema de ecuaciones es un conjunto de dos o más ecuaciones que contienen las mismas variables.

**Example** $\begin{cases} 2x - 3y = -13 \\ 4x + 5y = 7 \end{cases}$

**System of linear inequalities** A system of linear inequalities is a set of two or more linear inequalities using the same variables.

**Sistema de desigualdades lineales** Un sistema de desigualdades lineales es un conjunto de dos o más desigualdades lineales que contienen las mismas variables.

**Example** $x + 2y < 5$
$3x - 2y > -1$

## T

**Transformation** A transformation of a function maps each point of its graph to a new location.

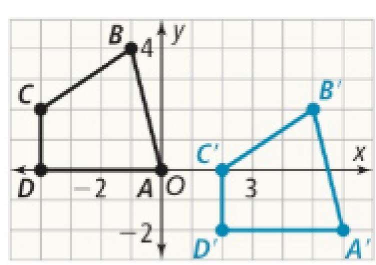
**Transformación** Una transformación de una función desplaza cada punto de su gráfica a una ubicación nueva.

**Example** $g(x) = 2(x - 3)$ is a transformation of $f(x) = x$.

**Translation** A translation shifts the graph of the parent function horizontally, vertically, or both.

**Traslación** Una traslación desplaza la gráfica de la función madre horizontalmente, verticalmente o en ambas direcciones.

**Example**




**Turning point** A turning point of the graph of a function is a point where the graph changes direction from upward to downward or from downward to upward.

**Punto de giro** Un punto de giro de la gráfica de una función es un punto donde la gráfica cambia de dirección de arriba hacia abajo o vice versa.

## V

**Vertex form of a quadratic function** The vertex form of a quadratic function is $f(x) = a(x - h)^2 + k$, where $a \neq 0$ and $(h, k)$ is the vertex of the function.

**Forma del vértice de una función cuadrática** La forma vértice de una función cuadrática es $f(x) = a(x - h)^2 + k$, donde $a \neq 0$ y $(h, k)$ es el vértice de la función.

**Example** $f(x) = x^2 + 2x - 1 = (x + 1)^2 - 2$
The vertex is $(-1, -2)$.

| English | Spanish |
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## English

## Spanish

**Variable matrix** A variable matrix is a one-column matrix that contains the variables of a system of equations.

**Matriz variable** Una matriz variable es una matriz de una sola columna que contiene las variables de un sistema de ecuaciones.

**Example** $\begin{cases} x + 2y = 5 \\ 3x + 5y = 14 \end{cases}$

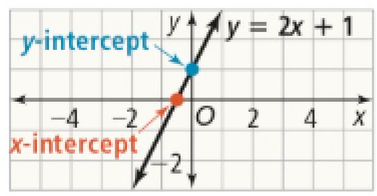
variable matrix $\begin{bmatrix} x \\ y \end{bmatrix}$

## X

**$x$-intercept, $y$-intercept** The point at which a line crosses the $x$-axis (or the $x$-coordinate of that point) is an $x$-intercept. The point at which a line crosses the $y$-axis (or the $y$-coordinate of that point) is a $y$-intercept.

**Intercepto en $x$, intercepto en $y$** El punto donde una recta corta el eje $x$ (o la coordenada $x$ de ese punto) es el intercepto en $x$. El punto donde una recta cruza el eje $y$ (o la coordenada $y$ de ese punto) es el intercepto en $y$.

**Example**




The $x$-intercept of $y = 2x + 1$ is $\left(-\frac{1}{2}, 0\right)$ or $-\frac{1}{2}$

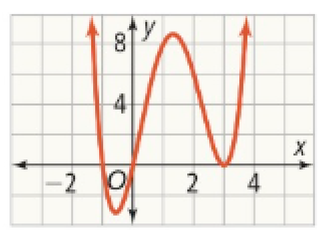
The $y$-intercept of $y = 2x + 1$ is $(0, 1)$ or 1.

## Z

**Zero of a function** A zero of a function is an $x$-intercept of the graph of a function.

**Cero de una función** Un cero de una función es un intercepto en $x$ de la gráfica de una función.

**Example**




**Zero matrix** The zero matrix $O$, or $O_{m \times n}$, is the $m \times n$ matrix whose elements are all zeros. It is the additive identity matrix for the set of all $m \times n$ matrices.

**Matriz cero** La matriz cero, $O$, o $O_{m \times n}$, es la matriz $m \times n$ cuyos elementos son todos ceros. Es la matriz de identidad aditiva para el conjunto de todas las matrices $m \times n$.

**Example** $\begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix} + O = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}$

**Zero-Product Property** If the product of two or more factors is zero, then at least one of the factors must be zero.

**Propiedad del cero del producto** Si el producto de dos o más factores es cero, entonces al menos uno de los factores debe ser cero.

**Example** $(x - 3)(2x - 5) = 0$

$x - 3 = 0$ or $2x - 5 = 0$

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