enVision® Florida

B.E.S.T. GEOMETRY

Student Edition

enVision® Florida

B.E.S.T. GEOMETRY

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Contents in Brief

enVision Florida
B.E.S.T. GEOMETRY

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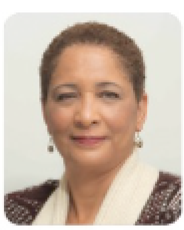

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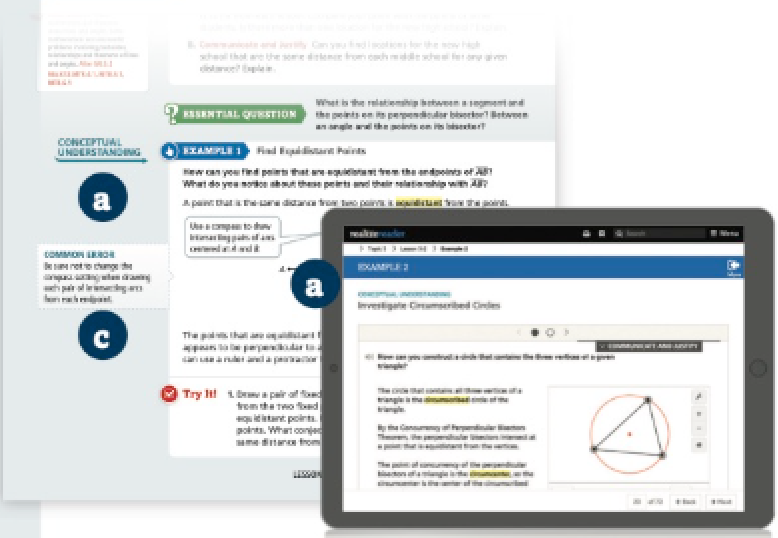
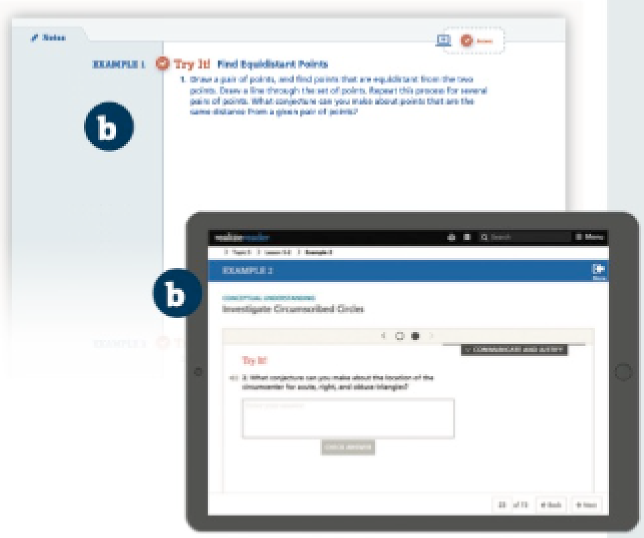
enVision® Florida B.E.S.T. Geometry offers a carefully constructed lesson design to help you succeed in math.

Step 1 At the start of each lesson, you and your classmates will work together to come up with a solution strategy for the problem or task posed. After a class discussion, you'll be asked to reflect back on the processes and strategies you used in solving the problem.

Step 2 Next, your teacher will guide you through new concepts and skills for the lesson.

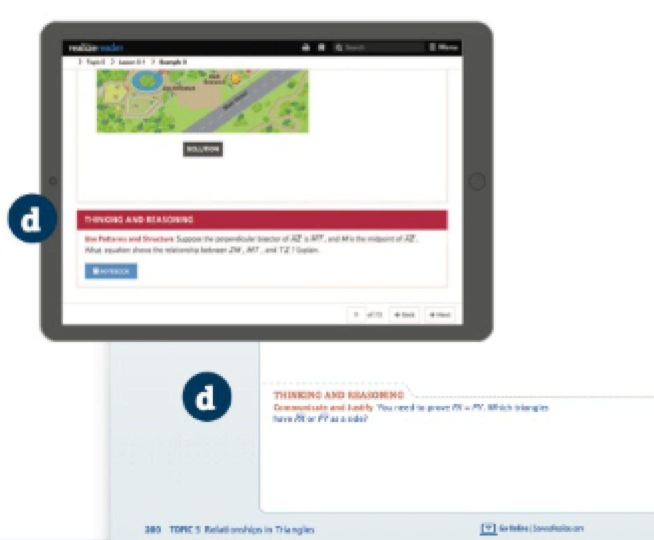


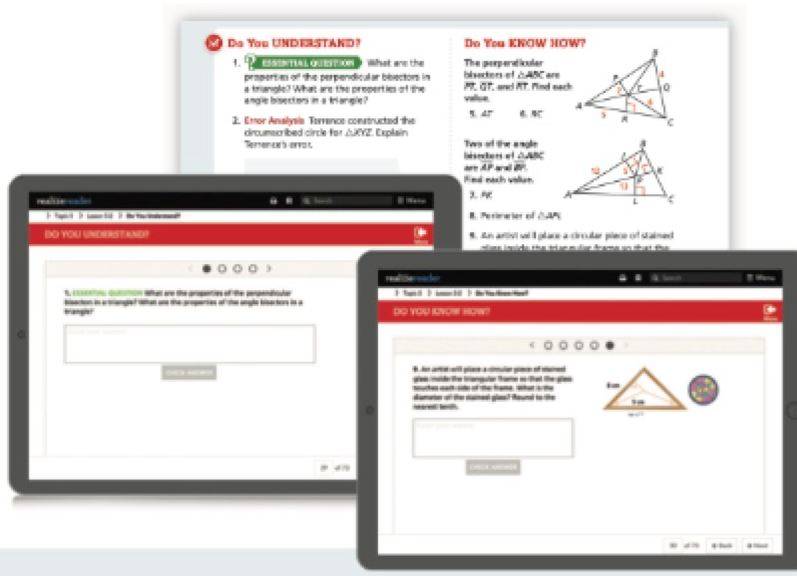


After each Example **a**, you work out a problem called the **Try It!** **b** to solidify your understanding of these concepts.

Side notes **c** help you with study tips, suggestions for avoiding common errors, and questions that support learning together and having a growth mindset.

In addition, you will periodically answer **Thinking and Reasoning** **d** questions to refine your thinking and problem-solving skills.

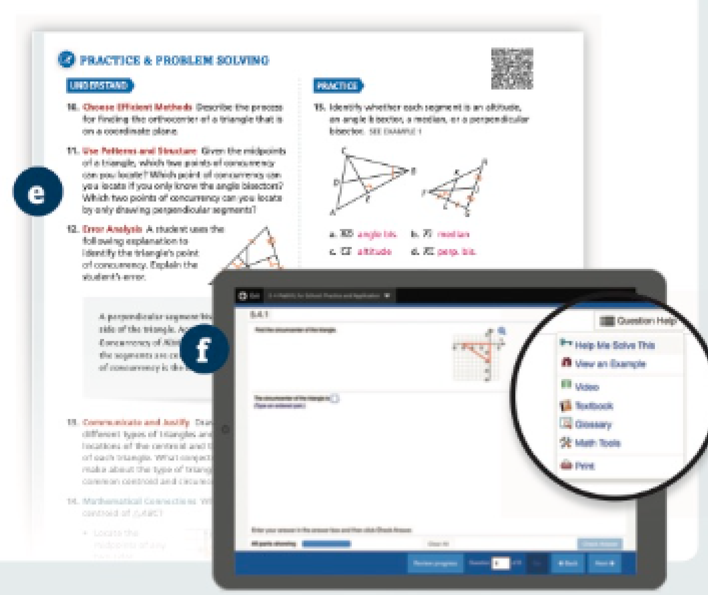




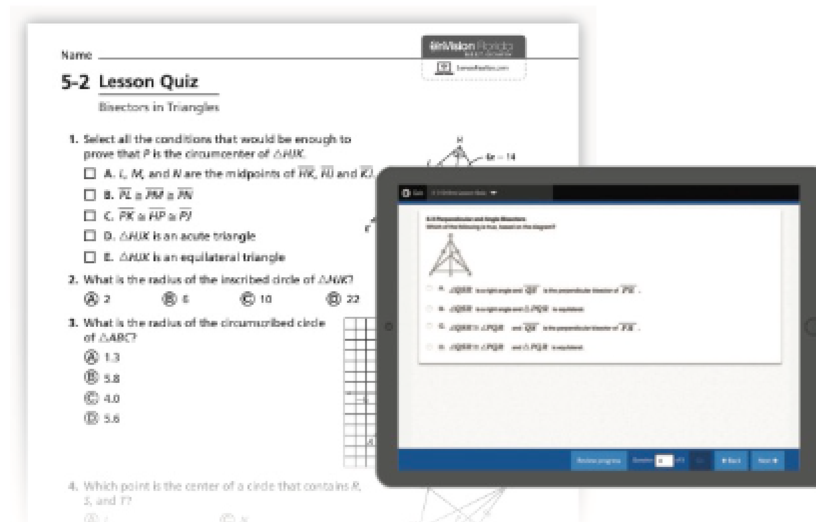
Step 2 cont. This part of the lesson concludes with a Lesson Check that helps you to know how well you are understanding the new content presented in the lesson. With the exercises in the **Do You Understand?** and **Do You Know How?**, you can gauge your understanding of the lesson concepts.

Step 3 In Step 3, you will find a balanced exercise set with **Understand** exercises that focus on conceptual understanding, **Practice** exercises that target procedural fluency, and **Apply** exercises for which you apply concept and skills to real-world situations **e**.

The **Assessment and Practice** **f** exercises offer practice for high stakes assessments. Your teacher may have you complete the assignment in your Student Edition, Student Companion, or online at SavvasRealize.com.



Step 4 Your teacher may have you take the Lesson Quiz after each lesson. You can take the quiz online or in print. To do your best on the quiz, review the lesson problems in that lesson.



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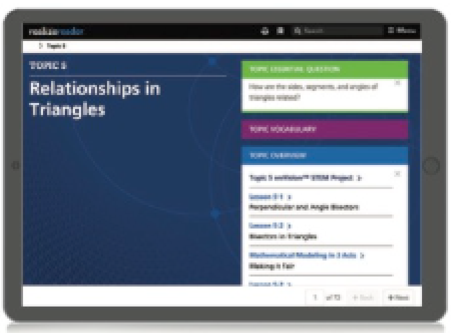
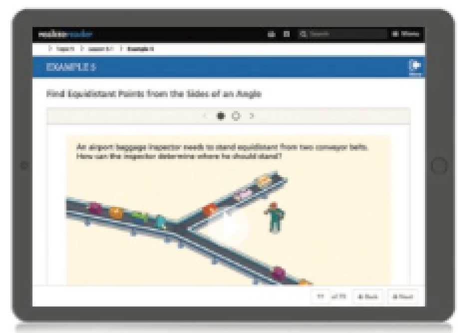
Everything you need for math, anytime, anywhere.

SavvasRealize.com is your gateway to all of the digital resources for **enVision® Florida B.E.S.T. Geometry.**

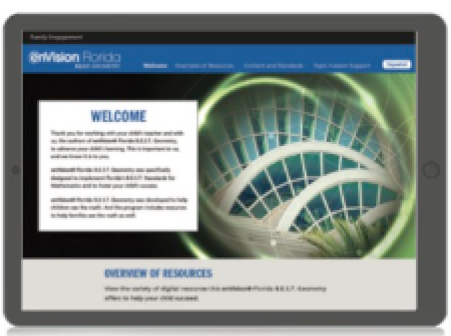
INTERACTIVE STUDENT EDITION Log in to access your interactive student edition, called Realize Reader.



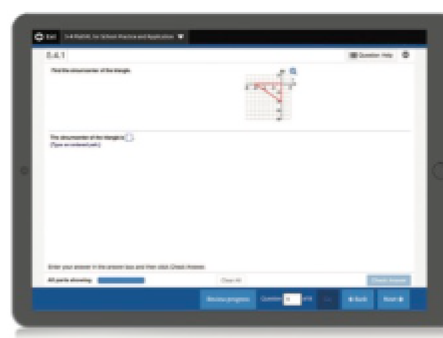
ANIMATION View and interact with real-world applications.



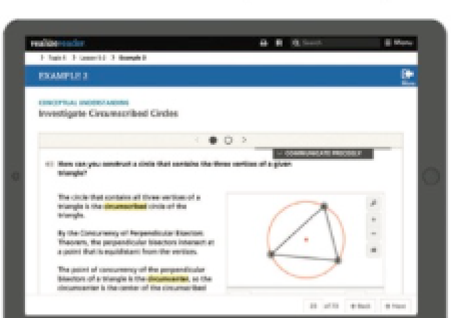
FAMILY ENGAGEMENT Involve family in your learning.



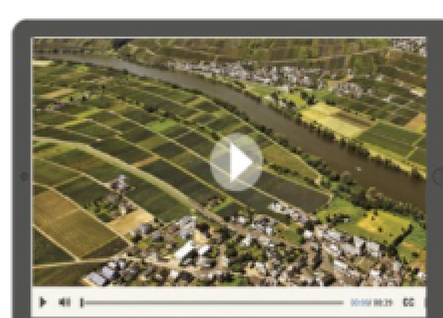
PRACTICE Practice what you've learned.



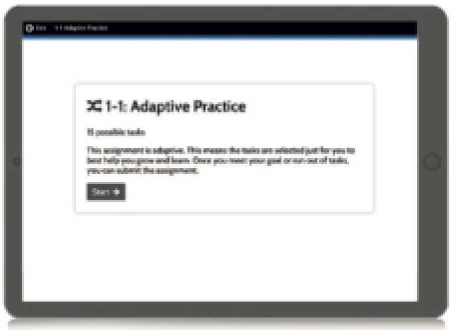
ACTIVITIES Complete *Explore & Reason*, *Model & Discuss*, *Critique & Explain* activities. Interact with *Examples* and *Try Its*.



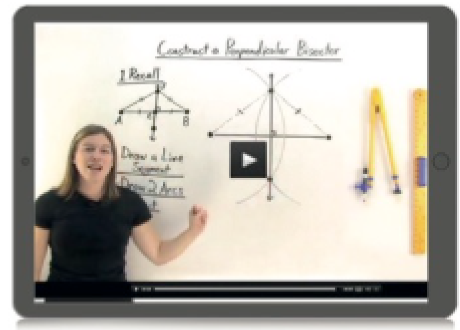
VIDEOS Watch clips to support Mathematical Modeling in 3 Acts Lessons and **enVision®** STEM Projects.



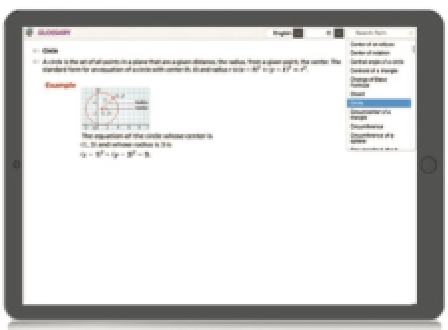
ADAPTIVE PRACTICE
Practice that is *just right* and *just for you.*



TUTORIALS Get help from Virtual Nerd, right when you need it.

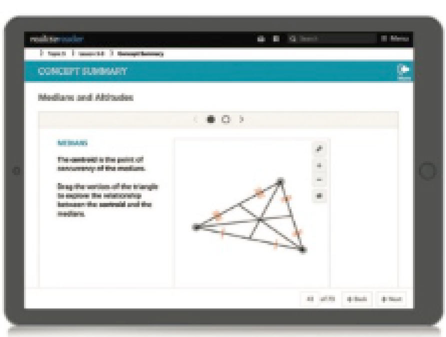


GLOSSARY Read and listen to English and Spanish definitions.

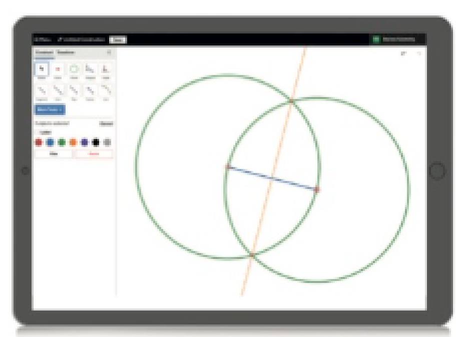


MATH TOOLS Explore math with digital tools and manipulatives.

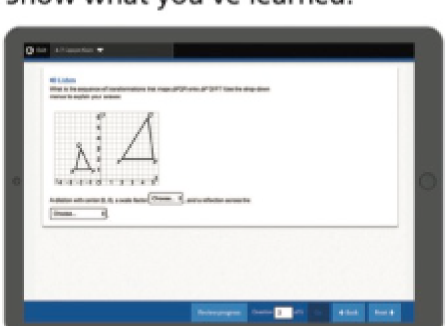
CONCEPT SUMMARY Review key lesson content through multiple representations.



DESMOS Use Anytime and as embedded Interactives in Lesson content.



ASSESSMENT
Show what you've learned.



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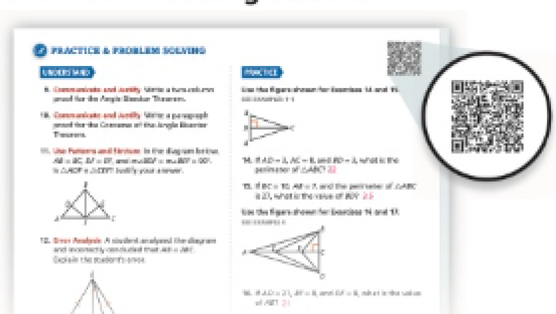


Geometric Reasoning

MA.912.GR.1 Prove and apply geometric theorems to solve problems.

MA.912.GR.1.1 Prove relationships and theorems about lines and angles. Solve mathematical and real-world problems involving postulates, relationships and theorems of lines and angles.

Clarification 1: Postulates, relationships and theorems include vertical angles are congruent; when a transversal crosses parallel lines, the consecutive angles are supplementary and alternate (interior and exterior) angles and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

Clarification 2: Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.

Clarification 3: Instruction focuses on helping a student choose a method they can use reliably.

MA.912.GR.1.2 Prove triangle congruence or similarity using Side-Side-Side, Side-Angle-Side, Angle-Side-Angle, Angle-Angle-Side, Angle-Angle and Hypotenuse-Leg.

Clarification 1: Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.

Clarification 2: Instruction focuses on helping a student choose a method they can use reliably.

MA.912.GR.1.3 Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles.

Clarification 1: Postulates, relationships and theorems include measures of interior angles of a triangle sum to 180°; measures of a set of exterior angles of a triangle sum to 360°; triangle inequality theorem; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

Clarification 2: Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.

Clarification 3: Instruction focuses on helping a student choose a method they can use reliably.

MA.912.GR.1.4 Prove relationships and theorems about parallelograms. Solve mathematical and real-world problems involving postulates, relationships and theorems of parallelograms.

Clarification 1: Postulates, relationships and theorems include opposite sides are congruent, consecutive angles are supplementary, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and rectangles are parallelograms with congruent diagonals.

Clarification 2: Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.

Clarification 3: Instruction focuses on helping a student choose a method they can use reliably.

MA.912.GR.1.5 Prove relationships and theorems about trapezoids. Solve mathematical and real-world problems involving postulates, relationships and theorems of trapezoids.

Clarification 1: Postulates, relationships and theorems include the Trapezoid Midsegment Theorem and for isosceles trapezoids: base angles are congruent, opposite angles are supplementary and diagonals are congruent.

Clarification 2: Instruction includes constructing two-column proofs, pictorial proofs, paragraph and narrative proofs, flow chart proofs or informal proofs.

Clarification 3: Instruction focuses on helping a student choose a method they can use reliably.

MA.912.GR.1.6 Solve mathematical and real-world problems involving congruence or similarity in two-dimensional figures.

Clarification 1: Instruction includes demonstrating that two-dimensional figures are congruent or similar based on given information.

MA.912.GR.2 Apply properties of transformations to describe congruence or similarity.

MA.912.GR.2.1 Given a preimage and image, describe the transformation and represent the transformation algebraically using coordinates.

Example: Given a triangle whose vertices have the coordinates (−3, 4), (2, 1.7) and (−0.4, −3). If this triangle is reflected across the *y*-axis the transformation can be described using coordinates as $(x, y) \rightarrow (-x, y)$ resulting in the image whose vertices have the coordinates (3, 4), (−2, 1.7) and (0.4, −3).

Clarification 1: Instruction includes the connection of transformations to functions that take points in the plane as inputs and give other points in the plane as outputs.

Clarification 2: Transformations include translations, dilations, rotations and reflections described using words or using coordinates.

Clarification 3: Within the Geometry course, rotations are limited to 90°, 180° and 270° counterclockwise or clockwise about the center of rotation, and the centers of rotations and dilations are limited to the origin or a point on the figure.

MA.912.GR.2.2 Identify transformations that do or do not preserve distance.

Clarification 1: Transformations include translations, dilations, rotations and reflections described using words or using coordinates.

Clarification 2: Instruction includes recognizing that these transformations preserve angle measure.

MA.912.GR.2.3 Identify a sequence of transformations that will map a given figure onto itself or onto another congruent or similar figure.

Clarification 1: Transformations include translations, dilations, rotations and reflections described using words or using coordinates.

Clarification 2: Within the Geometry course, figures are limited to triangles and quadrilaterals and rotations are limited to 90°, 180° and 270° counterclockwise or clockwise about the center of rotation.

Clarification 3: Instruction includes the understanding that when a figure is mapped onto itself using a reflection, it occurs over a line of symmetry.

MA.912.GR.2.4 Determine symmetries of reflection, symmetries of rotation and symmetries of translation of a geometric figure.

Clarification 1: Instruction includes determining the order of each symmetry.

Clarification 2: Instruction includes the connection between tessellations of the plane and symmetries of translations.

MA.912.GR.2.5 Given a geometric figure and a sequence of transformations, draw the transformed figure on a coordinate plane.

Clarification 1: Transformations include translations, dilations, rotations and reflections described using words or using coordinates.

Clarification 2: Instruction includes two or more transformations.

MA.912.GR.2.6 Apply rigid transformations to map one figure onto another to justify that the two figures are congruent.

Clarification 1: Instruction includes showing that the corresponding sides and the corresponding angles are congruent.

MA.912.GR.2.7 Justify the criteria for triangle congruence using the definition of congruence in terms of rigid transformations.

MA.912.GR.2.8 Apply an appropriate transformation to map one figure onto another to justify that the two figures are similar.

Clarification 1: Instruction includes showing that the corresponding sides are proportional, and the corresponding angles are congruent.

MA.912.GR.2.9 Justify the criteria for triangle similarity using the definition of similarity in terms of non-rigid transformations.

MA.912.GR.3 **Use coordinate geometry to solve problems or prove relationships.**

MA.912.GR.3.1 Determine the weighted average of two or more points on a line.

Clarification 1: Instruction includes using a number line and determining how changing the weights moves the weighted average of points on the number line.

MA.912.GR.3.2 Given a mathematical context, use coordinate geometry to classify or justify definitions, properties and theorems involving circles, triangles or quadrilaterals.

Example: Given Triangle ABC has vertices located at $(-2, 2)$, $(3, 3)$ and $(1, -3)$, respectively, classify the type of triangle ABC is.

Example: If a square has a diagonal with vertices $(-1, 1)$ and $(-4, -3)$, find the coordinate values of the vertices of the other diagonal and show that the two diagonals are perpendicular.

Clarification 1: Instruction includes using the distance or midpoint formulas and knowledge of slope to classify or justify definitions, properties and theorems.

MA.912.GR.3.3 Use coordinate geometry to solve mathematical and real-world geometric problems involving lines, circles, triangles and quadrilaterals.

Example: The line $x + 2y = 10$ is tangent to a circle whose center is located at $(2, -1)$. Find the tangent point and a second tangent point of a line with the same slope as the given line.

Example: Given $M(-4, 7)$ and $N(12, -1)$, find the coordinates of point P on $\overline{MN}$ so that P partitions $\overline{MN}$ in the ratio 2:3.

Clarification 1: Problems involving lines include the coordinates of a point on a line segment including the midpoint.

Clarification 2: Problems involving circles include determining points on a given circle and finding tangent lines.

Clarification 3: Problems involving triangles include median and centroid.

Clarification 4: Problems involving quadrilaterals include using parallel and perpendicular slope criteria.

MA.912.GR.3.4 Use coordinate geometry to solve mathematical and real-world problems on the coordinate plane involving perimeter or area of polygons.

Example: A new community garden has four corners. Starting at the first corner and working counterclockwise, the second corner is 200 feet east, the third corner is 150 feet north of the second corner and the fourth corner is 100 feet west of the third corner. Represent the garden in the coordinate plane, and determine how much fence is needed for the perimeter of the garden and determine the total area of the garden.

MA.912.GR.4 **Use geometric measurement and dimensions to solve problems.**

MA.912.GR.4.1 Identify the shapes of two-dimensional cross-sections of three-dimensional figures.

Clarification 1: Instruction includes the use of manipulatives and models to visualize cross-sections.

Clarification 2: Instruction focuses on cross-sections of right cylinders, right prisms, right pyramids and right cones that are parallel or perpendicular to the base.

MA.912.GR.4.2 Identify three-dimensional objects generated by rotations of two-dimensional figures.

Clarification 1: The axis of rotation must be within the same plane but outside of the given two-dimensional figure.

MA.912.GR.4.3 Extend previous understanding of scale drawings and scale factors to determine how dilations affect the area of two-dimensional figures and the surface area or volume of three-dimensional figures.

Example: Mike is having a graduation party and wants to make sure he has enough pizza. Which option would provide more pizza for his guests: one 12-inch pizza or three 6-inch pizzas?

MA.912.GR.4.4 Solve mathematical and real-world problems involving the area of two-dimensional figures.

Example: A town has 23 city blocks, each of which has dimensions of 1 quarter mile by 1 quarter mile, and there are 4500 people in the town. What is the population density of the town?

Clarification 1: Instruction includes concepts of population density based on area.

MA.912.GR.4.5 Solve mathematical and real-world problems involving the volume of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres.

Example: A cylindrical swimming pool is filled with water and has a diameter of 10 feet and height of 4 feet. If water weighs 62.4 pounds per cubic foot, what is the total weight of the water in a full tank to the nearest pound?

Clarification 1: Instruction includes concepts of density based on volume.

Clarification 2: Instruction includes using Cavalieri's Principle to give informal arguments about the formulas for the volumes of right and non-right cylinders, pyramids, prisms and cones.

MA.912.GR.4.6 Solve mathematical and real-world problems involving the surface area of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres.

MA.912.GR.5 **Make formal geometric constructions with a variety of tools and methods.**

MA.912.GR.5.1 Construct a copy of a segment or an angle.

Clarification 1: Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

MA.912.GR.5.2 Construct the bisector of a segment or an angle, including the perpendicular bisector of a line segment.

Clarification 1: Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

MA.912.GR.5.3 Construct the inscribed and circumscribed circles of a triangle.

Clarification 1: Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

MA.912.GR.5.4 Construct a regular polygon inscribed in a circle. Regular polygons are limited to triangles, quadrilaterals and hexagons.

Clarification 1: When given a circle, the center must be provided.

Clarification 2: Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

MA.912.GR.5.5 Given a point outside a circle, construct a line tangent to the circle that passes through the given point.

Clarification 1: When given a circle, the center must be provided.

Clarification 2: Instruction includes using compass and straightedge, string, reflective devices, paper folding or dynamic geometric software.

MA.912.GR.6 Use properties and theorems related to circles.

MA.912.GR.6.1 Solve mathematical and real-world problems involving the length of a secant, tangent, segment or chord in a given circle.

Clarification 1: Problems include relationships between two chords; two secants; a secant and a tangent; and the length of the tangent from a point to a circle.

MA.912.GR.6.2 Solve mathematical and real-world problems involving the measures of arcs and related angles.

Clarification 1: Within the Geometry course, problems are limited to relationships between inscribed angles; central angles; and angles formed by the following intersections: a tangent and a secant through the center, two tangents, and a chord and its perpendicular bisector.

MA.912.GR.6.3 Solve mathematical problems involving triangles and quadrilaterals inscribed in a circle.

Clarification 1: Instruction includes cases in which a triangle inscribed in a circle has a side that is the diameter.

MA.912.GR.6.4 Solve mathematical and real-world problems involving the arc length and area of a sector in a given circle.

Clarification 1: Instruction focuses on the conceptual understanding that for a given angle measure the length of the intercepted arc is proportional to the radius, and for a given radius the length of the intercepted arc is proportional is the angle measure.

MA.912.GR.6.5 Apply transformations to prove that all circles are similar.

MA.912.GR.7 Apply geometric and algebraic representations of conic sections.

MA.912.GR.7.2 Given a mathematical or real-world context, derive and create the equation of a circle using key features.

Clarification 1: Instruction includes using the Pythagorean Theorem and completing the square.

Clarification 2: Within the Geometry course, key features are limited to the radius, diameter and the center.

MA.912.GR.7.3 Graph and solve mathematical and real-world problems that are modeled with an equation of a circle. Determine and interpret key features in terms of the context.

Clarification 1: Key features are limited to domain, range, eccentricity, center and radius.

Clarification 2: Instruction includes representing the domain and range with inequality notation, interval notation or set-builder notation.

Clarification 3: Within the Geometry course, notations for domain and range are limited to inequality and set-builder.

Trigonometry

MA.912.T.1 Define and use trigonometric ratios, identities or functions to solve problems.

MA.912.T.1.1 Define trigonometric ratios for acute angles in right triangles.

Clarification 1: Instruction includes using the Pythagorean Theorem and using similar triangles to demonstrate that trigonometric ratios stay the same for similar right triangles.

Clarification 2: Within the Geometry course, instruction includes using the coordinate plane to make connections to the unit circle.

Clarification 3: Within the Geometry course, trigonometric ratios are limited to sine, cosine and tangent.

MA.912.T.1.2 Solve mathematical and real-world problems involving right triangles using trigonometric ratios and the Pythagorean Theorem.

Clarification 1: Instruction includes procedural fluency with the relationships of side lengths in special right triangles having angle measures of 30°–60°–90° and 45°–45°–90°.

MA.912.T.1.3 Apply the Law of Sines and the Law of Cosines to solve mathematical and real-world problems involving triangles.

MA.912.T.1.4 Solve mathematical problems involving finding the area of a triangle given two sides and the included angle.

Clarification 1: Problems include right triangles, heights inside of a triangle and heights outside of a triangle.

Logic and Discrete Theory

MA.912.LT.4 **Develop an understanding of the fundamentals of propositional logic, arguments and methods of proof.**

MA.912.LT.4.3 Identify and accurately interpret "if…then," "if and only if," "all" and "not" statements. Find the converse, inverse and contrapositive of a statement.

Clarification 1: Instruction focuses on recognizing the relationships between an "if…then" statement and the converse, inverse and contrapositive of that statement.

Clarification 2: Within the Geometry course, instruction focuses on the connection to proofs within the course.

MA.912.LT.4.8 Construct proofs, including proofs by contradiction.

Clarification 1: Within the Geometry course, proofs are limited to geometric statements within the course.

MA.912.LT.4.10 Judge the validity of arguments and give counterexamples to disprove statements.

Clarification 1: Within the Geometry course, instruction focuses on the connection to proofs within the course.

Mathematical Thinking and Reasoning Standards

MA.K12.MTR.1.1 Actively participate in effortful learning both individually and collectively.

Mathematicians who participate in effortful learning both individually and with others:

- Analyze the problem in a way that makes sense given the task.
- Ask questions that will help with solving the task.
- Build perseverance by modifying methods as needed while solving a challenging task.
- Stay engaged and maintain a positive mindset when working to solve tasks.
- Help and support each other when attempting a new method or approach.

MA.K12.MTR.2.1 Demonstrate understanding by representing problems in multiple ways.

Mathematicians who demonstrate understanding by representing problems in multiple ways:

- Build understanding through modeling and using manipulatives.
- Represent solutions to problems in multiple ways using objects, drawings, tables, graphs and equations.
- Progress from modeling problems with objects and drawings to using algorithms and equations.
- Express connections between concepts and representations.
- Choose a representation based on the given context or purpose.

MA.K12.MTR.3.1 Complete tasks with mathematical fluency.

Mathematicians who complete tasks with mathematical fluency:

- Select efficient and appropriate methods for solving problems within the given context.
- Maintain flexibility and accuracy while performing procedures and mental calculations.
- Complete tasks accurately and with confidence.
- Adapt procedures to apply them to a new context.
- Use feedback to improve efficiency when performing calculations.

MA.K12.MTR.4.1 Engage in discussions that reflect on the mathematical thinking of self and others.

Mathematicians who engage in discussions that reflect on the mathematical thinking of self and others:

- Communicate mathematical ideas, vocabulary and methods effectively.
- Analyze the mathematical thinking of others.
- Compare the efficiency of a method to those expressed by others.
- Recognize errors and suggest how to correctly solve the task.
- Justify results by explaining methods and processes.
- Construct possible arguments based on evidence.

MA.K12.MTR.5.1 Use patterns and structure to help understand and connect mathematical concepts.

Mathematicians who use patterns and structure to help understand and connect mathematical concepts:

- Focus on relevant details within a problem.
- Create plans and procedures to logically order events, steps or ideas to solve problems.
- Decompose a complex problem into manageable parts.
- Relate previously learned concepts to new concepts.
- Look for similarities among problems.
- Connect solutions of problems to more complicated large-scale situations.

MA.K12.MTR.6.1 Assess the reasonableness of solutions.

Mathematicians who assess the reasonableness of solutions:

- Estimate to discover possible solutions.
- Use benchmark quantities to determine if a solution makes sense.
- Check calculations when solving problems.
- Verify possible solutions by explaining the methods used.
- Evaluate results based on the given context.

MA.K12.MTR.7.1 Apply mathematics to real-world contexts.

Mathematicians who apply mathematics to real-world contexts:

- Connect mathematical concepts to everyday experiences.
- Use models and methods to understand, represent and solve problems.
- Perform investigations to gather data or determine if a method is appropriate.
- Redesign models and methods to improve accuracy or efficiency.

ELA Expectations

ELA.K12.EE.1.1 Cite evidence to explain and justify reasoning.

Clarifications:
K–1 Students include textual evidence in their oral communication with guidance and support from adults. The evidence can consist of details from the text without naming the text. During 1st grade, students learn how to incorporate the evidence in their writing.

2–3 Students include relevant textual evidence in their written and oral communication. Students should name the text when they refer to it. In 3rd grade, students should use a combination of direct and indirect citations.

4–5 Students continue with previous skills and reference comments made by speakers and peers. Students cite texts that they've directly quoted, paraphrased, or used for information. When writing, students will use the form of citation dictated by the instructor or the style guide referenced by the instructor.

6–8 Students continue with previous skills and use a style guide to create a proper citation.

9–12 Students continue with previous skills and should be aware of existing style guides and the ways in which they differ.

ELA.K12.EE.2.1 Read and comprehend grade-level complex texts proficiently.

Clarifications:
See Text Complexity for grade-level complexity bands and a text complexity rubric.

ELA.K12.EE.3.1 Make inferences to support comprehension.

Clarifications:
Students will make inferences before the words infer or inference are introduced. Kindergarten students will answer questions like "Why is the girl smiling?" or make predictions about what will happen based on the title page. Students will use the terms and apply them in 2nd grade and beyond.

ELA.K12.EE.4.1 Use appropriate collaborative techniques and active listening skills when engaging in discussions in a variety of situations.

Clarifications:
In kindergarten, students learn to listen to one another respectfully.

In grades 1–2, students build upon these skills by justifying what they are thinking. For example: "I think ________ because _______." The collaborative conversations are becoming academic conversations.

In grades 3–12, students engage in academic conversations discussing claims and justifying their reasoning, refining and applying skills. Students build on ideas, propel the conversation, and support claims and counterclaims with evidence.

ELA.K12.EE.5.1 Use the accepted rules governing a specific format to create quality work.

Clarifications:
Students will incorporate skills learned into work products to produce quality work. For students to incorporate these skills appropriately, they must receive instruction. A 3rd grade student creating a poster board display must have instruction in how to effectively present information to do quality work.

ELA.K12.EE.6.1 Use appropriate voice and tone when speaking or writing.

Clarifications:
In kindergarten and 1st grade, students learn the difference between formal and informal language. For example, the way we talk to our friends differs from the way we speak to adults. In 2nd grade and beyond, students practice appropriate social and academic language to discuss texts.

English Language Development for English Language Learners

ELD.K12.ELL.MA.1 English language learners communicate information, ideas and concepts necessary for academic success in the content area of Mathematics.

TOPIC 1

Foundations of Geometry

TOPIC 2

Parallel and Perpendicular Lines

TOPIC 3

Transformations

TOPIC 4

Triangle Congruence

TOPIC 5

Relationships in Triangles

TOPIC 6

Quadrilaterals

TOPIC 7

Similarity

TOPIC 8

Right Triangles and Trigonometry

TOPIC 9

Coordinate Geometry

TOPIC 10 Circles

TOPIC 11

Two- and Three-Dimensional Models

TOPIC
1

Foundations of Geometry

TOPIC ESSENTIAL QUESTION

What are some of the fundamentals of geometry?

Topic Overview

enVision® STEM Project
Design a Tablet

1-1 Measuring Segments and Angles
GR.1.1, MTR.4.1, MTR.1.1, MTR.3.1

1-2 Basic Constructions
GR.5.1, MTR.2.1, MTR.1.1, MTR.4.1

1-3 Midpoint and Distance
GR.3.3, GR.3.1, MTR.5.1, MTR.1.1, MTR.7.1

Mathematical Modeling in 3 Acts:
The Mystery Spokes
GR.1.1, GR.5.1, MTR.7.1

1-4 Conditional Statements
LT.4.3, LT.4.10, MTR.4.1, MTR.5.1, MTR.7.1

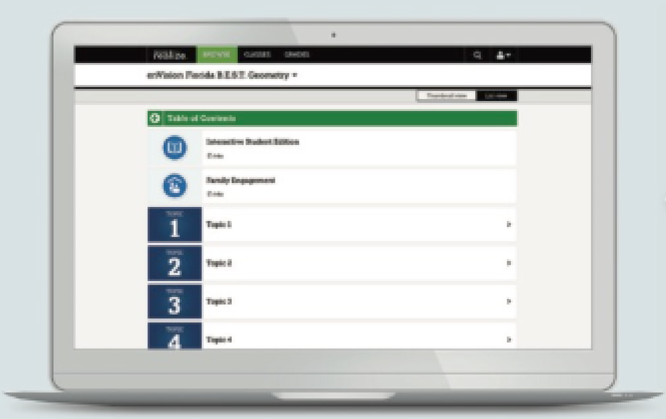
1-5 Deductive Reasoning
LT.4.3, MTR.5.1, MTR.1.1, MTR.6.1

1-6 Writing Proofs
GR.1.1, LT.4.3, MTR.4.1, MTR.5.1, MTR.6.1

1-7 Indirect Proof
LT.4.8, MTR.5.1, MTR.4.1, MTR.2.1

Topic Vocabulary

- angle bisector
- biconditional
- conditional
- conjecture
- construction
- contrapositive
- converse
- counterexample
- deductive reasoning
- inverse
- negation
- proof
- perpendicular bisector
- postulate
- theorem
- truth table
- truth value
- weighted average



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Digital Experience

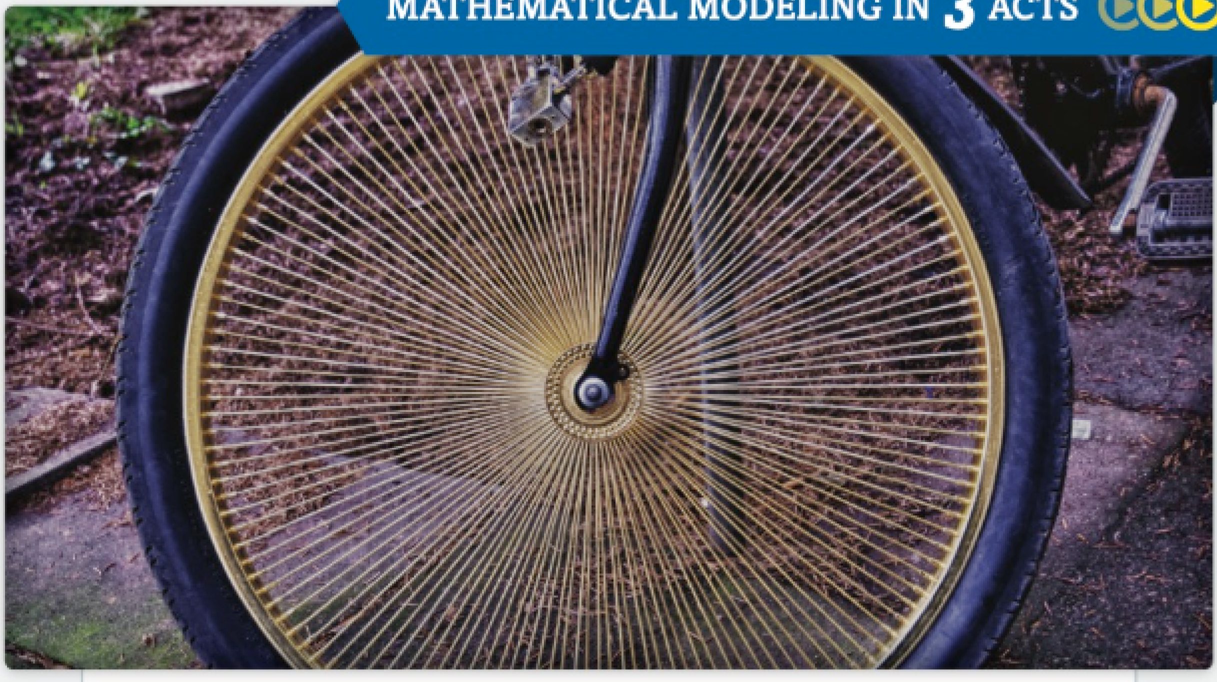
INTERACTIVE STUDENT EDITION
Access online or offline.

FAMILY ENGAGEMENT
Involve family in your learning.

ACTIVITIES Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its***.

ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.



The Mystery Spokes

Some photos are taken in such a way that it is difficult to determine exactly what the picture shows. Sometimes this is because the photo is a close up of an object, and you do not see the entire object. Other times, it might be because the photographer used special effects when taking the photo.

You can often use clues from the photo to determine what is in the photo and also what the rest of the object might look like. What clues would you look for? Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you.*

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

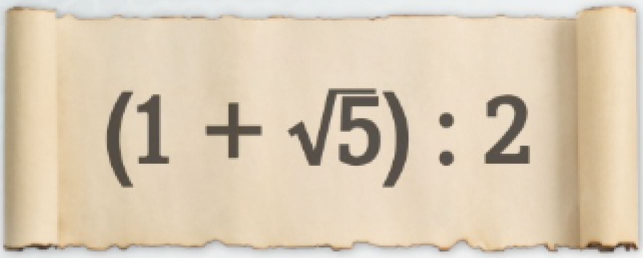
MATH TOOLS Explore math with digital tools and manipulatives.

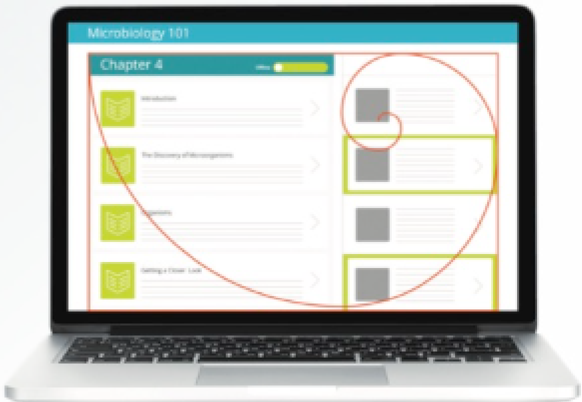
DESMOS Use Anytime and as embedded Interactives in Lesson content.

QR CODES Scan with your mobile device for ***Virtual Nerd Video Tutorials*** and ***Math Modeling Lessons***.

Did You Know?

The **golden ratio**, $(1 + \sqrt{5}) : 2$, has been explored in mathematics for over 2400 years. A golden rectangle has side lengths in the golden ratio.





Golden rectangles are used in webpage design to allocate space for content areas.

The main door of the Taj Mahal, in Agra, India, is in the shape of a **golden rectangle**.

Your Task: Design a Tablet

Tablets have become popular devices for folks who do not wish to own desktop computers. They allow people of any age to readily access their desired software and the Internet. In this project, you will design a new tablet using the golden ratio.

1-1 Measuring Segments and Angles

I CAN… use properties of segments and angles to find their measures.

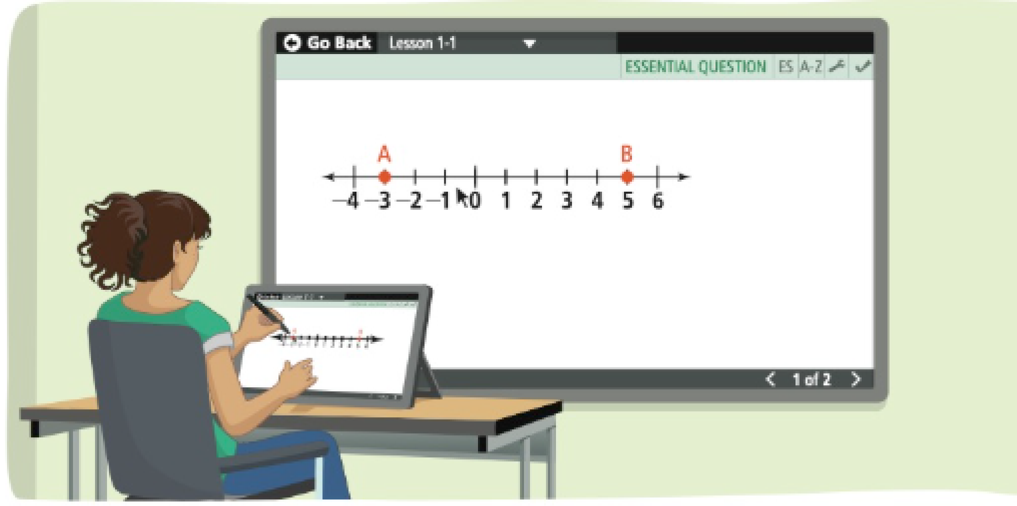
VOCABULARY

- collinear points
- line
- plane
- point
- postulate

MA.912.GR.1.1–Prove relationships and theorems about lines and angles. Solve mathematical and real-world problems involving postulates, relationships and theorems of lines and angles.
MA.K12.MTR.4.1, MTR.1.1, MTR.3.1

EXPLORE & REASON

A teacher labels two points on the number line.



A. What are some methods for finding the distance between points *A* and *B*?

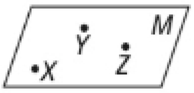
B. Communicate and Justify Which method of finding the distance is the best? Explain.

ESSENTIAL QUESTION

How are the properties of segments and angles used to determine their measures?

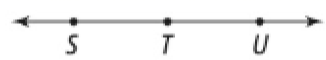
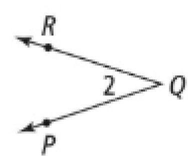
CONCEPT Undefined Terms

Undefined terms are terms whose meanings are accepted without formal definition. The terms point, line, and plane are undefined terms that are the basic building blocks of geometry.

Description	Diagram	Notation
A **point** is a location and has no size.	•*P*	*P*
A **line** is an infinite number of points on a straight path that extends in two opposite directions with no end and has no thickness.		line ℓ $\overleftrightarrow{AB}$
A **plane** is an infinite number of points and lines on a flat surface that extends without end and has no thickness.		plane *M* plane *XYZ*

CONCEPT Defined Terms

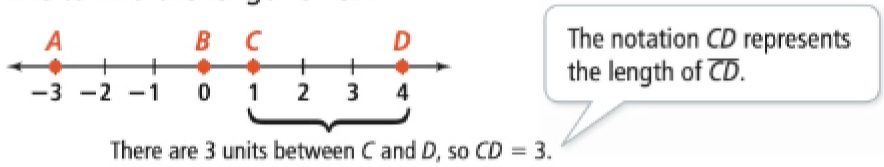
In geometry, new terms are defined using previously defined or known terms.

Description	Diagram	Notation
A segment is the part of a line that consists of two points, called *endpoints* and all points between them.		$\overline{AB}$
A ray is the part of a line that consists of one *endpoint* and all the points of the line on one side of the endpoint.		$\overrightarrow{MN}$
Opposite rays are rays with the same endpoint that lie on the same line.		$\overrightarrow{TS}$ and $\overrightarrow{TU}$
An angle is formed by two rays with the same endpoint. Each ray is a side of the angle and the common endpoint is the vertex of the angle.		$\angle Q$ $\angle PQR$ $\angle 2$

EXAMPLE 1 Find Segment Lengths

How can you find the length of $\overline{CD}$?

The length of a segment is a positive real number. You can use the number line to find the length of $\overline{CD}$.



COMMUNICATE AND JUSTIFY
Think about how notation and symbols are used. How might the notation for a segment and for the length of a segment help you remember their meaning?

To find the length of a segment, count the units of length between the endpoints. The length of $\overline{CD}$ is 3.

Try It! **1.** Refer to the figure in Example 1. How can you find the length of $\overline{AC}$?

POSTULATE 1-1 Ruler Postulate

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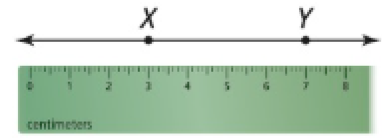
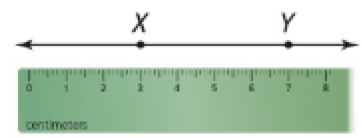
Every point on a line can be paired with a unique real number. This number is called the *coordinate* of the point.



The coordinate of X is 3.
The coordinate of Y is 7.

CONCEPT Distance on a Line

The distance between any two points X and Y is the absolute value of the difference of their coordinates.



$XY = |7 - 3| = 4$

$XY = |3 - 7| = 4$

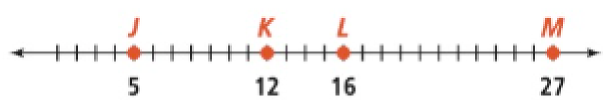
CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Find the Length of a Segment

What is *KL*?

A **postulate** is a statement that is assumed to be true.

Use the Ruler Postulate to find the coordinates of K and L.



$KL = |16 - 12| = 4$ or $KL = |12 - 16| = 4$

STUDY TIP
Remember, when finding distance between two points, take the absolute value of the difference because distance is positive.

Try It! **2.** Refer to the figure in Example 2.

a. What is JK?

b. What is KM?

POSTULATE 1-2 Segment Addition Postulate

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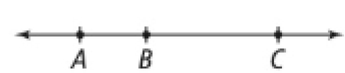
If points A, B, and C are on the same line with B between A and C, then $AB + BC = AC$.

If...



Then... $AB + BC = AC$

EXAMPLE 3 Use the Segment Addition Postulate

Points *F*, *G*, and *H* are collinear. If *GH* = 16, what is *FH*?

Collinear points lie on the same line.

Step 1 Use the expression for *GH* to find *x*.

$GH = 16$

$2x + 2 = 16$

$2x = 14$

$x = 7$

Step 2 Find *FH*.

$FH = FG + GH$

$= 5x + 1$

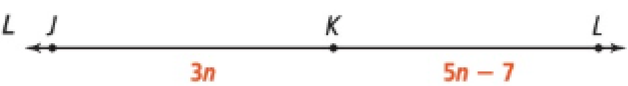
$= 5(7) + 1$

$= 36$

Apply the Segment Addition Postulate.

COMMON ERROR
Be sure to answer the question that is posed. You may state the value of the variable *x* as the answer, but you use this answer to find *FH*.

Try It! 3. Points *J*, *K*, and *L* are collinear.



a. If $JL = 25$, what is *n*?

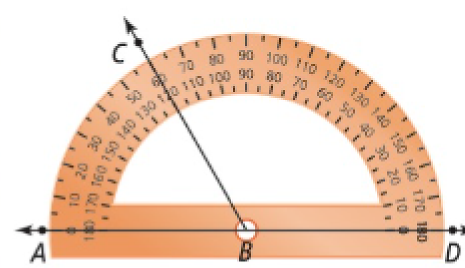
b. What is *JK*? *KL*?

POSTULATE 1-3 Protractor Postulate

Given $\overrightarrow{BA}$ and a point *C* not on $\overrightarrow{BA}$, a unique real number from 0 to 180 can be assigned to $\overrightarrow{BC}$.

0 is assigned to $\overrightarrow{BA}$.

180 is assigned to $\overrightarrow{BD}$.



EXAMPLE 4 Use the Protractor Postulate to Measure an Angle

STUDY TIP
Remember, the measure of ∠*BEC* is denoted as *m*∠*BEC*.

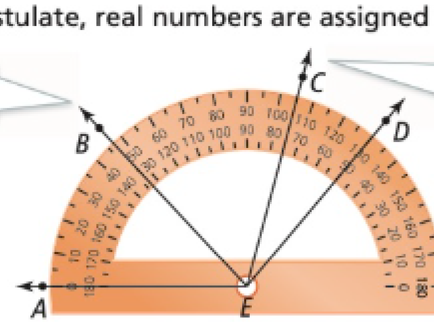
What is *m*∠*BEC*?

Since $\overrightarrow{EA}$ lines up with 0 on the top scale, use the top scale for all of the other rays in the figure.

By the Protractor Postulate, real numbers are assigned to $\overrightarrow{EB}$ and $\overrightarrow{EC}$.

47 is assigned to $\overrightarrow{EB}$.

105 is assigned to $\overrightarrow{EC}$.



You can subtract *m*∠*AEB* from *m*∠*AEC* to find *m*∠*BEC*.

$m\angle BEC = |105 - 47| = 58$

CONTINUED ON THE NEXT PAGE

Try It! **4.** Refer to the figure in Example 4.

a. What is $m\angle AEC$?

b. What is $m\angle BED$?

POSTULATE 1-4 Angle Addition Postulate

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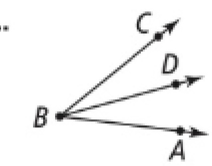
If point D is in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$.

If...

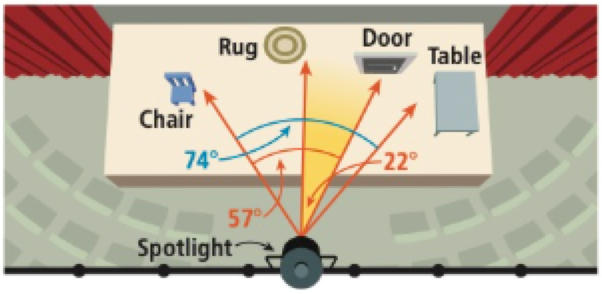


Then... $m\angle ABD + m\angle DBC = m\angle ABC$

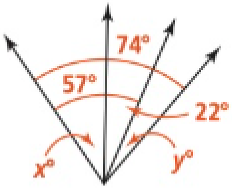
APPLICATION

EXAMPLE 5 Use the Angle Addition Postulate to Solve Problems

A lighting designer is finalizing the lighting plan for an upcoming production. The spotlight can rotate 25° to the left or right from the shown starting position. The beam of light from the spotlight forms a 22° angle. Can the designer use the spotlight to light each of the objects on the stage?



Formulate ◀ Draw and label a diagram to represent the beam angle, the angles given and the unknown angles.



Use the Angle Addition Postulate to find the angles the light must rotate to the left and right to light the chair and table.

Compute ◀ Write and solve equations to find x and y.

$x + 22 = 57$ $\qquad$ $y + 57 = 74$

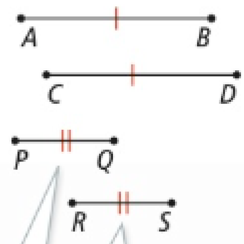
$x = 35$ $\qquad$ $y = 17$

Interpret ◀ The spotlight can rotate 25° to the right or left, so the designer can use the spotlight to light the table but cannot light the chair.

Try It! **5.** Refer to Example 5. Can the lighting designer use a spotlight with a 33° beam angle that can rotate 25° to the left and right to light all of the objects on the stage?

CONCEPT Congruent Segments and Congruent Angles

Segments that have the same length are congruent segments.

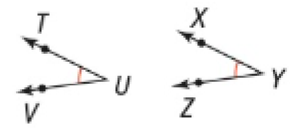


$\overline{AB} \cong \overline{CD}$

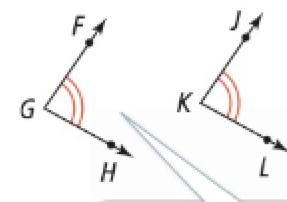
$\overline{PQ} \cong \overline{RS}$

The same number of *tick marks* shows congruent segments.

Angles that have the same measure are congruent angles.



$\angle TUV \cong \angle XYZ$



$\angle FGH \cong \angle JKL$

The same number of *arc marks* shows congruent angles.

Some properties of equality have equivalent versions for congruence. Let $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ be any line segments. Let $\angle A$, $\angle B$, and $\angle C$ be any angles.

Reflexive Property of Congruence $\overline{AB} \cong \overline{AB}$ $\quad \angle A \cong \angle A$

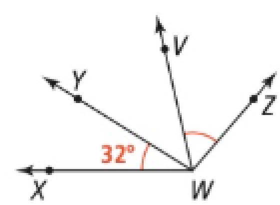
Symmetric Property of Congruence If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.
If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.

Transitive Property of Congruence If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.
If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

EXAMPLE 6 Use Congruent Angles and Congruent Segments

STUDY TIP
Be sure to read the congruency marks in the diagram carefully. Important information about a figure may be shown by using angle and segment congruence markings.

A. If $m\angle XWZ = 127$, what is $m\angle YWV$?



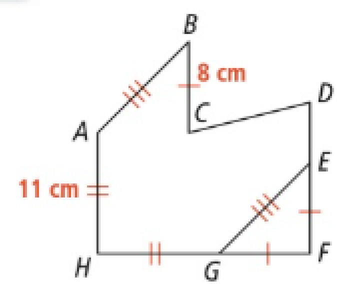
$$m\angle XWY + m\angle YWV + m\angle VWZ = m\angle XWZ$$

$$32 + m\angle YWV + 32 = 127$$

$$m\angle YWV = 63$$

Apply the Angle Addition Postulate.

B. What is *HF*?



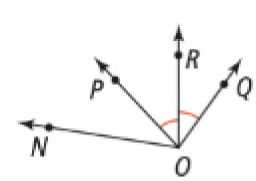
Apply the Segment Addition Postulate and substitute congruent segment lengths.

$$HF = HG + GF$$

$$HF = AH + BC$$

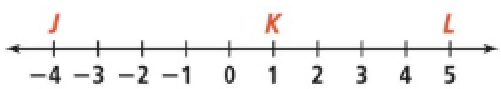
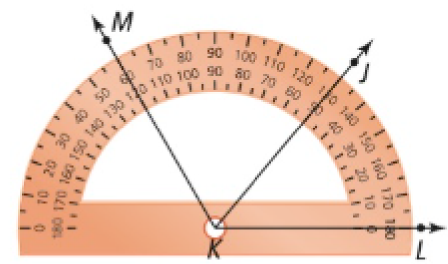
$$HF = 11 + 8 = 19 \text{ cm}$$

Try It! **6. a.** If $m\angle NOP = 31$ and $m\angle NOQ = 114$, what is $m\angle ROQ$?



b. In the figure in Part B above, suppose $CD = 11.5$ cm, $DE = 5.3$ cm, and the perimeter of *polygon ABCDEFGH* is 73.8 cm. What is *GE*?

CONCEPT SUMMARY Measuring Segments and Angles

	Ruler Postulate	Protractor Postulate
WORDS	Every point on a line can be paired with a unique real number. This number is called the *coordinate* of the point.	Given $\overrightarrow{KL}$ and a point J not on $\overrightarrow{KL}$, a unique real number from 0 to 180 can be paired with $\overrightarrow{KJ}$.
DIAGRAM		
SYMBOLS	$JK = 5$ $KL = 4$ $JL = 9$ $JK + KL = JL$	$m\angle JKL = 50$ $m\angle JKM = 70$ $m\angle JKL + m\angle JKM = 120$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the properties of segments and angles used to determine their measures?

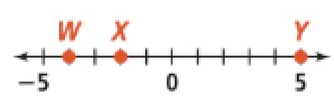
2. **Error Analysis** Ella wrote $AB = |-1 + 5| = 4$. Explain Ella's error.

3. **Vocabulary** What does it mean for segments to be congruent? What does it mean for angles to be congruent?

4. **Analyze and Persevere** Suppose M is a point in the interior of $\angle JKL$. If $m\angle MKL = 42$ and $m\angle JKL = 84$, what is $m\angle JKM$?

Do You KNOW HOW?

Find the length of each segment.

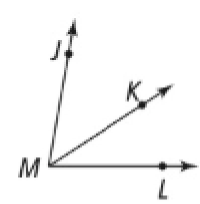


5. $\overline{WX}$

6. $\overline{WY}$

7. Points A, B, and C are collinear and B is between A and C. Given $AB = 12$ and $AC = 19$, what is BC?

8. Given $m\angle JML = 80$ and $m\angle KML = 33$, what is $m\angle JMK$?



PRACTICE & PROBLEM SOLVING

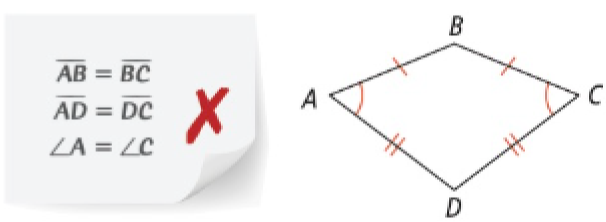
UNDERSTAND

9. Choose Efficient Methods The coordinate of point M on a number line is 11. If $MN = 12$, what are the possible coordinates for N on the number line?

10. Communicate and Justify How can you use the Segment Addition Postulate to show that $AE = AB + BC + CD + DE$?

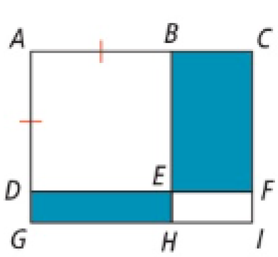
11. Higher Order Thinking If points C, D, and E are on a line and $CD = 20$ and $CE = 32$, what are the possible values of DE?

12. Error Analysis Benito wrote the equations shown about the figure. Explain Benito's errors.

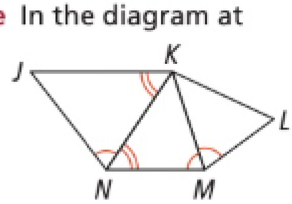


13. Analyze and Persevere Point Y is in the interior of $\angle XWZ$. Given that $\overrightarrow{WX}$ and $\overrightarrow{WZ}$ are opposite rays, and $m\angle XWY = 4(m\angle YWZ)$, what is $m\angle YWZ$?

14. Mathematical Connections The area of $ABED$ is 49 square units. Given $AG = 9$ units and $AC = 10$ units, what fraction of the area of rectangle $ACIG$ is represented by the shaded region? All figures in $ACIG$ are rectangles. Give your answer in simplest form.

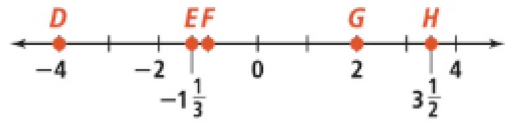


15. Use Patterns and Structure In the diagram at the right, $m\angle LMN = 116$, $m\angle JKM = 122$, and $m\angle JNM = 103$. What is $m\angle NKM$?



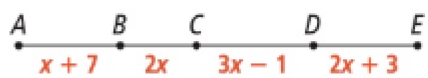
PRACTICE

Find the length of each segment. SEE EXAMPLES 1 AND 2



16. $\overline{DF}$	**17.** $\overline{DE}$	**18.** $\overline{FG}$
19. $\overline{FH}$	**20.** $\overline{GH}$	**21.** $\overline{EH}$

Points *A*, *B*, *C*, *D*, and *E* are collinear. SEE EXAMPLE 3



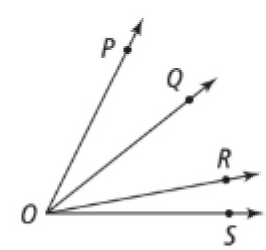
22. If $AC = 16$, what is x?

23. What is AB?

24. What is BD?

25. What is CE?

Use the figure shown for Exercises 26–28.
SEE EXAMPLES 4 AND 5

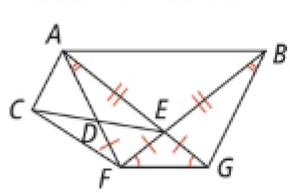


26. If $m\angle POQ = 24$ and $m\angle POR = 59$, what is $m\angle QOR$?

27. If $m\angle POQ = 19$, $m\angle QOR = 31$, and $m\angle ROS = 15$, what is $m\angle POS$?

28. If $m\angle QOS = 46$, $m\angle POR = 61$, and $m\angle POQ = 28$, what is $m\angle ROS$?

Suppose $EG = 3$, $EB = 8$, $AF = 7$, $m\angle EBG = 19$, $m\angle EGF = 28$, and $m\angle CAE = 51$. Find each value. SEE EXAMPLE 6



29. EF	**30.** AG	**31.** AD
32. $m\angle EFG$	**33.** $m\angle CAF$	**34.** DF

35. Points P, Q, R, and S are collinear. Point Q is between P and R, R is between Q and S, and $\overline{PQ} \cong \overline{RS}$. If $PS = 18$ and $PR = 15$, what is the value of QR?

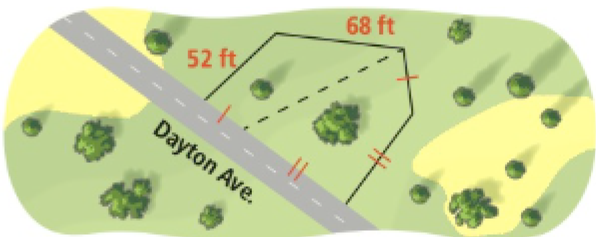
PRACTICE & PROBLEM SOLVING

APPLY

36. Analyze and Persevere Dave is driving to Gilmore to visit his friend. If he wants to stop for lunch when he is about halfway there, in which town should he plan to stop? Explain.

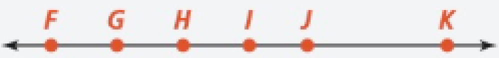
37. Apply Math Models A city planning commission must determine whether to approve the construction of a new building. The company wants to build in an area of the city that has a height limitation of 310 feet. The plans show that the first floor of the building is 20 ft high and each of the next 15 floors have a height of 11 ft, including the space between each floor needed for electrical, plumbing, and other systems. If the plan meets all other city code requirements, should the city commission approve the building plan? Explain.

38. Choose Efficient Methods The city planning committee wants one tree planted every 20 ft along Dayton Avenue. If the perimeter of the plot of land is 234 ft, about how many trees will be planted? Explain.



ASSESSMENT PRACTICE

39. In the diagram, $FH = 2FG$, $GH = HI$, and $FI = IK$. Which of the following statements must be true? Select all that apply. GR.1.1



- ☐ **A.** $FG = HI$
- ☐ **B.** $HI = IJ$
- ☐ **C.** $IK = 3FG$
- ☐ **D.** $FH = GI$
- ☐ **E.** $HJ = JK$
- ☐ **F.** $HK = 2GI$

40. SAT/ACT Point C is in the interior of $\angle ABD$, and $\angle ABC \cong \angle CBD$. If $m\angle ABC = \left(\frac{5}{2}x + 18\right)$ and $m\angle CBD = (4x)$, what is $m\angle ABD$?

Ⓐ 12 Ⓑ 36 Ⓒ 48 Ⓓ 72 Ⓔ 96

41. Performance Task The American Institute of Architects is located in a historical building called "The Octagon" in Washington, DC. Octagonal houses became popular in the United States in the mid-1800s.

Part A Design your own plan for one floor of an octagonal-shaped house. Your plan should include at least four rooms, two walls of equal length, and two angles with equal measure. Draw your floor plan using the scale 1 cm = 1 m. Write the measures of the angles and lengths of the walls on your plan, and use appropriate marks to show congruent angles and segments. Label all the points in your diagram where the walls intersect.

Part B Write equations that show congruent angles and segments in your plan.

1-2 Basic Constructions

I CAN… use a straightedge and compass to construct basic figures.

VOCABULARY

- angle bisector
- construction
- perpendicular bisector

MA.912.GR.5.1–Construct a copy of a segment or an angle.
Also GR.5.2
MA.K12.MTR.2.1, MTR.1.1, MTR.4.1

EXPLORE & REASON

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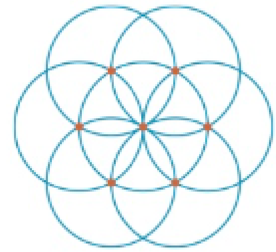
Using a compass, make a design using only circles like the one shown.



A. What instructions can you give to another student so they can make a copy of your design?

B. **Analyze and Persevere** Use a ruler to draw straight line segments to connect points where the circles intersect. Are any of the segments that you drew the same length? If so, why do you think they are?

ESSENTIAL QUESTION

How are a straightedge and compass used to make basic constructions?

EXAMPLE 1 Copy a Segment

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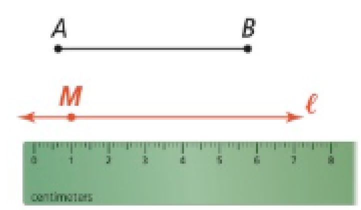
CONCEPTUAL UNDERSTANDING

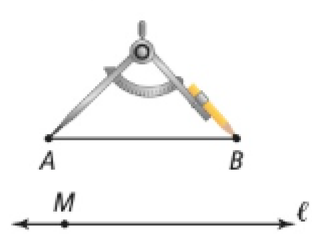
How can you copy a segment using only a straightedge and compass?

A straightedge is a tool for drawing straight lines. A compass is a tool for drawing arcs and circles of different sizes and can be used to copy lengths.

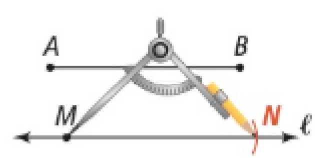
Step 1 To copy $\overline{AB}$, first use a straightedge to draw line ℓ. Mark point M on line ℓ.



Step 2 Place the compass point at A, and open the compass to length AB.



Step 3 Using the same setting, place the compass point at M, and draw an arc through line ℓ. Mark point N at the intersection.



STUDY TIP
Remember, with constructions, only use a ruler as a straightedge, not as a measuring tool.

The constructed segment MN is a copy of $\overline{AB}$. A copy of a line segment is a type of *construction*. A **construction** is a geometric figure made with only a straightedge and compass.

Try It! **1.** How can you construct a copy of $\overline{XY}$?

EXAMPLE 2 Copy an Angle

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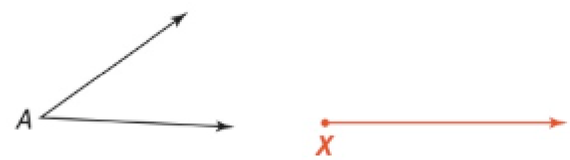
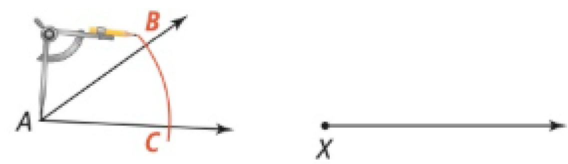
How can you construct a copy of ∠*A*?

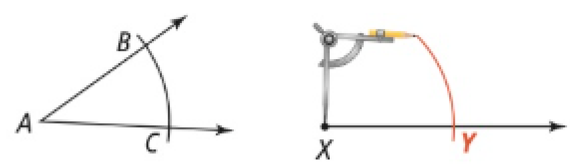
Step 1 Mark a point *X*. Use a straightedge to draw a ray with endpoint *X*.



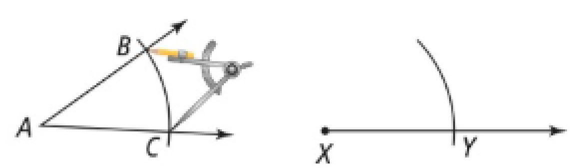
Step 2 Place the compass point at *A*. Draw an arc that intersects both rays of ∠*A*. Label the points of intersection *B* and *C*.



Step 3 Without changing the setting, place the compass point at *X* and draw an arc intersecting the ray. Mark the point *Y* at the intersection.

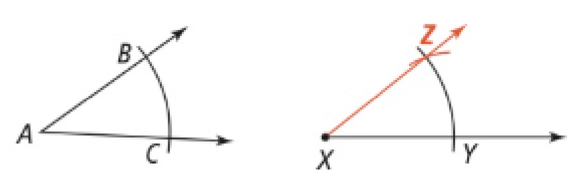


Step 4 Place the compass point at *C*, and open the compass to the distance between *B* and *C*.



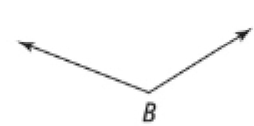
Step 5 Without changing the setting, place the compass point at *Y* and draw an arc. Label the point *Z* where the two arcs intersect. Use a straightedge to draw $\overrightarrow{XZ}$.

STUDY TIP
You can use a protractor to confirm that the two angles are congruent.



The constructed angle, ∠*YXZ*, is a copy of ∠*A*.

Try It! **2.** How can you construct a copy of ∠*B*?



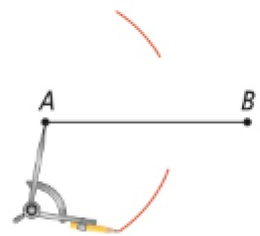
EXAMPLE 3 Construct a Perpendicular Line

A How can you construct the perpendicular bisector of $\overline{AB}$?

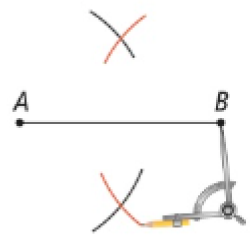
A **perpendicular bisector** of a segment is a line, segment, or ray that is perpendicular to the segment and divides the segment into two congruent segments.

You can use a straightedge and compass to construct the perpendicular bisector of a segment.

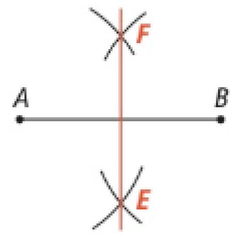
Step 1 With a setting greater than $\frac{1}{2}AB$, place the compass point at A. Draw arcs above and below $\overline{AB}$.



Step 2 With the same setting, place the compass point at B. Draw arcs above and below $\overline{AB}$.



Step 3 Label the points of intersection of the arcs E and F. Use a straightedge to draw $\overleftrightarrow{EF}$.



The constructed line, $\overleftrightarrow{EF}$, is the perpendicular bisector of $\overline{AB}$.

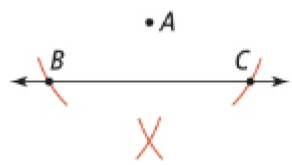
REPRESENT AND CONNECT
Consider the tools you can use to verify that a segment bisects another segment. What tool can you use?

B. How can you construct a perpendicular line from a point?

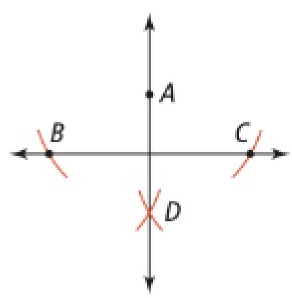
Use a straightedge and a compass to construct a line perpendicular to a given line from a point that is not on the line.

Step 1 First, with Point *A* as the center, construct an arc that intersects the line at two points. Label the points *B* and *C*.

Step 2 With the same setting, construct arcs from points *B* and *C* that intersect on the opposite side of the line.



Step 3 Label the point of intersection *D*. Use a straightedge to draw *AD*.

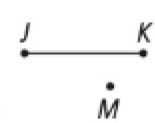


HAVE A GROWTH MINDSET
In what ways do you give your best effort and persist?

The constructed line *AD* is perpendicular to *BC*.

Try It! **3a.** How can you construct the perpendicular bisector of $\overline{JK}$?

b. How can you construct a line perpendicular to *JK* from *M*?



EXAMPLE 4 Construct an Angle Bisector

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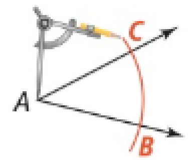
How can you construct the *angle bisector* of ∠*A*?

An **angle bisector** is a ray that divides an angle into two congruent angles. You can use a straightedge and compass to construct an angle bisector.

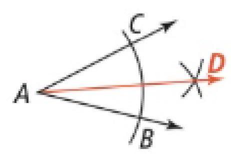
Step 1 Place the compass point at *A*. Draw an arc intersecting both rays of ∠*A*. Label the points of intersection *B* and *C*.



COMMON ERROR
Be sure to set the compass greater than $\frac{1}{2}$ the distance from *A* to *C*.

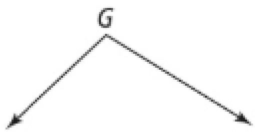
Step 2 Place the compass point at *B*. Draw an arc in the interior of ∠*A*. With the same setting, place the compass point at *C* and draw an arc intersecting the arc drawn from *B*.

Step 3 Label the point of intersection of the two arcs *D*. Use a straightedge to draw $\overrightarrow{AD}$.



The constructed ray, $\overrightarrow{AD}$, is the bisector of ∠*A*.

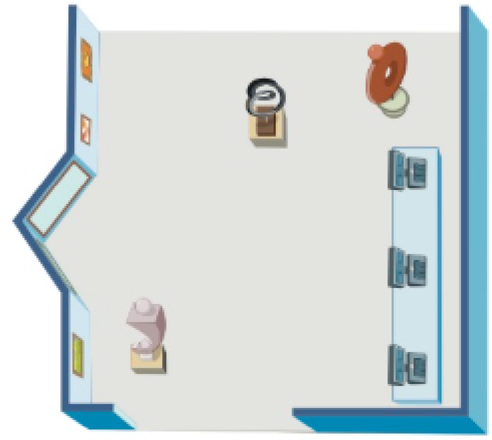
Try It! **4.** How can you construct the angle bisector of ∠*G*?



APPLICATION

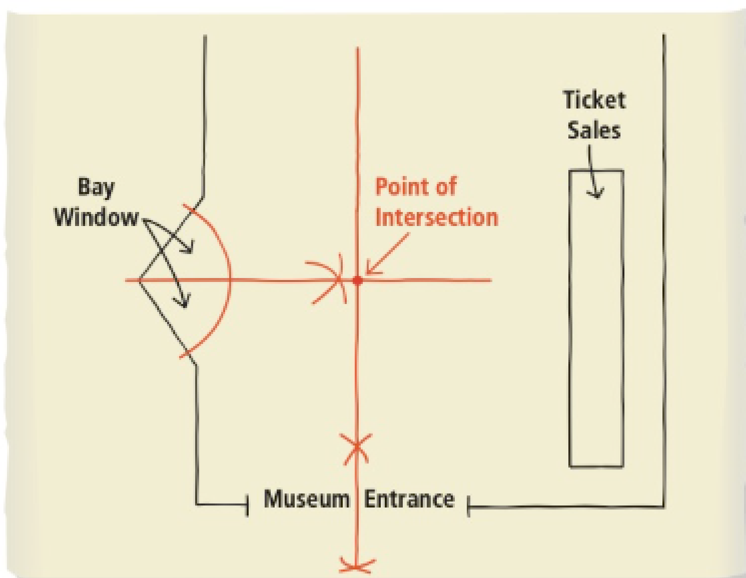
EXAMPLE 5 Use Constructions

An artist wants to center-align a new sculpture with the bay window in the museum lobby. He also wants to center-align it with the entrance. Where should the sculpture be placed?



Formulate ◀ If the sculpture is center-aligned with the bay window, it lies on the angle bisector of the bay window. If it is center-aligned with the entrance, it lies on the perpendicular bisector of the entrance.

Compute ◀ Construct the angle bisector of the bay window and the perpendicular bisector of the entrance.



Interpret ◀ The center of the sculpture should be placed at the point of intersection of the angle bisector of the bay window and the perpendicular bisector of the museum entrance.

Try It! 5. Where should the sculpture be placed if it is to be center-aligned with the museum entrance and the center of the ticket sales desk?

CONCEPT SUMMARY Constructions

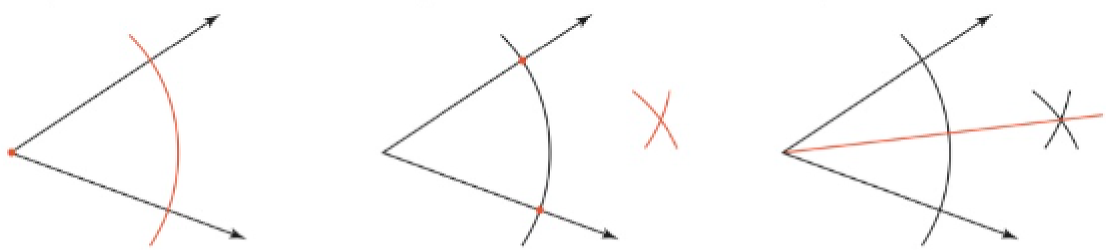
WORDS A **construction** is a geometric figure that can be made using only a straightedge and compass.

Straightedge
- is used to draw segments, lines and rays.

Compass
- is used to draw circles and arcs.
- is used to measure and copy length.

DIAGRAMS Construction of an Angle Bisector

Step 1 **Step 2** **Step 3**

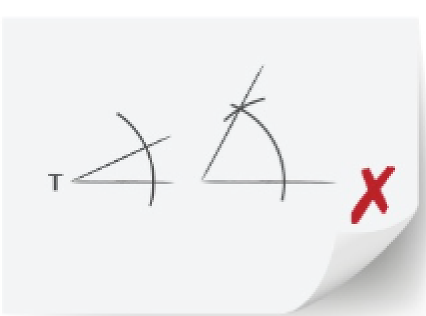


Use a compass to make arcs. Use a straightedge to draw the bisector.

Do You UNDERSTAND?

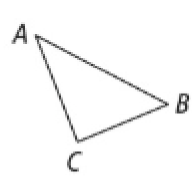
1. **ESSENTIAL QUESTION** How are a straightedge and compass used to make basic constructions?

2. **Error Analysis** Chris tries to copy $\angle T$ but is unable to make an exact copy. Explain Chris's error.



3. **Vocabulary** What is the difference between a line that is perpendicular to a segment and the perpendicular bisector of a segment?

4. **Use Patterns and Structure** Darren is copying $\triangle ABC$. First, he constructs $\overline{DE}$ as a copy of $\overline{AB}$. Next, he constructs $\angle D$ as a copy of $\angle A$, using $\overline{DE}$ as one of the sides. Explain what he needs to do to complete the copy of the triangle.



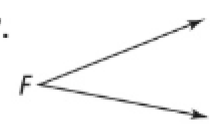
Do You KNOW HOW?

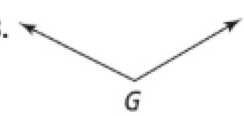
Construct a copy of each segment, and then construct its perpendicular bisector.

5. 

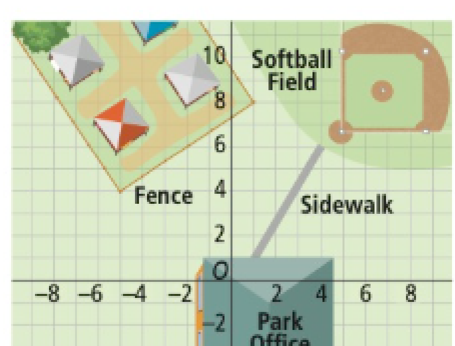
6. 

Construct a copy of each angle, and then construct its bisector.

7. 

8. 

9. A new sidewalk is perpendicular to and bisecting the existing sidewalk. At the point where new sidewalk meets the fence around the farmer's market, a gate is needed. At about what point should the gate be placed?



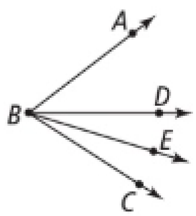
PRACTICE & PROBLEM SOLVING

UNDERSTAND

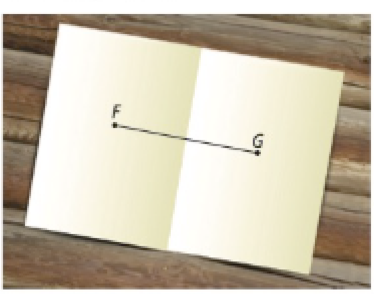
10. Represent and Connect How could you use a compass to construct a segment that is twice the length of a given segment?

11. Higher Order Thinking You can divide a segment into *n* congruent segments by bisecting segments repeatedly. What are some of the possible values of *n*? Give a rule for *n*.

12. Analyze and Persevere In the figure shown, suppose $m\angle ABC = n$ and $m\angle ABD = 2(m\angle DBC)$. The angle bisector of $\angle DBC$ is $\overrightarrow{BE}$. What is $m\angle EBC$?



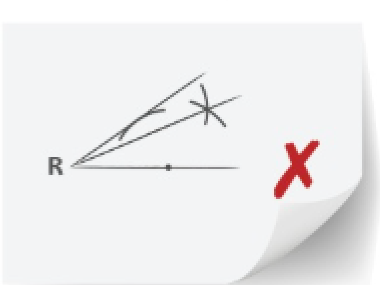
13. Analyze and Persevere There are other methods for making constructions, such as paper folding. Follow the steps to use paper folding to construct the perpendicular bisector of a segment.



- On a sheet of paper, draw $\overline{FG}$.
- Fold the paper so that *F* is on top of *G*.
- Crease the paper along the fold.
- Unfold the paper. The crease line represents the perpendicular bisector.

Why must *F* and *G* be aligned when you fold the paper?

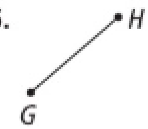
14. Error Analysis Adam is asked to construct the bisector of $\angle R$. Explain the error in Adam's work.



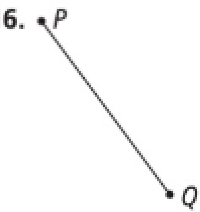
PRACTICE

Copy the segments. SEE EXAMPLE 1

15.

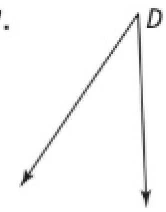


16.



Copy the angles. SEE EXAMPLE 2

17.



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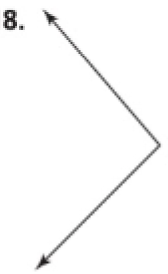
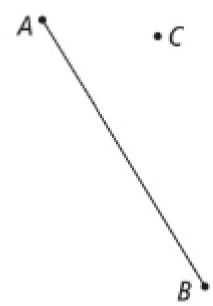


Copy and bisect the segments. Then copy each segment with the corresponding point, and draw the line that is perpendicular to the segment from the point. SEE EXAMPLE 3

19.



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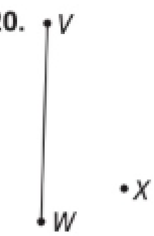
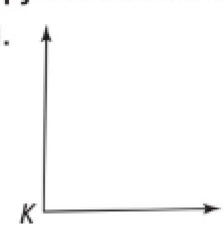


Copy and bisect the angles. SEE EXAMPLE 4

21.



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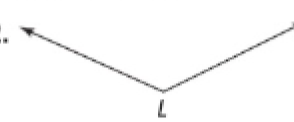
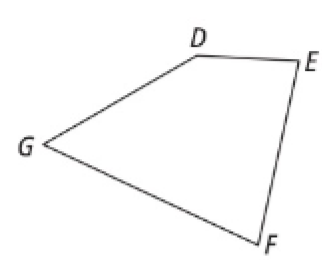


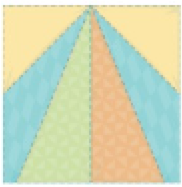
23. Where is the intersection of the perpendicular bisector of $\overline{GF}$ and the angle bisector of $\angle E$?
SEE EXAMPLE 5

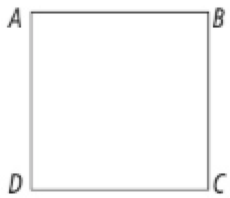


PRACTICE & PROBLEM SOLVING

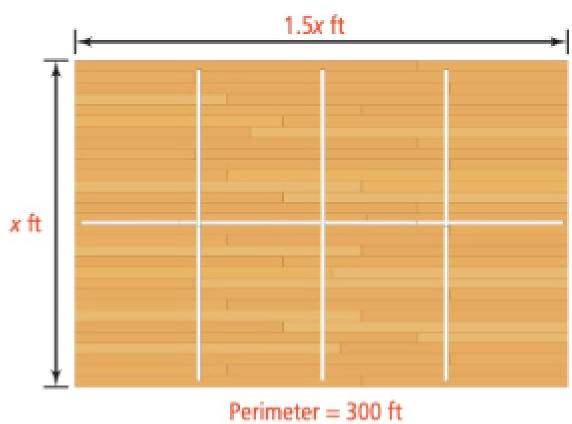
APPLY

24. Communicate and Justify The quilt block is designed from a square using only perpendicular bisectors and angle bisectors. Write instructions for constructing the pattern in square *ABCD*. You may find it helpful to name some additional points.

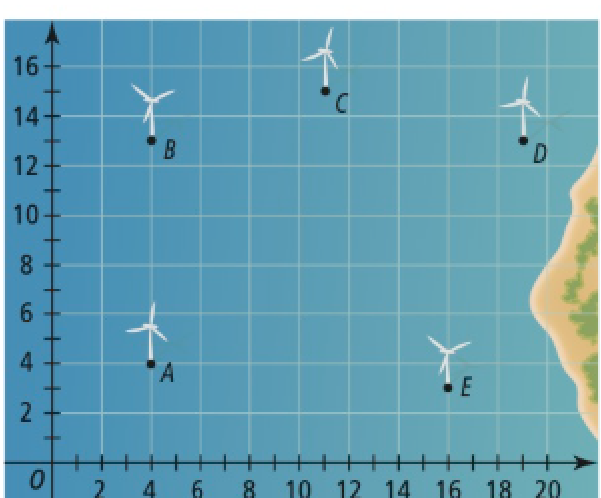




25. Mathematical Connections A school gym is divided for a fair by bisecting its width and its length. Each half of the length is then bisected, forming 8 sections in all. What are the dimensions and area of each section?

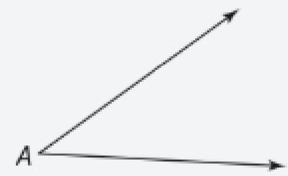


26. Apply Math Models A sixth wind turbine will be placed near the intersections of the bisector of $\angle BCD$ and the perpendicular bisectors of $\overline{AE}$ and $\overline{ED}$. What is a possible location for the sixth turbine?



ASSESSMENT PRACTICE

27. Explain the steps needed to construct the angle bisector of $\angle A$. GR.5.1



28. SAT/ACT A perpendicular bisector of $\overline{DC}$ is $\overleftrightarrow{AB}$, and a perpendicular bisector of $\overline{AB}$ is $\overline{DC}$. The intersection of $\overline{AB}$ and $\overline{DC}$ is at *E*. Which equation is true?

Ⓐ $AB = CD$

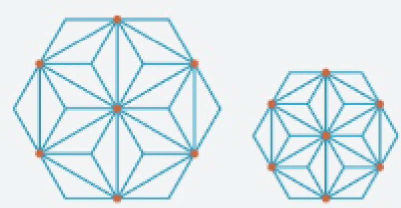
Ⓑ $CE = CD$

Ⓒ $DE = CE$

Ⓓ $AE = DE$

Ⓔ $EB = CD$

29. Performance Task Reducing or enlarging images can be useful when you need a smaller or larger version of a picture or graph for a report or poster.



Part A Use a compass and straightedge to draw a polygon with at least 3 sides.

Part B Make a reduced version of your figure with sides that are half the length of the original figure. First, select one of the sides, bisect it, and then copy one of the halves. Next, copy one of the angles that is adjacent. Repeat until you have a reduced version of your figure.

Part C Think about how you can double the length of the line segment. Make an enlarged version of your figure with sides that are twice the length of the original figure. Describe how you made the enlarged figure.

1-3
Midpoint and Distance

I CAN… use the midpoint and distance formulas to solve problems.

VOCABULARY
- midpoint
- weighted average

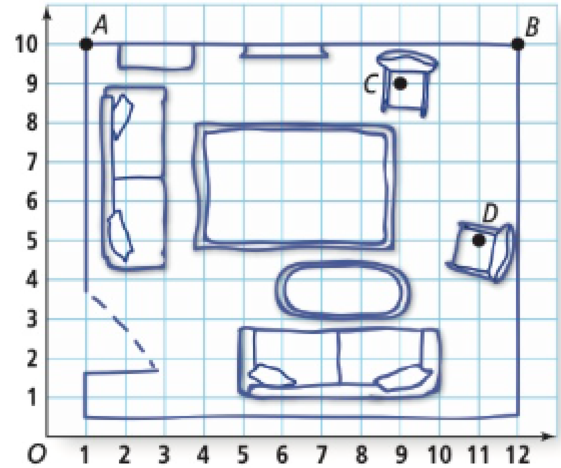
MA.912.GR.3.3–Use coordinate geometry to solve mathematical and real-world geometric problems involving lines, circles, triangles and quadrilaterals.
MA.K12.MTR.5.1, MTR.1.1, MTR.7.1

MODEL & DISCUSS

LaTanya is decorating her living room and draws a floorplan to help look at placement.

A. LaTanya wants to hang a picture at the center of the back wall. How do you find the point at the center between *A* and *B*?

B. Analyze and Persevere
LaTanya wants to place a lamp halfway between the chairs at points *C* and *D*. How can you find the point where the lamp should go?



ESSENTIAL QUESTION

How are the midpoint and length of a segment on the coordinate plane determined?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Determine the Weighted Average of Two Points

What number *w* is three times as close to 2 as it is to 10?

The number *w* is a *weighted average*. A **weighted average** is an average that favors numbers, or points, with greater weight. Each number, or point, has a weight, and the sum of weights is 1.

Step 1 Find the weights at 2 and 10.

Let x be the weight at 10.

$$(\text{weight at 2}) + (\text{weight at 10}) = 1$$
$$3x + x = 1$$
$$x = \frac{1}{4}$$

The weight at 2 is three times greater than the weight at 10.

The weight at 2 is $\frac{3}{4}$, and the weight at 10 is $\frac{1}{4}$.

Step 2 Find the weighted average, w.

$$w = \frac{3}{4}(2) + \frac{1}{4}(10)$$
$$= \frac{3}{2} + \frac{5}{2}$$
$$= 4$$

Multiply 2 and 10 by their weights.

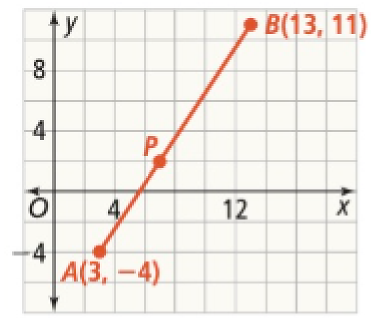
The weighted average is 4, so 4 is three times as close to 2 as it is 10.

CHECK FOR REASONABLENESS
How can you verify that 4 is 3 times as close to 2 as it is to 10?

Try It! **1.** What number n is 4 times as close to 10 than 2?

EXAMPLE 2 Use Weighted Average to Partition a Segment

A. What are the coordinates of point *P* that partitions $\overline{AB}$ in the ratio 2:3?



You can find *P* by finding the weighted average of *A* and *B*.

The weights $\frac{2}{5}$ and $\frac{3}{5}$ are in a ratio 2:3. Since *P* is closer to *A*, point *A* is the larger weight, $\frac{3}{5}$.

Find point *P*.

$$P = \frac{3}{5}A + \frac{2}{5}B$$
$$= \frac{3}{5}(3, -4) + \frac{2}{5}(13, 11)$$

Multiply the coordinates of *A* and *B* by their weight.

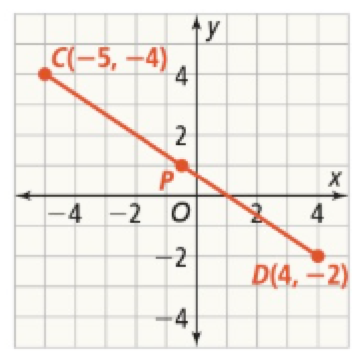
$$= \left(\frac{9}{5}, \frac{-12}{5}\right) + \left(\frac{26}{5}, \frac{22}{5}\right)$$
$$= \left(\frac{9 + 26}{5}, \frac{-12 + 22}{5}\right)$$

Add the *x*-coordinates and *y*-coordinates.

$$= \left(\frac{35}{5}, \frac{10}{5}\right)$$
$$= 7, 2$$

The coordinates at point *P* are 7, 2.

B. What are the coordinates of *P* that divides $\overline{CD}$ into two congruent segments?



Since $\overline{CP} \cong \overline{DP}$, *P* partitions $\overline{CD}$ in the ratio of 1:1. So, *P* is the weighted average of $\overline{CD}$ with equal weights of $\frac{1}{2}$ at both *C* and *D*.

Find point *P*.

$$P = \frac{1}{2}C + \frac{1}{2}D$$
$$= \frac{1}{2}(-5, -4) + \frac{1}{2}(4, -2)$$

Multiply the coordinates of *C* and *D* by their weight.

$$= \left(\frac{-5}{2}, \frac{-4}{2}\right) + \left(\frac{4}{2}, \frac{-2}{2}\right)$$
$$= \left(\frac{-5 + 4}{2}, \frac{-4 - 2}{2}\right)$$
$$= \left(-\frac{1}{2}, -3\right)$$

Add the *x*-coordinates and the *y*-coordinates.

Therefore, $P\left(-\frac{1}{2}, -3\right)$ divides $\overline{CD}$ into congruent segments. We say that *P* is the *midpoint* of $\overline{CD}$.

Try It! **2.** For *E*(−3, −6) and *F*(−8, 4), what are the coordinates of *P* that partitions $\overline{EF}$ in the ratio of 1:4?

CONCEPT Midpoint Formula

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A **midpoint** of a segment is the point that divides the segment into two congruent segments. The midpoint of $\overline{PQ}$ with $P(x_1, y_1)$ and $Q(x_2, y_2)$, is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

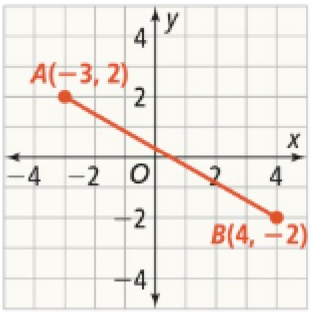
EXAMPLE 3 Find a Midpoint

What is the midpoint of $\overline{AB}$?

Substitute the coordinates of the endpoints of $\overline{AB}$ into the Midpoint Formula.

$$M = \left(\frac{-3 + 4}{2}, \frac{2 + (-2)}{2}\right)$$
$$= \left(\frac{1}{2}, 0\right)$$

The midpoint of $\overline{AB}$ is $\left(\frac{1}{2}, 0\right)$.



COMMON ERROR
Finding the midpoint is like finding the average of the x-coordinates and the y-coordinates, so be sure to add the coordinates before dividing by 2.

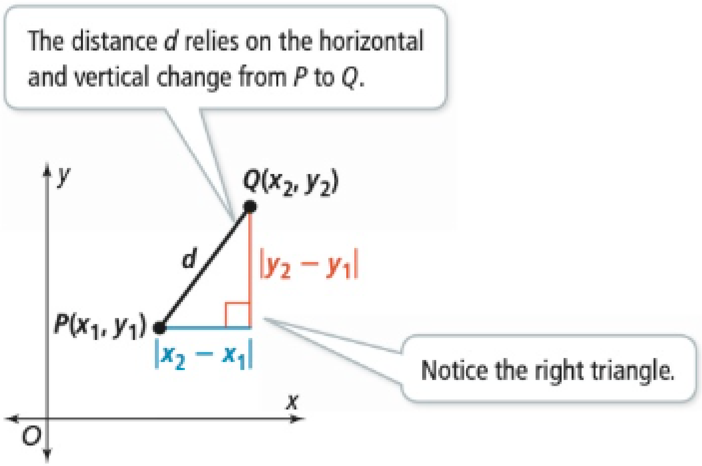
Try It! 3. Find the midpoint for each segment with the given endpoints.

a. $C(-2, 5)$ and $D(8, -12)$ b. $E(2.5, -7)$ and $F(-6.2, -3.8)$

CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Derive the Distance Formula

How can you find the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the coordinate plane?



$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

Apply the Pythagorean Theorem, $c^2 = a^2 + b^2$.

The length of $\overline{PQ}$ is the distance between points P and Q, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

STUDY TIP
Recall that the square of any quantity is always nonnegative, so absolute value bars are not needed when an expression is squared.

Try It! 4. Tavon claims that $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ can also be used to find distance between two points. Is he correct? Explain.

CONCEPT Distance Formula

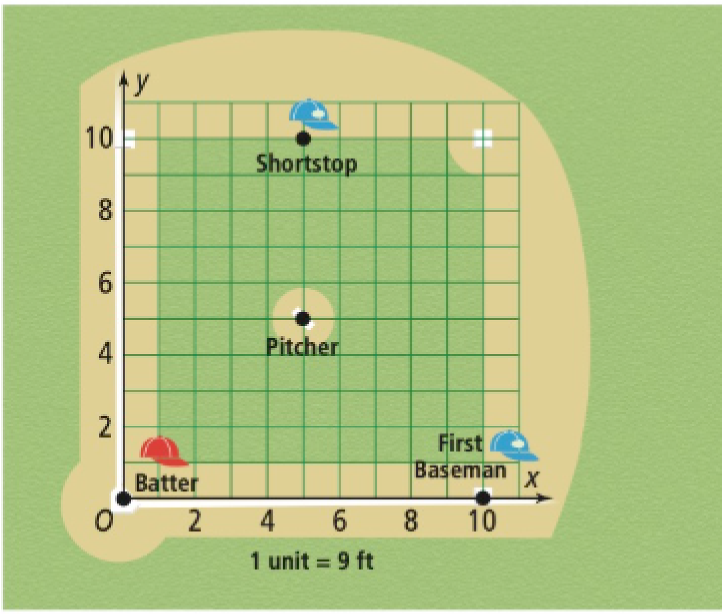
The distance d between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is:

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

APPLICATION

EXAMPLE 5 Find the Distance

A pitcher throws a ball to a batter, who hits the ball to the shortstop. If the ball travels in a straight line between each, what is the total distance traveled by the ball? Round your answer to the nearest tenth of a foot.



Formulate: Represent the pitcher at point $P(5, 5)$, the batter at point $B(0, 0)$, and the shortstop at point $S(5, 10)$.

Compute: Use the Distance Formula to find each distance.

$$\begin{aligned} d\text{ (pitcher to batter)} &= \sqrt{(0-5)^2 + (0-5)^2} \\ &= \sqrt{(-5)^2 + (-5)^2} \\ &= \sqrt{25 + 25} \\ &= \sqrt{50} \\ &\approx 7.1 \end{aligned}$$

Use the Distance Formula with $P(5, 5)$ and $B(0, 0)$.

$$\begin{aligned} d\text{ (batter to shortstop)} &= \sqrt{(5-0)^2 + (10-0)^2} \\ &= \sqrt{5^2 + 10^2} \\ &= \sqrt{25 + 100} \\ &= \sqrt{125} \\ &\approx 11.2 \end{aligned}$$

Use the Distance Formula with $B(0, 0)$ and $S(5, 10)$.

Interpret: The total distance the ball traveled is about $7.1 + 11.2 = 18.3$ units, or about $(18.3)(9) = 164.7$ ft.

Try It! **5.** How far does the shortstop need to throw the ball to reach the first baseman? Round your answer to the nearest tenth of a foot.

CONCEPT SUMMARY Midpoint and Distance on the Coordinate Plane

MIDPOINT

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

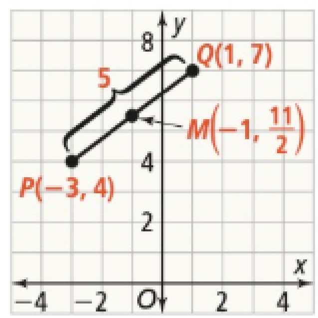
DISTANCE

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE

The endpoints of $\overline{PQ}$ are $P(-3, 4)$ and $Q(1, 7)$.

$$M = \left(\frac{-3 + 1}{2}, \frac{4 + 7}{2}\right)$$
$$= \left(-1, \frac{11}{2}\right)$$



$$d = \sqrt{(-3 - 1)^2 + (4 - 7)^2}$$
$$= \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{25}$$
$$= 5$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the midpoint and length of a segment on the coordinate plane determined?

2. **Error Analysis** Corey calculated the midpoint of $\overline{AB}$ with $A(-3, 5)$ and $B(1, 7)$. What is Corey's error?

$M\left(\frac{-3 + 5}{2}, \frac{1 + 7}{2}\right)$

$M(1, 4)$ ✗

3. **Vocabulary** If M is the midpoint of $\overline{PQ}$, what is the relationship between PM and MQ? Between PM and PQ?

4. **Generalize** Is it possible for $\overline{PQ}$ to have two distinct midpoints, $M_1(a, b)$ and $M_2(c, d)$? Explain.

Do You KNOW HOW?

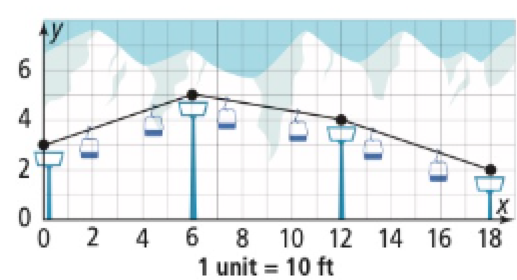
$\overline{PQ}$ has endpoints at $P(-5, 4)$ and $Q(7, -5)$.

5. What is the midpoint of $\overline{PQ}$?

6. What are the coordinates of the point $\frac{2}{3}$ of the way from P to Q?

7. What is the length of $\overline{PQ}$?

8. A chair lift at a ski resort travels along the cable as shown.



How long is the cable? Round your answer to the nearest whole foot.

PRACTICE & PROBLEM SOLVING

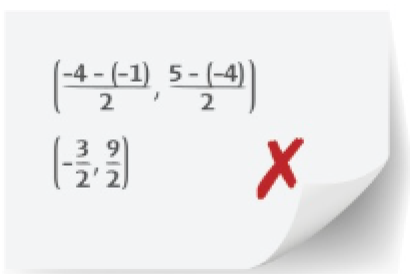
UNDERSTAND

9. **Use Patterns and Structure** Point K is $\frac{1}{n}$ of the way from $J(4, -5)$ to $L(0, -7)$.

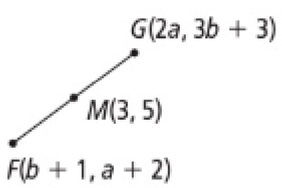
a. What are the coordinates of K if $n = 4$?

b. What is a formula for the coordinates of K for any n?

10. **Error Analysis** Describe and correct the error a student made in finding the midpoint of $\overline{CD}$ with $C(-4, 5)$ and $D(-1, -4)$.



11. **Mathematical Connections** Point M is the midpoint of $\overline{FG}$. Can you determine the value of a? Explain.



12. **Analyze and Persevere** Suppose $\overline{PQ}$ has one endpoint at $P(0, 0)$.

a. If $(2, 5)$ is the midpoint of $\overline{PQ}$, what are the coordinates of point Q?

b. How would you find Q if $(2, 5)$ is $\frac{1}{4}$ of the way from P to Q?

13. **Higher Order Thinking** $\overline{PQ}$ has a length of 17 units with $P(-4, 7)$. If the x- and y-coordinates of Q are both greater than the x- and y-coordinates of P, what are possible integer value coordinates of Q? Explain.

14. **Analyze and Persevere** Suppose $\overline{PQ}$ has $P(a, b)$ and midpoint $M(c, d)$. What is an expression for PM? Use the expression for PM to find an expression for PQ.

15. The endpoints of $\overline{RS}$ are $R(-4, 3)$ and $S(8, -5)$. Complete each statement using a fraction.

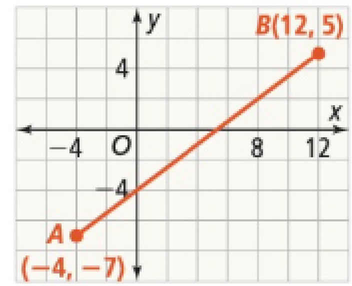
a. $(-1, 1)$ is the point ■ of the way from R to S.

b. $(5, -3)$ is the point ■ of the way from R to S.

PRACTICE

Find the coordinates of each given point

SEE EXAMPLES 1 AND 2



16. The point that partitions $\overline{AB}$ in the ratio 3:7.

17. The point that is 3 times as close to A as it is to B.

18. Point B partitions $\overline{AC}$ in the ratio 1:3. What are the coordinates of C?

Find the midpoint of $\overline{PQ}$. SEE EXAMPLE 3

19. $P(3, 5)$, $Q(-2, 13)$

20. $P(-2, 2.5)$, $Q(1.4, 4)$

21. $P\left(4\frac{1}{3}, 3\frac{1}{6}\right)$, $Q\left(-2\frac{1}{5}, 3\frac{2}{3}\right)$

Cameron, Arthur, and Jamie are playing soccer. Their locations are recorded by a motion tracking system. The distance between grids is 5 meters.

SEE EXAMPLES 4 AND 5

22. How far apart are Arthur and Jamie? Round to the nearest tenth of a meter.

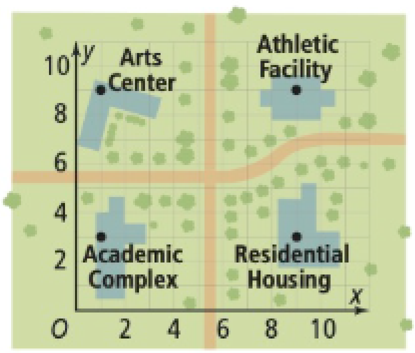
23. Who is closer to Cameron? Explain.

24. The soccer ball is located at the point $(35, 60)$. Who is closest to the soccer ball?

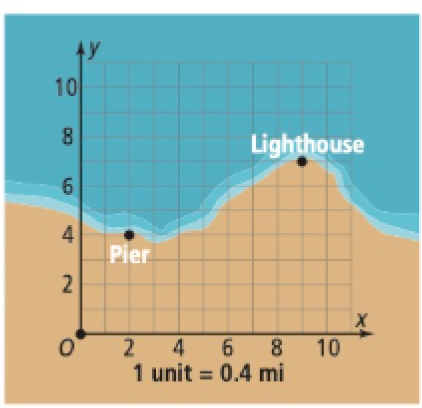
PRACTICE & PROBLEM SOLVING

APPLY

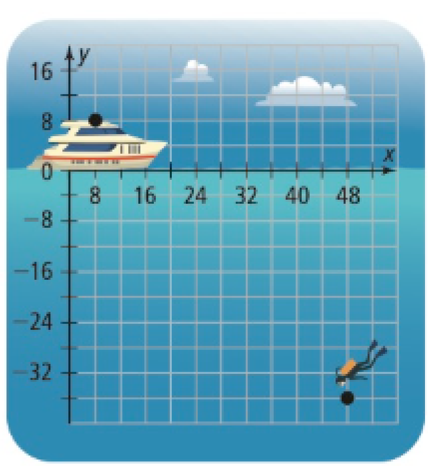
25. Apply Math Models A university is building a new student center that is two-thirds the distance from the arts center to the residential complex. What are the coordinates of the new center? Explain.



26. Mathematical Connections A lighthouse casts a revolving beam of light as far as the pier. What is the area that the light covers?



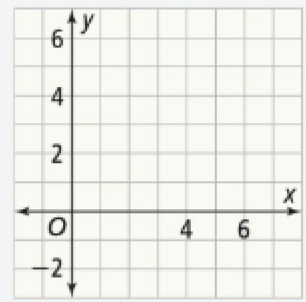
27. Analyze and Persevere A ship captain is attempting to contact a deep sea diver.



The map is in meters. If the maximum range for communication is 60 meters, will he be able to communicate with the diver based on their current positions? Explain.

ASSESSMENT PRACTICE

28. Copy the graph and plot a point C that partitions the segment into the ratio $AC : AB = \frac{3}{4}$. GR.3.3



29. SAT/ACT $\overline{RS}$ has an endpoint at $R(6, -4)$ and length 17. Which of the following cannot be the coordinates of S?

Ⓐ (14, 11)

Ⓑ (6, 13)

Ⓒ (−9, −12)

Ⓓ (23, 13)

Ⓔ (23, −4)

30. Performance Task A parade route must start and end at the intersections shown on the map. The city requires that the total distance of the route cannot exceed 3 miles. A proposed route is shown.

Part A Why does the proposed route not meet the requirement?

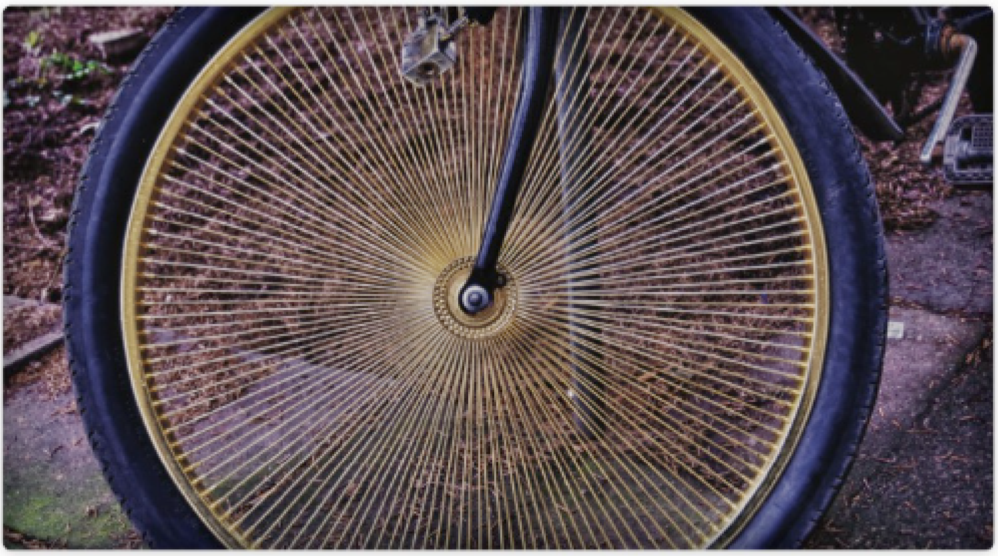
Part B Assuming that the roads used for the route are the same and the end point is the same, at what intersection could the parade start so the total distance is as close to 3 miles as possible?

Part C The city wants to station video cameras halfway down each road in the parade. Using your answer to Part B, what are the coordinates of the locations for the cameras?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.1.1–Prove relationships and theorems about lines and angles. Solve mathematical and real-world problems involving postulates, relationships and theorems of lines and angles. Also GR.5.1

MA.K12.MTR.7.1



The Mystery Spokes

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Some photos are taken in such a way that it is difficult to determine exactly what the picture shows. Sometimes it's because the photo is a close up part of an object, and you do not see the entire object. Other times, it might be because the photographer used special effects when taking the photo.

You can often use clues from the photo to determine what is in the photo and also what the rest of the object might look like. What clues would you look for? Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. Write a number that you know is too small.
6. Write a number that you know is too large.

ACT 2 Develop a Model

7. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

8. Is your refined conjecture between the highs and lows you set up earlier?
9. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

1-4 Conditional Statements

I CAN… write conditionals and biconditionals and find their truth values.

VOCABULARY

- biconditional
- conclusion
- conditional
- contrapositive
- converse
- hypothesis
- inverse
- negation
- truth table
- truth value

Prepares for **MA.912.LT.4.3**– Identify and accurately interpret "if…then," "if and only if," "all" and "not" statements. Find the converse, inverse and contrapositive of a statement. **Also LT.4.10**

MA.K12.MTR.4.1, MTR.5.1, MTR.7.1

EXPLORE & REASON

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If-then statements show a cause and effect. The table shows some if-then statements.

Cause	Effect
If it is raining	then it is spring.
If x and y are whole numbers	then their difference, $x - y$, is a whole number.
If water is heated	then it boils.
If a triangle has a right angle	then it is a right triangle.
If your favorite color is blue	then you are a good speller.

A. Communicate and Justify Determine whether each effect is always true for the given cause, or is not necessarily true for the given cause. For the effects that are not necessarily true, how could you change them to make them always true?

B. Write some if-then statements of your own. Write two statements that are always true and two statements that are not necessarily true.

ESSENTIAL QUESTION

How do if-then statements describe mathematical relationships?

CONCEPT Conditional Statement

A **conditional** is an *if-then* statement that relates a **hypothesis**, the part that follows *if*, to a **conclusion**, the part that follows *then*.

Conditionals can be represented as $p \rightarrow q$, read as "If p, then q," where p represents the hypothesis and q represents the conclusion.

EXAMPLE 1 Write a Conditional Statement

Write each statement as a conditional.

A. You can register to vote if you are at least 18 years old.

Identify the hypothesis and conclusion.

The conclusion gives the outcome or result.

The hypothesis follows "if" and gives the condition.

You can register to vote if you are at least 18 years old.

Conditional: If you are at least 18 years old, then you can register to vote.

COMMON ERROR
Remember that in everyday language, the hypothesis does not necessarily come before the conclusion.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

B. A square must have four congruent sides.

Identify the hypothesis and conclusion.

The hypothesis is that a polygon is a square.

The conclusion is that the polygon has four congruent sides.

A square must have four congruent sides.

Conditional: If a polygon is a square, then it has four congruent sides.

Try It! **1.** Write each statement as a conditional.

a. A triangle with all angles congruent is equilateral.

b. Alberto can go to the movies if he washes the car.

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Find a Truth Value of a Conditional

The truth value of a statement is "true" (T) or "false" (F) according to whether the statement is true or false, respectively. A truth table lists all the possible combinations of truth values for two or more statements.

Truth Table for $p \rightarrow q$

Hypothesis p	Conclusion q	Conditional $p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional with a false hypothesis has a value of true, regardless of the conclusion.

Only a conditional with a true hypothesis and a false conclusion has a value of false.

How can you determine the truth value of each conditional?

A. If a number is even, then it is divisible by 2.

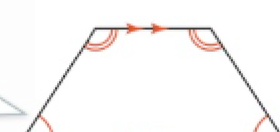
An even number is always divisible by two, so when the hypothesis is true, the conclusion is always true.

The conditional is true.

B. If a quadrilateral has two pairs of congruent angles, then it is a parallelogram.

Assume the hypothesis, a quadrilateral that has two pairs of congruent angles, is true. To decide whether the conclusion is true, determine whether the quadrilateral must be a parallelogram.

An isosceles trapezoid has two pairs of congruent angles, but is not a parallelogram. The conclusion is false.



In this example, the hypothesis of the conditional is true and the conclusion is false, so this conditional is false.

ANALYZE AND PERSEVERE

To determine the truth value of a conditional, consider all of the options for the hypothesis and for the conclusion. For example, assume the hypothesis is true, then determine whether the conclusion must also always be true.

CONTINUED ON THE NEXT PAGE

Try It! **2.** What is the truth value of each conditional? Explain your reasoning.

a. If a quadrilateral has a right angle, then it is a rectangle.

b. If X is the midpoint of $\overline{AB}$, then X lies on $\overline{AB}$.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Find a Counterexample to Show a Conjecture is False

How can a *counterexample* show that a conjecture is false?

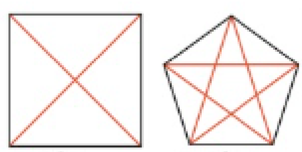
Conjecture: If a figure is a polygon then there are two fewer diagonals as sides.

A **conjecture** is an unproven conditional statement.

To find a counterexample, you must find a polygon that has a number of diagonals that is not two fewer than the number of its sides.

A **counterexample** is an example that shows a statement or conjecture is false.

CHECK FOR REASONABLENESS
For a conjecture to be true, it must be true for every possible case, so if a counterexample is found, the conjecture is false.



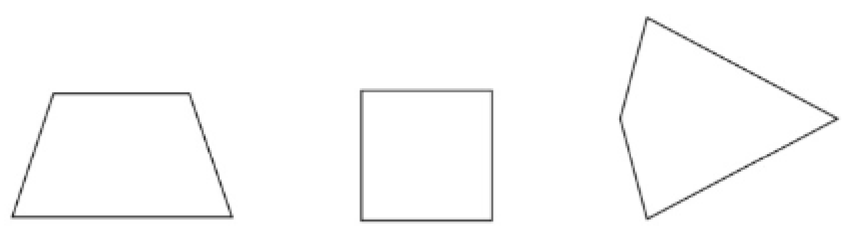
4 sides
2 diagonals

5 sides
5 diagonals

You only need to find one counterexample to show that a statement is false. A counterexample exists, so the conjecture is false.

B. Can you find a counterexample that shows this conjecture is false?

Conjecture: If a figure is a convex quadrilateral, then at least one angle is greater than or equal to 90°.



The figure has two obtuse angles.

The figure has four right angles.

The figure has one obtuse angle.

COMMUNICATE AND JUSTIFY`
Why is a lack of a counterexample not enough to conclude that the conjecture is true?

None of the figures are a counterexample to the conjecture. However, just because a couneterexample cannot be found, you cannot conclude that the conjecture is true.

Try It! **3.** What is a counterexample that shows the statement, *the sum of two composite numbers must be a composite number,* is false?

CONCEPT Related Conditional Statements

Definition	Symbols	Words
A conditional has a hypothesis and a conclusion.	$p \to q$	If *p*, then *q*.
The **converse** reverses the hypothesis and the conclusion of a conditional.	$q \to p$	If *q*, then *p*.
The **negation** of a statement has the opposite meaning of the original statement.	$\sim p$	not *p*
The **inverse** is obtained by negating both the hypothesis and the conclusion of a conditional.	$\sim p \to \sim q$	If not *p*, then not *q*.
The **contrapositive** is obtained by negating and reversing both the hypothesis and the conclusion of a conditional.	$\sim q \to \sim p$	If not *q*, then not *p*.

EXAMPLE 4 Write and Evaluate the Truth Value of a Converse

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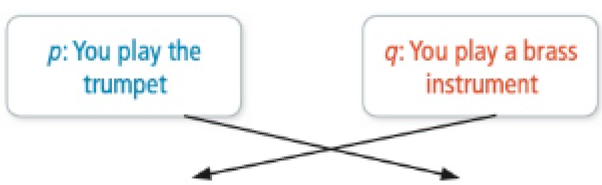
Write and determine the truth value of the converse of the conditional.

If you play the trumpet, then you play a brass instrument.

To write the converse, reverse the hypothesis and conclusion.



If you play a brass instrument, then you play the trumpet.

If you play a brass instrument, then you may play a brass instrument that is not a trumpet. The converse is false.

STUDY TIP
To remember that the converse switches the order back and forth, remember that a *conversation* goes back and forth between two people.

Try It! **4.** Write and determine the truth value of the converse of the conditional.

a. If a polygon is a quadrilateral, then it has four sides.

b. If two angles are complementary, then their angle measures add to 90.

EXAMPLE 5 Write and Evaluate the Truth Value of an Inverse and a Contrapositive

Write and determine the truth value of the inverse and contrapositive of the conditional.

If two whole numbers are both even, then their sum is even.

GENERALIZE
Consider how you can use clear and accurate reasoning to determine a truth value. What can you reason about a conditional if the conclusion is not true?

p: Two whole numbers are both even.

q: The sum of two whole numbers is even.

$\sim p$: Two whole numbers are **not** both even.

$\sim q$: The sum of two whole numbers is **not** even.

Inverse
$\sim p \rightarrow \sim q$:
If two whole numbers are **not** both even, then their sum is **not** even.

Contrapositive
$\sim q \rightarrow \sim p$:
If the sum of two whole numbers is **not** even, then the numbers are **not** both even.

The inverse is false because a counterexample exists.

$3 + 5 = 8$

Whole numbers 3 and 5 are **not** both even, but their sum is even.

The contrapositive is true. Whenever the sum of two numbers is **not** even, one of the numbers must be odd.

$7 = 1 + 6; 7 = 2 + 5; 7 = 3 + 4$

Since no counterexample exists, the contrapositive is true.

Try It! 5. Write the converse, the inverse, and the contrapositive. What is the truth value of each?

If today is a weekend day, then tomorrow is Monday.

CONCEPT Biconditional Statements

A **biconditional** is the combination of a conditional, $p \rightarrow q$, and its converse, $q \rightarrow p$. The resulting compound statement $p \leftrightarrow q$ is read as "p if and only if q."

When p and q have the same truth value, the biconditional is true. When they have opposite truth values, it is false.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

APPLICATION

EXAMPLE 6 Write and Evaluate a Biconditional

A marine biologist writes this conditional: "If a seahorse gives birth, then it is a male." Since it is true that, among seahorses, only the males can become pregnant and give birth, should the marine biologist state this as a biconditional in a paper she is writing?

Formulate ◀ Identify the hypothesis p and the conclusion q of the conditional.

Combine the conditionals $p \rightarrow q$ and $q \rightarrow p$ in the form $p \leftrightarrow q$ to write the biconditional.

Then evaluate the truth value of the biconditional.

Compute ◀ **p: A seahorse gives birth.**

q: A seahorse is male.

Biconditional $p \leftrightarrow q$: A seahorse gives birth if and only if it is male.

Determine the truth value of the biconditional.

$p \rightarrow q$: If a seahorse gives birth, then it is male. T

$q \rightarrow p$: If a seahorse is male, then it gives birth. F

If each of the combined conditionals is true, then the biconditional is true.

Interpret ◀ The biconditional is not true; the biologist should not include the statement as a biconditional in her paper.

Try It! 6. Write a biconditional for the following conditional. What is its truth value?

If two lines intersect at right angles, then they are perpendicular.

EXAMPLE 7 Identify the Conditionals in a Biconditional

What are the two conditionals implied by the biconditional?

A triangle is equilateral if and only if it has three congruent sides.

Identify the two statements in the biconditional

p: A triangle is equilateral.

q: A triangle has three congruent sides.

Write the two conditionals.

$p \rightarrow q$: If a triangle is equilateral, then it has three congruent sides.

$q \rightarrow p$: If a triangle has three congruent sides, then it is equilateral.

STUDY TIP
Remember that because the conditionals that form a true biconditional are also true, you can choose either part of the biconditional as the hypothesis and the other part as the conclusion.

Try It! 7. What are the two conditionals implied by the biconditional?

The product of two numbers is negative if and only if the numbers have opposite signs.

CONCEPT SUMMARY Conditional Statements

STATEMENT	Conditional	Converse	Inverse	Contrapositive	Biconditional
SYMBOLS	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$	$p \leftrightarrow q$
WORDS	If p, then q.	If q, then p.	If not p, then not q.	If not q, then not p.	p if and only if q.

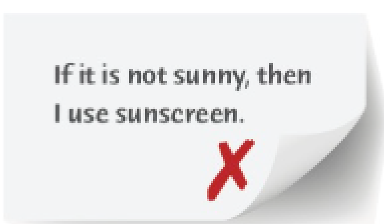
Do You UNDERSTAND?

1. ESSENTIAL QUESTION How do if-then statements describe mathematical relationships?

2. **Error Analysis** Allie was asked to write the inverse of the following conditional.

   If it is sunny, then I use sunscreen.

   What error did Allie make?



3. **Vocabulary** Which term is used to describe the opposite of a statement?

4. **Generalize** How do you write the converse of a conditional? How do you write the contrapositive of a conditional?

5. **Communicate and Justify** Explain how the inverse and the contrapositive of a conditional are alike and how they are different.

Do You KNOW HOW?

6. Write the following statement as a biconditional.

   A prime number has only 1 and itself as factors.

For Exercises 7–9, use the following conditional.

If a rectangle has an area of 12 m^2, then it has sides of length 3 m and 4 m.

7. What is the hypothesis? What is the conclusion?

8. Assume the hypothesis is false. What is the truth value of the conditional? Assume the hypothesis is true. What would be a counterexample?

9. What are the converse, the inverse, and the contrapositive? What are their truth values?

10. What two conditionals are implied by the following biconditional?

PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. Communicate and Justify Why is the following conditional logically true?

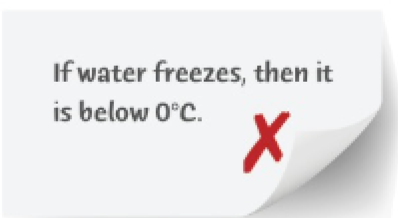
If 20 is a multiple of 3, then 101 is a perfect square.

12 Higher Order Thinking Write a true biconditional and show that both implied conditionals are true.

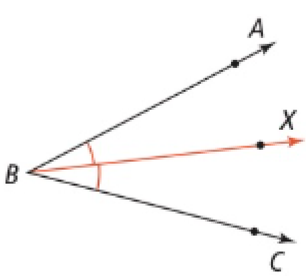
13. Error Analysis Jacy was asked to write the following statement as a conditional.

Water freezes if it is below 0°C.

What error did she make? What is the correct conditional?



14. Higher Order Thinking Write a true biconditional about angle bisectors.

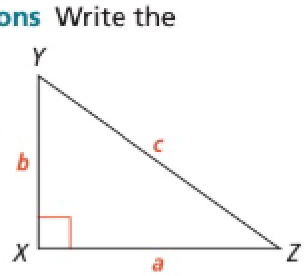


15. Communicate and Justify Can an inverse and the contrapositive both have false truth values? Explain.

16. Communicate and Justify If a biconditional is true, what are the truth values of the hypothesis and conclusion of the conditional? Explain.

17. Use Patterns and Structure Emma found a counterexample to a given conditional. What are the truth values of the hypothesis and the conclusion? Explain.

18. Mathematical Connections Write the Pythagorean Theorem as a conditional. Then write a biconditional to include the Converse of the Pythagorean Theorem.



PRACTICE

Write each statement as a conditional.
SEE EXAMPLE 1

19. My skin will get wet if I go swimming.

20. A number that is divisible by 6 is divisible by 3.

21. Movie tickets are half-price on Tuesdays.

Find the truth value of each conditional. Explain your reasoning or show a counterexample.
SEE EXAMPLES 2 and 3

22. If a pair of lines is parallel, then they do not intersect.

23. If the product of two numbers is positive, then the numbers are both positive.

Write the negation of the hypothesis and the negation of the conclusion for each conditional.
SEE EXAMPLES 4 and 5

24. If the sum of the interior angle measures of a polygon is 180, then the polygon is a triangle.

25. If one whole number is odd and the other whole number is even, then the sum of the two numbers is odd.

Write each related conditional and determine each truth value for the following conditional.
SEE EXAMPLES 4 and 5

If an angle measures 100, then it is obtuse.

26. converse

27. contrapositive

28. inverse

29. An employee at an animal shelter wrote the true conditional "If 47% of the dogs at the shelter are female, then 53% of the dogs are male." Can he rewrite this as a true biconditional? Explain. SEE EXAMPLE 6

Write two conditionals from each biconditional.
SEE EXAMPLE 7

30. A month has exactly 28 days if and only if it is February.

31. Two angles are complementary if and only if their measures add up to 90.

32. The area of a square is s^2 if and only if the perimeter of the square is $4s$.

PRACTICE & PROBLEM SOLVING

APPLY

33. Use Patterns and Structure In general, a person is 1% shorter in the evening than in the morning. Use your height to write a conditional that uses this fact.

34. Communicate and Justify In the year 1881, three different men were president of the United States—Rutherford B. Hayes, James Garfield, and Chester A. Arthur.

a. Use this fact to write a conditional and a biconditional.

b. There was one other year in which three different men were president of the United States. In 1841, Martin Van Buren, William Henry Harrison, and John Tyler were president. Using this information, determine the truth value of the conditional and the biconditional you wrote for part (a).

35. Apply Math Models The sign shows the hours for the Museum of Contemporary Art in Jacksonville.

a. Write a conditional to describe the hours of the museum on Mondays.

b. Write a conditional to describe the hours of the museum on Thursdays.

c. Write the converse, inverse, and contrapositive of the conditional you wrote in part (b). Then give the truth value for each statement.

d. Can each conditional you wrote for parts (a) and (b) be written as a true biconditional? Why or why not? If so, give each biconditional.

ASSESSMENT PRACTICE

36. Write the two conditional statements that make up the following biconditional statement.
G-CO.3.9
Two lines are parallel if and only if they do not intersect.

37. SAT/ACT Which represents the contrapositive of $p \to q$?

Ⓐ $p \leftrightarrow q$
Ⓑ $q \to p$
Ⓒ $\sim p \to \sim q$
Ⓓ $\sim q \to \sim p$
Ⓔ $\sim q \leftrightarrow \sim p$

38. Performance Task A group of students drew several different right triangles and found the measures of the two non-right angles. Their findings are shown in the table.

Angle Measure	Angle Measure	Sum
27	63	90
41	49	90
70	20	90
33	57	90

Part A Make a conjecture about the sum of two non-right angles in a right triangle. Write the conjecture in the form of a conditional.

Part B Construct several right triangles, and then measure the angles of each triangle. Do your measurements support your conjecture, or were you able to find a counterexample?

Part C Write the converse, the inverse, and the contrapositive of your conditional. Then, write a biconditional. Is the biconditional true? Explain.

1-5 Deductive Reasoning

I CAN… use deductive reasoning to draw conclusions.

VOCABULARY

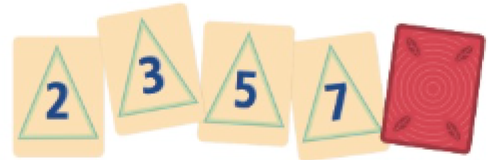
- deductive reasoning

Prepares for **MA.912.LT.4.3**–Identify and accurately interpret "if…then," "if and only if," "all" and "not" statements. Find the converse, inverse and contrapositive of a statement.

MA.K12.MTR.5.1, MTR.1.1, MTR.6.1

CRITIQUE & EXPLAIN

A deck of 60 game cards are numbered from 1 to 15 on one of four different shapes (triangle, circle, square, and pentagon). A teacher selects five cards and displays four of the cards.



She tells her class that all of the cards she selected have the same shape and asks them to draw a conclusion about the fifth card.

Chen

The fifth card is 11.

Carolina

The fifth card has a circle.

A. Describe how each student might have reached his or her conclusion. Is each student's conclusion valid? Explain.

B. **Analyze and Persevere** What are other possibilities of the fifth card? What could the teacher say to narrow the possibilities?

ESSENTIAL QUESTION How is deductive reasoning used to draw conclusions?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Determine Whether a Statement Is True

Given that a conditional and its conclusion are true, can you use deductive reasoning to determine whether the hypothesis is true?

STUDY TIP
Recall that in a conditional $p \rightarrow q$, p is the hypothesis and q is the conclusion.

You are given the facts that $p \rightarrow q$ is true and q is true. Make a truth table for the conditional $p \rightarrow q$.

Deductive reasoning is a process of reasoning using given and previously known facts to reach a logical conclusion.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Apply deductive reasoning to prove that theorems in if-then form are true.

You cannot determine whether the hypothesis is true.

Try It! **1.** Given that a conditional and its hypothesis are true, can you determine whether the conclusion is true?

CONCEPT Conclude a Statement is True

If a conditional statement and its hypothesis are true, then its conclusion is also true.

If... $p \rightarrow q$ and p are true.

Then... q is true.

EXAMPLE 2 Draw Real-World and Mathematical Conclusions

Assume that each set of given information is true.

A. If Alicia scores 85 or greater on her test, she will earn an A as her final grade. Alicia scores 89 on her test. What can you logically conclude?

Determine the truth value of $p \rightarrow q$ and p.

$p \rightarrow q$: If Alicia scores 85 or greater on her test, then she will earn an A as her final grade.

This given conditional is true.

p: Alicia scores 85 or greater on her test.

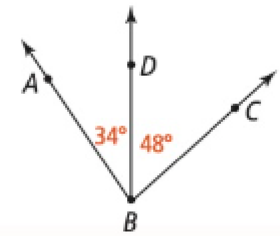
The hypothesis is true because $89 > 85$.

The conditional and its hypothesis are true, so the conclusion q is true.

You can conclude that Alicia will earn an A as her final grade.

GENERALIZE
If you cannot assume that given information is true, you can use deductive reasoning to determine whether the conditional and the hypothesis are true.

B. If point D is in the interior of $\angle ABC$, then $m\angle ABC = m\angle ABD + m\angle DBC$. What can you logically conclude about $m\angle ABC$?



Determine the truth value of $p \rightarrow q$ and p.

$p \rightarrow q$: If point D is in the interior of $\angle ABC$, then $m\angle ABC = m\angle ABD + m\angle DBC$.

The conditional is true by the Angle Addition Postulate.

p: Point D is in the interior of $\angle ABC$.

This given hypothesis is true.

The conditional and its hypothesis are true, so the conclusion q is true.

You can conclude that $m\angle ABC = m\angle ABD + m\angle DBC$.

Try It! **2.** Assume that each set of given information is true.

a. If two angles are congruent, then the measures of the two angles are equal to each other. Angle 1 is congruent to $\angle 2$. What can you logically conclude about the measures of $\angle 1$ and $\angle 2$?

b. If you finish the race in under 30 minutes, then you win a prize. You finished the race in 26 minutes. What can you logically conclude?

CONCEPT Conclude a Conditional is True

Given two true conditionals with the conclusion of the first being the hypothesis of the second, there exists a third true conditional having the hypothesis of the first and the conclusion of the second.

If... $p \rightarrow q$ and $q \rightarrow r$ are true.

Then... $p \rightarrow r$ is true.

EXAMPLE 3 Use Multiple Conditionals to Draw Real-World and Mathematical Conclusions

Assume that each set of conditionals is true. What can you conclude?

A. If Kenji plays the trumpet, then he plays a brass instrument. If he plays a brass instrument, he is a member of the marching band.

Determine whether the conclusion of one statement is the hypothesis of the other statement.

$p \rightarrow q$: If Kenji plays the trumpet, then he plays a brass instrument.

$q \rightarrow r$: If he plays a brass instrument, then he is a member of the marching band.

The conclusion of one statement is the hypothesis of the other statement.

Conclusion: If Kenji plays the trumpet, then he is a member of the marching band.

B. If points A, B, and C are collinear and B is between A and C, then $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays. If $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays, then $AB + BC = AC$. What can you conclude?

Determine whether the conclusion of one statement is the hypothesis of the other statement.

$p \rightarrow q$: If points A, B, and C are colinear and B is between A and C, then $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays.

$q \rightarrow r$: If $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays, then $AB + BC = AC$.

The conclusion of one statement is the hypothesis of the other statement.

Conclusion: If points A, B, and C are colinear and B is between A and C, then $AB + BC = AC$.

COMMON ERROR
You may confuse the hypotheses and conclusions of the given conditionals in writing the third conditional. Recall that the statement that is part of each conditional is not part of the conclusion.

Try It! **3.** Assume that each set of conditionals is true.

a. If an integer is divisible by 6, it is divisible by 2. If an integer is divisible by 2, then it is an even number. What can you logically conclude?

b. If it is a holiday, then you do not have to go to school. If it is Labor Day, then it is a holiday. What can you logically conclude?

APPLICATION

EXAMPLE 4 Draw Conclusions

What conclusions can you draw from the following true statements?

> **If you are climbing a mountain at an altitude of 28,500 feet or higher, then you are on the tallest mountain above sea level on Earth. If you are on the tallest mountain above sea level on Earth, then you are on Mount Everest. You are climbing a mountain at an altitude of 29,000 feet.**

Step 1 Identify conditional statements and use logic to draw a conclusion.

$p \rightarrow q$: If you are climbing a mountain at an altitude of 28,500 feet or higher, then you are on the tallest mountain above sea level on Earth.

This given conditional is true.

p: You are climbing a mountain at an altitude of 28,500 feet or higher.

p is true because $29{,}000 > 28{,}500$.

The conditional and the hypothesis are true, so the conclusion q is true.

You are on the tallest mountain above sea level on Earth.

LEARN TOGETHER
How do you listen actively as others share ideas?

Step 2 Use the conditional statements to conclude a new conditional statement.

$p \rightarrow q$
If you are climbing a mountain at an altitude of 28,500 feet or higher, then you are on the tallest mountain above sea level on Earth.

$p \rightarrow q$ is true.

$q \rightarrow r$
If you are on the tallest mountain above sea level on Earth, then you are on Mount Everest.

$q \rightarrow r$ is true.

$p \rightarrow r$
If you are climbing a mountain at an altitude of 28,500 feet or higher, then you are on Mount Everest.

$p \rightarrow r$ is true.

Step 3 Use $p \rightarrow r$ to draw a conclusion.

$p \rightarrow r$: If you are climbing a mountain at an altitude of 28,500 feet or higher, then you are on Mount Everest.

$p \rightarrow r$ is true.

p: You are climbing a mountain at an altitude of 28,500 feet or higher.

p is true because $29{,}000 > 28{,}500$.

The conditional and its hypothesis are true, so r: is true

You are on Mount Everest.

Try It! **4.** Martin walks his dog before dinner every day. Martin is now eating his dinner. What conclusions can you draw from these true statements?

CONCEPT SUMMARY Deductive Reasoning

WORDS

Conclude a Statement is True

If a conditional statement and its hypothesis are true, then its conclusion is also true.

Conclude a Conditional is True

Given two true conditionals with the conclusion of the first being the hypothesis of the second, there exists a third true conditional having the hypothesis of the first and the conclusion of the second.

SYMBOLS

If... $p \rightarrow q$ and p are true.

Then... q is true.

If... $p \rightarrow q$ and $q \rightarrow r$ are true.

Then... $p \rightarrow r$ is true.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is deductive reasoning used to draw conclusions?

2. **Error Analysis** Dakota writes the following. What is her error?

   If my favorite team wins more than 55 games, they win the championship. My team won the championship, so they won more than 55 games. ✗

3. **Generalize** If $p \rightarrow q$, $q \rightarrow r$, and $r \rightarrow s$ are true statements, what conditional statement can you conclude is true?

4. **Use Patterns and Structure** How can representing sentences and phrases with symbols help you draw conclusions from conditional statements?

Do You KNOW HOW?

Assume that each set of given information is true.

5. If you have a temperature above 100.4°F, then you have a fever. Casey has a temperature of 101.2°F. What can you conclude about Casey? What rule of inference did you use?

6. If points A, B, and C are collinear with B between A and C, then $AB + BC = AC$. Use the information in the figure shown. What can you conclude about AC?

Assume that each set of conditionals is true. Then write a true conditional.

7. If you eat too much, you get a stomach ache. If you get a stomach ache, you want to rest.

8. If two numbers are odd, the sum of the numbers is even. If a number is even, then the number is divisible by 2.

PRACTICE & PROBLEM SOLVING

UNDERSTAND

9. **Error Analysis** Samantha writes the following as an example of using the Law of Syllogism. of a true conditional that can be determined from two given conditionals.

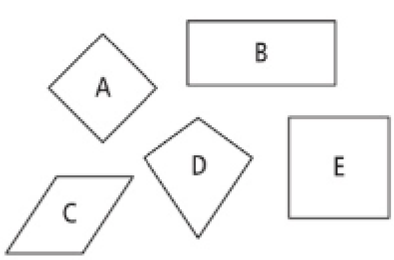
> If an animal is a dog, then it is a mammal.
> If an animal is a dog, then it has four legs.
>
> I can conclude that if an animal has four legs, then it is a mammal. ✗

10. **Use Patterns and Structure** Make a truth table with statements p, q, r, $p \to q$, $q \to r$, and $p \to r$. How does the truth table support the validity of the following statement: If $p \to q$ and $q \to r$ are true, then $p \to r$ is true.

11. **Mathematical Connections** Consider the following conditional.

    If all four sides of a quadrilateral are of equal length, then its diagonals intersect at right angles.

    Which of the following figures can you use with the conditional to draw the conclusion that its diagonals intersect at right angles?



12. **Higher Order Thinking** Suppose you are given that a conditional is true but its conclusion is false. What can you conclude about the hypothesis? Explain your answer.

13. **Generalize** Can you draw any conclusions if a conditional is true but **its** hypothesis is false? Explain.

PRACTICE

Determine the truth value of the following conditional. SEE EXAMPLE 1

14. If two adjacent angles form a right angle, then the angles are complementary.

Assume that each set of statements is true. Then draw a conclusion. If a conclusion cannot be made, explain why. SEE EXAMPLE 2

15. If you can play the piano, then you can play a musical instrument. Kiyo can play a musical instrument.

16. If the endpoints of a segment are $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the coordinates of the midpoint are $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. The endpoints of $\overline{CD}$ are $C(2, -4)$ and $D(-3, 5)$.

17. If the graph of a linear function has a positive slope, then the function is not decreasing. The slope of the graph is 0.

Assume that each set of conditionals is true. Use the conditionals to write a true conditional. If you cannot write a true conditional, explain why. SEE EXAMPLE 3

18. If $\overrightarrow{BD}$ bisects $\angle ABC$, then $\angle ABD \cong \angle DBC$. If $\angle ABD \cong \angle DBC$, then $m\angle ABD = m\angle DBC$.

19. If a whole number is even, then it is divisible by 2. If the sum of the digits of a whole number is divisible by 3, then the whole number is divisible by 3.

20. If Zachary eats fish for dinner, then he goes to bed early. If it is Tuesday night, then Zachary eats fish for dinner.

Draw conclusions from each set of true statements. SEE EXAMPLE 4

21. If it is Thursday, then Charles has baseball practice. If Charles has baseball practice, then he eats grilled chicken for dinner. It is the day after Wednesday.

22. If the length of a segment is PQ, then the distance from P to the midpoint of $\overline{PQ}$ is $\frac{1}{2}PQ$. If the endpoints of a segment are $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the length of the segment is $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The endpoints of $\overline{PQ}$ are $P(3, -4)$ and $Q(-3, -12)$.

PRACTICE & PROBLEM SOLVING

APPLY

23. **Use Patterns and Structure** Represent each true statement with symbols. Draw a truth table showing the truth values for each symbol. Then use the truth table to draw a conclusion.

    If Avery draws a numbered card from 4 to 10, then his game piece moves to home base. If his game piece moves to home base, he wins the game. Avery does not win.

24. **Mathematical Connections** The chart describes the number of tickets needed to win prizes at a family fun center.

Numbers of Tickets	Prize Level	Sample Prizes
0–100	A	magnet, stickers
101–200	B	keychain, flashlight
201–300	C	earbuds, MP3 speaker

   a. Write conditionals with sample prizes as the hypothesis and prize level as the conclusion. Write conditionals that relate the prize level to the number of tickets.

   b. Ines wins an MP3 speaker. Write true statements about Ines based on the conditionals you wrote in part (a).

25. **Analyze and Persevere** The table shows the main dishes served each day at a cafeteria.

Monday	hamburger, salad, pizza
Tuesday	hamburger, stir fry, pizza
Wednesday	fish and chips, stir fry, salad
Thursday	stir fry, salad, tacos
Friday	fish and chips, tacos, pizza

Suppose you know that Joshua has a salad and Nora has stir fry. Is that enough information to determine what day it is? If not, what other piece of information can help you?

ASSESSMENT PRACTICE

26. Make a conclusion based on the following statements. LT.4.3

    - If a triangle has three 60° angles, then the triangle is an equilateral triangle.
    - $\triangle EFG$ has three 60° angles.

27. **SAT/ACT** Which statement can you conclude from the given true statements?

    If you ride your bike to school, you exercise. If you exercise, you are happy.

    Ⓐ If you are happy, you exercise.
    Ⓑ You exercise.
    Ⓒ If you exercise, you ride your bike.
    Ⓓ If you bike to school, you are happy.

28. **Performance Task** In a game of exploration, rolling a cube numbered from 1 to 6 simulates asset acquisition. Some rules are listed.

    - If you roll an even number, you get 1 red chip.
    - If you roll a factor of 6, you get 1 blue chip.
    - If you roll a number greater than 3, you get 1 green chip.
    - If you get 2 green chips, then you exchange the 2 green chips for 1 purple chip.
    - If you get 2 red chips, then you exchange the 2 red chips for 1 purple chip.

Note that a roll can earn more than one chip.

Part A Jacinta rolls a 2, 5, 1, and then 3. What chips does she have?

Part B After four rolls, Kimberly has 1 purple chip, 1 green chip, 3 blue chips, and no red chips. What numbers could she have rolled?

1-6
Writing Proofs

I CAN… use deductive reasoning to prove theorems.

VOCABULARY

- paragraph proof
- proof
- theorem
- two-column proof

MA.912.GR.1.1–Prove relationships and theorems about lines and angles. Solve mathematical and real-world problems involving postulates, relationships and theorems of lines and angles. Also LT.4.3
MA.K12.MTR.4.1, MTR.5.1, MTR.6.1

CRITIQUE & EXPLAIN

William solved an equation for x and wrote justifications for each step of his solution.

$6(14 + x) = 108$	Given
$84 + 6x = 108$	Distributive Property
$6x = 108 - 84$	Subtraction Property of Equality
$6x = 24$	Simplify
$x = 4$	Multiplication Property of Equality

A. **Analyze and Persevere** Are William's justifications valid at each step? If not, what might you change? Explain.

B. Can you justify another series of steps that result in the same solution for x?

ESSENTIAL QUESTION How is deductive reasoning used to prove a theorem?

THEOREM 1-1 Vertical Angles Theorem

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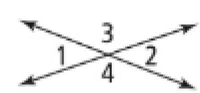
If two angles are vertical angles, then the angles are congruent.

PROOF: SEE EXAMPLE 1.

If...

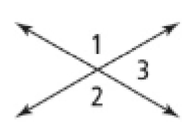


Then... $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Write a Two-Column Proof

A theorem is a conjecture that is proven. Prove the Vertical Angles Theorem.



Given: $\angle 1$ and $\angle 2$ are vertical angles

Prove: $\angle 1 \cong \angle 2$

The "If…" statement in the theorem is the Given statement for the proof, and the "then.. " statement is what you need to Prove.

A **proof** is a convincing argument that uses deductive reasoning. A **two-column proof**, in which the statements and reasons are aligned in columns, is one way to organize and present a proof.

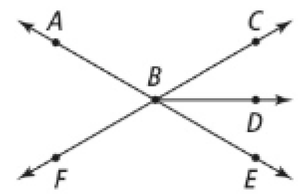
Statements	Reasons
1) $\angle 1$ and $\angle 2$ are vertical angles	**1)** Given
2) $m\angle 1 + m\angle 3 = 180$ and $m\angle 2 + m\angle 3 = 180$	**2)** Supplementary Angles
3) $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 3$	**3)** Transitive Property of Equality
4) $m\angle 1 = m\angle 2$	**4)** Subtraction Property of Equality
5) $\angle 1 \cong \angle 2$	**5)** Definition of congruent angles

COMMON ERROR
You may think that the proof is complete by stating that the measures of the angles are equal. You must explicitly state that the angles are congruent in order to complete the proof.

Try It! **1.** Write a two-column proof.

Given: $\overrightarrow{BD}$ bisects $\angle CBE$.

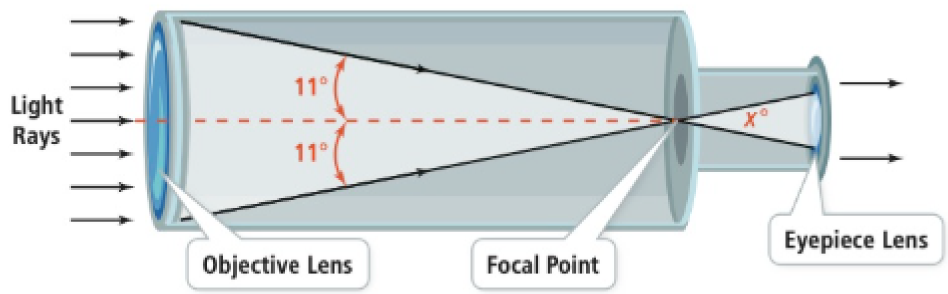
Prove: $\angle ABD \cong \angle FBD$



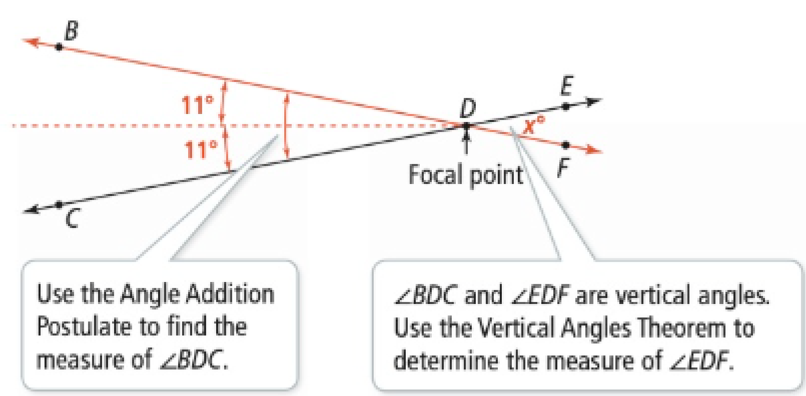
APPLICATION

EXAMPLE 2 Apply the Vertical Angles Theorem

The diagram shows how glass lenses change the direction of light rays passing through a telescope. What is the value of *x*, the angle formed by the crossed outermost light rays through the focal point?



Formulate ◀ Draw and label a diagram to represent the telescope.

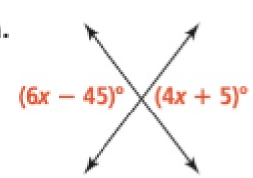


Compute ◀ $m\angle BDC = 11 + 11 = 22$ $m\angle EDF = m\angle BDC = 22$

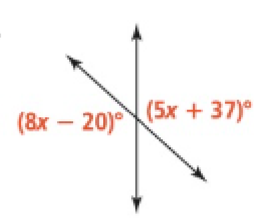
Interpret ◀ The outermost light rays form a 22° angle as they leave the focal point, so the value of *x* is 22.

Try It! **2.** Find the value of *x* and the measure of each labeled angle.

a.



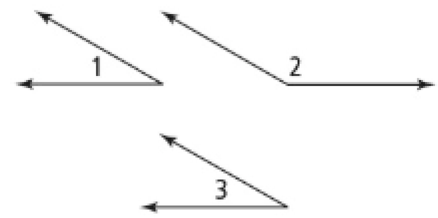
b.



THEOREM 1-2 Congruent Supplements Theorem

If two angles are supplementary to congruent angles (or to the same angle), then they are congruent.

If... $m\angle 1 + m\angle 2 = 180$ and
$m\angle 3 + m\angle 2 = 180$



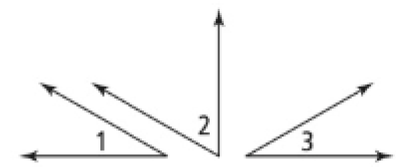
Then... $\angle 1 \cong \angle 3$

PROOF: SEE EXAMPLE 3.

THEOREM 1-3 Congruent Complements Theorem

If two angles are complementary to congruent angles (or to the same angle), then they are congruent.

If... $m\angle 1 + m\angle 2 = 90$ and
$m\angle 3 + m\angle 2 = 90$



Then... $\angle 1 \cong \angle 3$

PROOF: SEE EXAMPLE 3 TRY IT.

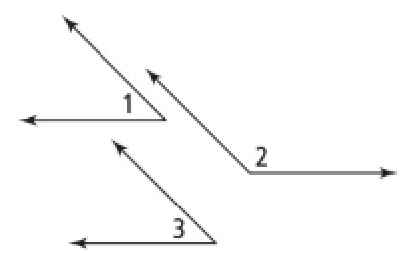
PROOF

EXAMPLE 3 Write a Paragraph Proof

Write a paragraph proof of the Congruent Supplements Theorem.

Given: $\angle 1$ and $\angle 2$ are supplementary.
$\angle 2$ and $\angle 3$ are supplementary.

Prove: $\angle 1 \cong \angle 3$



Another way to write a proof is a paragraph proof. In a **paragraph proof**, the statements and reasons are connected in sentences.

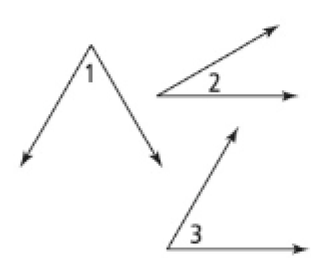
Proof: By the definition of supplementary angles, $m\angle 1 + m\angle 2 = 180$ and $m\angle 2 + m\angle 3 = 180$. Since both sums equal 180°, $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$. Subtract $m\angle 2$ from each side of this equation to get $m\angle 1 = m\angle 3$. By the definition of congruent angles, $\angle 1 \cong \angle 3$.

STUDY TIP
It may be helpful to confirm that a paragraph proof is complete by underlining each statement and then circling the corresponding reason.

Try It! 3. Write a paragraph proof of the Congruent Complements Theorem.

Given: $\angle 1$ and $\angle 2$ are complementary.
$\angle 2$ and $\angle 3$ are complementary.

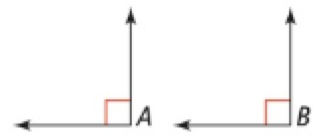
Prove: $\angle 1 \cong \angle 3$



THEOREM 1-4

All right angles are congruent.

If...



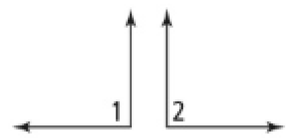
Then... $\angle A \cong \angle B$

PROOF: SEE EXERCISE 9.

THEOREM 1-5

If two angles are congruent and supplementary, then each is a right angle.

If... $\angle 1 \cong \angle 2$ and $m\angle 1 + m\angle 2 = 180$



Then... $\angle 1$ and $\angle 2$ are right angles

PROOF: SEE EXERCISE 11.

THEOREM 1-6 Linear Pairs Theorem

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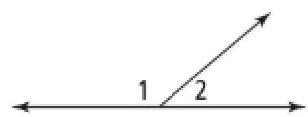
The sum of the measures of a linear pair is 180.

If... $\angle 1$ and $\angle 2$ form a linear pair.



Then... $m\angle 1 + m\angle 2 = 180$

PROOF: SEE EXERCISE 12.

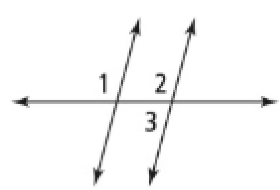
PROOF

EXAMPLE 4 Write a Proof Using a Theorem

Write a two-column proof.

Given: $m\angle 1 = m\angle 2$, $m\angle 1 = 105$

Prove: $m\angle 3 = 75$



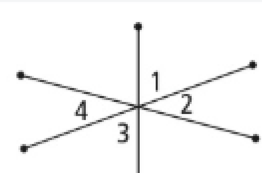
Statements	Reasons
1) $m\angle 1 = m\angle 2$	**1)** Given
2) $m\angle 1 = 105$	**2)** Given
3) $m\angle 2 = 105$	**3)** Transitive Property of Equality
4) $\angle 2$ and $\angle 3$ are a linear pair	**4)** Definition of a linear pair
5) $m\angle 2 + m\angle 3 = 180$	**5)** Linear Pairs Theorem
6) $105 + m\angle 3 = 180$	**6)** Substitution Property of Equality
7) $m\angle 3 = 75$	**7)** Subtraction Property of Equality

COMMUNICATE AND JUSTIFY
Consider the logical flow for writing a proof. How can you be sure that each step in a proof follows logically from the preceding step or steps?

Try It! **4.** Write a two-column proof.

Given: $m\angle 4 = 35$, $m\angle 1 = m\angle 2 + m\angle 4$

Prove: $m\angle 3 = 70$

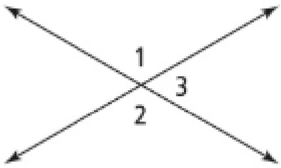


CONCEPT SUMMARY Proofs

Proofs use given information and logical steps justified by **definitions, postulates, theorems,** and **properties** to reach a conclusion.

Given: $\angle 1$ and $\angle 2$ and are vertical angles

Prove: $\angle 1 \cong \angle 2$



PROOF Two-Column Proof

Statements	Reasons
1) $\angle 1$ and $\angle 2$ are vertical angles	1) Given
2) $m\angle 1 + m\angle 3 = 180$ and $m\angle 2 + m\angle 3 = 180$	2) Supplementary Angles
3) $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 3$	3) Subst. Prop. of Equality
4) $m\angle 1 = m\angle 2$	4) Subtr. Prop. of Equality
5) $\angle 1 \cong \angle 2$	5) Def. $\cong$ angles

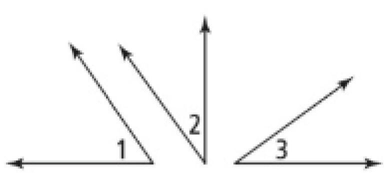
PROOF Paragraph Proof

By Supplementary Angles, $m\angle 1 + m\angle 3 = 180$ and $m\angle 2 + m\angle 3 = 180$. By the Substitution Property of Equality, $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 3$. Subtracting $m\angle 3$ from each side of the equation gives $m\angle 1 = m\angle 2$. Then by the definition of congruent angles, $\angle 1 \cong \angle 2$.

Do You UNDERSTAND?

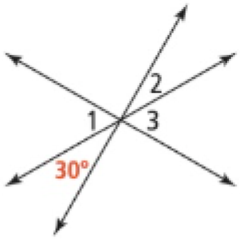
1. **ESSENTIAL QUESTION** How is deductive reasoning used to prove a theorem?

2. **Error Analysis** Jayden states that based on the Congruent Supplements Theorem, if $m\angle 1 + m\angle 2 = 90$ and if $m\angle 1 + m\angle 3 = 90$, then $\angle 2 \cong \angle 3$. What is the error in Jayden's reasoning?



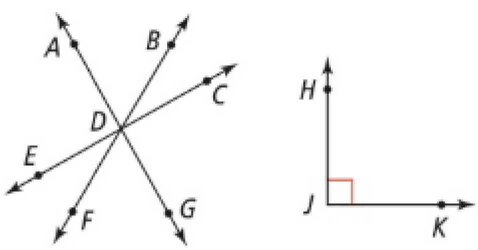
3. **Vocabulary** How is a theorem different from a postulate? How is a theorem different from a conjecture?

4. **Analyze and Persevere** If $\angle 2$ and $\angle 3$ are complementary, how could you use the Vertical Angles Theorem to find $m\angle 1$?



Do You KNOW HOW?

Use the figures to answer Exercises 5–7.

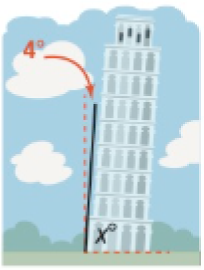


5. What statement could you write in a proof for $m\angle ADC$ using the Angle Addition Postulate as a reason?

6. Could you use the Vertical Angles Theorem as a reason in a proof to state $m\angle ADC = m\angle EDG$ or to state $\angle ADC \cong \angle EDG$? Explain.

7. Given $m\angle ADC = 90$, what reason could you give in a proof to state $\angle ADC \cong \angle HJK$?

8. The Leaning Tower of Pisa leans at an angle of about 4° from the vertical, as shown. What equation for the measure of x, the angle it makes from the horizontal, could you use in a proof?



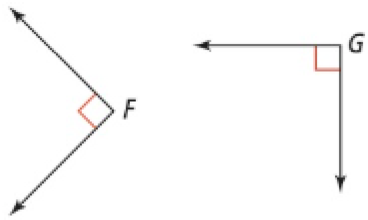
PRACTICE & PROBLEM SOLVING

UNDERSTAND

9. **Communicate and Justify** Fill in the missing reasons for the proof of Theorem 1-4.

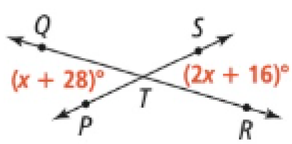
Given: $\angle F$ and $\angle G$ are right angles.

Prove: $\angle F \cong \angle G$



Statements	Reasons
1) $\angle F$ and $\angle G$ are right angles	1) Given
2) $m\angle F = 90$ and $m\angle G = 90$	2)
3) $m\angle F = m\angle G$	3)
4) $\angle F \cong \angle G$	4)

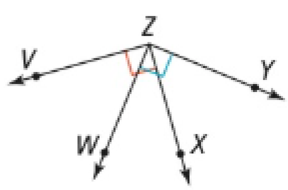
10. **Error Analysis** A student uses the Vertical Angles Theorem and the definition of complementary angles to conclude $m\angle PTR = 50$ in the figure. What mistake did the student make?



11. **Use Patterns and Structure** Write a paragraph proof of Theorem 1-5. Given that $\angle N$ and $\angle M$ are congruent and supplementary, prove that $\angle N$ and $\angle M$ are right angles.

12. **Use Patterns and Structure** Write a two-column proof of Theorem 1-6. Given that $\angle ABC$ and $\angle CBD$ are a linear pair, prove that $\angle ABC$ and $\angle CBD$ are supplementary.

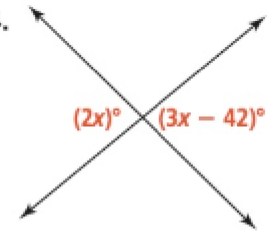
13. **Higher Order Thinking** Explain how the Congruent Complements Theorem applies to the figure shown.



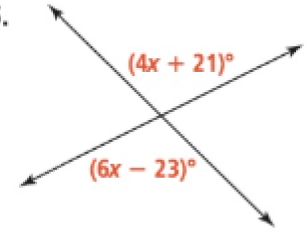
PRACTICE

Find the value of each variable and the measure of each labeled angle. SEE EXAMPLES 1 AND 2

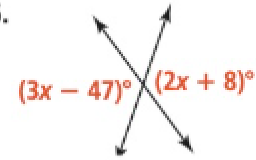
14.



15.



16.



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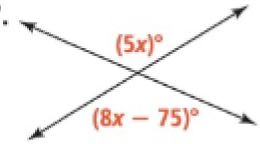
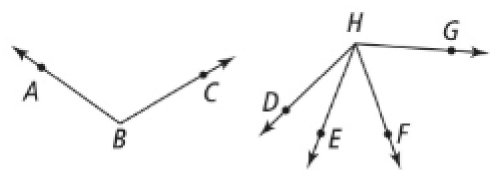


18. Write a paragraph proof. SEE EXAMPLE 3

Given: $m\angle ABC = 114$; $m\angle DHE = 25$; $m\angle EHF = 41$; $\angle ABC$ and $\angle GHF$ are supplementary.

Prove: $m\angle DHF \cong m\angle GHF$

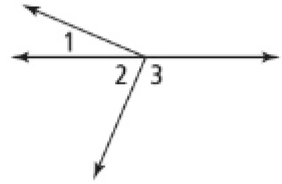


Write a two-column proof for each statement.
SEE EXAMPLE 4

19. **Given:** $\angle 1$ and $\angle 2$ are complementary.

$m\angle 1 = 23$

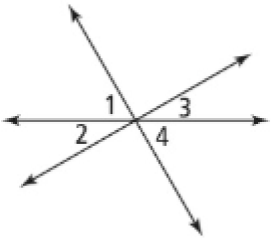
Prove: $m\angle 3 = 113$



20. **Given:** $m\angle 2 = 30$

$m\angle 1 = 2m\angle 2$

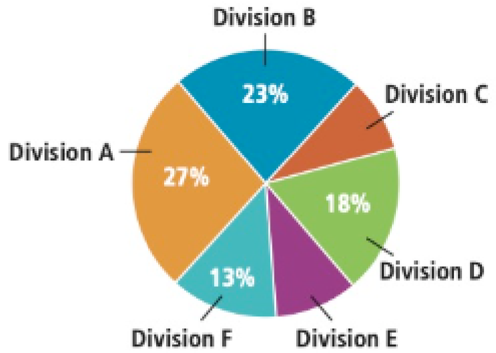
Prove: $m\angle 3 + m\angle 4 = 90$



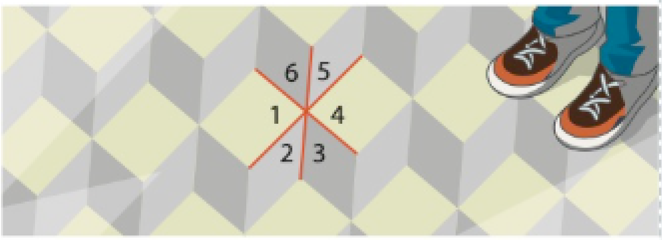
PRACTICE & PROBLEM SOLVING

APPLY

21. Mathematical Connections The graph shows percentages of sales made by various divisions of a company in one year. What are the angles formed by the segments for each division? What are the missing percentages? Explain how you were able to determine each percentage.



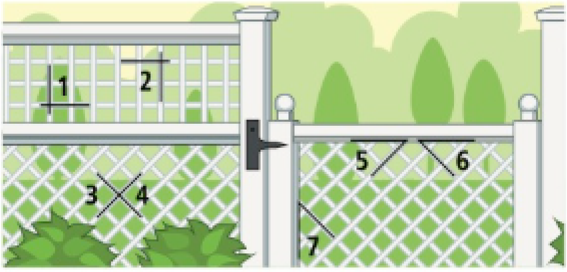
22. Analyze and Persevere A type of floor tiling is designed to give the illusion of a three-dimensional figure. Given that $m\angle 1 = 85$ and $m\angle 3 = 45$, what are the measures of the remaining angles?



23. Choose Efficient Methods Consider the angles formed by the garden gate. Using theorems from this lesson, what can you conclude from each of the following statements? State which theorem you applied to reach your conclusion.

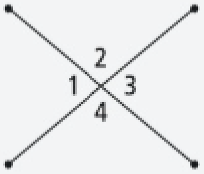
a. $m\angle 1 = 90$ and $m\angle 2 = 90$.

b. $\angle 3$ and $\angle 4$ are vertical angles.



ASSESSMENT PRACTICE

24. Consider the figure shown.



Select the statement that is never true. GR.1.1

Ⓐ $m\angle 1 + m\angle 2 = 180$

Ⓑ $m\angle 1 + m\angle 2 + m\angle 3 = 180$

Ⓒ $m\angle 2 + m\angle 4 = 180$

Ⓓ $m\angle 2 = m\angle 4$

25. SAT/ACT Given $\angle ABC$ and $\angle DEF$ are supplementary and $\angle ABC$ and $\angle GHJ$ are supplementary, what can you conclude about the angles?

Ⓐ $m\angle DEF = m\angle GHJ$

Ⓑ $m\angle DEF + m\angle GHJ = 90$

Ⓒ $m\angle DEF + m\angle GHJ = 180$

Ⓓ $m\angle ABC = m\angle DEF$ and $m\angle ABC = m\angle GHJ$

26. Performance Task The figure shows lines that divide a designer window into different parts.

Part A Copy the figure onto a sheet of paper. Label each of the inner angles. Use a protractor to measure any two of the inner angles in the figure. Using your measurements, determine the measurements of the other angles.

Part B Choose two of the inner angles that you did not actually measure. How do you know the angle measures for these two angles? Write a two-column proof to show how you know their measures are correct.

1-7 Indirect Proof

I CAN… use indirect reasoning to write a proof.

VOCABULARY
- indirect proof

MA.912.LT.4.8–Construct proofs, including proofs by contradiction.
MA.K12.MTR.5.1, MTR.4.1, MTR.2.1

CRITIQUE & EXPLAIN

Philip presents the following number puzzle to his friends.

A. **Analyze and Persevere** Philip states that the number must be 7. Explain why this cannot be true.

B. Write your own number puzzle that has an answer of 5. Your friend says the answer is not 5. How do you use the statements of your puzzle to identify the contradiction?

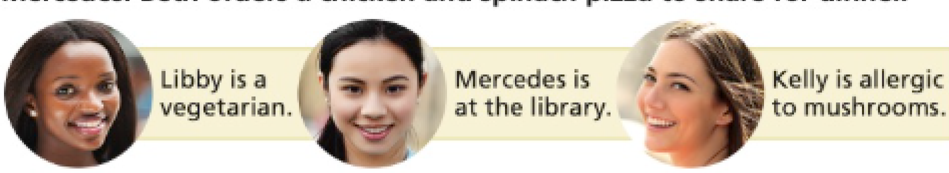
ESSENTIAL QUESTION

What can you conclude when valid reasoning leads to a contradiction?

APPLICATION

EXAMPLE 1 Use Indirect Reasoning

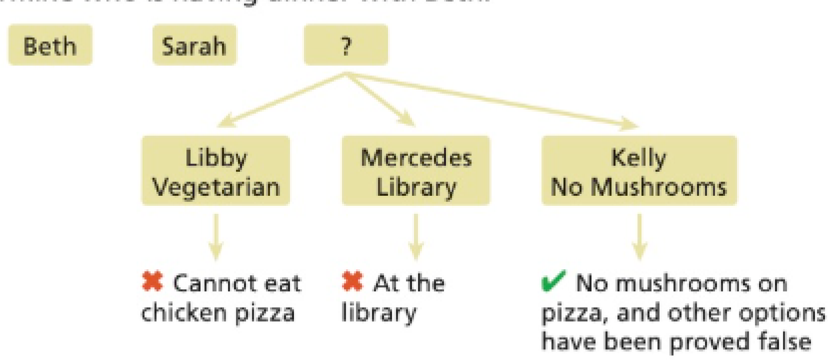
Beth is having dinner with Sarah and one of Sarah's friends—Libby, Kelly, or Mercedes. Beth orders a chicken and spinach pizza to share for dinner.



Who is having dinner with Beth?

Indirect reasoning is a type of reasoning where the conclusion is based on a pattern of specific examples or a past event. Use indirect reasoning to determine who is having dinner with Beth.

STUDY TIP
Use a flow chart or table to keep track of details if you have difficulty following the logic in an indirect reasoning problem.



Beth must be having dinner with Sarah and Kelly.

Try It! **1.** Use indirect reasoning to draw a conclusion in the following situation.

A bagel shop gives customers a free bagel on their birthday. Thato went to the bagel shop today but did not get a free bagel.

CONCEPT Writing an Indirect Proof by Contradiction

A proof that uses indirect reasoning is an **indirect proof**. Two types of indirect proof are proof by contradiction and proof by contrapositive.

Proof by Contradiction

A statement is given as a conditional $p \rightarrow q$.

Step 1 Assume p and $\sim q$ are true.

Step 2 Show that the assumption $\sim q$ leads to a contradiction.

Step 3 Conclude that q must be true.

PROOF

EXAMPLE 2 Write an Indirect Proof by Contradiction

Write an indirect proof of the following statement using proof by contradiction.

If Alani walks more than 8 kilometers over a two-day period, then she walks more than 4 kilometers on one or both days.

Identify the hypothesis and conclusion.

p: Alani walks more than 8 kilometers over a two-day period.

q: She walks more than 4 kilometers on one or both days.

The statement has the form $p \rightarrow q$.

Step 1 Assume p and $\sim q$ are true.

$\sim q$: Alani does not walk more than 4 kilometers on either day.

Step 2 Show that the assumption $\sim q$ leads to a contradiction.

If Alani does not walk more than 4 kilometers on either day, then the total distance she walks over the two-day period must be less than or equal to 8 kilometers. This contradicts the hypothesis, p.

Step 3 Conclude that q must be true.

Because the assumption leads to a contradiction, q must be true.

Alani walks more than 4 kilometers on one or both days.

USE PATTERNS AND STRUCTURE
Consider what you can logically conclude from your assumption. What can you conclude from the negation?

Try It! **2.** Write an indirect proof for each statement using proof by contradiction.

a. If today is a weekend day, then it is Saturday or Sunday.

b. If you draw an angle that is greater than 90°, it must be obtuse.

CONCEPT Writing an Indirect Proof by Contrapositive

A conditional $p \to q$ and its contrapositive $\sim q \to \sim p$ are logically equivalent, so they have the same truth value.

If you prove the contrapositive, you have also proven the conditional.

Proof by Contrapositive

Step 1 Assume $\sim q$ is true.

Step 2 Show that the assumption leads to $\sim p$, which shows $\sim q \to \sim p$.

Step 3 Conclude that $p \to q$ must be true.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Write an Indirect Proof by Contrapositive

Write an indirect proof of the following statement using proof by contrapositive.

For two positive integers *n* and *m*, if $nm > 16$, then either *n* or *m* is greater than 4 or both are greater than 4.

Write the negations of p and q.

p: $nm > 16$

$\sim p$: $nm \leq 16$

This is the part of the contrapositive you prove.

q: $n > 4$ or $m > 4$ or both are greater than 4

$\sim q$: $n \leq 4$ and $m \leq 4$

This is the part of the contrapositive you assume.

COMMON ERROR
Be careful not to make assumptions that are not given in the statements.

Step 1 Assume $\sim q$ is true.

Assume $n \leq 4$ and $m \leq 4$.

Step 2 Show that the assumption leads to $\sim p$.

$n \leq 4$	$m \leq 4$
$nm \leq 4m$	$4m \leq 16$

Use properties of inequality to write equivalent expressions of $n \leq 4$ and $m \leq 4$ as $nm \leq 16$.

By the Transitive Property, $nm \leq 16$.

Therefore, $\sim q \to \sim p$

Step 3 Conclude that $p \to q$ must be true.

Proving the contrapositive proves the conditional. Therefore, for two positive integers *n* and *m*, if $nm > 16$, then either *n* or *m* is greater than 4 or both are greater than 4.

Try It! **3.** Write an indirect proof of each statement using proof by contrapositive.

a. If today is Wednesday, then tomorrow is Thursday.

b. If a whole number is between 1 and 4, it is a factor of 6.

CONCEPT SUMMARY Indirect Proof of $p \to q$

BY CONTRADICTION

Steps

1. Assume p and $\sim q$ are true.
2. Show that the assumption $\sim q$ leads to a contradiction.
3. Conclude that q must be true.

BY CONTRAPOSITIVE

Steps

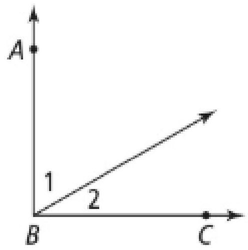
1. Assume $\sim q$ is true.
2. Show that the assumption leads to $\sim p$, which shows $\sim q \to \sim p$.
3. Conclude that $p \to q$ must be true.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What can you conclude when valid reasoning leads to a contradiction?

2. **Vocabulary** What are the two types of indirect proof? How are they similar and how are they different?

3. **Error Analysis** Consider the figure below.



Consider the following conditional.

If $\angle ABC$ is a right angle and, $m\angle 1 < 60$, then $m\angle 2 > 30$.

A student will prove the contrapositive as a way of proving the conditional. The student plans to assume $m\angle 2 < 30$ and then prove $m\angle 1 > 60$. Explain the error in the student's plan.

4. **Analyze and Persevere** How do truth tables explain why proving the contrapositive also prove the original conditional statement?

5. **Generalize** Explain how you can identify the statement you assume and the statement you try to prove when writing a proof by contrapositive.

Do You KNOW HOW?

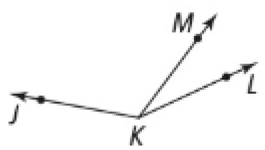
Use indirect reasoning to draw a conclusion in each situation.

6. Tamira only cuts the grass on a day that it does not rain. She cut the grass on Thursday.

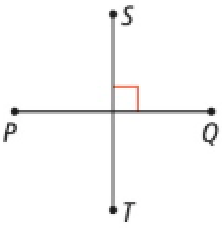
7. Gabriela works at the library every Saturday morning. She did not work at the library this morning.

Write the first step of an indirect proof for each of the following statements.

8. $m\angle JKM = m\angle JKL - m\angle MKL$



9. $\overline{PQ}$ is perpendicular to $\overline{ST}$.



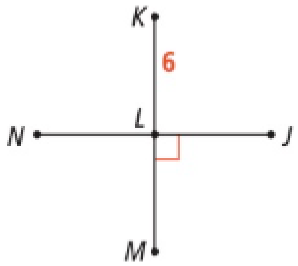
10. What can you conclude from the following situation using indirect reasoning? Explain.
    - Nadeem spent more than \$10 but less than \$11 for a sandwich and drink.
    - He spent \$8.49 on his sandwich.
    - The cost for milk is \$1.49.
    - The cost for orange juice is \$2.49.
    - The cost for a tropical smoothie is \$2.89.
    - The cost for apple juice is \$2.59.

PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. **Use Patterns and Structure** Write an indirect proof for this conditional statement about the given figure.

If $\overline{NJ}$ is the perpendicular bisector of $\overline{KM}$, then $LM = 6$.

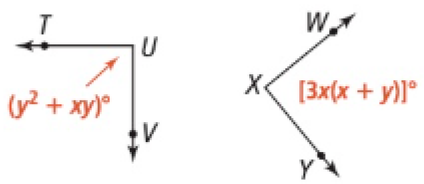


12. **Higher Order Thinking** Write an indirect proof about the given conditional using either contradiction or contrapositive. What is an advantage of the method you chose?

Given that x is a whole number, if x and $3x$ are both less than 10, then $x \leq 3$.

13. **Mathematical Connections** Write a proof by contrapositive to prove the following conditional about the figures.

If $\angle TUV \cong \angle WXY$, then $x \neq y$.



14. **Error Analysis** Consider the conditional, "If x^2 is even, then x is even." Anna uses the contrapositive to prove the conditional. What is Anna's error?

> Assume x is even. Then x can be written in the form $2k$, where k is an integer. Substitute this into x^2 to get $(2k)^2$, or $4k^2$. This expression can be written in the form $2m$, where m is an integer, which proves x^2 is even. ✗

PRACTICE

Use indirect reasoning to draw a conclusion in each situation. SEE EXAMPLE 1

15. Every student in Mr. Green's 2nd period class got an A on the math test. Paige got a B on the test.

16. Only students who studied at least 3 hours for the history test got an A on it. Derek studied 2 hours for the test.

Write the first step of an indirect proof of each statement. SEE EXAMPLE 2

17. $ST + TU + UV = 150$

18. Ray DE is the angle bisector of $\angle ADC$.

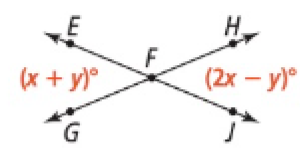
Identify the two statements that contradict each other in each set. SEE EXAMPLE 2

19. I. $m\angle K + m\angle L = 150$
II. $m\angle K - m\angle L = 20$
III. $m\angle K = 180$

20. I. $\angle S$ is an acute angle.
II. $m\angle S = 80$
III. $m\angle S + m\angle T = 40$

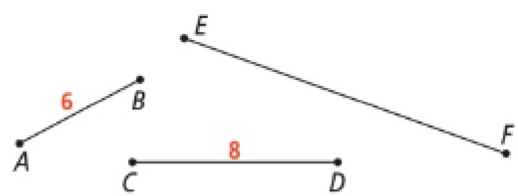
21. **Write an indirect proof for the following conditional about the figure.** SEE EXAMPLE 2

If $\angle EFG$ and $\angle HFJ$ are vertical angles, then $x \neq 3y$.



22. Write a proof of the contrapositive to prove the following conditional about the figures. SEE EXAMPLE 3

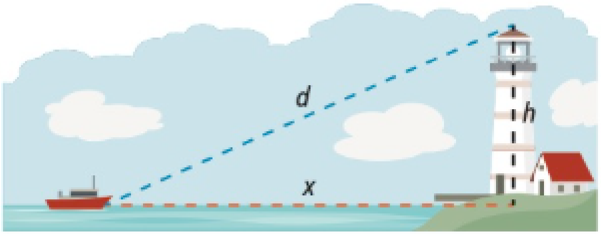
If $AB + CD = EF$, then $EF = 14$.



PRACTICE & PROBLEM SOLVING

APPLY

23. **Represent and Connect** The lighthouse forms a right angle with the path of the boat.



a. Write an equation relating *h, d,* and *x*.

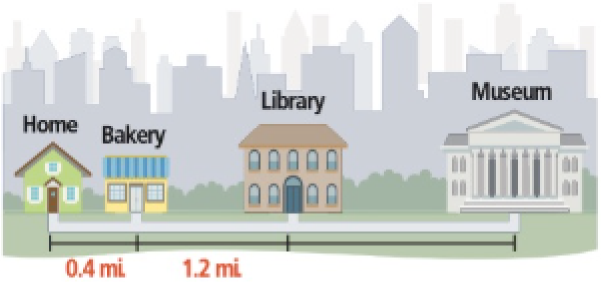
b. Write an indirect proof of the statement by proving the contrapositive.

If *x* decreases, then *d* decreases.

24. **Communicate and Justify** Friends eat the entire 6 slices of a key lime pie. No slices are shared. Prove the following conditional by proving the contrapositive for the conditional.

If four friends share the pie, then at most two of the friends will have more than one slice of key lime pie each.

25. **Communicate and Justify** The library is at the midpoint between Nicky's home and the museum.



Nicky begins at her home and walks toward the museum. Write an indirect proof for this conditional: When she gets to the library, she will have less than 2 miles left to go.

ASSESSMENT PRACTICE

26. Does this pair of statements contradict each other? Explain. LT.4.8

$\angle P$ and $\angle Q$ are both obtuse angles.

$\angle P$ and $\angle Q$ are supplementary.

27. **SAT/ACT** If you write a proof of the following conditional by proving the contrapositive, what should your assumption be?

If $\overrightarrow{JK}$ is the angle bisector of $\angle HJL$, then $m\angle HJK + m\angle KJL = 90$.

Ⓐ $\overrightarrow{JK}$ is the angle bisector of $\angle HJL$.

Ⓑ $\overrightarrow{JK}$ is not the angle bisector of $\angle HJL$.

Ⓒ $m\angle HJK + m\angle KJL = 90$

Ⓓ $m\angle HJK + m\angle KJL \neq 90$

28. **Performance Task** Customers who eat lunch at a diner have the following meal choices:

Part A Write an indirect proof of the following conditional.

If a customer chooses a meal with cabbage, then the customer also has stewed beef.

Part B Write another conditional statement related to which meal a customer has at the diner. Then write an indirect proof of your conditional statement.

TOPIC
1

Topic Review

? TOPIC ESSENTIAL QUESTION

1. What are the fundamental building blocks of geometry?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A statement accepted without proof is a ________.
3. A ________ is the combination of a conditional and its converse.
4. If both a conditional statement and its ________ are true, then you can conclude that its conclusion is true.
5. A conjecture that has been proven is a ________.
6. A statement of the form *if not q, then not p* is a ________ of the conditional *if p, then q*.
7. You use ________ when you logically come to a valid conclusion based on given statements.

- biconditional
- conjecture
- contrapositive
- converse
- deductive reasoning
- hypothesis
- postulate
- theorem

Concepts & Skills Review

LESSON 1-1 Measuring Segments and Angles

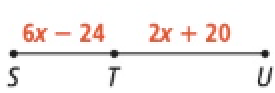
Quick Review

If a **line segment** is divided into parts, the length of the whole segment is the sum of the lengths of its individual parts. **Congruent segments** have the same length.

Similarly, if an **angle** is divided into parts, the measure of the whole angle is the sum of the measures of the individual angles. **Congruent angles** have the same measure.

Example

Given $SU = 60$, find x.

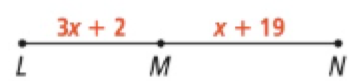


$$ST + TU = SU$$
$$(6x - 24) + (2x + 20) = 60$$
$$8x - 4 = 60$$
$$8x = 64$$
$$x = 8$$

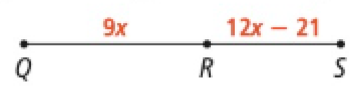
Practice & Problem Solving

Find each value.

8. $LN = 45$. Find x.



9. $RS = 27$. Find QS.

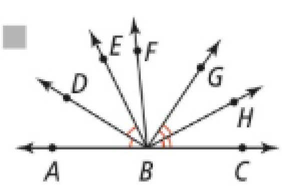


Find the measure of each angle.

10. $m\angle EBG = 60$; $m\angle FBG = 2m\angle EBF$; $m\angle EBF = ■$

11. $m\angle ABE = 64$; $m\angle DBE = ■$

12. $m\angle GBH = 28$; $m\angle GBC = ■$



13. **Choose Efficient Methods** Point K is located at 7 on a number line, and $JK = KL$. If the coordinate of L is 23, what is the coordinate of point J?

LESSON 1-2 Basic Constructions

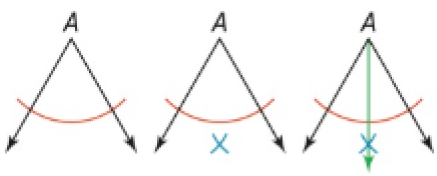
Quick Review

You can use a compass and a straightedge to copy segments and angles, and to construct the **angle bisector** of a given angle and the **perpendicular bisector** of a given line segment.

Any geometric figure that can be constructed using a compass and straightedge is a **construction**.

Example

Construct the angle bisector of ∠A.



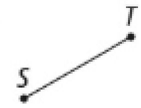
From the vertex, draw an arc that intersects both sides of the angle.

Next, using the same compass setting at each intersection, draw intersecting arcs within the angle.

Finally, draw the angle bisector from the vertex through the intersecting arcs.

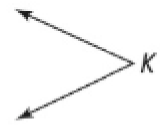
Practice & Problem Solving

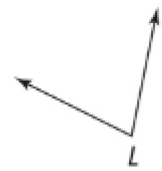
Copy each segment, and construct the perpendicular bisector.

14. 

15. 

Copy each angle, and construct its bisector.

16. 

17. 

18. **Analyze and Persevere** Why must the compass width be larger than half the segment width to draw a perpendicular bisector?

19. **Represent and Connect** The sides of a roof meet at a 120° angle. A strip of wood extends down from the vertex so that it bisects the angle. Draw a diagram of the roof with the bisecting strip of wood.

LESSON 1-3 Midpoint and Distance

Quick Review

The midpoint formula gives the coordinates of the **midpoint** between two points.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The distance formula gives the distance between two points on a coordinate plane.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example

What are the coordinates of point K that is $\frac{2}{3}$ the distance from $J(8, 3)$ to $L(2, 6)$?

Horizontal distance: $\frac{2}{3}(x_2 - x_1) = \frac{2}{3}(2 - 8) = -4$

Vertical distance: $\frac{2}{3}(y_2 - y_1) = \frac{2}{3}(6 - 3) = 2$

$K(x, y) = (x_1 + (-4), y_1 + 2) = (8 - 4, 3 + 2) = (4, 5)$

Practice & Problem Solving

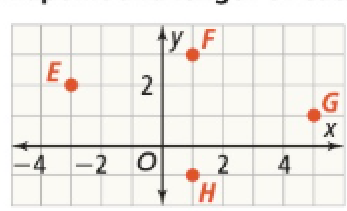
Find the midpoint and length of each segment.

20. $\overline{EF}$

21. $\overline{FG}$

22. $\overline{GH}$

23. $\overline{EH}$



24. What are the coordinates of the point that partitions $\overline{HE}$ in the ratio 2:3?

25. **Analyze and Persevere** Sadie models her neighborhood on a coordinate plane so that her school is at (8, 12) and a store is at (14, 3). What are the coordinates of the point halfway between the school and the store?

LESSON 1-4 Conditional Statements

Quick Review

A **conditional** $p \rightarrow q$ relates a **hypothesis** p to a **conclusion** q. The **converse** is $q \rightarrow p$, the **inverse** is $\sim p \rightarrow \sim q$, the **contrapositive** is $\sim q \rightarrow \sim p$ and a **biconditional** is $p \leftrightarrow q$.

Statements are logically equivalent if they have the same **truth value**. A conditional and its contrapositive are logically equivalent. A converse and an inverse are also logically equivalent.

Example

Find the truth value of the following conditional: All quadrilaterals have four congruent angles.

A parallelogram that is not a rectangle is an example of a quadrilateral. It is a counterexample because its angles are not all congruent. The truth value for the conditional is false.

Practice & Problem Solving

For each statement, write a conditional and the converse, inverse, and contrapositive.

26. A number that is a multiple of 4 is a multiple of 2.

27. Kona jogs 5 miles every Saturday morning.

Find the truth value of each conditional. Explain your reasoning or show a counterexample.

28. If a number is less than 4, then it is prime.

29. If $3x - 7 < 14$, then $x < 8$.

30. If it snows, then school will be cancelled.

31. **Analyze and Persevere** A cafeteria only offers pudding on Tuesdays. Use this fact to write a biconditional about the pudding.

LESSON 1-5 Deductive Reasoning

Quick Review

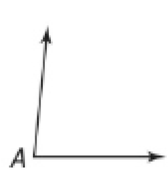
Deductive reasoning uses logical steps based on given facts to reach a conclusion and can be applied through laws of logic.

- If $p \rightarrow q$ and p are true, then q is true.
- If $p \rightarrow q$ and $q \rightarrow r$ are true, then $p \rightarrow r$ is true.

Example

Given the following, write a true conditional.

- If $m\angle A < 90$, then $\angle A$ is acute.
- If $\angle A$ is acute, then it is not a right angle.



Conditional: If $m\angle A < 90$, then $\angle A$ is not a right angle.

Practice & Problem Solving

Assume the given information is true.

32. If $AB = BC$, then $DE = 2(AB)$. $AB = 6$ and $BC = 6$. What can you conclude?

33. If it is a sunny day, the water park is filled with people. If the water park is filled with people, the lines for each ride are long. Assume the conditionals are true. Then write a true conditional.

34. **Communicate and Justify** An advertisement says if you use their toothpaste for more than a week, you will have fresher breath. You use the toothpaste ten days. If the advertisement is true, what can you conclude?

LESSON 1-6 & 1-7 Writing Proofs and Indirect Proof

Quick Review

A **proof** uses deductive reasoning to explain why a conjecture is true. A conjecture that has been proven is a **theorem**.

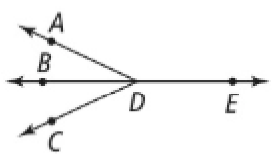
For an indirect proof, assume the negation of what is to be proven, and then show that the assumption leads to a contradiction.

Example

Write a paragraph proof.

Given: $m\angle BDC + m\angle ADE = 180$

Prove: $\angle ADB \cong \angle BDC$

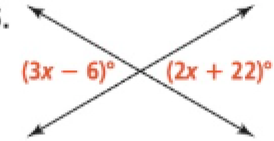


Proof: By definition of supplementary angles, $m\angle ADB + m\angle ADE = 180$. Since it is given that $m\angle BDC + m\angle ADE = 180$, by the Congruent Supplements Theorem, $\angle ADB \cong \angle BDC$.

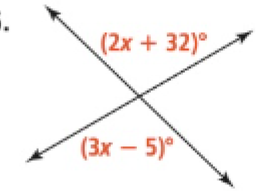
Practice & Problem Solving

Find the value of each variable and the measure of each labeled angle.

35.



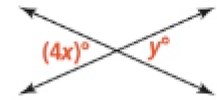
36.

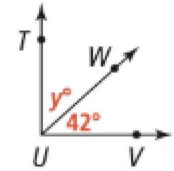


37. Communicate and Justify Write a proof.

Given: $m\angle TUV = 90$

Prove: $x = 12$





38. Communicate and Justify Write an indirect proof by proving the contrapositive.

Given: $GJ = 48$

Prove: $x \neq 12$

TOPIC 2

Parallel and Perpendicular Lines

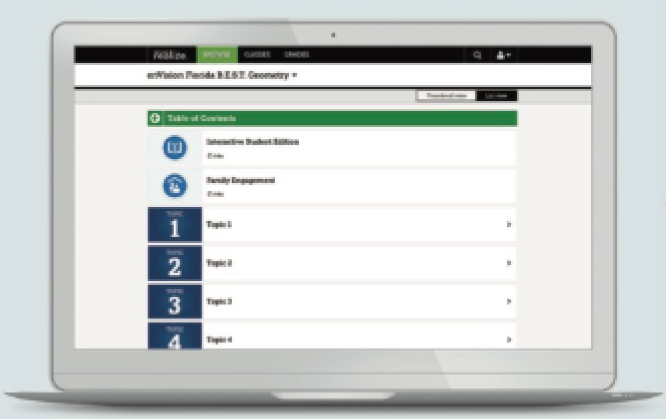
TOPIC ESSENTIAL QUESTION

What properties are specific to parallel lines and perpendicular lines?

Topic Overview

Topic Vocabulary

- flow proof



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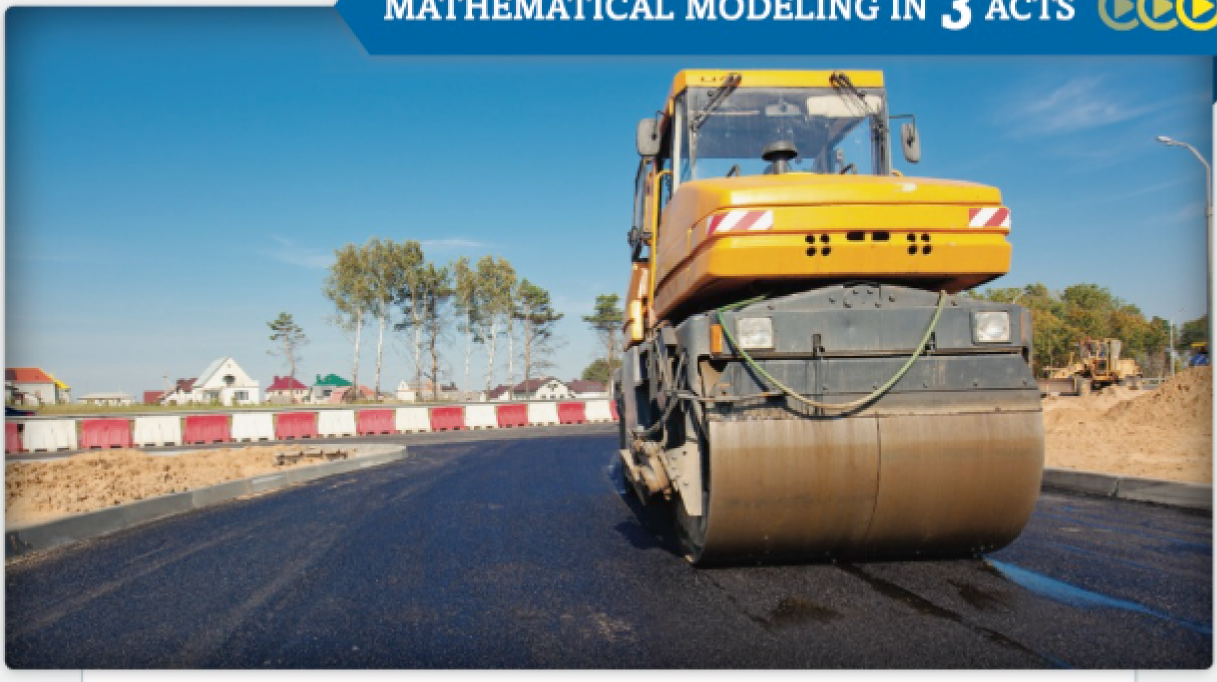
MATHEMATICAL MODELING IN 3 ACTS



Parallel Paving Company

Building roads consists of many different tasks. Once civil engineers have designed the road, they work with surveyors and construction crews to clear and level the land. Sometimes specialists have to blast away rock in order to clear the land. Once the land is leveled, the crews bring in asphalt pavers to smooth out the hot asphalt.

Sometimes construction crews will start work at both ends of the new road and meet in the middle. Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

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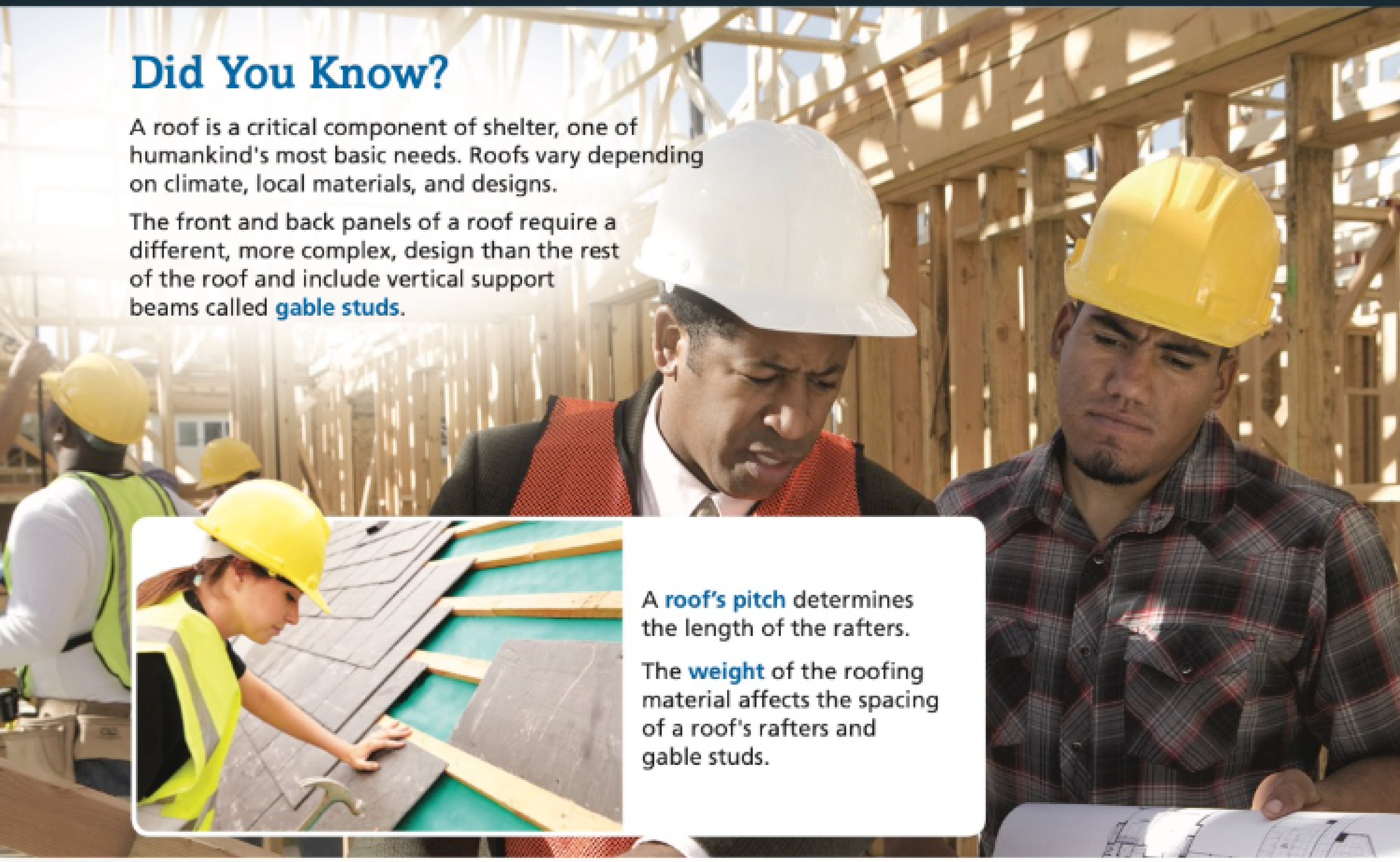

Did You Know?

A roof is a critical component of shelter, one of humankind's most basic needs. Roofs vary depending on climate, local materials, and designs.

The front and back panels of a roof require a different, more complex, design than the rest of the roof and include vertical support beams called **gable studs**.

A **roof's pitch** determines the length of the rafters.

The **weight** of the roofing material affects the spacing of a roof's rafters and gable studs.

Bermuda has no fresh water other than falling rain, so roofs are designed to **funnel rain** down into underground holding tanks.

The weight of snow on a roof can be up to **21 lbs per square foot**.

A **green roof** is topped with earth and plants, which cools the building in the summer and insulates it in the winter.

Your Task: Build a Roof

You and your classmates will plan the construction of a roof, including the location and cost of its ridge-board, rafters, and gable studs. How does the cost of the roof change based on on its pitch and the spacing between rafters?

2-1 Parallel Lines

I CAN… determine the measures of the angles formed when parallel lines are intersected by a transversal.

MA.912.GR.1.1–Prove relationships and theorems about lines and angles. Solve mathematical and real-world problems involving postulates, relationships and theorems of lines and angles.
MA.K12.MTR.5.1, MTR.2.1, MTR.7.1

EXPLORE & REASON

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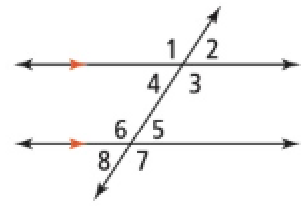
The diagram shows two parallel lines cut by a transversal.



A. Use Patterns and Structure What relationships among the measures of the angles do you see?

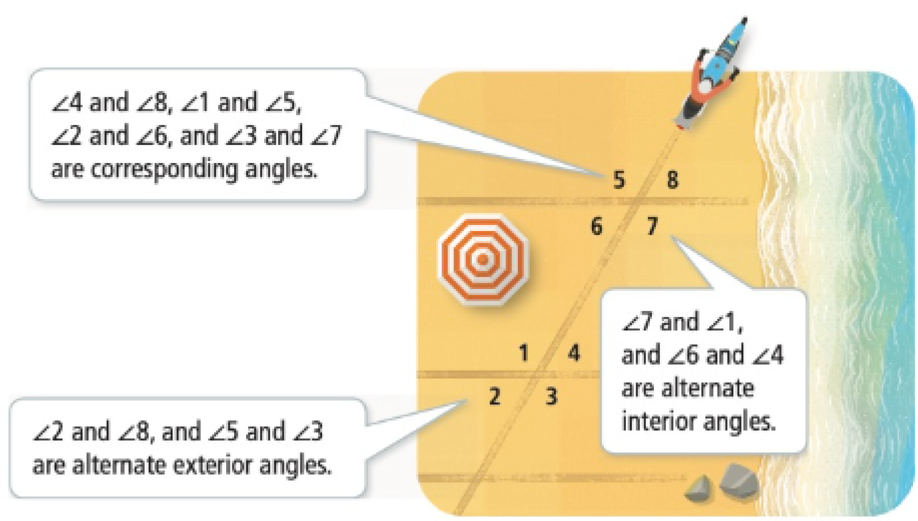
B. Suppose a different transversal intersects the parallel lines. Would you expect to find the same relationships with the measures of those angles? Explain.

ESSENTIAL QUESTION

What angle relationships are created when parallel lines are intersected by a transversal?

EXAMPLE 1 Identify Angle Pairs

Identify the pairs of angles of each angle type made by the motorcycle tracks.

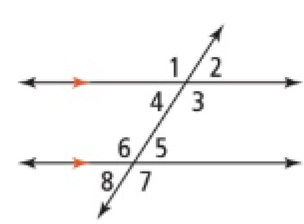


STUDY TIP
Transversals can intersect either parallel or nonparallel lines. The types of angle pairs remain the same.

Try It!

1. Which angle pairs include the named angle?

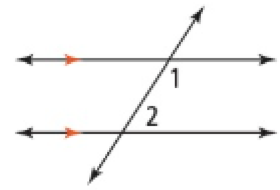
a. ∠4 **b.** ∠7



POSTULATE 2-1 Same-Side Interior Angles Postulate

If a transversal intersects two parallel lines, then same-side interior angles are supplementary.

If...

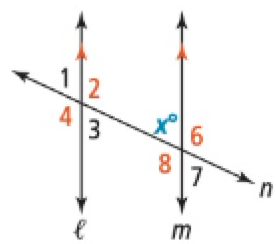


Then... $m\angle 1 + m\angle 2 = 180$

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Explore Angle Relationships

How can you express each of the numbered angles in terms of *x*?



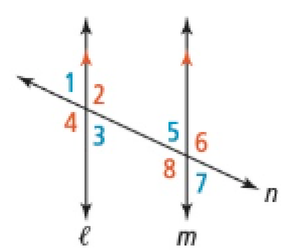
USE PATTERNS AND STRUCTURE
What patterns do you notice about the angles formed by two parallel lines cut by a transversal?

Angle 7 and the angle with measure x are vertical angles. Both $\angle 6$ and $\angle 8$ each form a linear pair with the angle with measure x and are therefore supplementary to it.

By Postulate 2-1 you know that $\angle 2$ and the angle with measure x are supplementary. From that you can make conclusions about $\angle 1$, $\angle 3$, and $\angle 4$ like you did with $\angle 6$, $\angle 7$, and $\angle 8$.

The angles equal to $x°$ are $\angle 1$, $\angle 3$, and $\angle 7$.
The angles that are supplementary to the angle with measure x have the measure $(180 - x)$. These are $\angle 2$, $\angle 4$, $\angle 6$, and $\angle 8$.

Try It! **2.** If $\angle 4 = 118°$, what is the measure of each of the other angles?

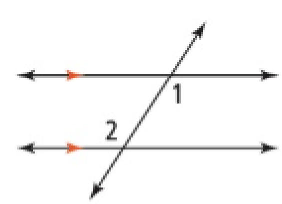


THEOREM 2-1 Alternate Interior Angles Theorem

If a transversal intersects two parallel lines, then alternate interior angles are congruent.

PROOF: SEE EXAMPLE 3.

If...



Then... $\angle 1 \cong \angle 2$

THEOREM 2-2 Corresponding Angles Theorem

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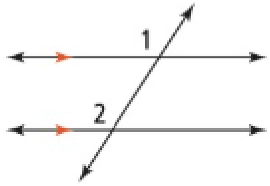
If a transversal intersects two parallel lines, then corresponding angles are congruent.

PROOF: SEE EXAMPLE 3 TRY IT.

If...



Then... $\angle 1 \cong \angle 2$

THEOREM 2-3 Alternate Exterior Angles Theorem

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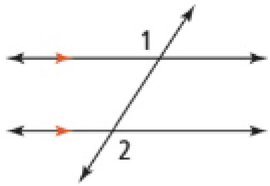
If a transversal intersects two parallel lines, then alternate exterior angles are congruent.

PROOF: SEE EXERCISE 10.

If...

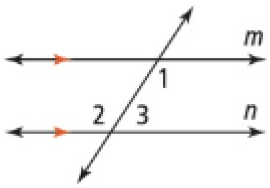


Then... $\angle 1 \cong \angle 2$

PROOF

EXAMPLE 3 Prove the Alternate Interior Angles Theorem

Prove the Alternate Interior Angles Theorem.



Given: $m \parallel n$

Prove: $\angle 1 \cong \angle 2$

Plan: Use the Same-Side Interior Angles Postulate to show $\angle 1$ is supplementary to $\angle 3$. Then show that angles 1 and 2 are congruent because they are both supplementary to the same angle.

Proof:

Statements	Reasons
1) $m \parallel n$	**1)** Given
2) $\angle 1$ and $\angle 3$ are supplementary	**2)** Same-Side Interior $\angle$s Postulate
3) $m\angle 1 + m\angle 3 = 180$	**3)** Def. of supplementary angles
4) $m\angle 2 + m\angle 3 = 180$	**4)** Linear Pairs Theorem
5) $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 3$	**5)** Transitive Property of Equality
6) $m\angle 1 = m\angle 2$	**6)** Subtraction Property of Equality
7) $\angle 1 \cong \angle 2$	**7)** Def. of congruence

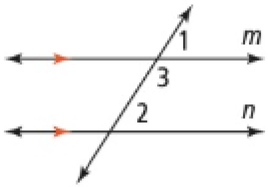
COMMON ERROR

Remember that for the proof to be complete, the last statement of the proof must match what you are trying to prove.

Try It! **3.** Prove the Corresponding Angles Theorem.

Given: $m \parallel n$

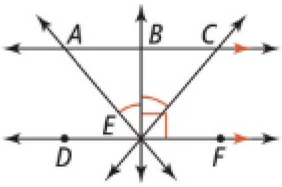
Prove: $\angle 1 \cong \angle 2$



PROOF

EXAMPLE 4 Use Parallel Lines to Prove an Angle Relationship

Use the diagram to prove the angle relationship.



Given: $\overline{AC} \parallel \overline{DF}$, and $\overline{BE} \perp \overline{DF}$, $\angle AEB \cong \angle CEB$

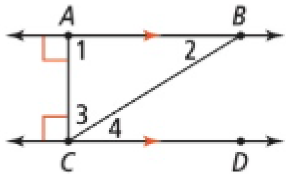
Prove: $\angle BAE \cong \angle BCE$

Proof:

Statements	Reasons
1) $\overline{AC} \parallel \overline{DF}$, $\overline{AC} \perp \overline{BE}$	1) Given
2) $\angle BED$, $\angle BEF$ are rt. angles	2) Def. of perpendicular
3) $m\angle BED = m\angle BEF = 90$	3) Def. of rt. angles
4) $m\angle AED + m\angle AEB = 90$, $m\angle CEF + m\angle CEB = 90$	4) Angle Addition Postulate
5) $\angle AEB \cong \angle CEB$	5) Given
6) $\angle AED \cong \angle CEF$	6) Congruent Complements Thm.
7) $\angle BAE \cong \angle AED$, $\angle BCE \cong \angle CEF$	7) Alt. Interior $\angle$s Thm.
8) $\angle BAE \cong \angle BCE$	8) Transitive Prop. of Congruence

ANALYZE AND PERSEVERE
Look for relationships in the diagram not listed as given information. What angle relationships are shown in the diagram?

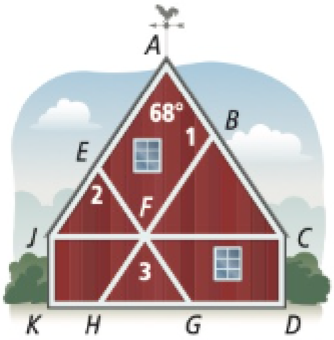
Try It! **4.** Given $\overline{AB} \parallel \overline{CD}$, prove that $m\angle 1 + m\angle 2 + m\angle 3 = 180$.

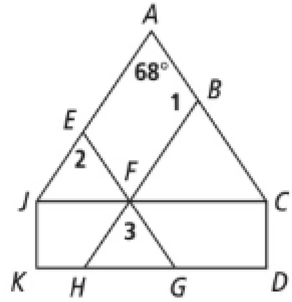


APPLICATION

EXAMPLE 5 Find Angle Measures

The white trim shown for the wall of a barn should be constructed so that $\overline{AC} \parallel \overline{EG}$, $\overline{JA} \parallel \overline{HB}$, and $\overline{JC} \parallel \overline{KG}$. What should $m\angle 1$ and $m\angle 3$ be?





Formulate ◀ Look for relationships among the angles.

Compute ◀ By the Same-Side Interior Angles Postulate, $m\angle 1 + 68 = 180$.

$m\angle 1 = 180 - 68 = 112$

By the Corresponding Angles Theorem, $\angle EAB \cong \angle 2$ and $\angle 2 \cong \angle 3$, so $\angle 3 \cong \angle EAB$ by the Transitive Property of Congruence.

$m\angle 3 = 68$

Interpret ◀ So, $m\angle 1 = 112$ and $m\angle 3 = 68$.

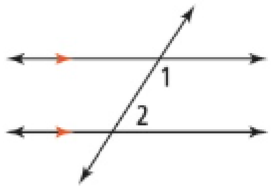
Try It! **5.** If $m\angle EJF = 56$, find $m\angle FHK$.

CONCEPT SUMMARY Parallel Lines and Angle Pairs

There are four special angle relationships formed when parallel lines are intersected by a transversal.

POSTULATE 2-1 Same-Side Interior Angles Postulate

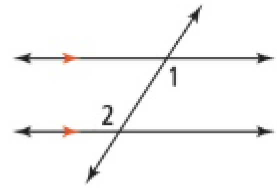
If...



Then... $m\angle 1 + m\angle 2 = 180$

THEOREM 2-1 Alternate Interior Angles Theorem

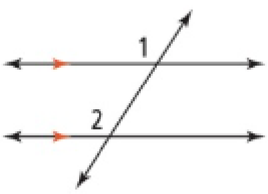
If...



Then... $\angle 1 \cong \angle 2$

THEOREM 2-2 Corresponding Angles Theorem

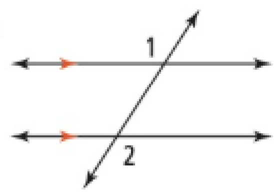
If...



Then... $\angle 1 \cong \angle 2$

THEOREM 2-3 Alternate Exterior Angles Theorem

If...



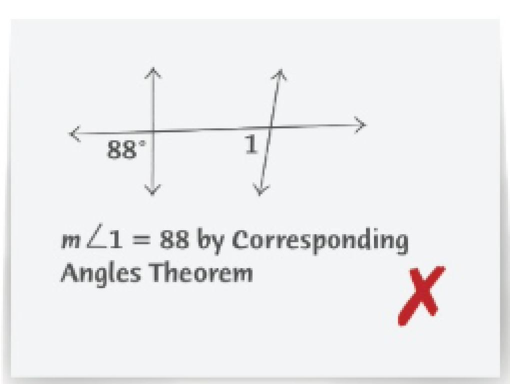
Then... $\angle 1 \cong \angle 2$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What angle relationships are created when parallel lines are intersected by a transversal?

2. **Vocabulary** When a transversal intersects two parallel lines, which angle pairs are congruent?

3. **Error Analysis** What error did Leah make?

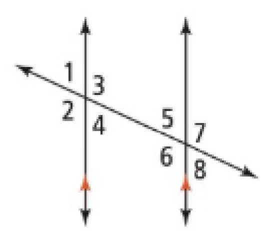


4. **Generalize** For any pair of angles formed by a transversal intersecting parallel lines, what are two possible relationships?

Do You KNOW HOW?

Use the diagram for Exercises 5–8.

Classify each pair of angles. Compare angle measures, and give the postulate or theorem that justifies it.



5. $\angle 2$ and $\angle 6$

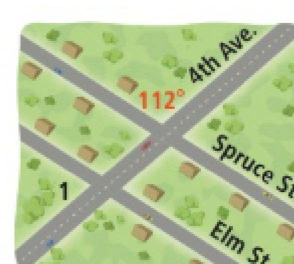
6. $\angle 3$ and $\angle 5$

If $m\angle 1 = 71$, find the measure of each angle.

7. $\angle 5$

8. $\angle 7$

9. Elm St. and Spruce St. are parallel. What is $m\angle 1$?



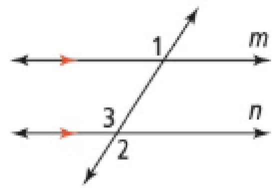
PRACTICE & PROBLEM SOLVING

UNDERSTAND

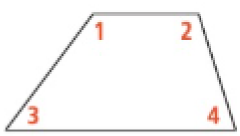
10. Communicate and Justify Write a two-column proof of the Alternate Exterior Angles Theorem.

Given: $m \parallel n$

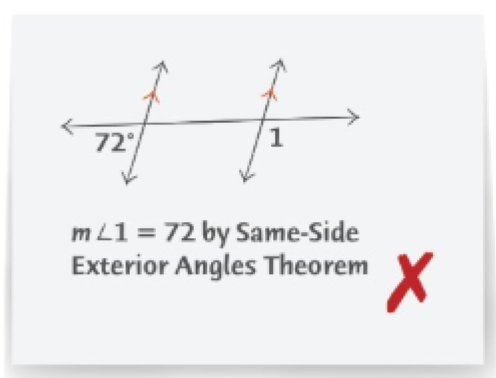
Prove: $\angle 1 \cong \angle 2$



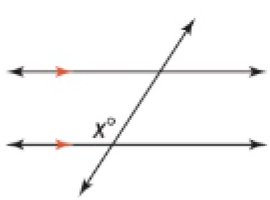
11. Higher Order Thinking Using what you know about angle pairs formed by parallel lines and a transversal, how are $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$ related in the trapezoid? Explain.



12. Error Analysis What error did Tyler make?



13. Generalize In the diagram shown, if $x + y = 180$, label the remaining angles as $x°$ or $y°$.



14. Mathematical Connections A transversal intersects two parallel lines. The measures of a pair of alternate interior angles are $5v$ and $2w$. The measures of a pair of same-side exterior angles are $10w$ and $5v$. What are the values of w and v?

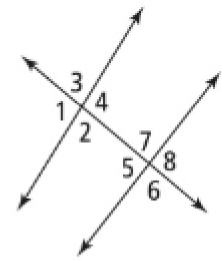
PRACTICE

Identify a pair of angles for each type. SEE EXAMPLE 1

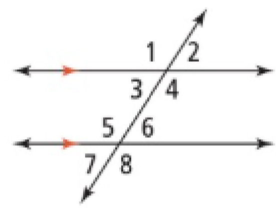
15. same-side interior

16. corresponding

17. alternate exterior



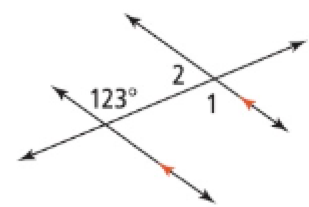
18. Which angles are supplementary to $\angle 1$? Which are congruent to $\angle 1$? SEE EXAMPLE 2



Find each angle measure. SEE EXAMPLE 3

19. $m\angle 1$

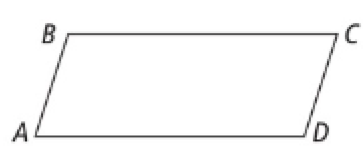
20. $m\angle 2$



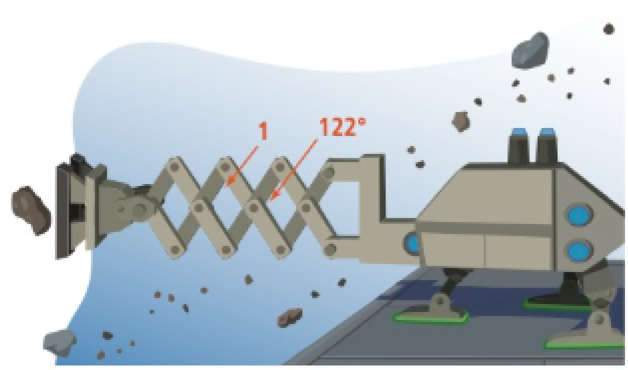
21. Opposite sides of a parallelogram are parallel. Prove that opposite angles of a parallelogram are congruent. SEE EXAMPLE 4

Given: $ABCD$ is a parallelogram

Prove: $\angle A \cong \angle C$, $\angle B \cong \angle D$



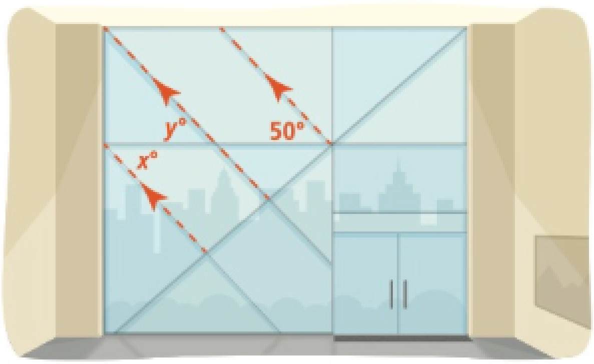
22. Three parallelograms are hinged at each vertex to create an arm that can extend and collapse for an exploratory spaceship robot. What is $m\angle 1$? Explain how you found the answer. SEE EXAMPLE 5



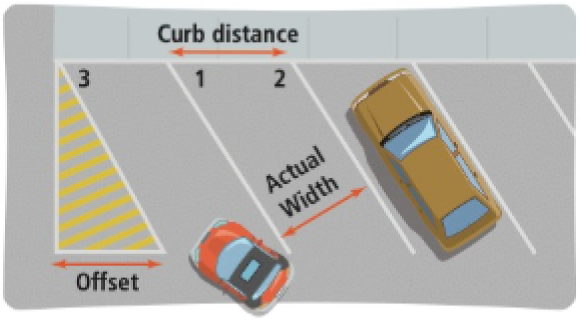
PRACTICE & PROBLEM SOLVING

APPLY

23. **Represent and Connect** A glazier is setting supports in parallel segments to prevent glass breakage during storms. What are the values of x and y? Justify your conclusions.



24. **Use Patterns and Structure** In the parking lot shown, all of the lines for the parking spaces should be parallel. If $m\angle 3 = 61$, what should $m\angle 1$ and $m\angle 2$ be? Explain.

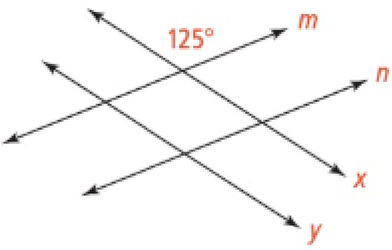


25. **Apply Math Models** Margaret is in a boat traveling due west. She turned the boat 50° north of due west for a couple of minutes to get around a peninsula. Then she resumed due west again.

    a. How many degrees would she turn the wheel to resume a due west course?

    b. Name the pair of angles she used. Are the angles congruent or supplementary?

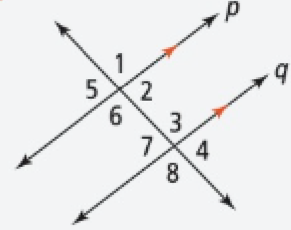
26. Parallel lines m and n intersect parallel lines x and y, representing two sets of intersecting railroad tracks. At what angles do the tracks intersect?



ASSESSMENT PRACTICE

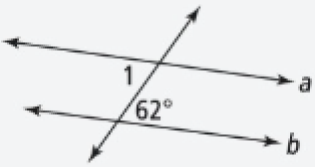
27. Select all true statements about the angles in the figure. GR.1.1

- ☐ A. $\angle 1 \cong \angle 3$
- ☐ B. $\angle 2 \cong \angle 5$
- ☐ C. $\angle 4 \cong \angle 8$
- ☐ D. $\angle 6 \cong \angle 7$
- ☐ E. $\angle 3 \cong \angle 6$



28. **SAT/ACT** In the diagram, $a \parallel b$. What is $m\angle 1$?

Ⓐ 28
Ⓑ 62
Ⓒ 90
Ⓓ 118



29. **Performance Task** Students on a scavenger hunt are given the map shown and several clues.

Part A The first clue states the following.

> Skyline Trail forms a transversal with Hood Path and Mission Path. Go to the corners that form same side exterior angles north of Skyline Trail.

Which two corners does the clue mean? Use intersections and directions to explain.

Part B If the second clue states the following, what trail marker should they go to?

> Hood and Mission Paths are parallel, and the northeast corner of Hood Path and Skyline Trail forms a 131° angle. The measure of the angle formed by the southwest corner of Skyline Trail and Mission Path is equal to the trail marker number on River Trail that you must go to.

2-2 Proving Lines Parallel

I CAN… use angle relationships to prove that lines are parallel.

VOCABULARY

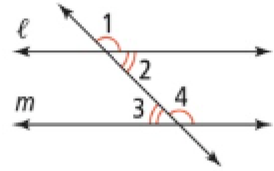
- flow proof

MA.912.GR.1.1–Prove relationships and theorems about lines and angles. Solve mathematical and real-world problems involving postulates, relationships and theorems of lines and angles.

MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

CRITIQUE & EXPLAIN

Juan analyzes the diagram to see if line ℓ is parallel to line *m*. His teacher asks if there is enough information to say whether the lines are parallel.



Yes, if a transversal intersects two parallel lines, then alternate interior angles are congruent and corresponding angles are congruent. I have both angle relationships here, so the lines are parallel.

A. **Analyze and Persevere** Why is Juan's statement correct or incorrect?

B. Can you use the Alternate Exterior Angles Theorem to prove that the lines are not parallel?

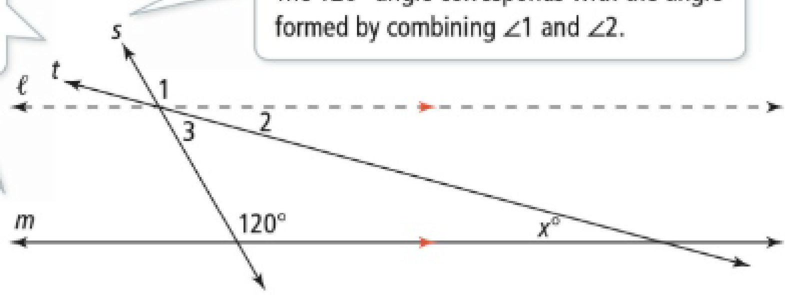
ESSENTIAL QUESTION

What angle relationships can be used to prove that two lines intersected by a transversal are parallel?

EXAMPLE 1 Understand Angle Relationships

Suppose two lines are not parallel. Can corresponding angles still be congruent?

Draw two nonparallel lines *t* and *m* and a transversal *s*. Draw line ℓ parallel to *m* that passes through the intersection of *s* and *t*.



Since ℓ ∥ *m*, $m\angle 1 = 120$ by the Corresponding Angles Theorem. By the Alternate Interior Angles Theorem, $m\angle 2 = x$. Since $m\angle 1 + m\angle 2 = 120 + x$, $m\angle 1 + m\angle 2 > 120$.

If two lines are not parallel, then corresponding angles are not congruent.

COMMUNICATE AND JUSTIFY
Consider the conclusion in the example. What statement can you write that is logically equivalent to this statement?

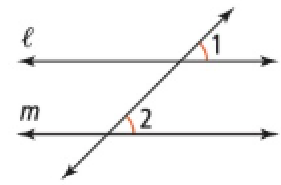
Try It! **1.** Could ∠3 be supplementary to a 120° angle? Explain.

THEOREM 2-4 Converse of the Corresponding Angles Theorem

If two lines and a transversal form corresponding angles that are congruent, then the lines are parallel.

PROOF: SEE EXERCISE 8.

If...



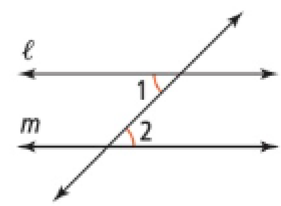
Then... $\ell \parallel m$

THEOREM 2-5 Converse of the Alternate Interior Angles Theorem

If two lines and a transversal form alternate interior angles that are congruent, then the lines are parallel.

PROOF: SEE EXAMPLE 2.

If...



Then... $\ell \parallel m$

PROOF

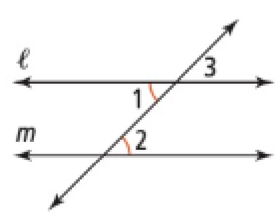
EXAMPLE 2 Write a Flow Proof of Theorem 2-5

Write a flow proof to prove the Converse of the Alternate Interior Angles Theorem.

In a **flow proof**, arrows show the logical connections between statements. Reasons are shown below the statements.

Given: $\angle 1 \cong \angle 2$

Prove: $\ell \parallel m$



Proof:

$\angle 1 \cong \angle 2$ — Given → $\angle 2 \cong \angle 3$

$\angle 1 \cong \angle 3$ — Vertical $\angle$s are $\cong$. → $\angle 2 \cong \angle 3$

$\angle 2 \cong \angle 3$ — Transitive Prop. of $\cong$ → $\ell \parallel m$ — Converse of the Corr. $\angle$s Thm.

COMMON ERROR

You may incorrectly write all information along one line of the flow proof. Remember that you should have two separate arrows when two statements are needed to justify the next statement in a proof.

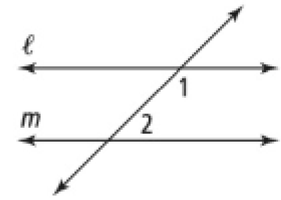
Try It! 2. Write a flow proof for Theorem 2-6, the Converse of the Same-Side Interior Angles Postulate.

THEOREM 2-6 Converse of the Same-Side Interior Angles Postulate

If two lines and a transversal form same-side interior angles that are supplementary, then the lines are parallel.

PROOF: SEE EXAMPLE 2 TRY IT.

If... $m\angle 1 + m\angle 2 = 180$



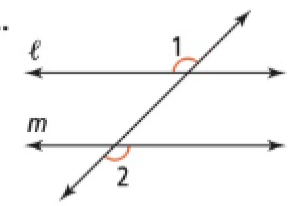
Then... $\ell \parallel m$

THEOREM 2-7 Converse of the Alternate Exterior Angles Theorem

If two lines and a transversal form alternate exterior angles that are congruent, then the lines are parallel.

PROOF: SEE EXERCISE 16.

If...



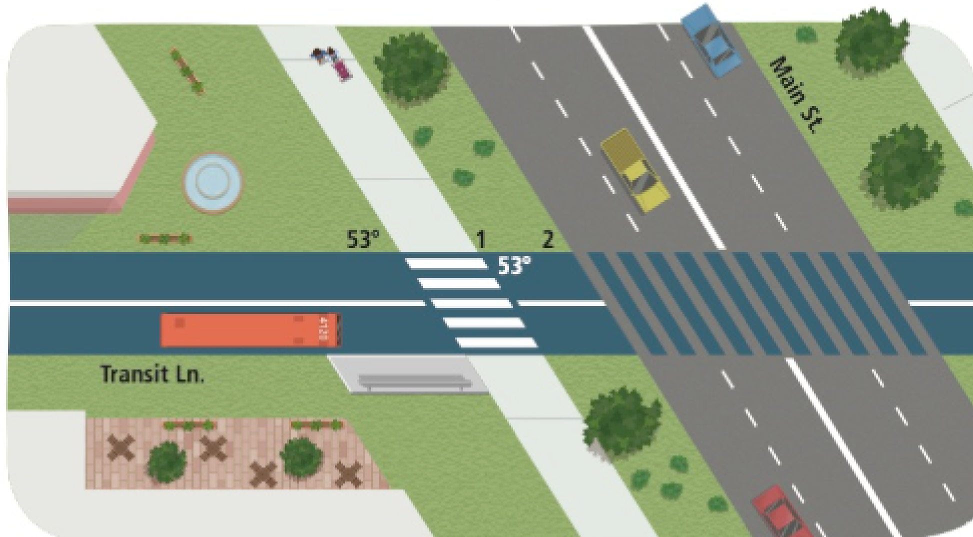
Then... $\ell \parallel m$

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Determine Whether Lines Are Parallel

The edges of a new sidewalk must be parallel in order to meet accessibility requirements. Concrete is poured between straight strings. How does an inspector know that the edges of the sidewalk are parallel?

The inspector can first measure the angles of corners of the sidewalk.



Since the two 53° angles are congruent, he can apply the Converse of the Alternate Exterior Angles Theorem. The edges of the sidewalk are parallel.

ANALYZE AND PERSEVERE
Think about what other theorems could be applied to determine parallel edges. What other measurements could the inspector make?

Try It! 3. What is $m\angle 1$? What should $\angle 2$ measure in order to guarantee that the sidewalk is parallel to Main Street? Explain.

THEOREM 2-8

If two lines are parallel to the same line, then they are parallel to each other.

PROOF: SEE EXERCISE 17.

If...

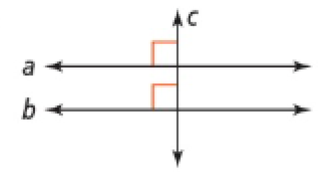
Then... $a \parallel b$

THEOREM 2-9

If two lines are perpendicular to the same line, then they are parallel to each other.

PROOF: SEE EXERCISE 18.

If...

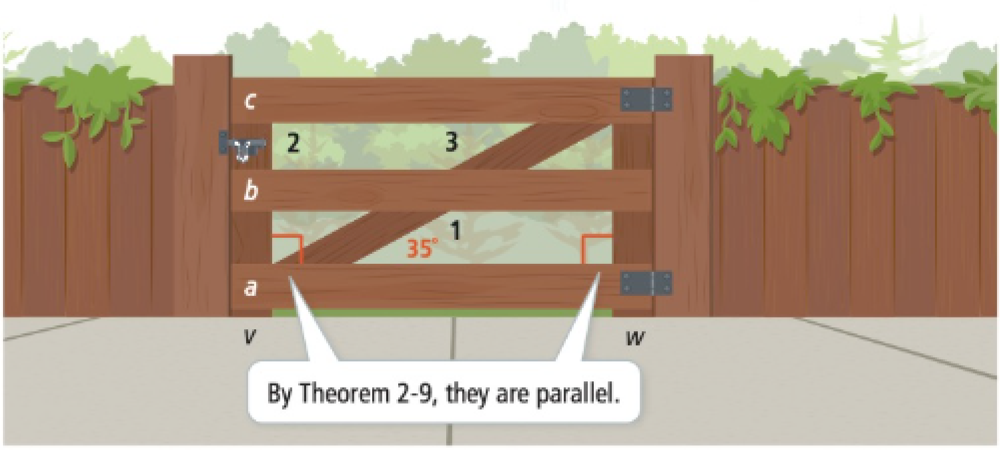


Then... $a \parallel b$

APPLICATION

EXAMPLE 4 Solve a Problem With Parallel Lines

A. When building a gate, how does Bailey know that the vertical boards *v* and *w* are parallel?



LEARN TOGETHER
How can you provide constructive feedback while being aware of the feelings and reactions of others?

B. What should ∠1 measure to ensure board *b* is parallel to board *a*?

Apply the Converse of the Same-Side Interior Angles Postulate.

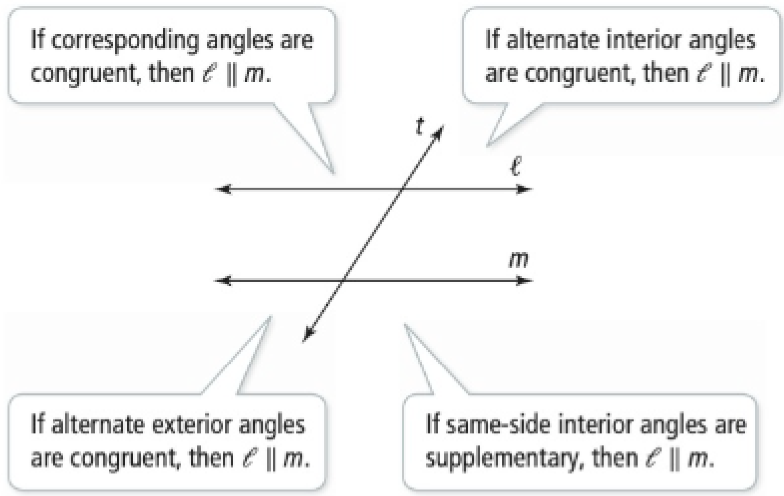
$$35 + m\angle 1 = 180$$
$$m\angle 1 = 145$$

Try It! **4. a.** Bailey also needs board *c* to be parallel to board *a*. What should ∠2 measure? Explain.

b. Is $b \parallel c$? Explain.

CONCEPT SUMMARY Criteria for Parallel lines

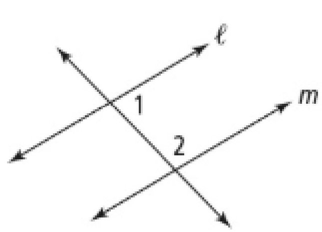
DIAGRAM



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What angle relationships can be used to prove that two lines intersected by a transversal are parallel?

2. **Error Analysis** Noemi wrote, "If $\angle 1 \cong \angle 2$, then by the Converse of the Same-Side Interior Angles Postulate, $\ell \parallel m$." Explain the error in Noemi's reasoning.

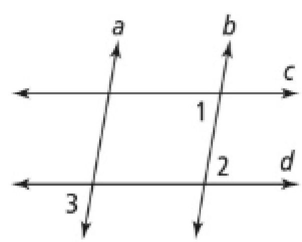


3. **Vocabulary** How does a *flow proof* show logical steps in the proof of a conditional statement?

4. **Analyze and Persevere** How is Theorem 2-9 a special case of the Converse of the Corresponding Angles Theorem?

Do You KNOW HOW?

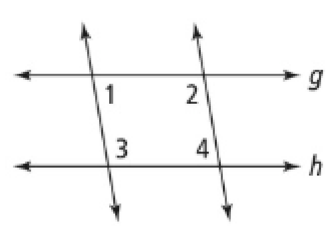
Use the figure shown for Exercises 5 and 6.



5. If $\angle 1 \cong \angle 2$, which theorem proves that $c \parallel d$?

6. If $m\angle 2 = 4x - 6$ and $m\angle 3 = 2x + 18$, for what value of x is $a \parallel b$? Which theorem justifies your answer?

7. Using the Converse of the Same-Side Interior Angles Postulate, what equation could be used to prove that $g \parallel h$?



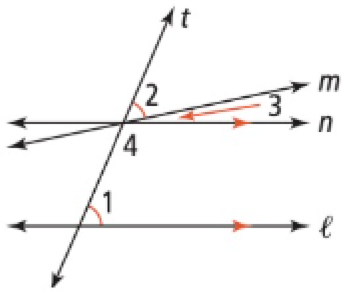
PRACTICE & PROBLEM SOLVING

UNDERSTAND

8. **Communicate and Justify** Write an indirect proof of the Converse of the Corresponding Angles Theorem following the outline below.

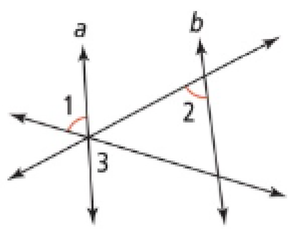
Given: $\angle 1 \cong \angle 2$

Prove: $\ell \parallel m$



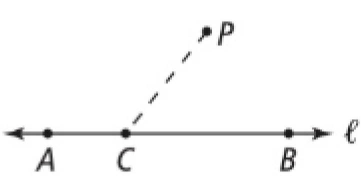
- Assume that lines ℓ and m are not parallel.
- Draw line n parallel to line ℓ.
- Conclude that $m\angle 3 > 0$.
- Use the Same-Side Interior Angles Postulate to arrive at the contradiction that $m\angle 1 \neq m\angle 2$.

9. **Error Analysis** What is the student's error?

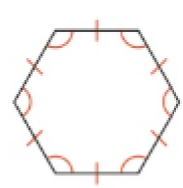


Given $\angle 1 \cong \angle 2$. By the Vertical Angles Thm., $\angle 1 \cong \angle 3$, so by the Transitive Property, $\angle 2 \cong \angle 3$. By the Converse of the Corresponding Angles Thm., $a \parallel b$. ✗

10. **Mathematical Connections** Copy the figure below. Construct a line through P parallel to ℓ. (*Hint*: Copy either $\angle PCA$ or $\angle PCB$ so that one of the sides of the angle is parallel to ℓ.) What theorem justifies your construction?

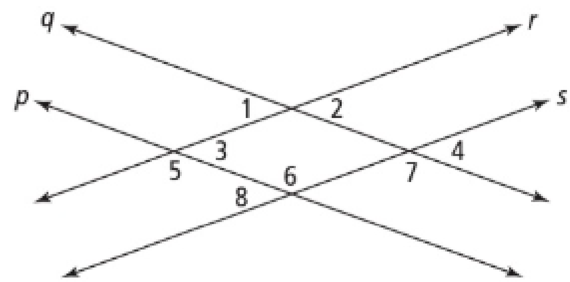


11. **Higher Order Thinking** The interior angles of a regular hexagon are congruent. Why are any pair of opposite sides parallel?



PRACTICE

For Exercises 12–15, use the given information. Which lines in the figure can you conclude are parallel? State the theorem that justifies each answer. SEE EXAMPLES 1 AND 3



12. $\angle 2 \cong \angle 3$

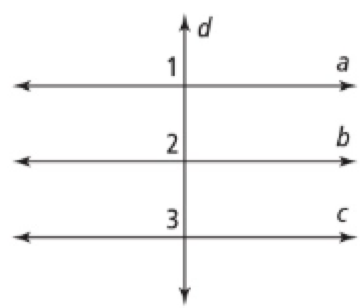
13. $\angle 6 \cong \angle 7$

14. $\angle 1 \cong \angle 4$

15. $m\angle 5 + m\angle 8 = 180°$

16. Write a flow proof of the Converse of the Alternate Exterior Angles Theorem. SEE EXAMPLE 2

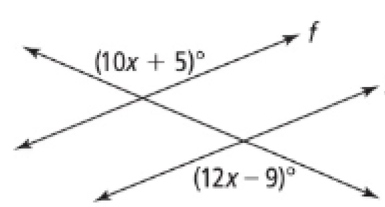
Use the figure for Exercises 17 and 18. SEE EXAMPLE 2



17. Given $a \parallel c$ and $b \parallel c$, write a flow proof of Theorem 2-8.

18. Given $a \perp d$ and $b \perp d$, write a flow proof of Theorem 2-9.

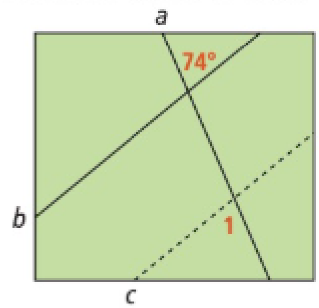
19. For what value of x is $f \parallel g$? Which theorem justifies your answer? SEE EXAMPLE 4



PRACTICE & PROBLEM SOLVING

APPLY

20. Use Patterns and Structure To make a puzzle, Denzel draws lines a and b to cut along on a square piece of posterboard. He wants to draw line c so that it is parallel to line b. What should the measure of $\angle 1$ be? Explain.

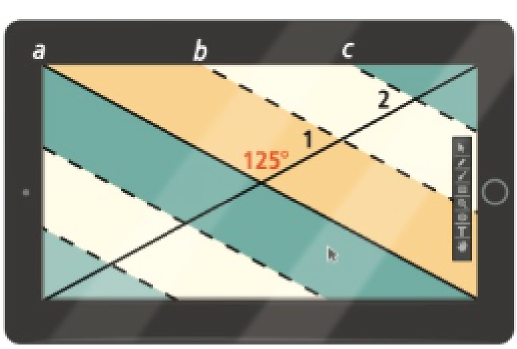


21. Communicate and Justify A downhill skier is fastest when her skis are parallel. What should $\angle 1$ be in order for the skier to maximize her speed through a gate? Which theorem justifies your answer?

22. Analyze and Persevere Malia makes a fabric design by drawing diagonals between opposite corners. She wants to draw other lines parallel to one of the diagonal lines, as shown by the dashed lines.

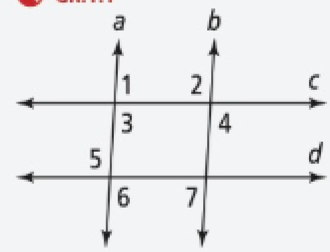
a. What should $\angle 1$ be in order for line b to be parallel to line a? Explain.

b. What should $\angle 2$ be in order for line c to be parallel to line b? Explain.

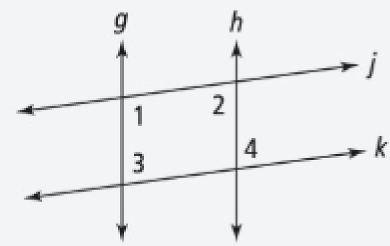


ASSESSMENT PRACTICE

23. In order for $c \parallel d$, how must $\angle 2$ and $\angle 7$ be related? How must $\angle 3$ and $\angle 5$ be related?
GR.1.1

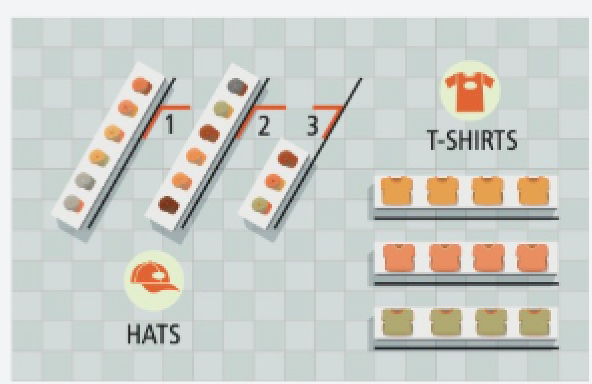


24. SAT/ACT Which statement must always be true?



Ⓐ If $\angle 1 \cong \angle 2$, then $g \parallel h$.
Ⓑ If $\angle 1 \cong \angle 3$, then $g \parallel h$.
Ⓒ If $\angle 2 \cong \angle 4$, then $j \parallel k$.
Ⓓ If $\angle 3 \cong \angle 4$, then $j \parallel k$.

25. Performance Task The diagram shows part of a plan to arrange aisles in a store.



Part A The aisles are arranged so that $m\angle 1 = 125$. What should be the measures of the other labeled angles so that all three aisles will be parallel? Explain.

Part B Describe how theorems can be applied to make sure that the T-shirt aisles are parallel.

2-3

Parallel Lines and Triangle Angle Sums

I CAN… solve problems using the measures of interior and exterior angles of triangles.

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles.

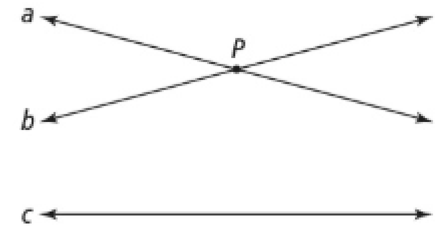
MA.K12.MTR.5.1, MTR.1.1, MTR.3.1

EXPLORE & REASON

Powered By desmos

Two parallel lines never intersect. But, can two lines that intersect ever be parallel to the same line?

Draw point *P*. Then draw lines *a* and *b* that intersect at point *P* as shown.



A. Place a pencil below the intersecting lines on your paper to represent line *c*. Rotate the pencil so that it is parallel to line *b*. Can you rotate the pencil so that it is parallel to line *a* at the same time as line *b*?

B. **Use Patterns and Structure** Can you adjust your drawing of the two intersecting lines so you can rotate the pencil to be parallel to both lines?

ESSENTIAL QUESTION **What is true about the interior and exterior angle measures of a triangle?**

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Investigate the Measures of Triangle Angles

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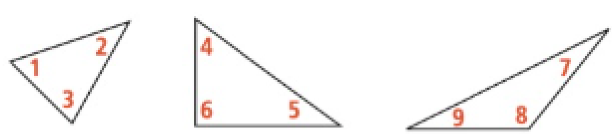
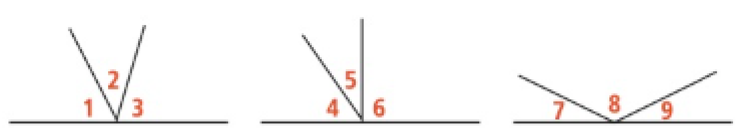
What appears to be the relationship between the angle measures of a triangle?

Using pencil and paper, or scissors and paper, construct several triangles of different types. Number the angles.



Trace the angles from each triangle and place the vertices together with the sides of the angles sharing a vertex and ray.



For each triangle shown, the angles combine to form a straight angle.

So, the sum of the angle measures of a triangle appears to be 180.

REPRESENT AND CONNECT
What tools might you use to confirm that the angles sum to 180°?

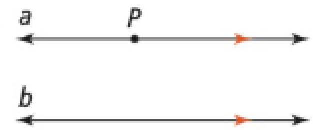
Try It! **1.** Given two angle measures in a triangle, can you find the measure of the third angle? Explain.

THEOREM 2-10

Through a point not on a line, there is one and only one line parallel to the given line.

PROOF: SEE EXERCISE 10.

If...



Then... line a is the only line parallel to line b through P.

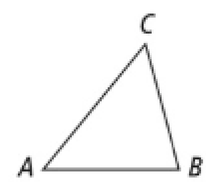
THEOREM 2-11 Triangle Angle-Sum Theorem

Powered By desmos

The sum of the measures of all the angles of a triangle is 180.

PROOF: SEE EXAMPLE 2.

If...



Then... $m\angle A + m\angle B + m\angle C = 180$

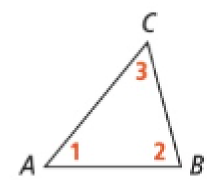
PROOF

EXAMPLE 2 Prove the Triangle Angle-Sum Theorem

Prove the Triangle Angle-Sum Theorem.

Given: $\triangle ABC$

Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180$



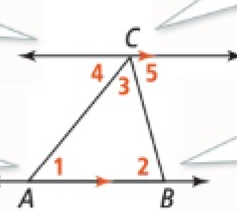
STUDY TIP
When using a geometric figure in a proof, you can construct additional parts to help with the proof, such as parallel lines, angle bisectors, and midpoints.

Plan: Draw a line through C, because a straight angle measures 180. This line should be parallel to the line containing $\overline{AB}$ so that an alternate interior angle relationship is formed.

Proof:

Use Theorem 2-10 to justify drawing a line through C parallel to $\overleftrightarrow{AB}$.

By applying the Angle Addition Postulate, $m\angle 4 + m\angle 3 + m\angle 5 = 180°$.



By the Alternate Interior Angles Theorem, $\angle 1 \cong \angle 4$, so $m\angle 1 = m\angle 4$.

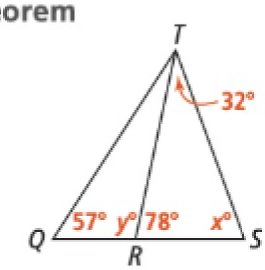
Also by the Alternate Interior Angles Theorem, $\angle 2 \cong \angle 5$, so $m\angle 2 = m\angle 5$.

By substitution, $m\angle 1 + m\angle 3 + m\angle 2 = 180$. Therefore, using the Commutative Property of Addition, $m\angle 1 + m\angle 2 + m\angle 3 = 180$.

Try It! 2. How does Theorem 2-10 justify the construction of the line through C that is parallel to $\overleftrightarrow{AB}$?

EXAMPLE 3 Use the Triangle Angle-Sum Theorem

What are the values of *x* and *y*?



Write and solve an equation that relates the measures of the angles of $\triangle TRS$.

$32 + 78 + x = 180$

$x = 70$

Use the Triangle Angle-Sum Theorem.

To find the value of *y*, notice that $\angle QRS$ is a straight angle.

$m\angle QRT + m\angle TRS = 180$

$y + 78 = 180$

$y = 102$

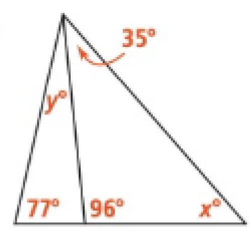
Apply the Angle Addition Postulate.

The value of *x* is 70 and the value of *y* is 102.

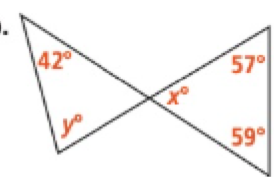
HAVE A GROWTH MINDSET
How can you take on challenges with positivity?

Try It! **3.** What are the values of *x* and *y* in each figure?

a.



b.



THEOREM 2-12 Triangle Exterior Angle Theorem

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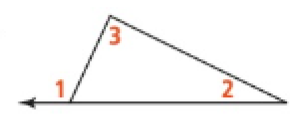
The measure of each exterior angle of a triangle equals the sum of the measures of its two remote interior angles.

PROOF: SEE EXERCISE 13.

If...

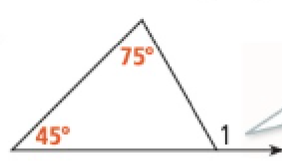


Then... $m\angle 1 = m\angle 2 + m\angle 3$

EXAMPLE 4 Apply the Triangle Exterior Angle Theorem

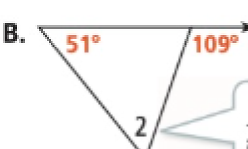
What is the missing angle measure in each figure?

A.



$\angle 1$ is an exterior angle.

B.



$\angle 2$ is a remote interior angle.

$m\angle 1 = 45 + 75$

$m\angle 1 = 120$

Use the Exterior Angles Theorem.

$109 = 51 + m\angle 2$

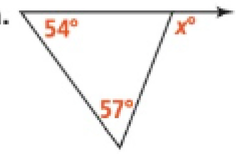
$58 = m\angle 2$

COMMON ERROR
Be careful when writing an equation to solve for the unknown angle measure. The unknown value can be an addend or the sum.

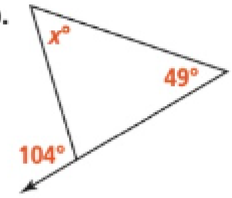
CONTINUED ON THE NEXT PAGE

Try It! **4.** What is the value of x in each figure?

a.



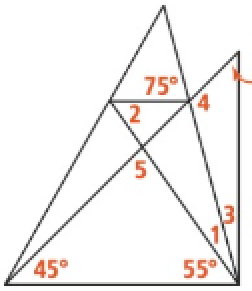
b.



APPLICATION

EXAMPLE 5 Apply the Triangle Theorems

Cheyenne built this display for her ornament collection. Each shelf is parallel to the base. She recalls only the angle measures shown in the diagram. Now she wants to build another just like it. What are the measures of ∠1, ∠2, and ∠3?



Formulate ◀ Begin by writing equations for the unknown angle measures. Since the bottom and top shelves are parallel, apply the Corresponding Angles Theorem.

$$55 + m\angle 1 = 75$$

Apply the Triangle Exterior Angle Theorem.

$$m\angle 1 + m\angle 2 = 75$$

Use the Triangle Angle-Sum Theorem.

$$45 + 45 + (m\angle 1 + m\angle 3 + 55) = 180$$

Compute ◀ Solve for $m\angle 1$, $m\angle 2$, and $m\angle 3$.

$$55 + m\angle 1 = 75 \qquad m\angle 1 + m\angle 2 = 75$$

$$m\angle 1 = 20 \qquad 20 + m\angle 2 = 75$$

$$m\angle 2 = 55$$

$$45 + 45 + (m\angle 1 + m\angle 3 + m\angle 2) = 180$$

$$45 + 45 + (20 + m\angle 3 + 55) = 180$$

$$m\angle 3 + 165 = 180$$

$$m\angle 3 = 15$$

Interpret ◀ The measures of the angles are $m\angle 1 = 20$, $m\angle 2 = 55$, and $m\angle 3 = 15$.

Try It! **5.** What are the measures of ∠4 and ∠5? Explain.

CONCEPT SUMMARY Angle Measures of Triangles

WORDS

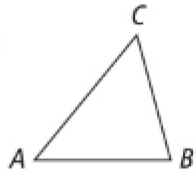
Interior Angle Measures

The sum of the measures of all the angles of a triangle is 180.

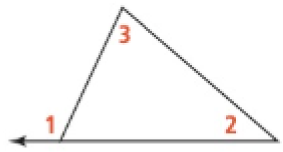
Exterior Angle Measure

The measure of each exterior angle of a triangle equals the sum of the measures of its two remote interior angles.

DIAGRAM



$m\angle A + m\angle B + m\angle C = 180$

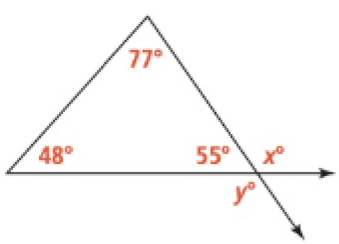


$m\angle 1 = m\angle 2 + m\angle 3$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What is true about the interior and exterior angle measures of a triangle?

2. **Error Analysis** Chiang determined that the value of x is 103 and the value of y is 132 in the figure below. What mistake did Chiang make?



3. **Vocabulary** The word *remote* means distant or far apart. What parts of a figure are *remote interior angles* distant from?

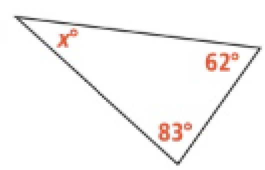
4. **Use Patterns and Structure** Use the Triangle Angle-Sum Theorem to answer the following questions. Explain your answers.

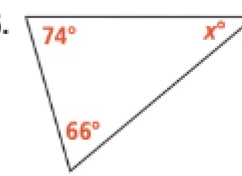
   a. What are the measures of each angle of an equiangular triangle?

   b. If one of the angle measures of an isosceles triangle is 90, what are the measures of the other two angles?

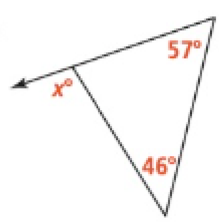
Do You KNOW HOW?

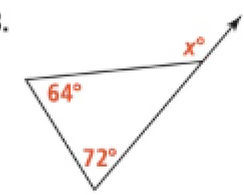
What is the value of *x* in each figure?

5. 

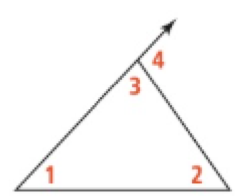
6. 

What is the value of *x* in each figure?

7. 

8. 

9. Write an equation relating the measures of $\angle 1$, $\angle 2$, and $\angle 3$. Write another equation relating the measures of $\angle 1$, $\angle 2$, and $\angle 4$.



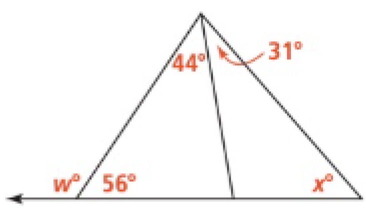
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. Communicate and Justify Write a proof of Theorem 2-10.

11. Higher Order Thinking Marisol claims that each pair of remote interior angles in a triangle has two exterior angles. Do you agree? Use a diagram to support your answer.

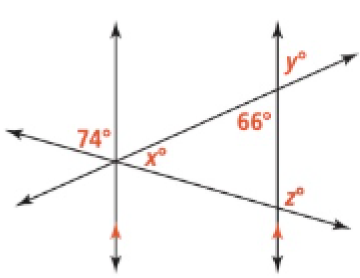
12. Error Analysis A student was asked to find the value of *x*. What error did the student make?



> By the Linear Pairs Theorem,
> $w + 56 = 180$, so $w = 124$.
> By the Triangle Exterior Angle Theorem, $w = x + 31$, or
> $124 = x + 31$, so $x = 93$. ✗

13. Choose Efficient Methods Prove the Triangle Exterior Angle Theorem.

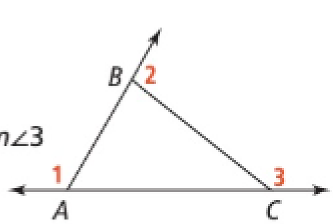
14. Mathematical Connections What are the values of *x*, *y*, and *z*? Use theorems to justify each answer.



15. Use Patterns and Structure Prove that the sum of the exterior angles of a triangle have a sum of 360°.

Given: ∠1, ∠2, and ∠3 are exterior angles

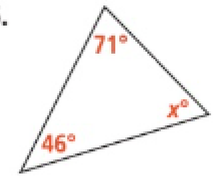
Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 360$



PRACTICE

What are the values of the variables in each figure? SEE EXAMPLES 1–3

16.



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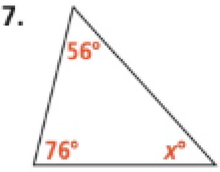

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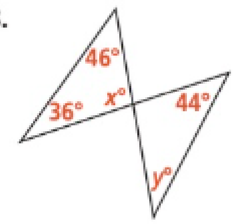
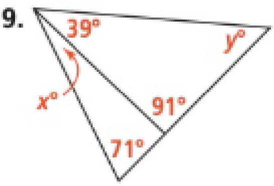
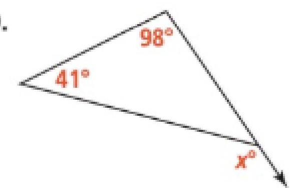


19.



What is the value of *x* in each figure? SEE EXAMPLE 4

20.



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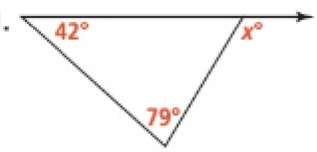

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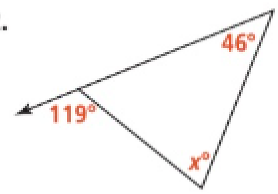

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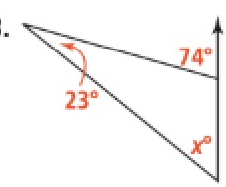
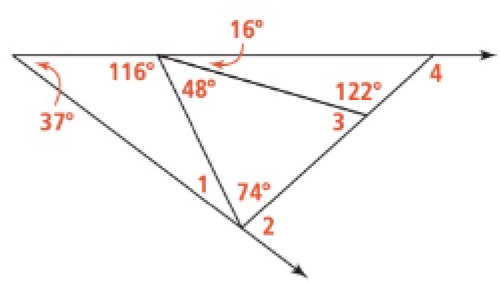


For Exercises 24–27, find the measure of each angle. SEE EXAMPLE 4



24. ∠1

25. ∠2

26. ∠3

27. ∠4

28. A flag on a golf course is in the shape of an isosceles triangle. One leg of the triangle is fastened to a pole and forms an 84° angle with the pole. What is the measure of each remote interior angle?

PRACTICE & PROBLEM SOLVING

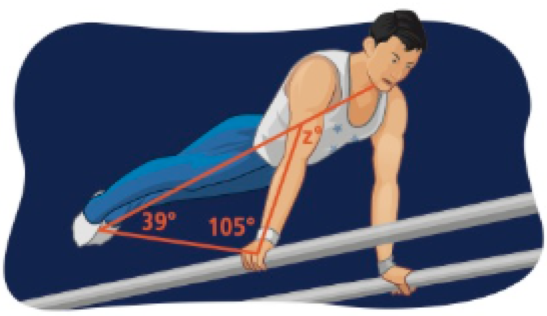
APPLY

29. **Apply Math Models** Pilar is making a replacement set of sails for a sailboat.

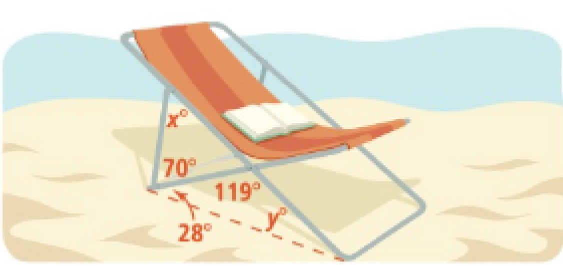
a. What equation can Pilar use that relates the values of w and x?

b. What equation can Pilar use that relates the values of y and z?

30. **Analyze and Persevere** An artist painting from a photo begins with a geometric sketch to match angle measures. What is the value of z?



31. **Use Patterns and Structure** Use the figure shown.



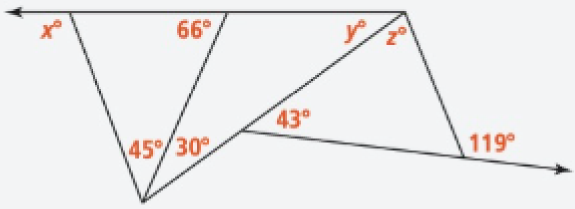
a. What is the value of x?

b. What is the value of y?

c. The chair can lay farther back so that the 70° angle changes to 86° and x° changes to 36°. How does this affect the 119° angle?

ASSESSMENT PRACTICE

32. What are the values of x, y, and z? GR.1.3



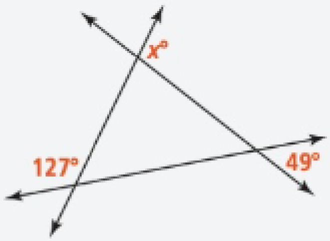
Ⓐ $x = 24°$, $y = 30°$, $z = 61°$

Ⓑ $x = 111°$, $y = 36°$, $z = 76°$

Ⓒ $x = 119°$, $y = 43°$, $z = 66°$

Ⓓ $x = 114°$, $y = 39°$, $z = 94°$

33. **SAT/ACT** What is the value of x?



Ⓐ 98 Ⓒ 102

Ⓑ 106 Ⓓ 176

34. **Performance Task** A tablet case is supported at the back. The measure of the slant angle of the tablet can be changed, but $m\angle 2 = m\angle 3$ for any slant that is chosen.

Part A A user adjusts the case so that $m\angle 2 = 42$. What are the measures of the other angles?

Part B Is it possible to slant the tablet case so that $m\angle 1 = m\angle 5$? If so, explain how. If not, explain why it is not possible.

Part C A user wants to slant the tablet case so that $m\angle 1 = 2(m\angle 5)$. What should the measure of each of the five angles be?

2-4 Slopes of Parallel and Perpendicular Lines

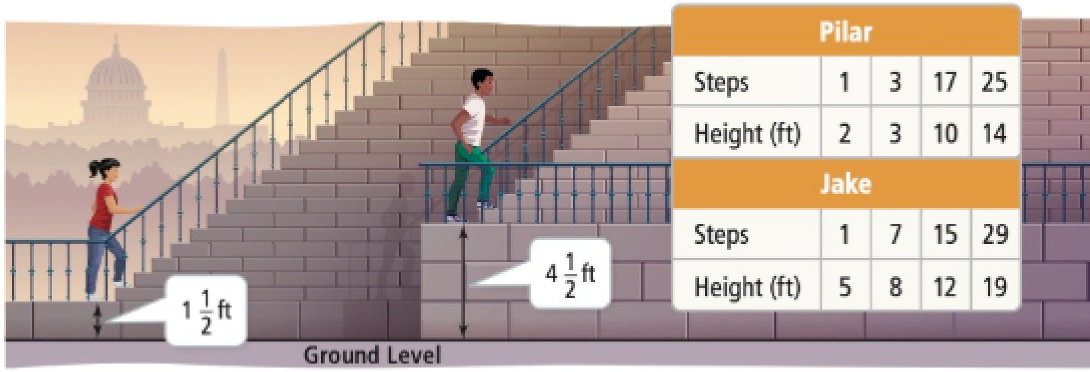
I CAN… use slope to solve problems about parallel and perpendicular lines.

Prepares for **MA.912.GR.3.3**–Use coordinate geometry to solve mathematical and real-world geometric problems involving lines, circles, triangles and quadrilaterals.

MA.K12.MTR.4.1, MTR.6.1, MTR.7.1

MODEL & DISCUSS

Pilar and Jake begin climbing to the top of a 100-ft monument at the same time along two different sets of steps at the same rate. The tables show their distances above ground level after a number of steps.



Pilar				
Steps	1	3	17	25
Height (ft)	2	3	10	14

Jake				
Steps	1	7	15	29
Height (ft)	5	8	12	19

A. How many feet does each student climb after 10 steps? Explain.

B. Will Pilar and Jake be at the same height after the same number of steps? Explain.

C. **Analyze and Persevere** What would you expect the graphs of each to look like given your answers to parts A and B? Explain.

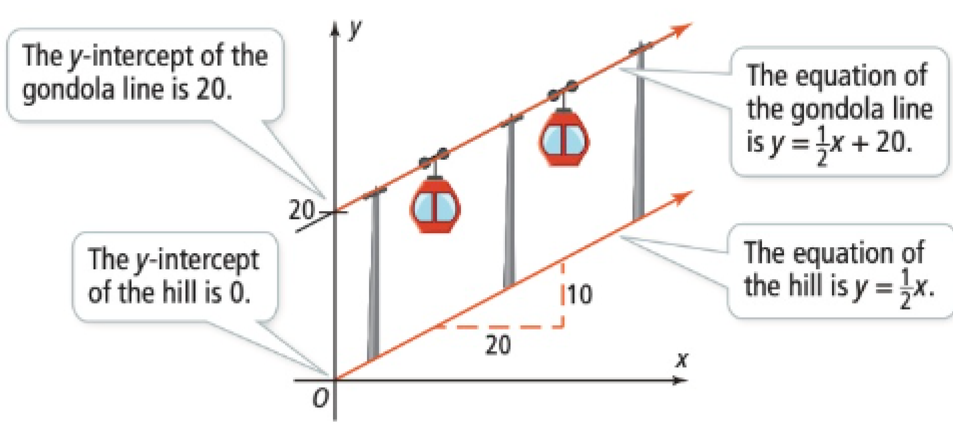
ESSENTIAL QUESTION How do the slopes of lines that are parallel to each other compare? How do the slopes of lines that are perpendicular to each other compare?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Slopes of Parallel Lines

A hill and a gondola line 20 ft above the ground that goes up the hill both have slope $\frac{1}{2}$. What is the geometric relationship between the hill and the gondola line?

Model the hill and gondola line on a coordinate plane where *x* represents the horizontal distance from the base of the hill and *y* represents the vertical distance from the base of the hill.



Because the slope of the hill is $\frac{1}{2}$, the hill gains one foot of height for every two feet of horizontal distance. The same is true for the gondola. It never gets any closer or farther away from the hill.

Conjecture: If two linear equations have the same slope, then the graphs of the equations are parallel.

APPLY MATH MODELS
Would you describe a different relationship between the slopes of parallel lines if the *y*-intercept for the hill were not at (0, 0)?

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Try It! **1.** Suppose another line for a chair lift is placed at a constant distance c below the gondola line. What is an equation of the new line? Is the new line also parallel to the hill? Explain.

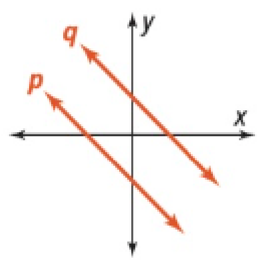
THEOREM 2-13

Two non-vertical lines are parallel if and only if their slopes are equal.

Any two vertical lines are parallel.

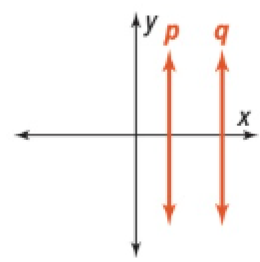
PROOF: SEE LESSON 7-5.

If... p and q are both not vertical



Then... $p \parallel q$ if and only if the slope of line p = slope of line q

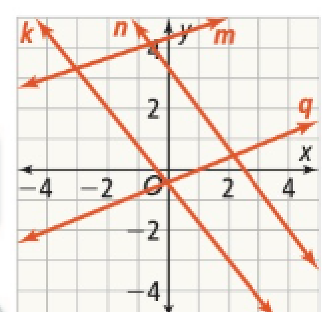
If... p and q are both vertical



Then... $p \parallel q$

EXAMPLE 2 Check Parallelism

Are lines k and n parallel?



COMMON ERROR
Be sure that the first numbers in both subtraction expressions are the coordinates of the same point.

Step 1 Find the slope of line k.

$$m = \frac{-3 - 2}{2 - (-2)} = -\frac{5}{4}$$

Line k passes through $(-2, 2)$ and $(2, -3)$.

Step 2 Find the slope of line n.

$$m = \frac{-2 - 2}{4 - 1} = -\frac{4}{3}$$

Line n passes through $(1, 2)$ and $(4, -2)$.

Step 3 Compare the slopes.

Parallel lines have equal slope, but $-\frac{5}{4} \neq -\frac{4}{3}$. Thus, lines k and n are not parallel.

Try It! **2.** Are lines m and q parallel?

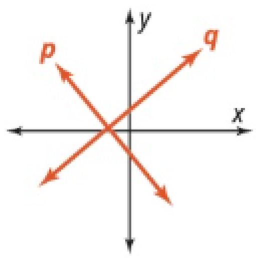
THEOREM 2-14

Two non-vertical lines are perpendicular if and only if the product of their slopes is −1.

A vertical line and a horizontal line are perpendicular to each other.

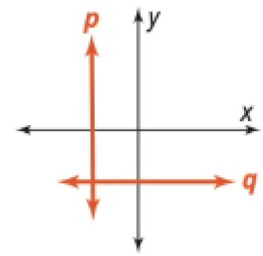
PROOF: SEE LESSON 7-4.

If... p and q are both not vertical



Then... $p \perp q$ if and only if the product of their slopes is −1

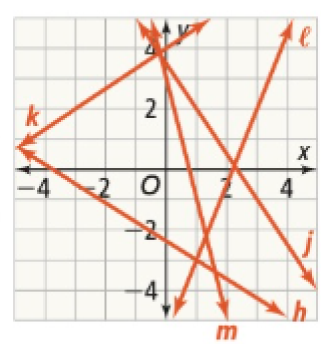
If... one of p and q is vertical and the other is horizontal



Then... $p \perp q$

EXAMPLE 3 Check Perpendicularity

Are lines j and k perpendicular?



STUDY TIP
Look for two points on each line where you can easily read the coordinates of the points from the graph.

Step 1 Find the slope of line j.

$$m = \frac{2-5}{1-(-1)} = -\frac{3}{2}$$

Line j passes through (−1, 5) and (1, 2).

Step 2 Find the slope of line k.

$$m = \frac{4-2}{0-(-3)} = \frac{2}{3}$$

Line k passes through (−3, 2) and (0, 4).

Step 3 Compare the slopes.

Perpendicular lines have slopes with a product of −1, and $-\frac{3}{2} \cdot \frac{2}{3} = -1$. Thus, lines j and k are perpendicular.

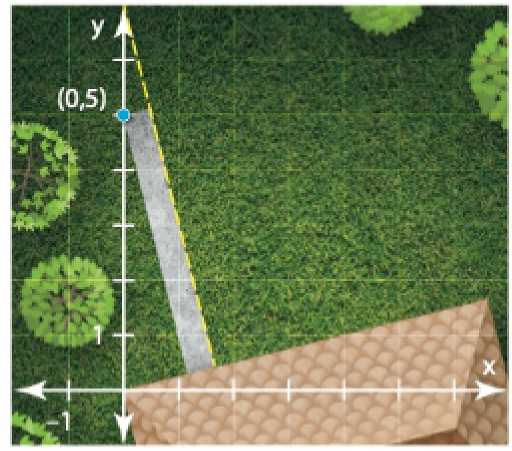
Try It! **3. a.** Are lines h and ℓ perpendicular?

b. Are lines k and m perpendicular?

APPLICATION

EXAMPLE 4 Use Slope to Solve a Problem

A landscape architect is designing her own back yard and she wants the edges of her walkway to be perpendicular to the back of the house. Can she use her design or will she need to revise it? Explain.



Step 1 Find the slope of each edge of the walkway.

One edge of the walkway passes through (0, 5) and (1, 1).

$m = \frac{1-5}{1-0} = -4$

The other edge of the walkway passes through (0, 7) and (1, 3).

$m = \frac{3-7}{1-0} = -4$

The slope of each side of the walkway are equal, so the sides are parallel.

Step 2 Find the slope of the edge of the house.

The edge of the house passes through (0, 0) and (8, 2).

$m = \frac{2-0}{8-0} = \frac{1}{4}$

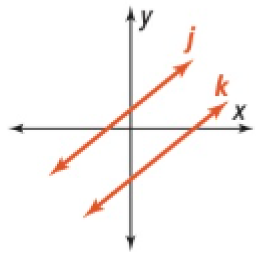
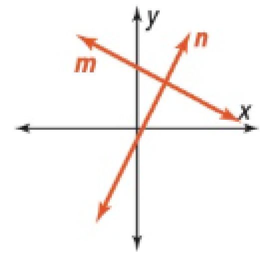
Step 3 Compare the slopes.

Since $-4 \cdot \frac{1}{4} = -1$, the edges of the walkway and the house are perpendicular. The architect can use her design.

ASSESS FOR REASONABLENESS
Before stating that two lines are perpendicular, always check that the product of the slopes equals –1.

Try It! 4. The architect wants the front of the shed in the back corner to be parallel to the back of the house. If one corner of the shed is at (0, 5), what point can the front line of the shed pass through?

CONCEPT SUMMARY Slopes of Parallel and Perpendicular Lines

	Parallel Lines	Perpendicular Lines
DIAGRAMS		
SYMBOLS	$j \parallel k$ if and only if the slopes are the same.	$m \perp n$ if and only if the product of the two slopes is -1.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do the slopes of lines that are parallel to each other compare? How do the slopes of lines that are perpendicular to each other compare?

2. **Represent and Connect** Give an equation for a line perpendicular to the line $y = 0$. Is there more than one such line? Explain.

3. **Communicate and Justify** What are two different if-then statements implied by Theorem 2-13?

4. **Error Analysis** Devin said that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ for $A(-2, 0)$, $B(2, 3)$, $C(1, -1)$, and $D(5, -4)$ are parallel. Explain and correct Devin's error.

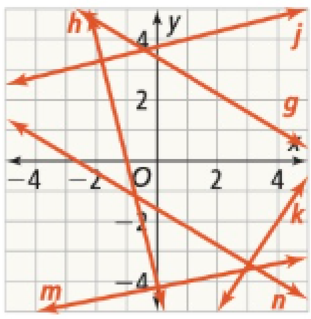
slope of $\overleftrightarrow{AB}$: $\frac{3-0}{2-(-2)} = \frac{3}{4}$

slope of $\overleftrightarrow{CD}$: $\frac{-1-(-4)}{5-1} = \frac{3}{4}$

slopes are equal, so $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ ✗

Do You KNOW HOW?

Use the diagram for Exercises 6–9.



5. Are lines g and n parallel?

6. Are lines j and m parallel?

7. Are lines n and k perpendicular?

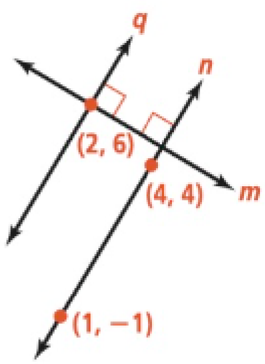
8. Are lines h and j perpendicular?

9. The graph of a roller coaster track goes in a straight line through coordinates (10, 54) and (42, 48), with coordinates in feet. A support beam runs parallel 12 feet below the track. What is the slope of the support beam?

PRACTICE & PROBLEM SOLVING

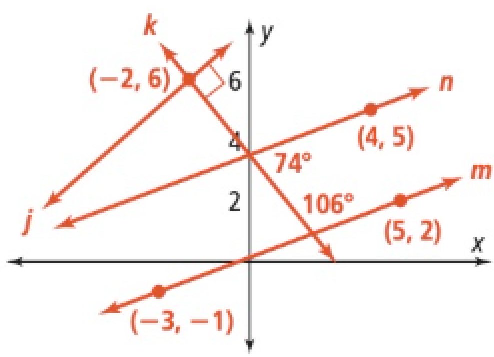
UNDERSTAND

10. Use Patterns and Structure What are the equations of lines m and q?



11. Generalize Why can you not say that two vertical lines have equal slope? Why can you not say that the product of the slopes of a vertical and horizontal line is −1?

12. Higher Order Thinking Lines k and n intersect on the y-axis.



a. What is the slope of line k?

b. What is the slope of line j?

13. Communicate and Justify Line m passes through points X and Y. Line n passes through points X and Z. If m and n have equal slope, what can you conclude about points X, Y, and Z? Explain.

14. Error Analysis Shannon says that lines with slopes -3, $-\frac{1}{3}$, -3, and $\frac{1}{3}$ could represent the sides of a rectangle. Explain Shannon's error.

PRACTICE

Compare the slopes of the lines for $y = f(x)$ and $y = g(x)$ to determine if each pair of lines is parallel.

SEE EXAMPLE 1

15.

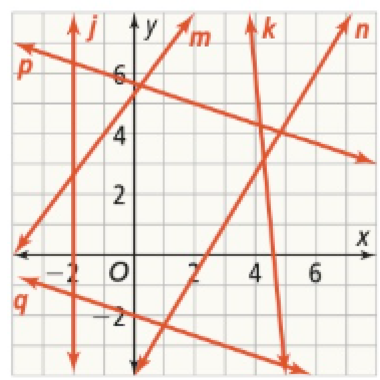
x	$f(x)$	$g(x)$
0	20	22
1	35	37
2	50	52
3	65	67

16.

x	$f(x)$	$g(x)$
0	5	10
1	7	15
2	9	20
3	11	25

Determine if each pair of lines is parallel.

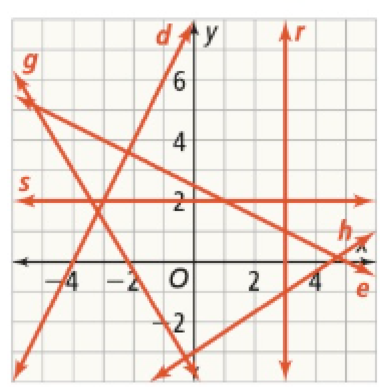
SEE EXAMPLE 2



17. j and k **18.** m and n **19.** p and q

Determine if each pair of lines is perpendicular.

SEE EXAMPLE 3



20. d and e **21.** g and h **22.** r and s

Line w passes through the points (−2, 6) and (1, −3).

23. Give a second pair of coordinates on the line parallel to w that passes through (2, 4).

24. Give a second pair of coordinates on the line perpendicular to w that passes through (3, 3).

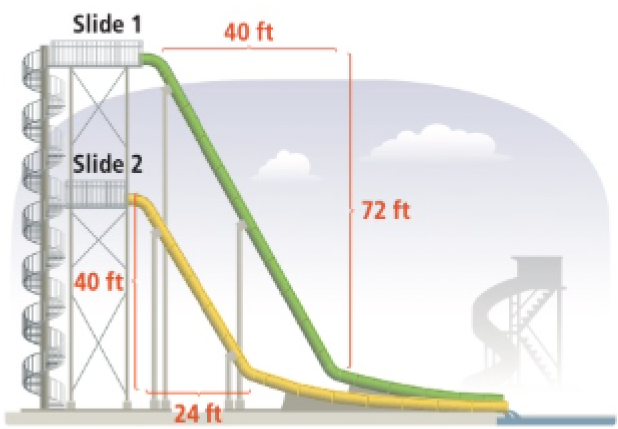
PRACTICE & PROBLEM SOLVING

APPLY

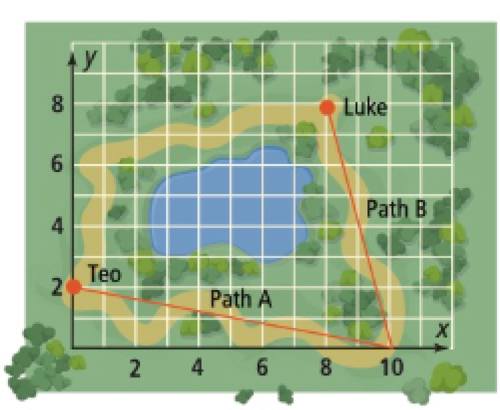
25. **Represent and Connect** The table shows locations of several sites at a high school campus. A landscaper wants to connect two sites with a path perpendicular to the path connecting the cafeteria and the library. Which two sites should he connect?

Locations	
Cafeteria (5, 5)	Library (11, 14)
Office (4, 12)	Gym (15, 8)
Woodshop (11, 6)	Art Studio (3, 16)

26. **Analyze and Persevere** Are the steepest parts of the two water slides parallel? Explain.



27. **Mathematical Connections** Teo rides his bike in a straight line from his location, perpendicular to path A, and Luke rides his bike in a straight line from his location, perpendicular to path B. What are the coordinates of the point where they meet?



ASSESSMENT PRACTICE

28. For the points $A(-3, 2)$, $B(-3, 6)$, and $C(-1, 6)$, find the slope of the line through B perpendicular to $\overleftrightarrow{AC}$. GR.3.3

Ⓐ -2

Ⓑ $-\frac{1}{2}$

Ⓒ $\frac{1}{2}$

Ⓓ 2

29. **SAT/ACT** Line k passes through $(2, -3)$ and $(8, 1)$. What would be the slope of a line that is parallel to k?

Ⓐ $-\frac{3}{2}$

Ⓑ $-\frac{2}{3}$

Ⓒ $\frac{2}{3}$

Ⓓ $\frac{3}{2}$

30. **Performance Task** A knight travels in a straight line from the starting point to Token 1. The knight can only make right-angle turns to get to Tokens 2 and 3.

Part A Since the knight can only make right-angle turns, what are the slopes of the straight line paths the knight can travel?

Part B What is the fewest number of turns that the knight can take in order to get all three tokens?

MATHEMATICAL MODELING IN 3 ACTS

Prepares for MA.912.GR.3.3–Use coordinate geometry to solve mathematical and real-world geometric problems involving lines, circles, triangles and quadrilaterals.
MA.K12.MTR.7.1

Parallel Paving Company

Powered By desmos

Building roads consists of many different tasks. Once civil engineers have designed the road, they work with surveyors and construction crews to clear and level the land. Sometimes specialists have to blast away rock in order to clear the land. Once the land is leveled, the crews bring in asphalt pavers to smooth out the hot asphalt.

Sometimes construction crews will start work at both ends of the new road and meet in the middle. Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC 2

Topic Review

? TOPIC ESSENTIAL QUESTION

1. What properties are specific to parallel lines and perpendicular lines?

Vocabulary Review

Choose the correct term to complete each sentence.

2. Angles that are outside the space between parallel lines and that lie on the same side of a transversal are ________.
3. A ________ intersects coplanar lines at distinct points.
4. Two angles inside a triangle that correspond to the nonadjacent exterior angle are the ________.
5. ________ lie on the same side of a transversal of parallel lines and are in corresponding positions relative to the parallel lines.
6. Angles between parallel lines that are nonadjacent and that lie on opposite sides of a transversal are ________.
7. Angles between parallel lines that are on the same side of a transversal are ________.

- alternate exterior angles
- alternate interior angles
- corresponding angles
- exterior angle of a triangle
- remote interior angles
- same-side exterior angles
- same-side interior angles
- transversal

Concepts & Skills Review

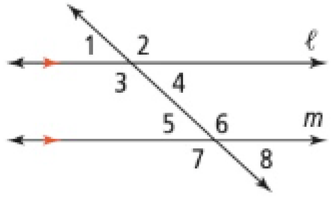
LESSONS 2-1 & 2-2 Parallel Lines and Proving Lines Parallel

Quick Review

When two **parallel lines** are intersected by a **transversal**, the angle pairs that are formed have special relationships. These angle pairs are either congruent or supplementary angles.

Example

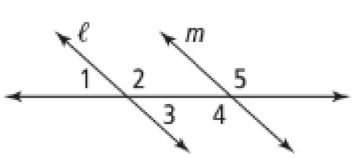
Which angles are supplementary to ∠3?



∠1, ∠4, ∠5, ∠8

Practice & Problem Solving

Use the figure for Exercises 8–10.



8. Suppose $\ell \parallel m$. What is the measure of each angle if $m\angle 2 = 138$?

   a. $m\angle 1$ b. $m\angle 3$ c. $m\angle 4$

9. If $m\angle 1 = 3x - 3$ and $m\angle 5 = 7x + 23$, for what value of x is $\ell \parallel m$?

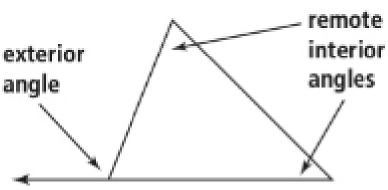
10. **Choose Efficient Methods** The transversal that intersects two parallel lines forms corresponding angles with measures $m\angle 1 = 3x - 7$ and $m\angle 2 = 2x + 12$. What is the measure of each angle?

LESSON 2-3 Parallel Lines and Triangle Angle Sums

Quick Review

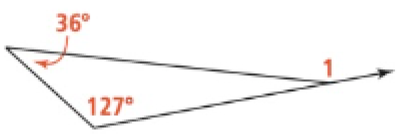
The interior and exterior angle measures of a triangle have the following properties.

- The sum of the interior angles of every triangle is 180°.
- The measure of each **exterior angle of a triangle** equals the sum of the measures of the two corresponding **remote interior angles**.



Example

What is $m\angle 1$?

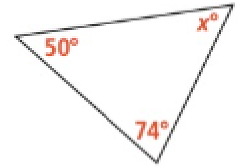


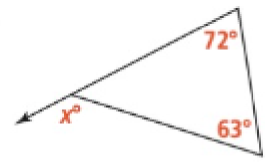
$m\angle 1 = 36 + 127$

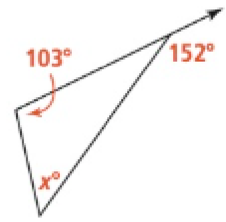
$m\angle 1 = 163$

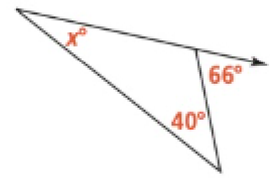
Practice & Problem Solving

What is the value of *x* in each figure?

11. 

12. 

13. 

14. 

15. **Analyze and Persevere** During a storm, a tree is blown against a building so that it forms a triangle with remote interior angles of 90° and 52°. What is the measure of the corresponding exterior angle formed by the leaning tree?

LESSON 2-4 Slopes of Parallel and Perpendicular Lines

Quick Review

Two non-vertical lines are parallel if they have the same slope. Two vertical lines are parallel to each other.

Two non-vertical lines are perpendicular if the product of the slopes is −1. A vertical line is perpendicular to a horizontal line.

Example

A contractor starts drawing a plan for a parking lot with a center line through the points (2, 10) and (4, 0). He wants to make shorter lines perpendicular to the center line to designate each parking spot. What should be the slope of the shorter lines?

Find the slope of the center line.

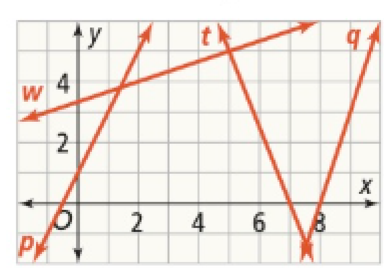
$$m = \frac{10 - 0}{2 - 4} = \frac{10}{-2}$$

The slope of the center line is −5.

The slope of the shorter lines should be $\frac{1}{5}$.

Practice & Problem Solving

Use the figure for Exercises 16–17. Show the calculations you use to answer each question.



16. Are lines *p* and *q* parallel?

17. Are lines *w* and *t* perpendicular?

18. **Communicate and Justify** Theorem 2-4 states that two non-vertical lines are perpendicular if and only if the product of their slopes is −1. Why are vertical lines excluded?

19. **Use Patterns and Structure** A blueprint has one path of a park passing through (1, 1) and (3, 6). If the planners want a second path to be perpendicular to the first, what should be the slope of the second path?

TOPIC
3
Transformations

TOPIC ESSENTIAL QUESTION

What are properties of the four types of rigid motion?

Topic Overview

enVision® STEM Project
Create an Animation

3-1 Reflections
GR.2.1, GR.2.2, GR.2.3, GR.2.5, MTR.5.1, MTR.7.1, MTR.3.1

3-2 Translations
GR.2.3, GR.2.1, GR.2.2, GR.2.5, MTR.4.1, MTR.2.1, MTR.7.1

3-3 Rotations
GR.2.2, GR.2.1, GR.2.3, GR.2.5, MTR.5.1, MTR.4.1, MTR.6.1

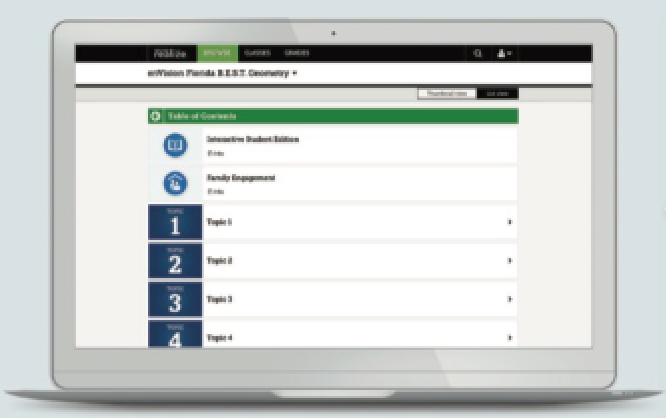
3-4 Classification of Rigid Motions
GR.2.2, GR.2.3, MTR.5.1, MTR.2.1, MTR.4.1

3-5 Symmetry
GR.2.3, GR.2.4, MTR.1.1, MTR.5.1, MTR.3.1

Mathematical Modeling in 3 Acts:
The Perplexing Polygon
GR.2.5, MTR.7.1

Topic Vocabulary

- composition of rigid motions
- glide reflection
- point symmetry
- reflectional symmetry
- rigid motion
- rotational symmetry
- tessellation
- translational symmetry



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Digital Experience

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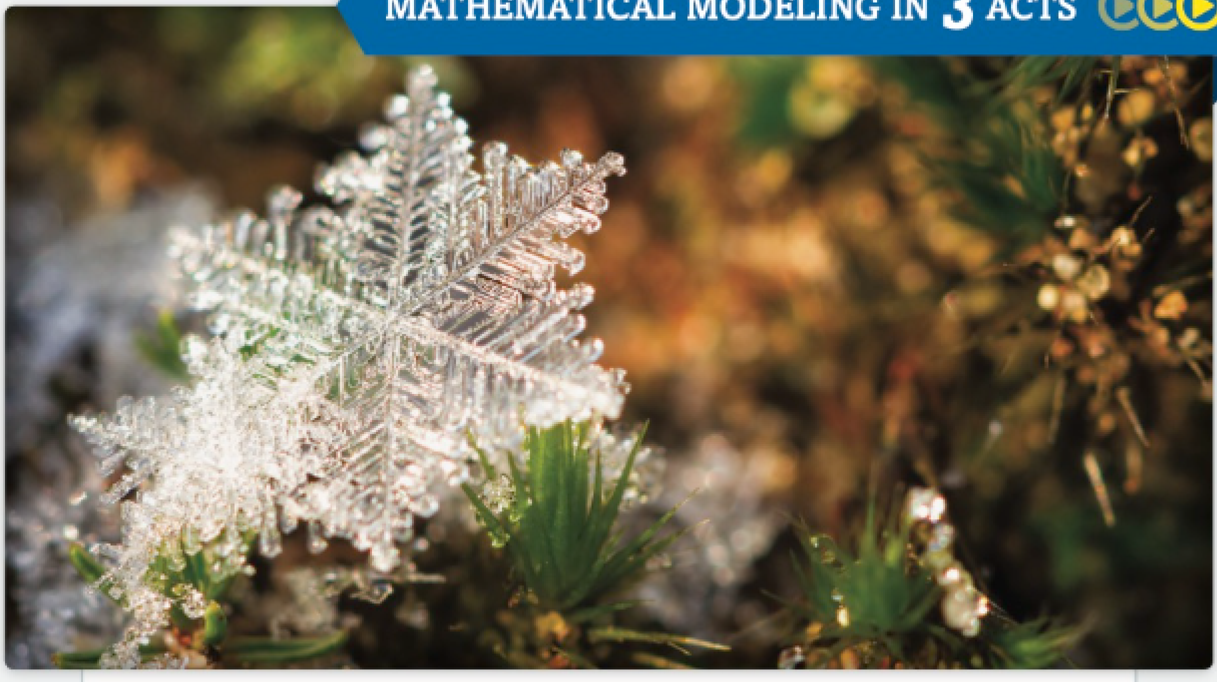
ACTIVITIES Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with Examples and Try Its.

ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



The Perplexing Polygon

Look around and you will see shapes and patterns everywhere you look. The tiles on a floor are often all the same shape and fit together to form a pattern. The petals on a flower often make a repeating pattern around the center of the flower. When you look at snowflakes under a microscope, you can notice that they are made up of repeating three-dimensional crystals. Think about the patterns you have seen during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects.***

CONCEPT SUMMARY Review key lesson content through multiple representations.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you.*

GLOSSARY Read and listen to English and Spanish definitions.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

MATH TOOLS Explore math with digital tools and manipulatives.

DESMOS Use Anytime and as embedded Interactives in Lesson content.

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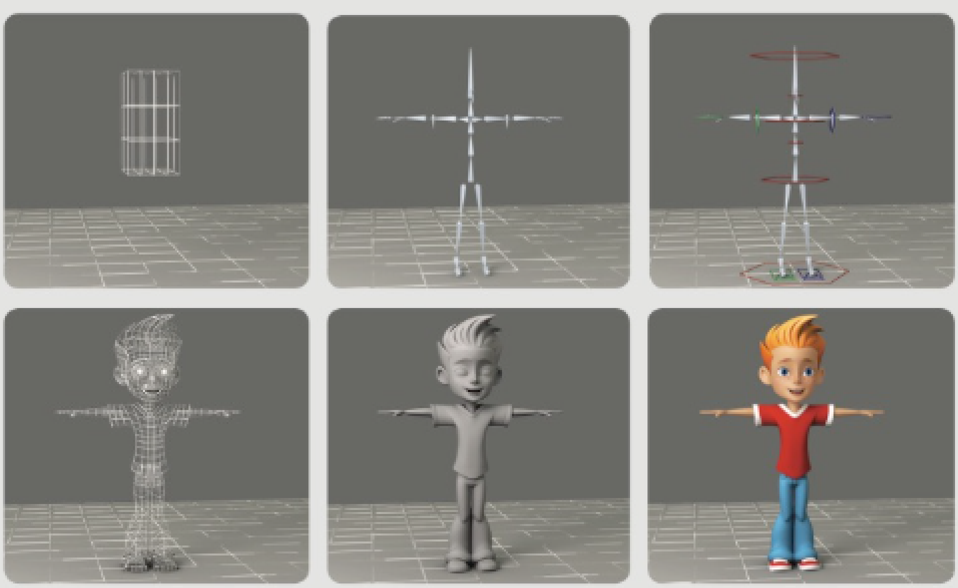
Did You Know?

Polygonal modeling uses polygons to model the surfaces of three-dimensional objects. Animators use vertices and edges to define polygons (usually triangles or quadrilaterals), and they use multiple polygons to create more complex shapes.

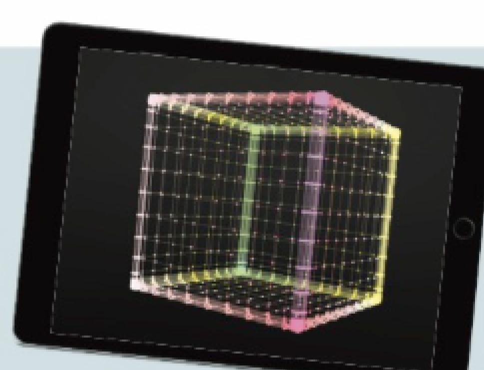
The phenakistoscope was invented nearly **200 years ago**. When the viewer looks through a slot, a sequence of images appears to show moving figures.



An animated character represents hundreds of hours of work. An animator builds a *mesh* of polygons connected through shared vertices and edges. A rigger links the mesh to a system of joints and control handles. To represent a curved surface in a realistic way, the animator uses a mesh of many small polygons. Then the animator programs the joints and handles so that the character moves realistically. Finally, an artist provides surface texture and shading.

Your Task: Create an Animation

Starting with the pixels (points) of a simple geometric figure, you and your classmates will use translations and reflections to move the figure through a series of frames.



3-1

Reflections

I CAN… draw and describe the reflection of a figure across a line of reflection.

VOCABULARY

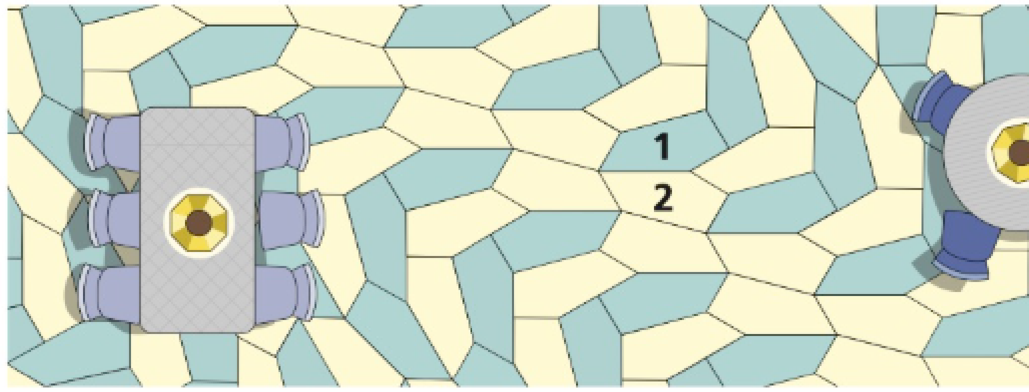
- rigid motion

MA.912.GR.2.1–Given a preimage and image, describe the transformation and represent the transformation algebraically using coordinates. Also GR.2.2, GR.2.3, GR.2.5

MA.K12.MTR.5.1, MTR.1.1, MTR.3.1

EXPLORE & REASON

The illustration shows irregular pentagon-shaped tiles covering a floor.



A. Which tiles are copies of tile 1. Explain.

B. Use Patterns and Structure If you were to move tile 1 from the design, what actions would you have to do so it completely covers tile 2?

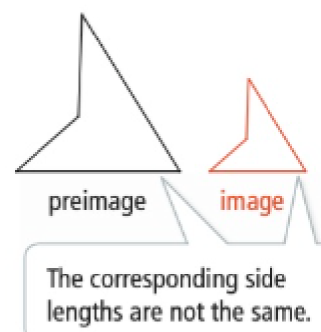
C. Which tiles are *not* copies of tile 1? Explain.

ESSENTIAL QUESTION

How are the properties of reflection used to transform a figure?

EXAMPLE 1 Identify Rigid Motions

A rigid motion is a transformation that preserves length and angle measure. Is the transformation a rigid motion? Explain.



Although angle measure is preserved, the image is smaller than the preimage, so the transformation involves a change in length.

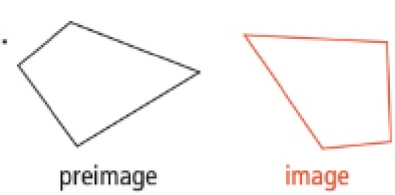
The transformation is not a rigid motion because the length is not preserved.

STUDY TIP
Recall that a transformation is a function that maps a given figure called the preimage onto the resulting figure, the image.

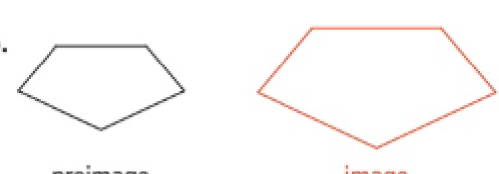
Try It!

1. Is each transformation a rigid motion? Explain.

a.

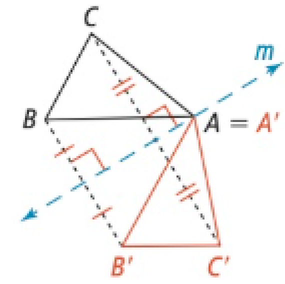


b.



CONCEPT Reflections

A reflection is a transformation that reflects each point in the preimage across a line of reflection.



A reflection has these properties:

- If a point A is on line m, then the point and its image are the same point (that is, $A' = A$).
- If a point B is not on line m, line m is the perpendicular bisector of $\overline{BB'}$.

The reflection of $\triangle ABC$ across line m can be written as $r_m(\triangle ABC) = \triangle A'B'C'$.

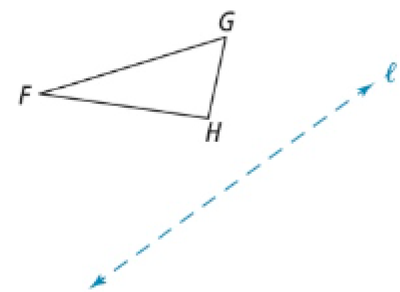
A reflection is a rigid motion so length and angle measures are preserved.

CONCEPTUAL UNDERSTANDING

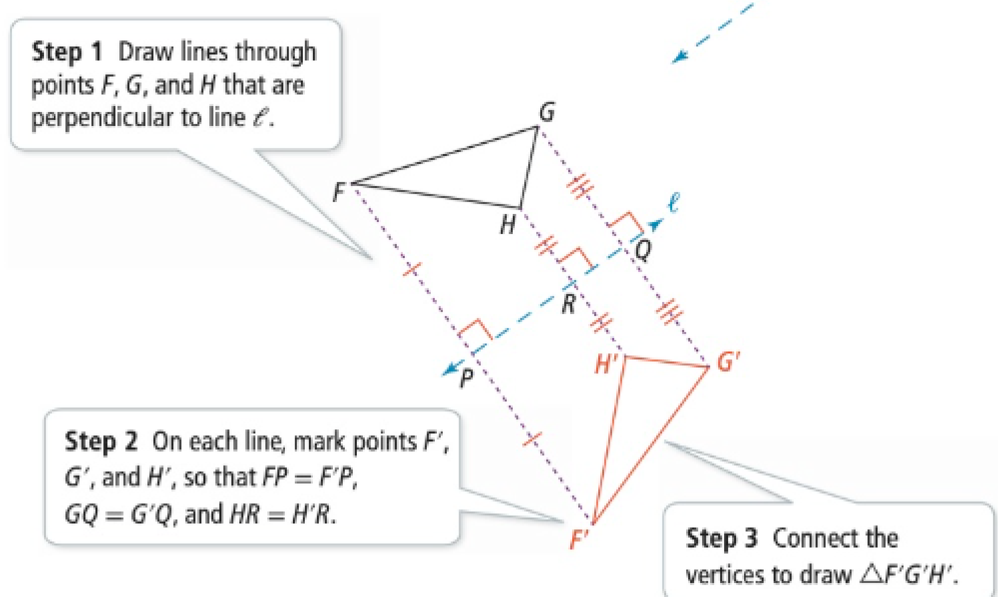
EXAMPLE 2 Reflect a Figure Across a Line

How can you reflect $\triangle FGH$ across line ℓ?

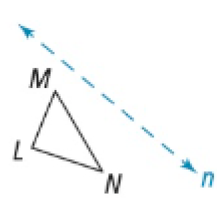
Use the properties of reflections to draw the image of $\triangle FGH$.



ANALYZE AND PERSEVERE
For any point not on the line of reflection, the line of reflection is the perpendicular bisector of the segment between corresponding preimage and image points.



Try It! **2.** What is the reflection of $\triangle LMN$ across line n?



EXAMPLE 3 Reflect a Figure on a Coordinate Plane

Quadrilateral *FGHJ* has coordinates *F*(0, 3), *G*(2, 4), *H*(4, 2), and *J*(−2, 0).

A. Graph and label *FGHJ* and $r_{x\text{-axis}}(FGHJ)$. What is a general rule for reflecting a point across the *x*-axis?

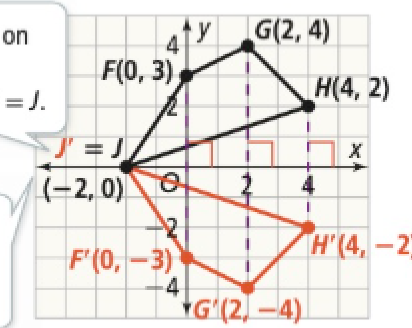
Step 1 Graph *FGHJ*.

Step 2 Find *F*′, *G*′, *H*′, and *J*′ and draw *F*′*G*′*H*′*J*′.

$r_{x\text{-axis}}(0, 3) = (0, -3)$ $\quad r_{x\text{-axis}}(2, 4) = (2, -4)$

$r_{x\text{-axis}}(4, 2) = (4, -2)$ $\quad r_{x\text{-axis}}(-2, 0) = (-2, 0)$

Because *J* is on the line of reflection, *J*′ = *J*.



The *x*-axis is the line of reflection, so it is the perpendicular bisector of $\overline{FF'}$, $\overline{GG'}$, and $\overline{HH'}$.

The reflection of any point (*x*, *y*) across the *x*-axis is the point (*x*, −*y*).

$r_{x\text{-axis}}(x, y) = (x, -y)$

B. Graph and label *FGHJ* and $r_{y\text{-axis}}(FGHJ)$. What is a general rule for reflecting a point across the *y*-axis?

Step 1 Graph *FGHJ*.

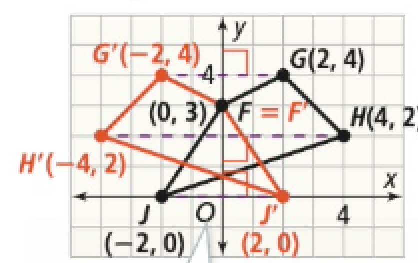
Step 2 Find *F*′, *G*′, *H*′, and *J*′ and draw *F*′*G*′*H*′*J*′.

$r_{y\text{-axis}}(0, 3) = (0, 3)$

$r_{y\text{-axis}}(2, 4) = (-2, 4)$

$r_{y\text{-axis}}(4, 2) = (-4, 2)$

$r_{x\text{-axis}}(-2, 0) = (2, 0)$



The *y*-axis is the line of reflection, so it is the perpendicular bisector of $\overline{GG'}$, $\overline{HH'}$, and $\overline{JJ'}$.

COMMON ERROR
Remember, when the *y*-axis is the line of reflection, the image points must have the same distances from the *x*-axis and on a line perpendicular to the *y*-axis, so the *y*-coordinate stays the same and the *x*-coordinate is the opposite.

The reflection of any point (*x*, *y*) across the *y*-axis is the point (−*x*, *y*).

$r_{y\text{-axis}}(x, y) = (-x, y)$

Try It! **3.** Triangle *ABC* has vertices *A*(−5, 6), *B*(1, −2), and *C*(−3, −4). What are the coordinates of the vertices of △*A*′*B*′*C*′ for each reflection?

a. $r_{x\text{-axis}}$ **b.** $r_{y\text{-axis}}$

CONCEPT Reflecting Points Across the x–axis and y-axis

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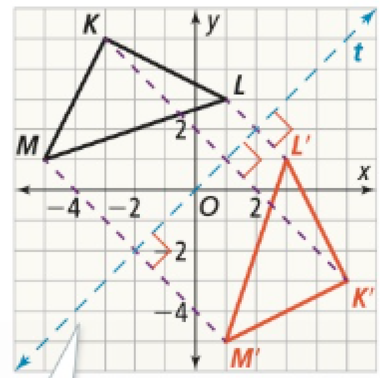
When any point $P(x, y)$ on the coordinate plane is reflected across the x-axis, its image is $P'(x, -y)$.

When any point $P(x, y)$ on the coordinate plane is reflected across the y-axis, its image is $P'(-x, y)$.

EXAMPLE 4 Describe a Reflection on the Coordinate Plane

What reflection maps △*KLM* to its image?



The line of reflection is the perpendicular bisector of the segments that connect corresponding vertices of the preimage and image.

Step 1 Write the coordinates of the preimage and the image.

$K(-3, 5)$ $L(1, 3)$ $M(-5, 1)$

$K'(5, -3)$ $L'(3, 1)$ $M'(1, -5)$

Step 2 Find the midpoints of the segments connecting two pairs of corresponding points.

Midpoint of $\overline{KK'}$:

$$\left(\frac{-3+5}{2}, \frac{5+(-3)}{2}\right) = (1, 1)$$

Midpoint of $\overline{MM'}$:

$$\left(\frac{-5+1}{2}, \frac{1+(-5)}{2}\right) = (-2, -2)$$

STUDY TIP
When providing a rule, you must clearly define the line of reflection. Be certain that the line of reflection is the perpendicular bisector of segments between preimage and image points.

Step 3 Write the equation of the line through the midpoints.

Find the slope.

$$m = \frac{1-(-2)}{1-(-2)}$$
$$= 1$$

Use point slope form.

$$y - 1 = 1 \cdot (x - 1)$$
$$y = x$$

The transformation is a reflection across the line $y = x$. You can write this reflection rule as $r_{y=x}(\triangle KLM) = (\triangle K'L'M')$ or $r_m(x, y) \rightarrow (y, x)$, where m is the line $y = x$.

Try It! **4.** What is a reflection rule that maps each triangle to its image?

a. $C(3, 8)$, $D(5, 12)$, $E(4, 6)$ and $C'(-8, -3)$, $D'(-12, -5)$, $E'(-6, -4)$

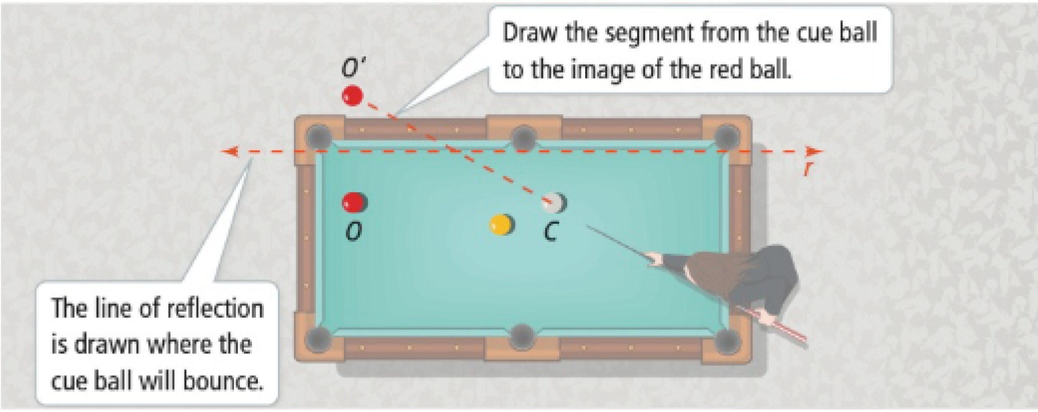
b. $F(7, 6)$, $G(0, -4)$, $H(-5, 0)$ and $F'(-5, 6)$, $G'(2, -4)$, $H'(7, 0)$

APPLICATION

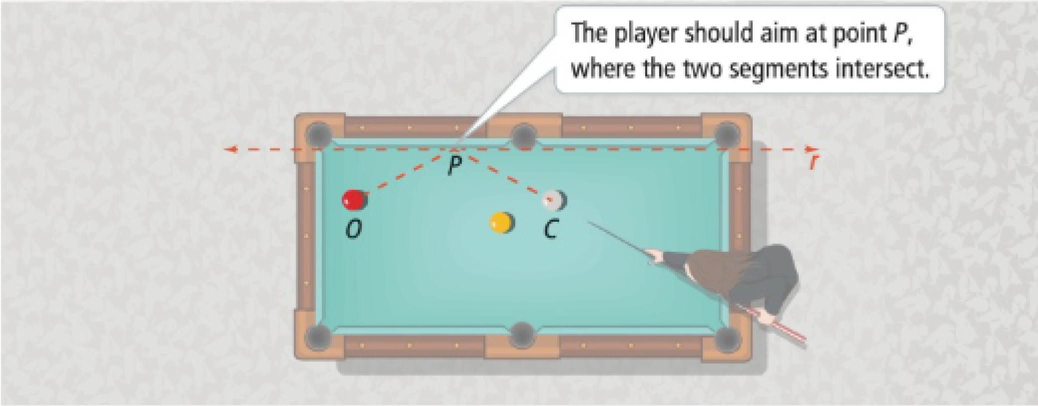
EXAMPLE 5 Use Reflections

In a billiards game, a player must hit the white cue ball so that the cue ball hits the red ball without touching the yellow ball. Where should the cue ball bounce off the top rail so that it hits the red ball?

Consider the top rail as a line of reflection and find the reflection of the red ball.

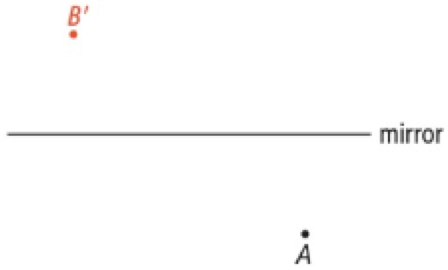


Reflect the segment from the cue ball to the red ball across the line of reflection.



ANALYZE AND PERSEVERE
What do you notice about the angles formed by the line of reflection and the path of the cue ball? Which are congruent?

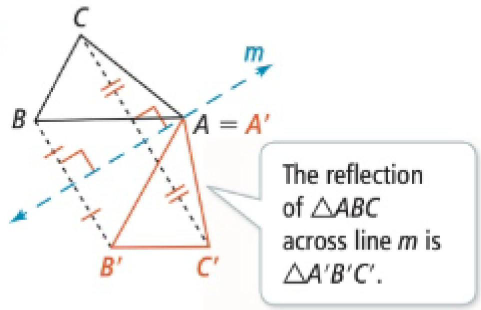
Try It! 5. Student A sees the reflected image across the mirror of another student who appears to be at B'. Trace the diagram and show the actual position of Student B.



CONCEPT SUMMARY Reflections

WORDS A reflection is a transformation that reflects each point in the preimage across a line of reflection.

DIAGRAM



SYMBOLS $r_m(\triangle ABC) = \triangle A'B'C'$

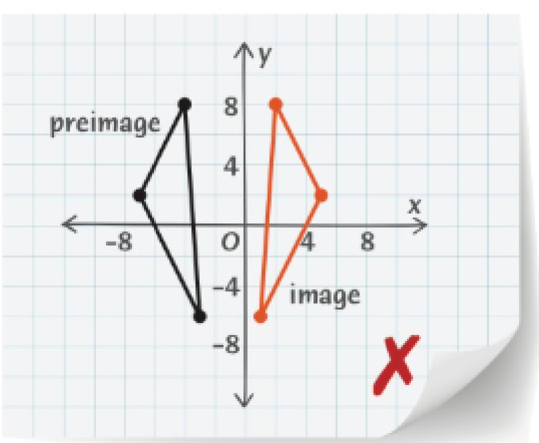
$r_m(A) = A'$

Line m is the perpendicular bisector of $\overline{BB'}$ and $\overline{CC'}$.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the properties of reflection used to transform a figure?

2. **Error Analysis** Oscar drew the image of a triangle reflected across the line $y = -1$. What mistake did Oscar make?



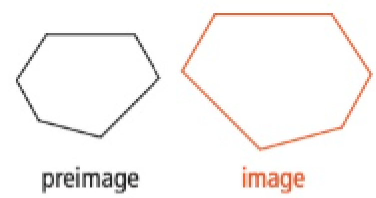
3. **Vocabulary** One meaning of the word *rigid* is "not bendable," and another is "unable to be changed." How do those meanings correspond to the definition of rigid motion?

4. **Choose Efficient Methods** How can you determine whether the transformation of a figure is a rigid motion?

5. **Generalize** Describe the steps you must take to identify the path an object will follow if it bounces off a surface and strikes another object.

Do You KNOW HOW?

6. Does the transformation shown appear to be a rigid motion? Explain.



What are the coordinates of each image?

7. $r_{x\text{-axis}}(-5, 3)$

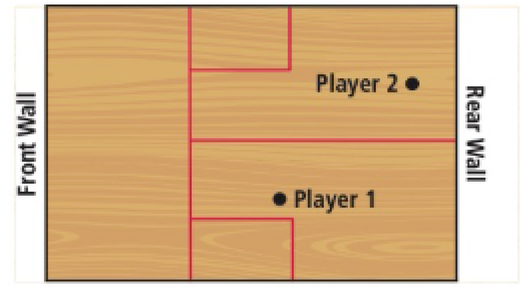
8. $r_{x\text{-axis}}(1, 6)$

9. Write a reflection rule that maps each triangle to its image.

   a. $J(1, 0)$, $K(-5, 2)$, $L(4, -4)$ and $J'(-9, 0)$, $K'(-3, 2)$, $L'(-12, -4)$

   b. $P(8, 6)$, $Q(-4, 12)$, $R(7, 7)$ and $P'(8, -20)$, $Q'(-4, -26)$, $R'(7, -21)$

10. Squash is a racket sport like tennis, except that the ball must bounce off a wall between returns. Trace the squash court. At what point on the front wall should player 1 aim in order to reach the rear wall as far from player 2 as possible?



(not to scale)

PRACTICE & PROBLEM SOLVING

UNDERSTAND

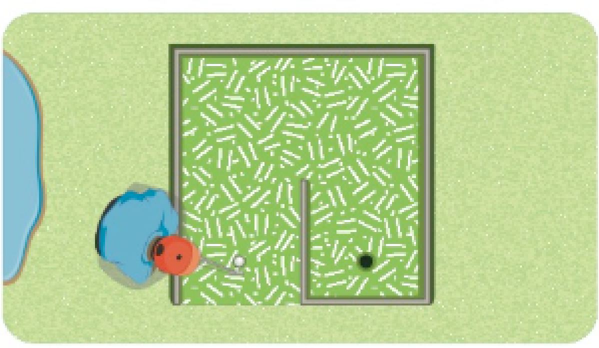
11. **Use Patterns and Structure** Becky draws a triangle with vertices $A(6,7)$, $B(9,3)$, and $C(4, -2)$ on a coordinate grid. She reflects the triangle across the line $y = 4$ to get $\triangle A'B'C'$. She then reflects the image across the line $x = 3$ to get $\triangle A''B''C''$.

    a. What are the coordinates of $\triangle A'B'C'$ and $\triangle A''B''C''$?

    b. Write a rule for each reflection.

12. **Use Patterns and Structure** Under a transformation, a preimage and its image are both squares with side length 3. The image, however, is rotated with respect to the preimage. Is the transformation a rigid motion? Explain.

13. **Error Analysis** Jacob is playing miniature golf. He states that he cannot hit the ball from the start, bounce it off the back wall once, and reach the hole in one shot. Is Jacob correct? Trace and label a diagram to support your answer.



14. **Higher Order Thinking** For the miniature golf hole in Exercise 13, Jacob wants to bounce the ball off the back wall and then the right wall. Draw a diagram to show how Jacob can hit the ball so that it reaches the hole after two bounces.

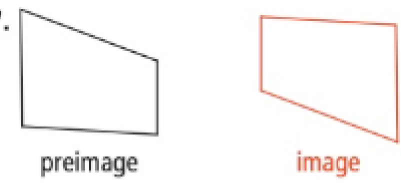
15. **Mathematical Connection** Dana reflects point $A(2, 5)$ across line ℓ to get image point $A'(6, 1)$. What is an equation for line ℓ?

16. **Communicate and Justify** Can a figure be reflected across three lines of reflection so the image is the original figure? Explain.

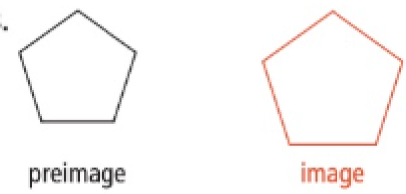
PRACTICE

For Exercises 17 and 18, does each transformation appear to be a rigid motion? Explain. SEE EXAMPLE 1

17.



18.



For Exercises 19–24, suppose m is the line with equation $x = -5$, line n is the line with equation $y = 1$, line g is the line with equation $y = x$, and line h is the line with equation $y = -2$. Given $A(9, -3)$, $B(6, 4)$, and $C(-1, -5)$, what are the coordinates of the vertices of $\triangle A'B'C'$ for each reflection? SEE EXAMPLES 2 AND 3

19. $r_{x\text{-axis}}$
20. $r_{y\text{-axis}}$
21. r_m
22. r_n
23. r_g
24. r_h

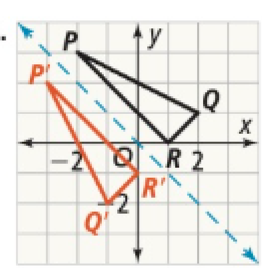
For Exercises 25–28, what is a reflection rule that maps each triangle and its image? SEE EXAMPLE 4

25. $D(3, 6)$, $E(-4, -3)$, $F(6, 1)$ and $D'(1, 6)$, $E'(8, -3)$, $F'(-2, 1)$

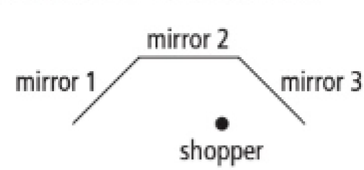
26. $G(9, 12)$, $H(-2, -15)$, $J(3, 8)$ and $G'(9, -2)$, $H'(-2, 25)$, $J'(3, 2)$

27. $K(7, -6)$, $L(9, -3)$, $M(-4, 6)$ and $K'(7, -4)$, $L'(9, -7)$, $M'(-4, -16)$

28.



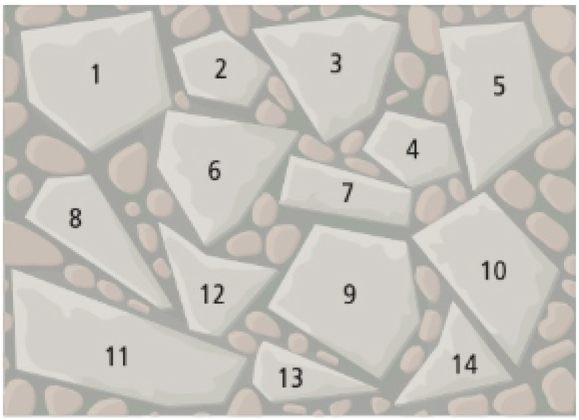
29. Trace the diagram below. Where does the shopper in a dressing room see her image in each mirror? SEE EXAMPLE 5



PRACTICE & PROBLEM SOLVING

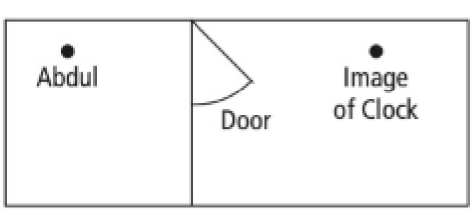
APPLY

30. Use Patterns and Structure Which of the numbered stones shown cannot be mapped to another with a rigid motion?



31. Use Patterns and Structure Reese is inside a shop and sees the sign on the window from the back. Draw the letters as they would appear from the outside of the shop. Is the transformation a rigid motion?

32. Analyze and Persevere Look at the floor plan below. Abdul sees the image of a clock in the mirror on the door.



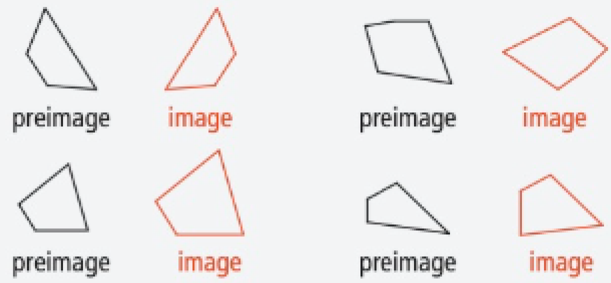
a. Trace the diagram. Where is the line of reflection? Explain.

b. Where is the clock located? Explain.

c. Find where Abdul's image is located relative to the line of reflection. Can Abdul see himself in the mirror? Explain.

ASSESSMENT PRACTICE

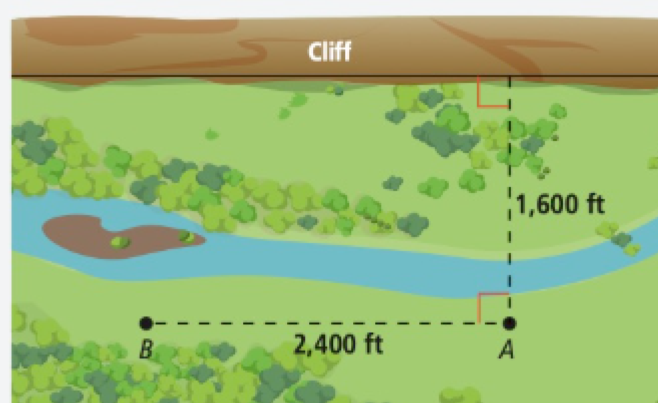
33. Classify whether each pair of figures appears to be a *rigid motion* or *not a rigid motion*. GR.2.1



34. SAT/ACT Consider the following reflection.
Preimage: $A(3, 9)$, $B(2, -7)$, $C(6, 14)$
Image: $A'(-25, 9)$, $B'(-24, -7)$, $C'(-28, 14)$
Suppose p is the line with equation $x = 11$, q is the line with equation $x = 22$, s is the line with equation $x = -11$, and t is the line with equation $x = -22$. What is the rule for the reflection?

Ⓐ $r_p(x, y)$
Ⓑ $r_q(x, y)$
Ⓒ $r_s(x, y)$
Ⓓ $r_t(x, y)$

35. Performance Task Sound echoes from a solid object in the same way that light reflects from a mirror. A hiker at point A shouts the word *hello*. The hiker at point B first hears the shout directly and later hears the echo.



Part A Trace the diagram. Show the path taken by the sound the hiker at point B hears echoing from the cliff.

Part B Sound travels at about 1,000 feet per second. After how long does the hiker at point B hear the shout directly? After how long does he hear the echo? Show your work.

3-2 Translations

I CAN… describe the properties of a figure before and after translation.

VOCABULARY

- composition of rigid motions

MA.912.GR.2.3–Identify a sequence of transformations that will map a given figure onto itself or onto another congruent or similar figure. Also GR.2.1, GR.2.2, GR.2.5

MA.K12.MTR.4.1, MTR.2.1, MTR.7.1

EXPLORE & REASON

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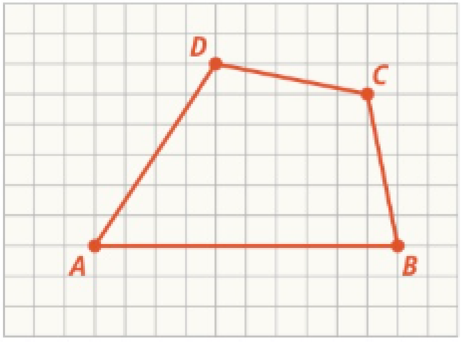
Draw a copy of *ABCD* on a grid. Using another color, draw a copy of *ABCD* on the grid in a different location with the same orientation, and label it *QRST*.



A. On another sheet of paper, write instructions that describe how to move *ABCD* to the location of *QRST*.

B. Exchange instructions with a partner. Follow your partner's instructions to draw a third shape *EFGH* in another color on the same grid. Compare your drawings. Do your drawings look the same? Explain.

C. **Communicate and Justify** What makes a set of instructions for this Explore & Reason a good set of instructions?

ESSENTIAL QUESTION

What are the properties of a translation?

CONCEPT Translations

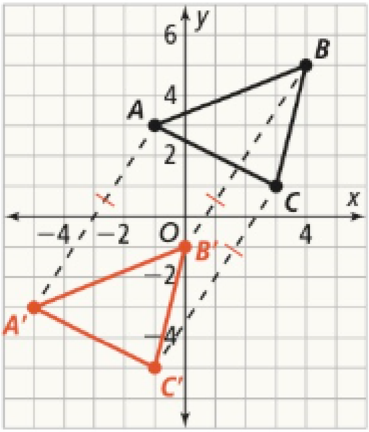
A translation is a transformation in a plane that maps all points of a preimage the same distance and in the same direction.

The translation of $\triangle ABC$ by x units along the x-axis and by y units along the y-axis can be written as $T_{\langle x,\, y\rangle}(\triangle ABC) = \triangle A'B'C'$.

A translation has the following properties:

If $T_{\langle x,\, y\rangle}(\triangle ABC) = \triangle A'B'C'$, then

- $\overline{AA'} \parallel \overline{BB'} \parallel \overline{CC'}$.
- $\overline{AA'} \cong \overline{BB'} \cong \overline{CC'}$.
- $\triangle ABC$ and $\triangle A'B'C'$ have the same orientation.



A translation is a rigid motion, so length and angle measure are preserved.

EXAMPLE 1 Find the Image of a Translation

What is the graph of $T_{\langle 7, -4\rangle}(\triangle EFG) = \triangle E'F'G'$?

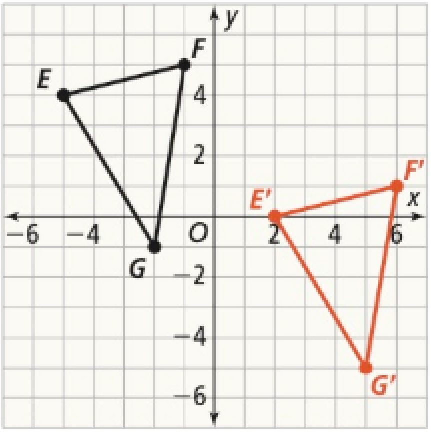
The subscript $\langle 7, -4\rangle$ indicates that each point of $\triangle EFG$ is translated 7 units right and 4 units down.

Find the coordinates of the vertices of the image. Then plot the points and draw $\triangle E'F'G'$.

$E(-5, 4) \rightarrow E'(-5 + 7, 4 - 4) = E'(2, 0)$

$F(-1, 5) \rightarrow F'(-1 + 7, 5 - 4) = F'(6, 1)$

$G(-2, -1) \rightarrow G'(-2 + 7, -1 - 4) = G'(5, -5)$



COMMON ERROR
Do not misunderstand that the translation only maps the vertices of the preimage to the vertices of the image. The translation maps $\triangle EFG$ onto $\triangle E'F'G'$.

Try It! 1. What are the vertices of $\triangle E'F'G'$ for each translation?

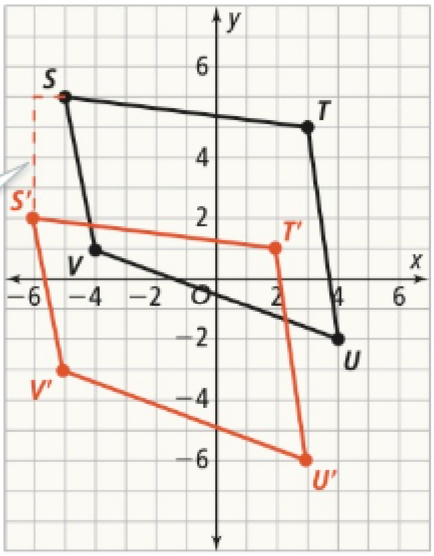
a. $T_{\langle 6, -7\rangle}(\triangle EFG) = \triangle E'F'G'$ b. $T_{\langle 11, 2\rangle}(\triangle EFG) = \triangle E'F'G'$

EXAMPLE 2 Write a Translation Rule

What translation rule maps $STUV$ onto $S'T'U'V'$?

Use one pair of corresponding vertices to determine the change in the horizontal and vertical directions between the preimage to its image.

Use the vertex $S(-5, 6)$ and its image $S'(-6, 2)$.



Change in the horizontal direction x:

$-6 - (-5) = -1$

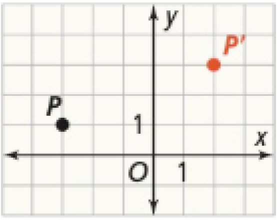
Change in the vertical direction y:

$2 - 6 = -4$

The translation maps every (x, y) point to $(x - 1, y - 4)$, so this translation rule is $T_{\langle -1, -4\rangle}$. You can verify the rule on the remaining vertices.

LEARN TOGETHER
What are ways to stay positive and work toward goals?

Try It! 2. What translation rule maps $P(-3, 1)$ to its image $P'(2, 3)$?



CONCEPT Composition of Rigid Motions

A **composition of rigid motions** is a transformation with two rigid motions in which the second rigid motion is performed on the image of the first rigid motion.

This notation uses a small open circle to indicate a composition of rigid motions on $\triangle ABC$.

Step 1 Translate $\triangle ABC$ left 2 units and up 5 units.

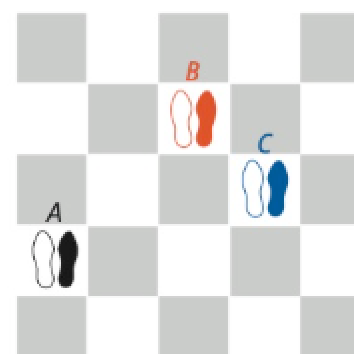
$$(r_{\ell} \circ T_{\langle -2,\, 5\rangle})(\triangle ABC)$$

Step 2 Reflect $\triangle A'B'C'$ across line ℓ.

APPLICATION

EXAMPLE 3 Compose Translations

In learning a new dance, Kyle slides from position A to position B and then to position C. What single transformation describes Kyle's move from position A to position C?



Formulate ◀ Let (0, 0) represent position A.

The translation from A to B is $T_{\langle 2,\, 2\rangle}(x, y)$.

The translation from B to C is $T_{\langle 1,\, -1\rangle}(x, y)$.

Kyle's final position is the composition of those two translations.

Compute ◀ Find $T_{\langle 1,\, -1\rangle} \circ T_{\langle 2,\, 2\rangle}(x, y)$.

$T_{\langle 2,\, 2\rangle}(x, y) = (x + 2, y + 2)$

$T_{\langle 1,\, -1\rangle}(x + 2, y + 2) = (x + 3, y + 1)$

First, apply $T_{\langle 2,\, 2\rangle}$. Then apply $T_{\langle 1,\, -1\rangle}$ to the result.

Interpret ◀ The translation $T_{\langle 3,\, 1\rangle}$(A) represents a single transformation that maps Kyle's move from position A to position C.

Try It! **3.** What is the composition of the transformations written as one transformation?

a. $T_{\langle 3,\, -2\rangle} \circ T_{\langle 1,\, -1\rangle}$

b. $T_{\langle -4,\, 0\rangle} \circ T_{\langle -2,\, 5\rangle}$

CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Relate Translations and Reflections

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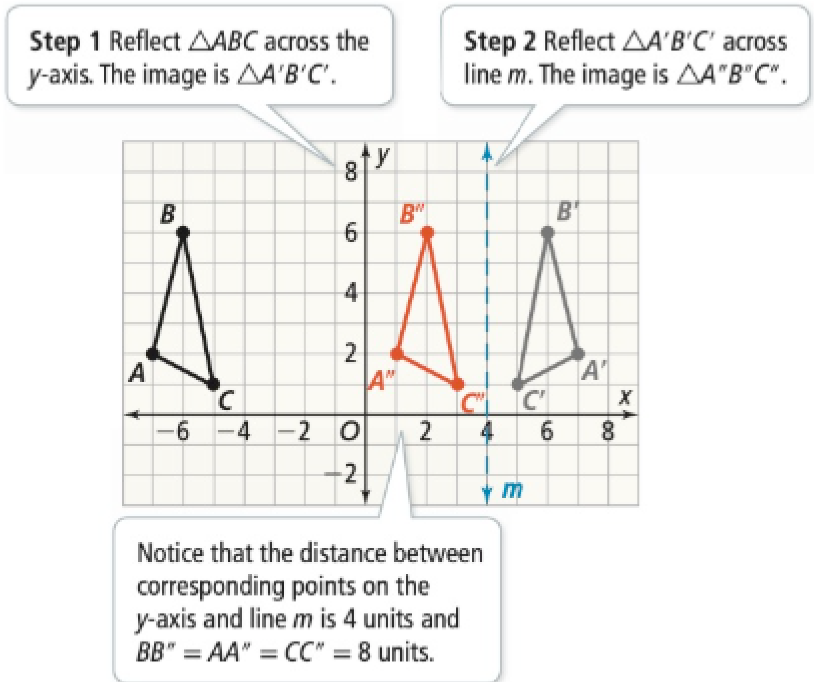
How is a composition of reflections across parallel lines related to a translation?



STUDY TIP
After the first reflection, the orientation of the figure is reversed. After the second reflection, the orientation of the figure returns to that of the preimage.

If $\triangle ABC$ is translated 8 units to the right, its image is also $\triangle A''B''C''$.

So, $(r_m \circ r_{y\text{-axis}})(\triangle ABC) = T_{\langle 8,\, 0\rangle}(\triangle ABC)$.

Try It! 4. Suppose n is the line with equation $y = 1$. Given $\triangle DEF$ with vertices $D(0, 0)$, $E(0, 3)$, and $F(3, 0)$, what translation image is equivalent to $(r_n \circ r_{x\text{-axis}})(\triangle DEF)$?

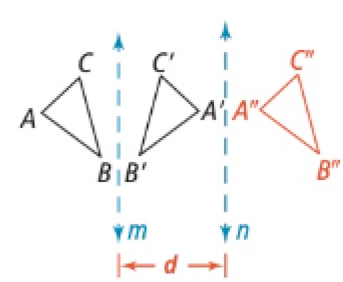
THEOREM 3-1

A translation is a composition of reflections across two parallel lines.

- Both reflection lines are perpendicular to the line containing a preimage point and its corresponding image point.
- The distance between the preimage and the image is twice the distance between the two reflection lines.

If... $T(ABC) = A''B''C''$

$AA'' = BB'' = CC'' = 2d$



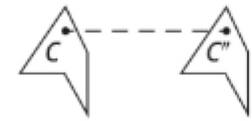
Then... $(r_n \circ r_m)(ABC) = A''B''C''$

PROOF: SEE EXAMPLE 5.

EXAMPLE 5 Prove Theorem 3-1

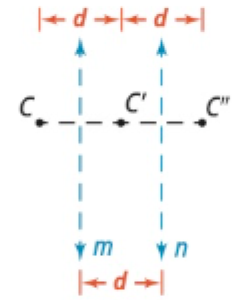
Given: A translation T, with $T(C) = C''$

Prove: There exist parallel lines m and n such that $T = r_n \circ r_m$.



Plan: The given information says the translation T maps C to C''. First find a composition of reflections that maps C to C''. Then show that this composition of reflections is equivalent to the translation T for *any* point. (There are several cases to consider. One case is shown below.)

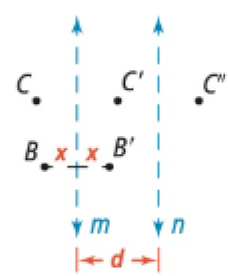
Proof: Let C' be the midpoint of $\overline{CC''}$, and let $CC' = C'C'' = d$. Let m be the perpendicular bisector of $\overline{CC'}$ and n be the perpendicular bisector of $\overline{C'C''}$.



By the properties of reflections $R_m(C) = C'$ and $r_n(C') = C''$, so $(r_n \circ r_m)(C) = C''$. Also, the distance between n and m is d and the distance between C and C'' is $2d$.

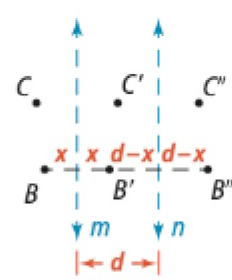
Now pick another point B and show that $(r_n \circ r_m)(B) = T(B)$. To do this, show that $BB'' = CC'' = 2d$, and $\overline{BB''} \parallel \overline{CC''}$.

First reflect across m. Call the image B'. Let the distance from B to m and the distance from m to B' be x.



Now reflect B' across n. Call the image B''. Since the distance between m and n is d, the distance between B' and n is $d - x$. By the properties of reflections, the distance between n and B'' is also $d - x$.

So $BB'' = x + x + (d - x) + (d - x) = 2d$.



Since $\overline{CC''}$ and $\overline{BB''}$ are both perpendicular to m and n, they are parallel to each other.

Therefore $(r_n \circ r_m)(B) = T(B)$.

COMMUNICATE AND JUSTIFY
The choice of C and C'' were arbitrary. Why does this mean that this proof is valid for *any* translation?

Try It! 5. Suppose the point B you chose in the Proof of Theorem 3-1 was between lines m and n. How would that affect the proof? What are the possible cases you need to consider?

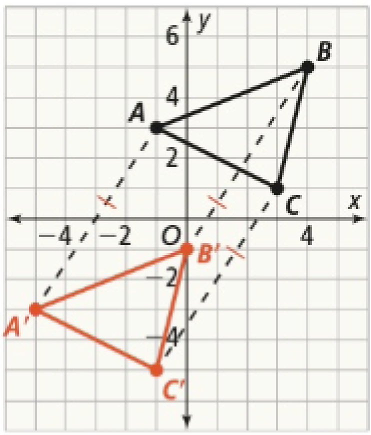
CONCEPT SUMMARY Translations and Compositions of Rigid Motions

WORDS

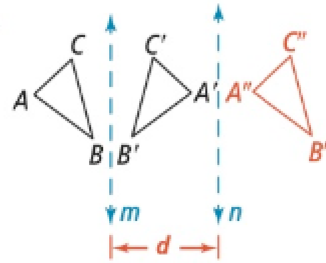
A translation is a transformation that maps all points the same distance and in the same direction.

A composition of two reflections across parallel lines is a translation.

GRAPH



DIAGRAM



SYMBOLS

$T_{\langle -4, -6\rangle}(\triangle ABC) = \triangle A'B'C'$

$\overline{AA'} \parallel \overline{BB'} \parallel \overline{CC'}$

$\overline{AA'} \cong \overline{BB'} \cong \overline{CC'}$

$T(ABC) = (r_n \circ r_m)(ABC)$

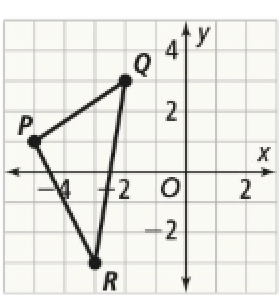
$AA'' = BB'' = CC'' = 2d$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are the properties of a translation?

2. **Error Analysis** Sasha says that for any $\triangle XYZ$, the reflection over the y-axis composed with the reflection over the x-axis is equivalent to a translation of $\triangle XYZ$. Explain Sasha's error.

3. **Vocabulary** Write an example of a composition of rigid motions for $\triangle PQR$.



4. **Analyze and Persevere** What are the values of x and y if $T_{\langle -2, 7\rangle}(x, y) = (3, -1)$?

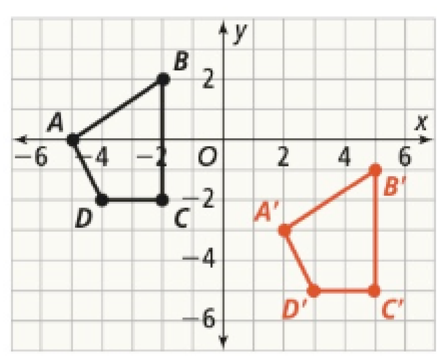
Do You KNOW HOW?

For Exercises 5 and 6, the vertices of $\triangle XYZ$ are $X(1, -4)$, $Y(-2, -1)$, and $Z(3, 1)$. For each translation, give the vertices of $\triangle X'Y'Z'$.

5. $T_{\langle -4, -3\rangle}(\triangle XYZ)$

6. $T_{\langle 5, -3\rangle}(\triangle XYZ)$

7. What is the rule for the translation shown?



For Exercises 8 and 9, write composition of translations as one translation.

8. $T_{\langle 7, 8\rangle} \circ T_{\langle -3, -4\rangle}$

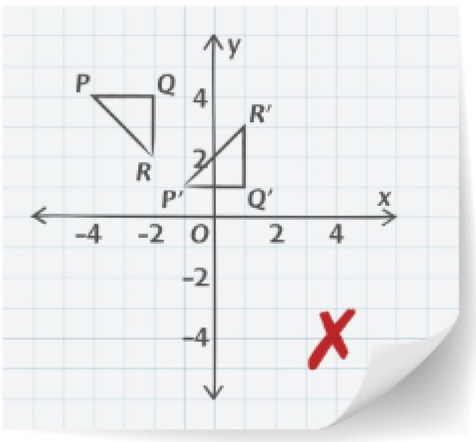
9. $T_{\langle 0, 3\rangle} \circ T_{\langle 4, 6\rangle}$

10. How far apart are two parallel lines m and n such that $T_{\langle 12, 0\rangle}(\triangle JKL) = (r_n \circ r_m)(\triangle JKL)$?

PRACTICE & PROBLEM SOLVING

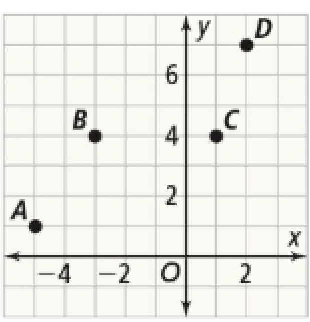
UNDERSTAND

11. **Error Analysis** Hugo graphed $\triangle PQR$ and $(r_t \circ T_{\langle 3, 1 \rangle})(\triangle PQR)$ where the equation of line t is $y = 2$. His translation and reflection were both correct. What mistake did Hugo make?

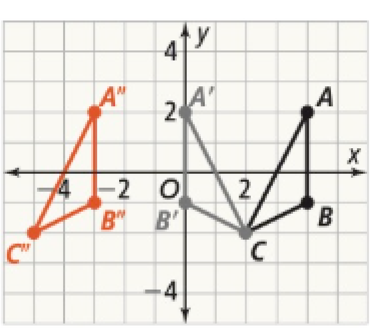


12. **Mathematical Connections** Suppose line k has equation $x = 3$. Compare the areas of $ABCD$ and $A''B''C''D'' = (T_{\langle 1, 2 \rangle} \circ r_k)(ABCD)$. Justify your answer.

13. **Analyze and Persevere** A robot travels from position A to B to C to D. What composition of rigid motions represents those moves?



14. **Higher Order Thinking** How can you describe the complete transformation to a person who cannot see the transformations below?



PRACTICE

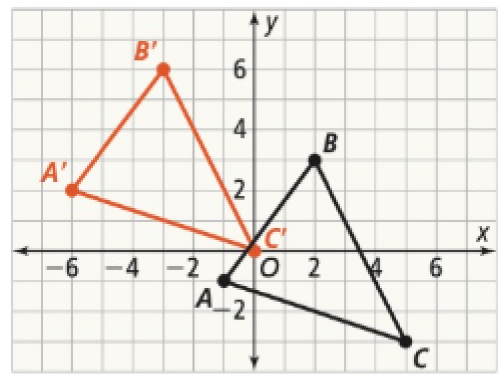
For Exercises 15–17, give the coordinates of the image. SEE EXAMPLE 1

15. $T_{\langle 3, -1 \rangle}(\triangle ABC)$ for $A(5, 0)$, $B(-1, 2)$, $C(6, -3)$

16. $T_{\langle -4, 0 \rangle}(\triangle DEF)$ for $D(3, 3)$, $E(-2, 3)$, $F(0, 2)$

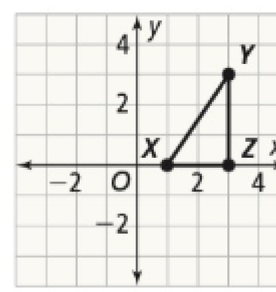
17. $T_{\langle -10, -5 \rangle}(\triangle GHJ)$ for $G(0, 0)$, $H(3, 6)$, $J(12, -1)$

18. What is the rule for the rigid motion?
SEE EXAMPLE 2



19. Write a composition of translations that is equivalent to $T_{\langle 8, -5 \rangle}(x, y)$. SEE EXAMPLE 3

20. Given $\triangle XYZ$, line n with equation $x = -2$, and line p with equation $x = 2$, write a translation that is equivalent to $r_n \circ r_p$.
SEE EXAMPLE 4



For Exercises 21–24, write each composition of translations as one translation. SEE EXAMPLE 3

21. $T_{\langle -3, 3 \rangle} \circ T_{\langle -2, 4 \rangle}$

22. $T_{\langle -4, -3 \rangle} \circ T_{\langle 3, 1 \rangle}$

23. $T_{\langle 5, -6 \rangle} \circ T_{\langle -7, 5 \rangle}$

24. $T_{\langle 8, -2 \rangle} \circ T_{\langle -4, 9 \rangle}$

For Exericses 25–28, write each composition of reflections as one translation. Suppose k is the line with equation $x = -3$, ℓ is the line with equation $x = -2$, m is the line with equation $x = 1$, n is the line with equation $x = -1$, p is the line with equation $y = 1$, q is the line with equation $y = 3$, s is the line with equation $y = 2$, and t is the line with equation $y = -4$. SEE EXAMPLE 4

25. $r_k \circ r_\ell$

26. $r_m \circ r_n$

27. $r_p \circ r_q$

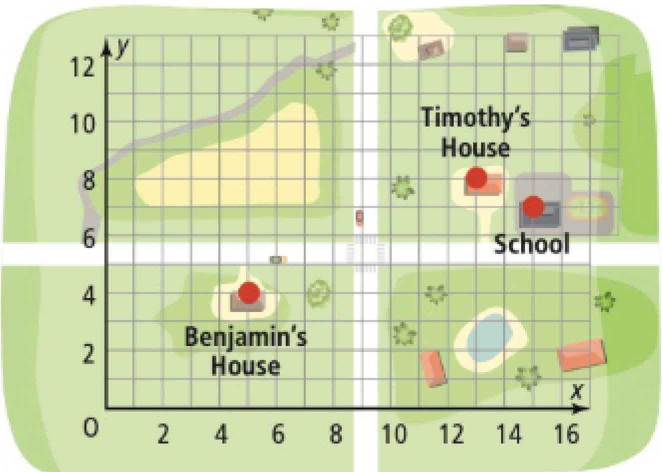
28. $r_s \circ r_t$

29. The distance between vertical lines a and b is 6 units and a is left of b. If $T_{\langle x, 0 \rangle}(\triangle JKL) = (r_b \circ r_a)(\triangle JKL)$, what is the value of x?
SEE EXAMPLE 5

PRACTICE & PROBLEM SOLVING

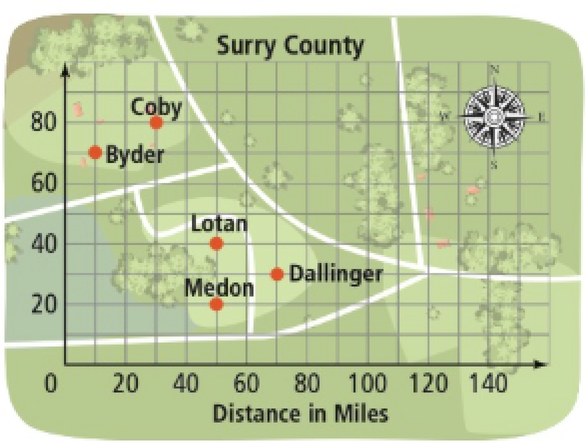
APPLY

30. Communicate and Justify Benjamin walks from his house to Timothy's house and then to school. Describe Benjamin's walk as a composition of translations. If Benjamin walks from his house directly to school, what translation describes his walk?



1 = 80ft

Use the map for Exercises 31 and 32.

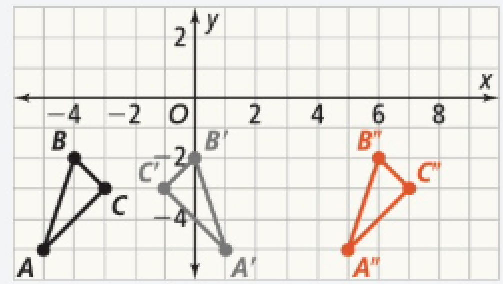


31. Represent and Connect The Surry County sheriff's patrol route starts in Coby. The composition of rigid motions $T_{\langle -20,\ 10\rangle} \circ T_{\langle 40,\ -50\rangle}$ describes her route. How would you describe the sheriff's route in words?

32. Apply Math Models What composition of rigid motions describes a car trip starting in Medon, stopping in Dallinger, and then going on to Byder?

ASSESSMENT PRACTICE

33. Select all expressions equal to $\triangle A''B''C''$. GR.2.1



Suppose a is the line with equation $x = 6$, b is the line with equation $x = 3$, and c is the line with equation $x = -2$.

☐ **A.** $T_{(0,10)}(\triangle ABC)$
☐ **B.** $T_{(10,0)}(\triangle ABC)$
☐ **C.** $(r_{y\text{-axis}} \circ r_a)\ (\triangle ABC)$
☐ **D.** $(r_b \circ r_c)\ (\triangle ABC)$
☐ **E.** $(r_b \circ T_{\langle -1,\ 0\rangle})\ (\triangle ABC)$

34. SAT/ACT Suppose the equation of line m is $x = -7$ and the equation of line n is $x = 7$. Which is the equivalent to the composition $T_{\langle -1,\ 3\rangle} \circ T_{\langle -6,\ 4\rangle}$?

Ⓐ r_m
Ⓑ $T_{\langle -7,\ 7\rangle}$
Ⓒ r_n
Ⓓ $T_{\langle -6,\ 4\rangle} \circ T_{\langle -1,\ 3\rangle}$

35. Performance Task Rectangle *WXYZ* has a perimeter of 16 units and an area of 15 square units.

Part A Graph *WXYZ* on a sheet of graph paper. Write a composition of rigid motions describing two reflections of *WXYZ* across parallel lines of your choosing. Graph and label the parallel lines *W′X′Y′Z′* and *W″X″Y″Z″*.

Part B Write a single rigid motion that is equivalent to the composition of rigid motions in Part B. Justify your answer.

Part C Compare the perimeter and area of *WXYZ* and *W″X″Y″Z″*. What can you conclude about the effect of translation on the properties of figures?

3-3 Rotations

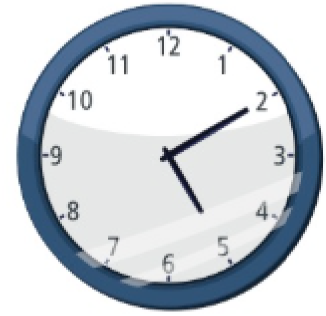
I CAN… draw and describe the rotation of a figure about a point of rotation for a given angle of rotation.

MA.912.GR.2.2–Identify transformations that do or do not preserve distance. **Also GR.2.1, GR.2.3, GR.2.5**

MA.K12.MTR.5.1, MTR.4.1, MTR.6.1

CRITIQUE & EXPLAIN

Filipe says that the next time one of the hands of the clock points to 7 will be at 7:00 when the hour hand points to 7. Nadia says that it will be at 5:35 when the minute hand points to 7.



A. Whose statement is correct? Explain.

B. **Apply Math Models** Suppose the numbers on the clock face are removed. Write instructions that another person could follow to move the minute hand from 2 to 6.

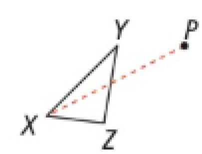
ESSENTIAL QUESTION

What are the properties that identify a rotation?

EXAMPLE 1 Draw a Rotated Image

How can you perform a 75° rotation of $\triangle XYZ$ about point P?

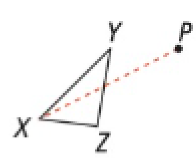
To rotate $\triangle XYZ$ 75° about point P, each point in the triangle must rotate 75°. Then the measure of the angle formed by each preimage point, point P, and the corresponding image point is 75°.



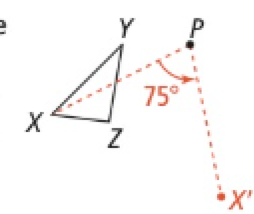
Use a ruler and protractor to draw 75° angles from each vertex on $\triangle XYZ$ with point P, and mark image points that are the same distance from P.

STUDY TIP
Unless otherwise stated, rotations are always performed counterclockwise.

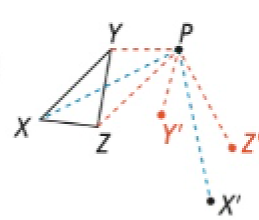
Step 1 To rotate X, draw $\overline{PX}$ to form one side of a 75° angle.



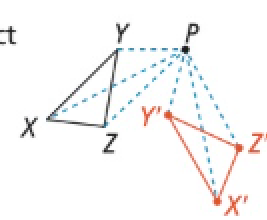
Step 2 Measure the angle and draw the other side, PX'. Mark point X' so $PX' = PX$.



Step 3 Repeat Step 1 and Step 2 for Y and Z in order to locate points Y' and Z'.



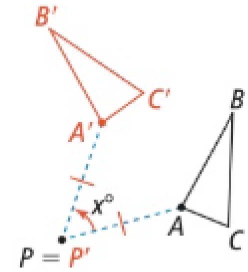
Step 4 Connect the image points to form $\triangle X'Y'Z'$.



Try It! **1.** Do you think a rotated image would ever coincide with the original figure? Explain.

CONCEPT Rotations

A rotation $R_{(x°,\ P)}$ is a transformation that rotates each point in the preimage about a point P, called the center of rotation, by an angle measure of $x°$, called the angle of rotation. A rotation has these properties:



- The image of P is P' (that is, $P' = P$).
- For a preimage point A, $PA = PA'$ and $m\angle APA' = x°$.

A rotation is a rigid motion, so length and angle measure are preserved. Note that a rotation is counterclockwise for a positive angle measure.

CONCEPT Rotations in the Coordinate Plane

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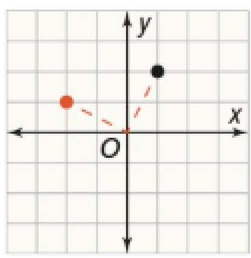
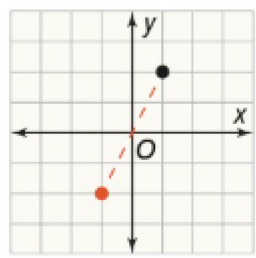
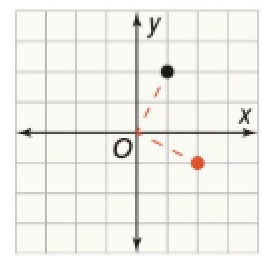
Rules can be used to rotate a figure 90°, 180°, and 270° about the origin O in the coordinate plane.

$R_{(90°,\ O)}(x, y) = (-y, x)$ $R_{(180°,\ O)}(x, y) = (-x, -y)$ $R_{(270°,\ O)}(x, y) = (y, -x)$







EXAMPLE 2 Draw Rotations in the Coordinate Plane

What is $R_{(90°,\ O)}\ ABCD$?

A rotation of 90° about the origin follows the rule $(x, y) \rightarrow (-y, x)$.

Determine the vertices of the image.

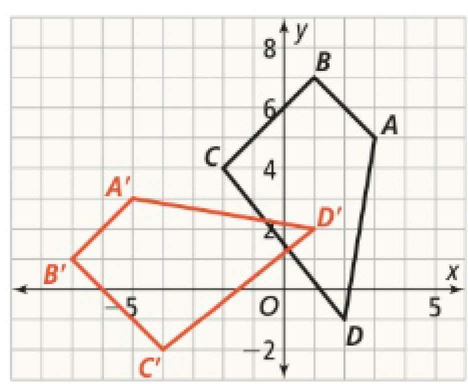
$A(3, 5) \rightarrow A'(-5, 3)$

$B(1, 7) \rightarrow B'(-7, 1)$

$C(-2, 4) \rightarrow C'(-4, -2)$

$D(2, -1) \rightarrow D'(1, 2)$

Draw $A'B'C'D'$ on the coordinate plane.



USE PATTERNS AND STRUCTURE
Compare the distance of each vertex from the origin for the preimage and image. What relationships must they have?

Try It! 2. The vertices of $\triangle XYZ$ are $X(-4, 7)$, $Y(0, 8)$, and $Z(2, -1)$.

a. What are the vertices of $R_{(180°,\ O)}(\triangle XYZ)$?

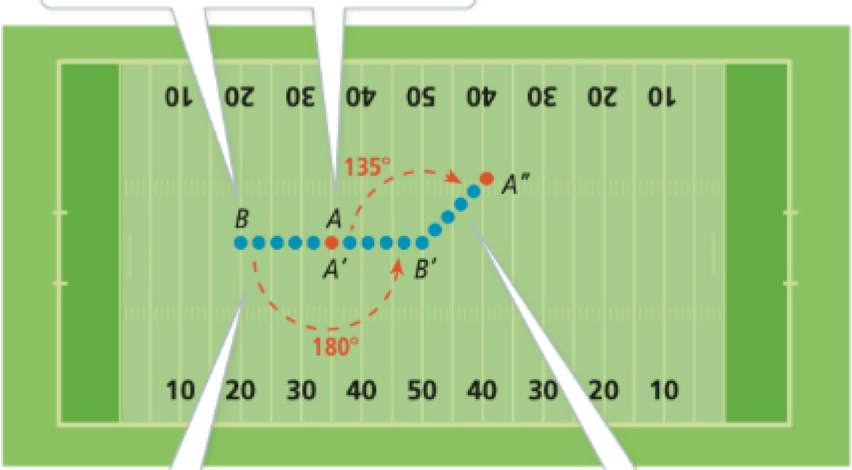
b. What are the vertices of $R_{(270°,\ O)}(\triangle XYZ)$?

APPLICATION

EXAMPLE 3 Use Rotations

The first drummer in a drumline is at the 20 yard line and the sixth drummer is at the 35 yard line. The drumline rotates counterclockwise 180° about the sixth drummer and then rotates 135° clockwise about the first drummer. Where does the sixth drummer stand after the rotations? Describe the change in position as a composition of rotations.

Represent the first drummer as point *B* and the sixth drummer as point *A*.



First, rotate the drumline 180° counterclockwise about point *A*.

Second, rotate the drumline 135° clockwise about point *B*.

COMMUNICATE AND JUSTIFY
Think about how the notation used represents the information shown. What information do you need to identify in order to use the notation?

The sixth drummer stands at the position labeled A''. The drumline is first transformed by the rotation $R_{(180°, A)}$ and then by the rotation $R_{(-135°, B')}$.

Try It! **3. a.** Suppose the drumline instead turns counterclockwise about B'. How many degrees must it rotate so that the sixth drummer ends in the same position?

b. Can the composition of rotations be described by $R_{(45°, A)}$ since $180° - 135° = 45°$? Explain.

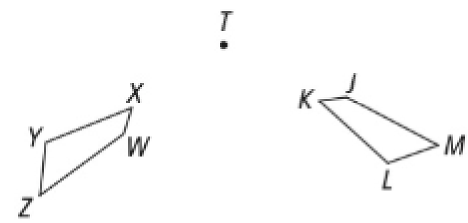
CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Investigate Reflections and Rotations

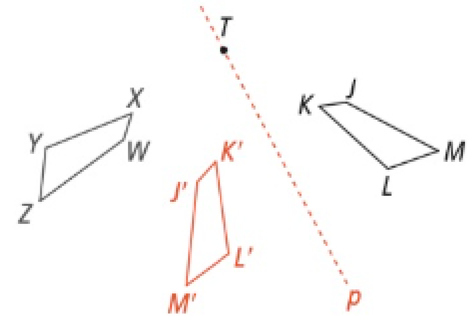
USE PATTERNS AND STRUCTURE
Consider how properties of the transformations are related. What properties would you consider in determining the number of reflections needed?

Can you find a sequence of reflections that result in the same image as a rotation?

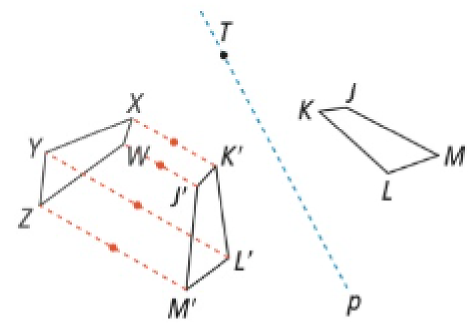
The image of *JKLM* rotated about point *T* is *WXYZ*. Try to reflect *JKLM* one or more times so that the image aligns with *WXYZ*.



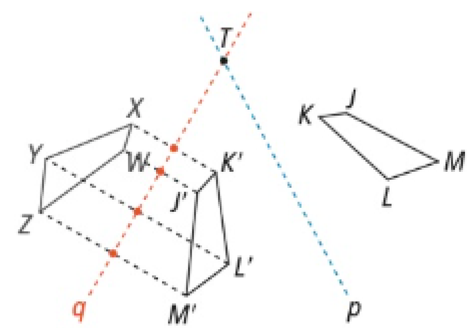
Step 1 Draw line *p* through point *T* and reflect *JKLM* across *p* to form *J′K′L′M′*.



Step 2 Think about how to reflect *J′K′L′M′* to form *WXYZ*. Connect corresponding points in *J′K′L′M′* and *WXYZ* and then find each midpoint.



Step 3 The midpoints appear to be collinear with *T*. Draw line *q* through the midpoints and *T*. The reflection of *J′K′L′M′* across *q* is *WXYZ*.



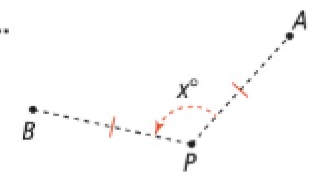
Try It! **4.** Perform the same constructions shown, except draw line *p* so that it does not pass through *T*. Do you get the same results? Explain.

THEOREM 3-2

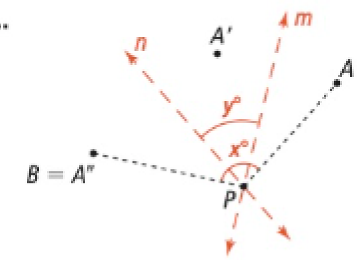
Any rotation is a composition of reflections across two lines that intersect at the center of rotation.

The angle of rotation is twice the angle formed by the lines of reflection.

If...



Then...



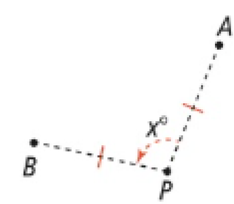
PROOF: SEE EXAMPLE 5.

$y = \frac{1}{2}x$

PROOF

EXAMPLE 5 Prove Theorem 3-2

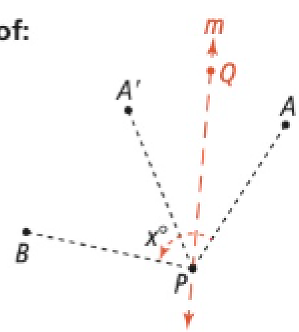
Prove Theorem 3-2.



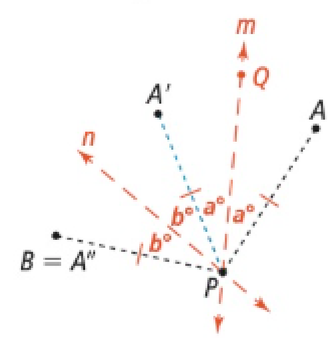
Given: $R_{(x°,\ P)}(A) = B$

Prove: There exist two lines m and n such that $(r_n \circ r_m)(A) = R_{(x°,\ P)}(A)$ equals B, and the measure of the angle formed by lines m and n is $\frac{1}{2}x$.

Proof:



Mark a point Q anywhere except on $\overrightarrow{PA}$. Then draw line m through Q and P. Reflect A across m to image A'. The reflection line is an angle bisector of $\angle APA'$. Let $m\angle APQ = m\angle A'PQ = a$.



Construct the angle bisector n of $\angle BPA'$. Reflect A' across n to image A''. Since a reflection is rigid motion, $PA = PA' = PA'' = PB$. So $A'' = B$. The congruent angles formed measure b. Therefore, $R_{(x°,\ P)}(A) = (r_n \circ r_m)(A)$.

The angle of rotation $x°$ has a measure equal to $a + a + b + b$, or $2(a + b)$. The angle formed by lines m and n has a measure equal to $a + b$, or $\frac{1}{2}x$.

USE PATTERNS AND STRUCTURE
Think about how to construct lines to help you in the proof. What properties does the angle bisector preserve as a line of reflection?

Try It! 5. Suppose point Q is closer to point B or even outside of $\angle APB$. Does the relationship still hold for the angle between the reflection lines and the angle between the preimage and the image? Explain.

CONCEPT SUMMARY Properties of Rotations

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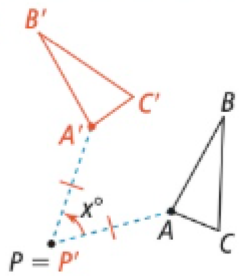
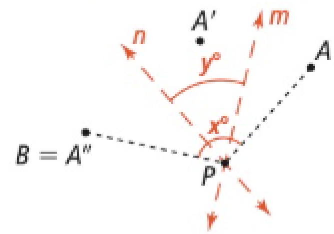
WORDS

A rotation is a transformation that rotates each point in the preimage about the center of rotation through the angle of rotation.

Any rotation is a composition of reflections across two intersecting lines.

DIAGRAMS





SYMBOLS

$R_{(x°,\,P)}(\triangle ABC) = \triangle A'B'C'$

$PA = PA',\ PB = PB',\ PC = PC'$

$m\angle APA' = m\angle BPB' = m\angle CPC' = x°$

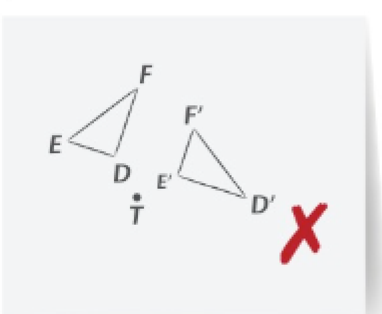
$R_{(x°,\,P)}(A) = (r_n \circ r_m)(A) = B$

$y° = \frac{1}{2}x°$

Do You UNDERSTAND?

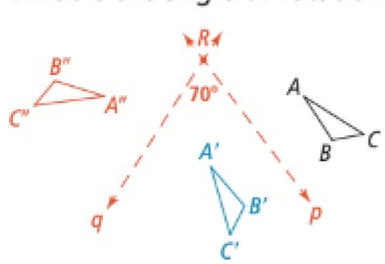
1. **ESSENTIAL QUESTION** What are the properties that identify a rotation?

2. **Error Analysis** Isabel drew the diagram below to show the rotation of $\triangle DEF$ about point T. What is her error?



3. **Vocabulary** How is the *center of rotation* related to the *center of a circle*?

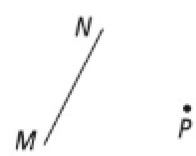
4. **Communicate and Justify** In the diagram, $\triangle A''B''C''$ is the image of reflections of $\triangle ABC$ across lines p and q. It is also the image of a rotation of $\triangle ABC$ about R. What is the angle of rotation? Explain.



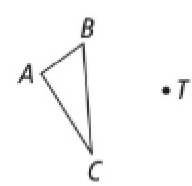
Do You KNOW HOW?

Trace each figure and draw its rotated image.

5. $R_{(90°,\,P)}(\overline{MN})$



6. $R_{(120°,\,T)}(\triangle ABC)$



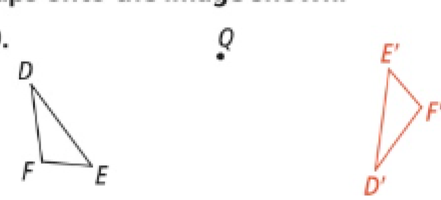
Give the coordinates of each image.

7. $R_{(180°,\,O)}(\overline{GH})$ for $G(2, -9)$, $H(-1, 3)$

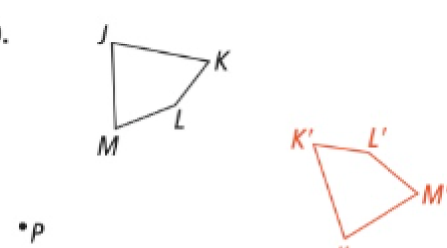
8. $R_{(90°,\,O)}(\triangle XYZ)$ for $X(0, 3)$, $Y(1, -4)$, $Z(5, 2)$

For exercises 9 and 10, trace each figure and construct two lines of reflection such that the composition of the reflections across the lines maps onto the image shown.

9.



10.

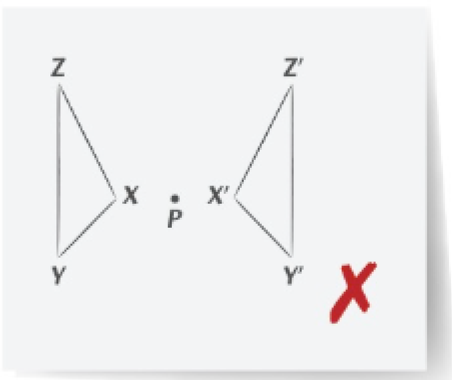


PRACTICE & PROBLEM SOLVING

UNDERSTAND

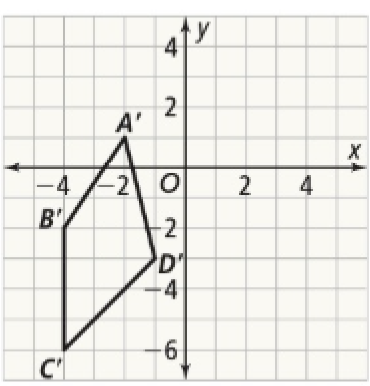
11. **Communicate and Justify** When you rotate a figure, does every point move the same distance? Explain.

12. **Error Analysis** Shannon says that $\triangle X'Y'Z'$ is a rotation of $\triangle XYZ$ about P. What is the correct transformation from $\triangle XYZ$ to $\triangle X'Y'Z'$?

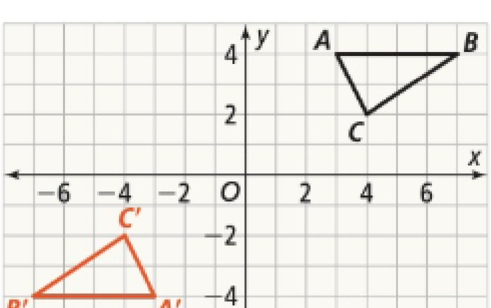


13. **Mathematical Connections** Points A' and B' are the images of points A and B after a 270° rotation about the origin. If the slope of $\overleftrightarrow{AB}$ is −3, what is the slope of $\overleftrightarrow{A'B'}$? Explain.

14. **Use Patterns and Structure** The diagram shows $R_{(90°,\, O)}(ABCD)$. What are the coordinates of $ABCD$?



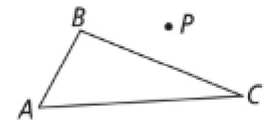
15. **Higher Order Thinking** In the diagram, $R_{(180°,\, O)}(\triangle ABC) = \triangle A'B'C'$. Describe a composition of a rotation and a translation that results in the same image.



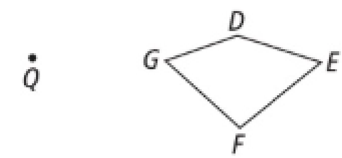
PRACTICE

For Exercises 16–18, trace each figure and draw its rotated image. SEE EXAMPLES 1 AND 3

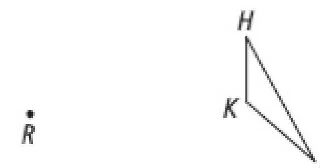
16. $R_{(80°,\, P)}(\triangle ABC)$



17. $R_{(110°,\, Q)}(\triangle DEFG)$



18. $R_{(175°,\, R)}(\triangle HJK)$



For Exercises 19–22, give the coordinates of each image. SEE EXAMPLE 2

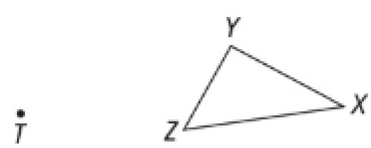
19. $R_{(90°,\, O)}(\triangle DEF)$ for $D(0, 5)$, $E(-2, 8)$, $F(-3, -5)$

20. $R_{(270°,\, O)}(WXYZ)$ for $W(4, -2)$, $X(7, 3)$, $Y(1, 11)$, $Z(-4, 6)$

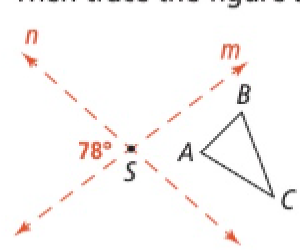
21. $R_{(180°,\, O)}(\triangle STU)$ for $S(-2, -6)$, $T(-5, 3)$, $U(1, 0)$

22. $R_{(360°,\, O)}(JKLM)$ for $J(-4, 7)$, $K(1, 5)$, $L(6, 1)$, $M(3, -9)$

23. Trace the point and triangle. Draw the image $R_{(160°,\, T)}(\triangle XYZ)$. Then draw two reflections that result in the same image. SEE EXAMPLES 4 AND 5



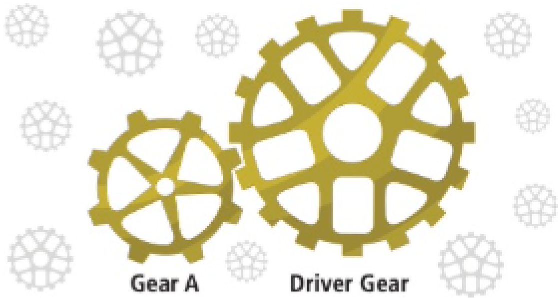
24. Find the angle of rotation for the rotation about point S that is the composition $r_n \circ r_m$. Then trace the figure and draw the image.



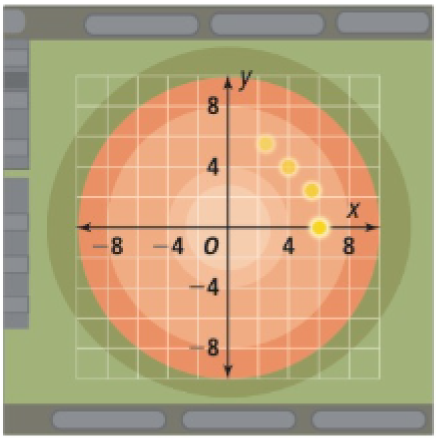
PRACTICE & PROBLEM SOLVING

APPLY

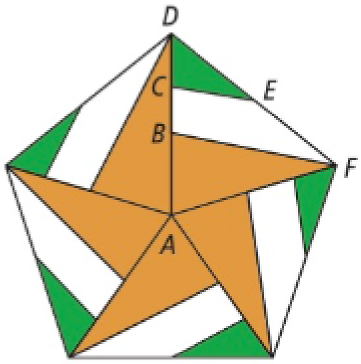
25. **Check for Reasonableness** What rotation must the driver gear make for gear A to rotate 90° clockwise? Explain how you found your answer.



26. **Use Patterns and Structure** Luis is programming an animation for a countdown timer where points flash in sequence, one at a time, around in a circle. He calculates that the coordinates of the first four points in his sequence are (6, 0), (5.5, 2.3), (4.2, 4.2), and (2.3, 5.5). He can find the rest of the coordinates by rotating the first four points by 90°, 180°, and 270°. What are the coordinates of the points that complete the sequence around in a circle?



27. **Communicate and Justify** Lourdes created the design below by rotating $\triangle ABF$, quadrilateral $BCEF$, and $\triangle CDE$. Describe the rotations she used. How do you determine the angles of rotation?



ASSESSMENT PRACTICE

28. Quadrilateral $J'K'L'M'$ is the image of $JKLM$ rotated about point P. Select all true statements that describe the relationship between $JKLM$ and $J'K'L'M'$. GR.2.2

☐ **A.** $m\angle J'K'L' \cong m\angle JKL$

☐ **B.** $\overline{LM} \cong \overline{JK}$

☐ **C.** $m\angle KLM \cong m\angle J'K'L'$

☐ **D.** $\overline{K'L'} \cong \overline{ML}$

☐ **E.** $\overline{J'K'} \cong \overline{JK}$

29. **SAT/ACT** A point is rotated 270° about the origin. The image of the point is (−11, 7). What are the coordinates of the preimage?

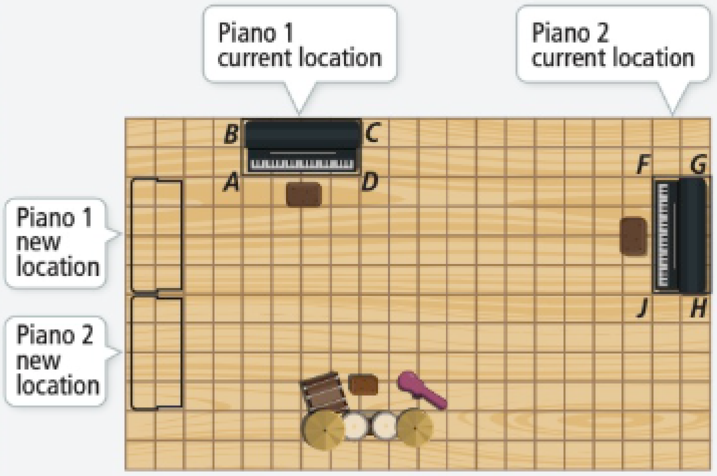
Ⓐ (7, −11)

Ⓑ (−7, −11)

Ⓒ (7, 11)

Ⓓ (11, 7)

30. **Performance Task** Movers need to move the pianos as shown in the diagram.



Part A Describe a sequence of rigid motions for each piano that maps the piano from its current location to the new location.

Part B Describe a single rotation for each piano that maps the piano from its current location to the new location. (*Hint*: You can find two reflections to determine the center of rotation.)

3-4 Classification of Rigid Motions

I CAN... identify different rigid motions used to transform two-dimensional shapes.

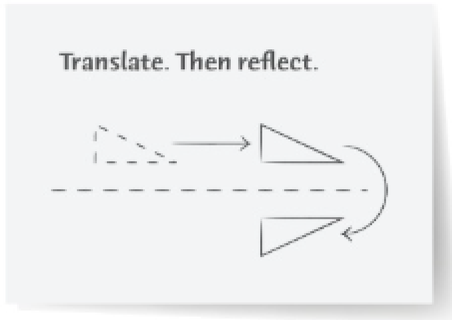
VOCABULARY

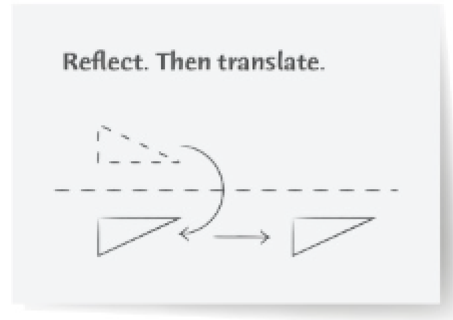
- glide reflection

MA.912.GR.2.2–Identify transformations that do or do not preserve distance. **Also GR.2.3**
MA.K12.MTR.5.1, MTR.2.1, MTR.4.1

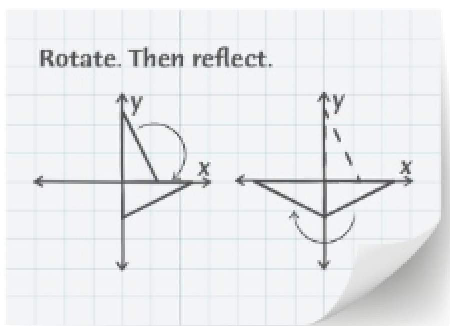
CRITIQUE & EXPLAIN

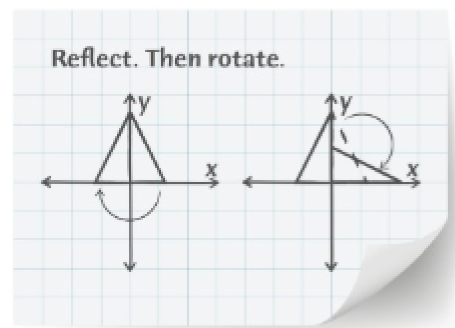
Two students are trying to determine whether compositions of rigid motions are commutative. Paula translates a triangle and then reflects it across a line. When she reflects and then translates, she gets the same image. She concludes that compositions of rigid motions are commutative.





Keenan rotates a triangle and then reflects it. When he changes the order of the rigid motions, he gets a different image. He concludes that compositions of rigid motions are not commutative.





A. Should Paula have used grid paper? Explain.

B. **Communicate and Justify** Do you agree with Paula or with Keenan? Explain.

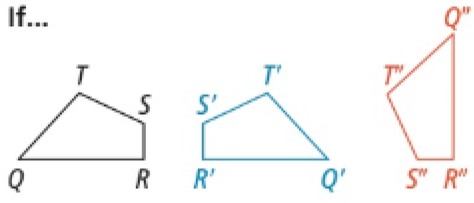
ESSENTIAL QUESTION

How can rigid motions be classified?

THEOREM 3-3

The composition of two or more rigid motions is a rigid motion.

If...



M: $QRST \to Q'R'S'T'$ and
N: $Q'R'S'T' \to Q''R''S''T''$ are rigid motions.

Then...

$(N \circ M)$: $QRST \to Q''R''S''T''$ is a rigid motion.

PROOF: SEE EXAMPLE 1.

PROOF

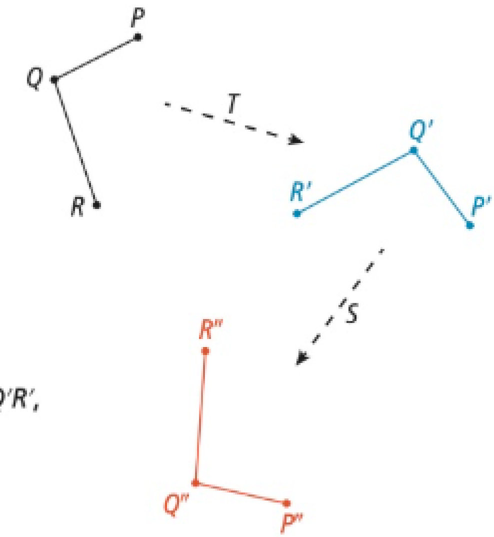
EXAMPLE 1 Prove Theorem 3-3

Write a paragraph proof of Theorem 3-3.

Given: T and S are rigid motions.

Prove: $S \circ T$ is a rigid motion.

Plan: Let P, Q, and R be any three noncollinear points in the preimage. You want to show that length and angle measure are preserved, so it is sufficient to show that $PQ = P''Q''$ and $m\angle PQR = m\angle P''Q''R''$.



Proof: Since T and S are rigid motions, $PQ = P'Q'$, $P'Q' = P''Q''$, $m\angle PQR = m\angle P'Q'R'$, and $m\angle P'Q'R' = m\angle P''Q''R''$.

By the Transitive Property of Equality, $PQ = P''Q''$ and $m\angle PQR = m\angle P''Q''R''$.

$S \circ T$ is a rigid motion because it preserves length and angle measure.

HAVE A GROWTH MINDSET
When it takes time to learn something new, how do you stick with it?

Try It! **1.** Describe how you can use the reasoning used to prove Theorem 3-3 to show that the theorem is true when composing three rigid motions. Can your strategy be extended to include any number of rigid motions?

CONCEPTUAL UNDERSTANDING

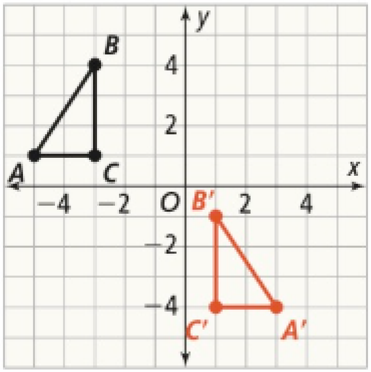
EXAMPLE 2 Explore Glide Reflections

A. Is there a rigid motion that maps $\triangle ABC$ to $\triangle A'B'C'$?

Observe that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, and $m\angle C = m\angle C'$.

Also, $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$.

Length and angle measure are preserved, so the transformation is a rigid motion.



B. Does a reflection, translation, or rotation map $\triangle ABC$ to $\triangle A'B'C'$?

Check whether a translation maps $\triangle ABC$ to $\triangle A'B'C'$.

Use the point $B(-3, 4)$ and its image $B'(1, -1)$.

The translation rule that maps B onto B' is $T_{\langle 4, -5 \rangle}$. This rule does not map A to A'.

The rigid motion is not a translation.

Check whether a rotation maps $\triangle ABC$ to $\triangle A'B'C'$.

Since orientation of the triangle is not preserved, the rigid motion is not a rotation.

Check whether a reflection maps $\triangle ABC$ to $\triangle A'B'C'$.

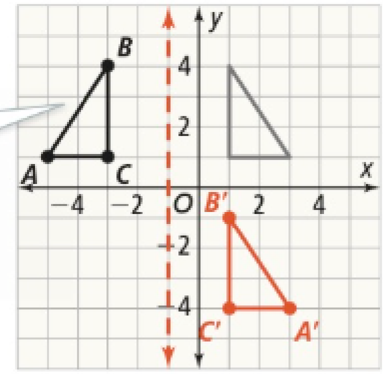
There is no line of reflection that produces the image, so the rigid motion is not a reflection.

USE PATTERNS AND STRUCTURE
Each rigid motion has specific properties. What properties should you consider for each type?

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

C. What composition of two rigid motions maps △*ABC* to △*A′B′C′*?



Because △*ABC* has a clockwise orientation and △*A′B′C′* has a counterclockwise orientation, orientation is not preserved. This means that one of the rigid motions must be a reflection.

Reflect △*ABC* across *k*, the line with equation $x = -1$.

Then translate the image down 5 units.

$$(T_{\langle 0,\,-5\rangle} \circ r_k)(\triangle ABC) = \triangle A'B'C'$$

The composition of a reflection followed by a translation in a direction parallel to the line of reflection is a rigid motion called a **glide reflection**.

USE PATTERNS AND STRUCTURE
Often there is more than one composition of rigid motions that maps a preimage to its image. What is another composition of rigid motions that maps △*ABC* to △*A′B′C′*?

Try It! 2. Draw the perpendicular bisector of $\overline{BB'}$. Is that line also the perpendicular bisector of $\overline{AA'}$ and $\overline{CC'}$? Use your answer to explain why a reflection alone can or cannot map △*ABC* to △*A′B′C′*.

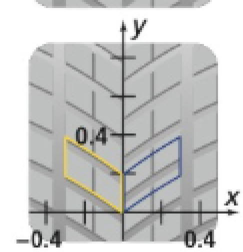
APPLICATION

EXAMPLE 3 Find the Image of a Glide Reflection

A digital artist is reproducing the inner pattern of a tire tread from a tire print taken from a crime scene by applying a glide reflection. She uses the rule $T_{\langle 0,\,0.1\rangle} \circ r_{y\text{-axis}}$ to generate the pattern. Confirm that her rule can be applied to the tire tread pattern taken from the crime scene.

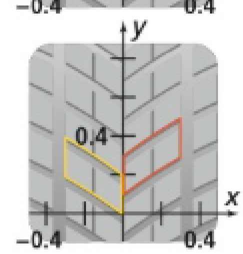
Step 1 Apply the first rigid motion.

Reflect the outlined preimage on the left side of the *y*-axis across the line $x = 0$.



Step 2 Apply the second rigid motion.

Translate the reflection 0.1 unit up.



The rule appears to reproduce the inner pattern of the tire tread.

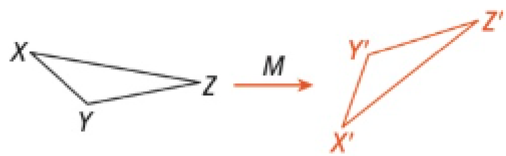
COMMON ERROR
The notation tells you the order in which you should use the transformations. Remember that in the composition $T_{\langle 0,\,0.1\rangle} \circ r_{y\text{-axis}}$, the reflection is performed first.

Try It! 3. Quadrilateral *RSTV* has vertices *R*(−3, 2), *S*(0, 5), *T*(4, −4), and *V*(0, −2). Use the rule $T_{\langle 1,\,0\rangle} \circ r_{x\text{-axis}}$ to graph and label the glide reflection of *RSTV*.

THEOREM 3-4

Any rigid motion is either a translation, reflection, rotation, or glide reflection.

If... M is a rigid motion



Then... $M = r_\ell$ or
$M = T_{\langle x,\, y \rangle}$ or
$M = R_{(n°,\, P)}$ or
$M = T_{\langle x,\, y \rangle} \circ r_\ell$

The proof is reserved for an advanced course.

COROLLARY TO THEOREM 3-4

Any rigid motion can be expressed as a composition of reflections.

PROOF: SEE EXERCISE 11.

If... M is a rigid motion

Then... $M = r_\ell$ or $M = r_\ell \circ r_m$ or $M = r_\ell \circ r_m \circ r_n$

EXAMPLE 4 Determine a Glide Reflection

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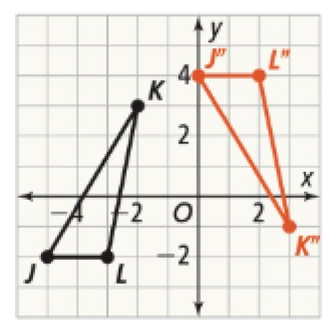
What is the glide reflection that maps $\triangle JKL$ to $\triangle J''K''L''$?



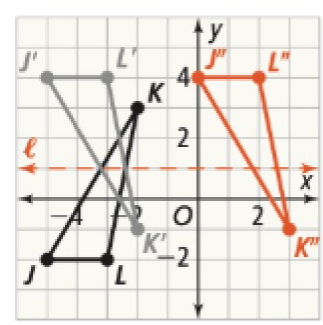
To determine the glide reflection, you can work backward.

Step 1 Determine the translation.

First determine the translation that vertically aligns J'' with J and L'' with L. It is 5 units horizontally and 0 units vertically.

STUDY TIP
The vertices of the preimage and of the image are equidistant from the line of reflection.

Step 2 Determine the line of reflection.



The vertices of an image and preimage are the same distance from the line ℓ with equation $y = 1$, so ℓ is the line of reflection. If you reflect $\triangle J'K'L'$ across ℓ, you map back to the original triangle, $\triangle JKL$.

Step 3 Write the complete glide reflection.

$$(T_{\langle 5,\, 0 \rangle} \circ r_\ell)(\triangle JKL) = \triangle J'K'L', \text{ where } \ell \text{ is the line } y = 1$$

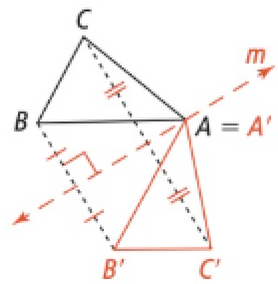
Try It! **4.** What is the glide reflection that maps each of the following?

a. $\triangle ABC \to \triangle A'B'C'$ given $A(-3, 4)$, $B(-4, 2)$, $C(-1, 1)$, $A'(1, 1)$, $B'(2, -1)$, and $C'(-1, -2)$.

b. $\overline{RS} \to \overline{R'S'}$ given $R(-2, 4)$, $S(2, 6)$, $R'(4, 0)$, and $S'(8, -2)$.

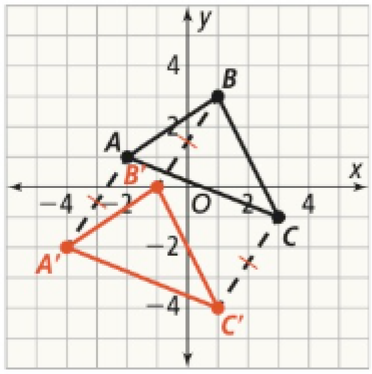
CONCEPT SUMMARY Types of Rigid Motions

REFLECTION



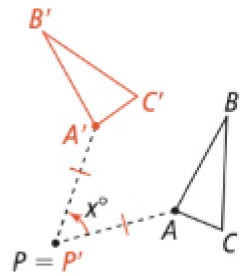
$r_m(\triangle ABC) = \triangle A'B'C'$

TRANSLATION



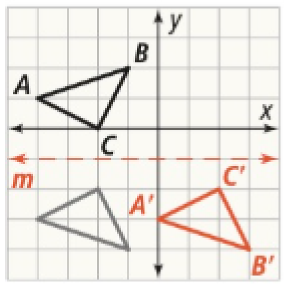
$T_{\langle -2, -3 \rangle}(\triangle ABC) = \triangle A'B'C'$

ROTATION



$R_{(x^\circ, P)}(\triangle ABC) = \triangle A'B'C'$

GLIDE REFLECTION



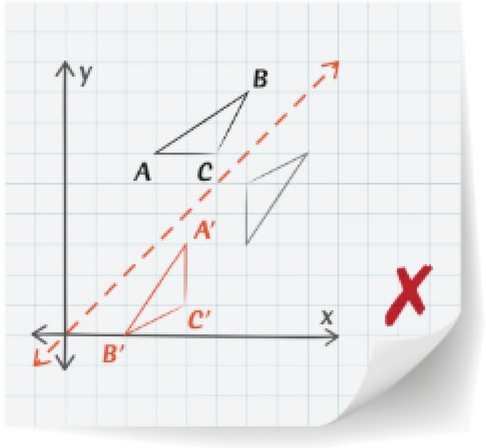
$(T_{\langle 4, 0 \rangle} \circ r_m)(\triangle ABC) = \triangle A'B'C'$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can rigid motions be classified?

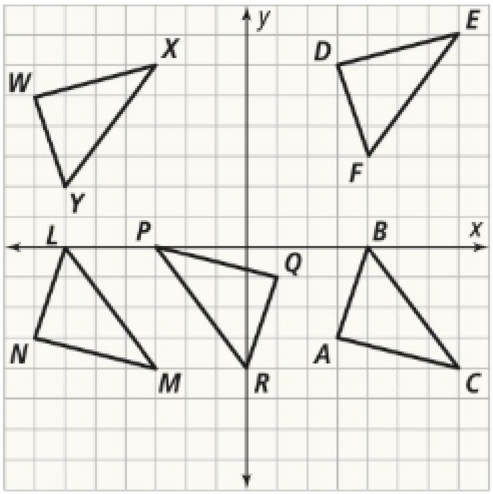
2. Is it correct to say that the composition of a translation followed by a reflection is a glide reflection? Explain.

3. **Error Analysis** Tamika draws the following diagram as an example of a glide reflection. What error did she make?



Do You KNOW HOW?

Use the figures for Exercises 4–7. Identify each rigid motion as a translation, a reflection, a rotation, or a glide reflection.



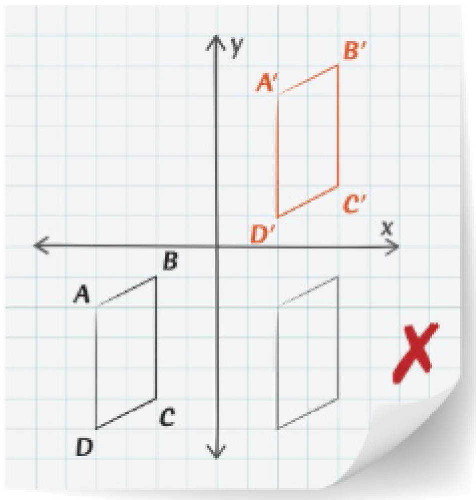
4. $\triangle WYX \rightarrow \triangle NLM$
5. $\triangle DFE \rightarrow \triangle WYX$
6. $\triangle WYX \rightarrow \triangle ABC$
7. $\triangle NLM \rightarrow \triangle QRP$

PRACTICE & PROBLEM SOLVING

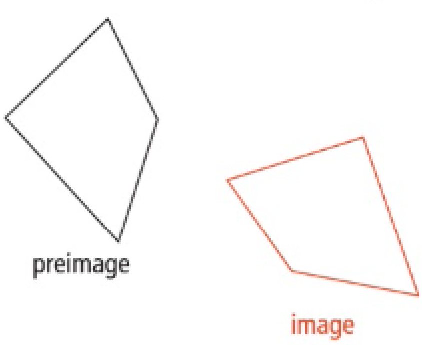
UNDERSTAND

8. **Use Patterns and Structure** Write a paragraph proof of the Corollary to Theorem 3-4.

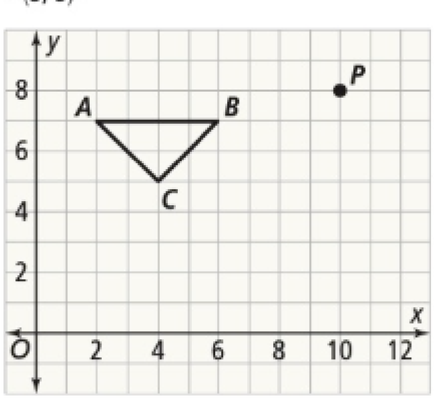
9. **Error Analysis** Damian draws the diagram for the glide reflection $(T_{\langle 0, 7\rangle} \circ r_{y\text{-axis}})(ABCD)$. What error did he make?



10. **Higher Order Thinking** What are the reflection and translation for the glide reflection shown? Sketch the intermediate image.

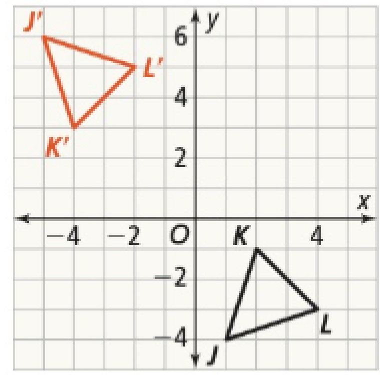


11. **Mathematical Connections** What are the coordinates of the vertices of $\triangle A'B'C'$ after a reflection across a line through point P with a y-intercept at $y = -2$, followed by translation $T_{\langle 3, 3\rangle}$?



PRACTICE

12. What are two rigid motions with a composition that maps $\triangle JKL$ to $\triangle J'K'L'$? SEE EXAMPLES 1 AND 2



For Exercises 13–17, given $A(6, -4)$, $B(3, 8)$, and $C(-7, 9)$, determine the coordinates of the vertices of $\triangle A'B'C'$ for each glide reflection. Suppose p is the line with equation $x = -3$, q is the line with equation $y = 9$, and s is the line with equation $y = -2$. SEE EXAMPLE 3

13. $(T_{\langle 0, -2\rangle} \circ r_{y\text{-axis}})(\triangle ABC) = \triangle A'B'C'$

14. $(T_{\langle 4, 0\rangle} \circ r_{x\text{-axis}})(\triangle ABC) = \triangle A'B'C'$

15. $(T_{\langle 0, 8\rangle} \circ r_p)(\triangle ABC) = \triangle A'B'C'$

16. $(T_{\langle -5, 0\rangle} \circ r_q)(\triangle ABC) = \triangle A'B'C'$

17. $(T_{\langle 7, 0\rangle} \circ r_s)(\triangle ABC) = \triangle A'B'C'$

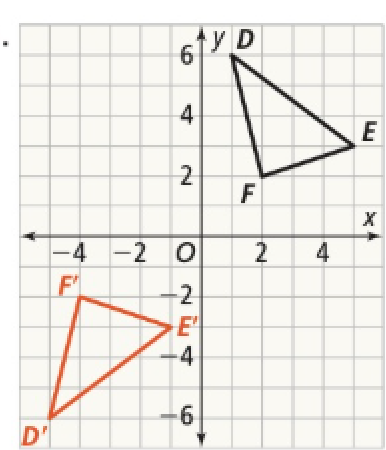
For Exercises 18–21, write a rule for each glide reflection that maps $\triangle DEF$ to $\triangle D'E'F'$. SEE EXAMPLE 4

18. $D(7, -2)$, $E(3, 9)$, $F(8, 6)$;
$D'(-5, 1)$, $E'(-1, 12)$, $F'(-6, 9)$

19. $D(-5, 8)$, $E(1, 4)$, $F(6, 3)$;
$D'(-3, 8)$, $E'(3, 12)$, $F'(8, 13)$

20. $D(0, 4)$, $E(6, 3)$, $F(9, 8)$;
$D'(-6, -8)$, $E'(0, -7)$, $F'(3, -12)$

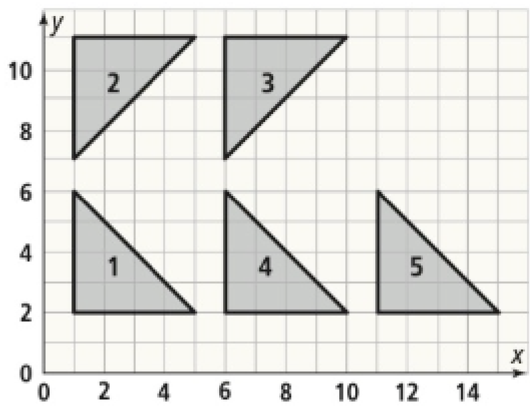
21.



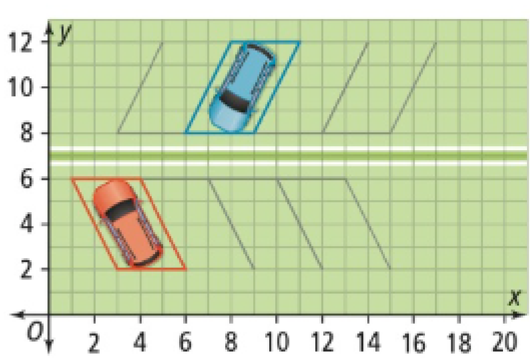
PRACTICE & PROBLEM SOLVING

APPLY

22. **Use Patterns and Structure** The diagram shows one section of concrete being stamped with a pattern. The design can be described by two glide reflections from triangle 1 to triangle 5. Write the rules for each glide reflection.



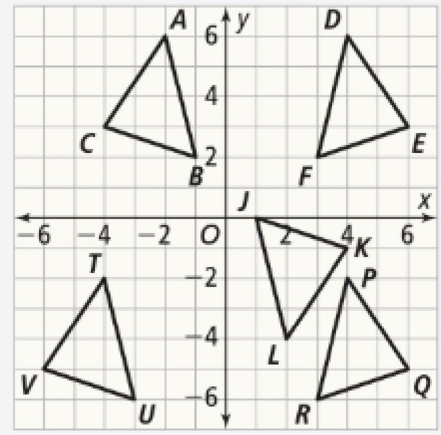
23. **Represent and Connect** Each parking space in the figure can be the image of another parking space as a glide reflection. What is the rule that maps the parking space where the red car is parked to the parking space where the blue car is parked?



24. **Use Patterns and Structure** Starting from tile 1, quadrilateral tiles are embedded into a wall following a pattern of glide reflections. If the pattern continues, what are the shapes and locations of the next two tiles the builder will place in the wall? Explain.

ASSESSMENT PRACTICE

25. Name each image. GR.2.1



a. $R_{(180°, (0, 1))}(\triangle ABC)$

b. $(T_{\langle 0, 8\rangle} \circ r_{y\text{-axis}})(\triangle TUV)$

c. $(T_{\langle -2, 2\rangle} \circ r_{x\text{-axis}})(\triangle DFE)$

d. $r_{y\text{-axis}}(\triangle TUV)$

26. **SAT/ACT** Suppose m is the line with equation $y = 3$. Given $A(7, 1)$, $B(2, 9)$, and $C(3, -5)$, what are the coordinates of the vertices of $\triangle A'B'C'$ for $(T_{\langle -4, 0\rangle} \circ r_m)(\triangle ABC) = \triangle A'B'C'$?

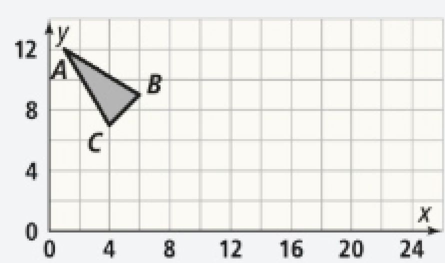
Ⓐ $A'(11, 4)$, $B'(6, 12)$, $C'(7, -2)$

Ⓑ $A'(11, 5)$, $B'(-6, -3)$, $C'(7, 11)$

Ⓒ $A'(3, 4)$, $B'(-2, 12)$, $C'(-1, -2)$

Ⓓ $A'(3, 5)$, $B'(-2, -3)$, $C'(-1, 11)$

27. **Performance Task** Glide reflections are used to print a design across a length of wrapping paper.



Part A Copy the diagram and draw image $\triangle A'B'C' = (T_{\langle 0, -2\rangle} \circ r_{\ell})(\triangle ABC)$, where ℓ is the line with equation $x = 5$.

Part B Translate $\triangle ABC$ 7 units to the right to print $\triangle DEF$ and 14 units to right to print $\triangle GHJ$. What glide reflections of $\triangle DEF$ and $\triangle GHJ$ result in the same arrangement of figures as in Part A? Draw these images to create the wrapping paper pattern.

3-5 Symmetry

I CAN… identify different types of symmetry in two-dimensional figures.

VOCABULARY

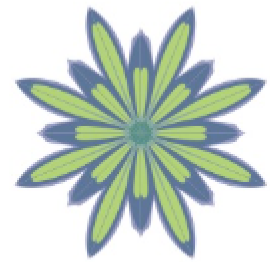
- point symmetry
- reflectional symmetry
- rotational symmetry
- tessellation
- translational symmetry

MA.912.GR.2.3–Identify a sequence of transformations that will map a given figure onto itself or onto another congruent or similar figure. Also GR.2.4

MA.K12.MTR.1.1, MTR.5.1, MTR.3.1

EXPLORE & REASON

Look at the kaleidoscope image shown. Then consider pieces A and B taken from the image.



Piece A Piece B

A. How are piece A and piece B related? Describe a rigid motion that you can use on piece B to produce piece A.

B. Analyze and Persevere Describe a composition of rigid motions that you can use on piece A to produce the image.

C. How many rigid motions did you need to produce the image from piece A? Can you think of another composition of rigid motions to produce the image starting with piece A?

ESSENTIAL QUESTION

How can you tell whether a figure is symmetric?

CONCEPTUAL UNDERSTANDING

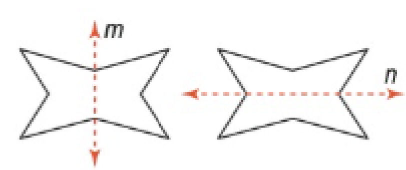
EXAMPLE 1 Identify Transformations for Symmetry

What transformations can be used to map the figure onto itself? How can figures be mapped onto themselves?

A figure has symmetry if a rigid motion can map the figure onto itself.

Reflectional symmetry is a symmetry for which a reflection maps the figure onto itself. The line of reflection for a reflection symmetry is called the line of symmetry.

The reflections r_m and r_n map the figure onto itself. Observe that lines m and n both divide the figure into two pieces with the same size and shape.



A figure has **rotational symmetry** if its image is mapped onto the preimage after a rotation of less than 360°.

The rotation $R_{(180°, P)}$ maps the figure onto itself.

STUDY TIP
To identify reflectional symmetry, fold a figure so one half lines up with the other.

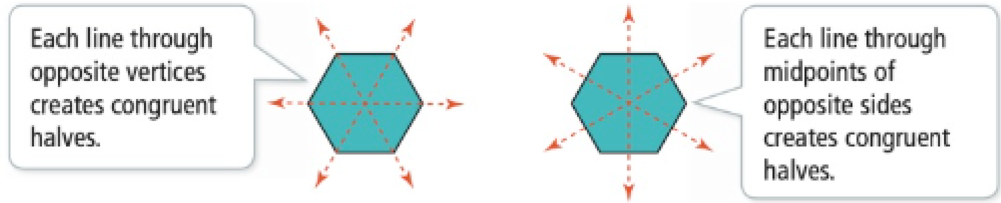
Try It! **1.** What transformations map each figure onto itself?

a.

b.

EXAMPLE 2 Identify Lines of Symmetry

How many lines of symmetry does a regular hexagon have?

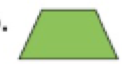


A regular hexagon has six lines of symmetry.

COMMON ERROR
Remember that lines of symmetry are not necessarily horizontal or vertical lines.

Try It! 2. How many lines of symmetry does each figure have? How do you know whether you have found them all?

a. 

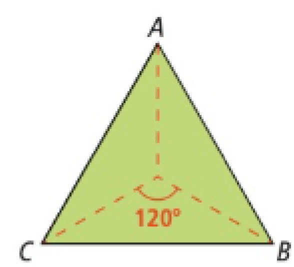
b. 

EXAMPLE 3 Identify Rotational Symmetry

For what angles of rotation does the figure map onto itself?

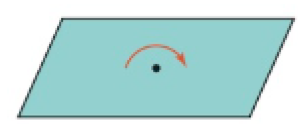
A. an equilateral triangle

Find the angles of rotation about the center that map $\triangle ABC$ onto itself.



- A rotation of 120° creates an identical image with vertex B at the top.
- A rotation of 240° creates an identical image with vertex C at the top.

B. a parallelogram



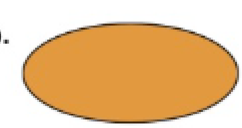
- Only a rotation of 180° maps the figure onto itself.

The type of symmetry for which there is rotation of 180° that maps a figure onto itself is called **point symmetry**. A parallelogram has 180° rotational symmetry, or point symmetry.

USE PATTERNS AND STRUCTURE
Think about how a regular polygon can be divided. How can you divide a regular polygon into pieces of the same size and shape?

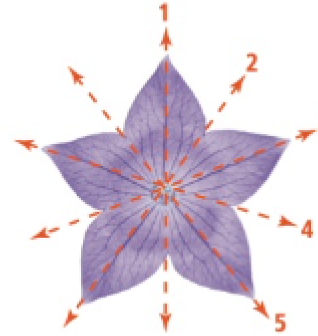
Try It! 3. What are the rotational symmetries that map the figure onto itself each figure? Does each figure have point symmetry?

a. 

b. 

EXAMPLE 4 Determine Symmetries

What type(s) of symmetry does each figure have?

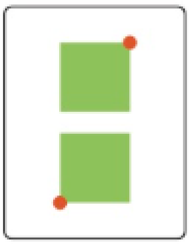


A. • A line through the center of a petal is a line of reflection. Since there are 5 petals, there are 5 lines of symmetry.

• Since 360° ÷ 5 = 72°, there is rotational symmetry at multiples of 72°.

The flower has reflectional symmetry with 5 lines of symmetry and rotational symmetry for angles of 72°, 144°, 216°, 288°, and 360°.

USE PATTERNS AND STRUCTURE
The *order of symmetry* of a transformation that maps a figure onto itself is the number of times that the transformation must be applied for each point to return to its original position. If each point never returns to the original position, then the order is infinity.



B. • No lines of symmetry can be drawn.

• Rotating the card 180° about its center creates an identical image.

The card has 180° rotational symmetry, or point symmetry.

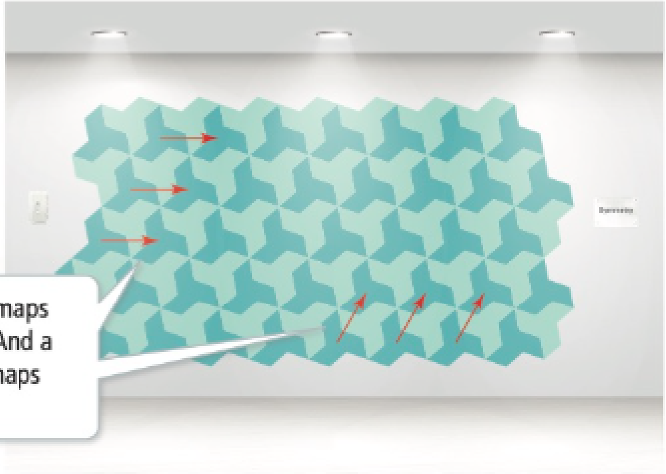
Try It! **4.** What symmetries does a square have?

APPLICATION

EXAMPLE 5 Identify Translational Symmetry

A local museum has commissioned a painter to create a mural on an interior wall. The only criteria is that the mural must have translational symmetry. Does the mural design meet the criteria?

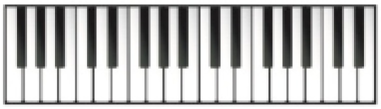
A figure consisting of a repeated pattern has **translational symmetry** if a translation maps the pattern onto itself.

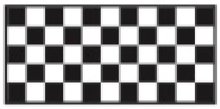


VOCABULARY
The mural is a representation of a *tessellation*. A **tessellation** is a pattern in a plane consisting of a shape that is repeated infinitely with no overlaps or gaps. Many tessellations have translational symmetry.

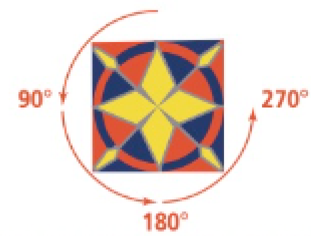
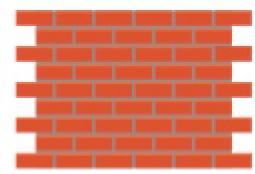
This mural has translational symmetry and meets the criteria.

Try It! **5.** Does the figure have translational symmetry? Explain.

a. 

b. 

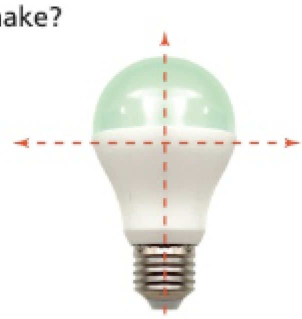
CONCEPT SUMMARY Symmetry

WORDS	DIAGRAM
Reflectional Symmetry • A figure that maps onto itself when it is reflected over a line has reflectional symmetry. • A line of symmetry is a line of reflection when a figure is reflected onto itself.	
Rotational Symmetry • A figure that maps onto itself when it is rotated about its center by an angle measuring less than 360° has **rotational symmetry**. • A figure with 180° rotational symmetry has **point symmetry**.	
Translational Symmetry • When a figure is translated horizontally, vertically, or diagonally on a plane, it creates a design with translational symmetry.	

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you tell whether a figure is symmetric?

2. **Error Analysis** For the figure below, Adam was asked to draw all lines of reflection. His work is shown. What error did Adam make?



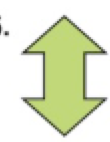
3. **Vocabulary** What type of symmetry does a figure have if it can be mapped onto itself by being flipped over a line?

4. **Use Patterns and Structure** What does it mean for a figure to have 60° rotational symmetry?

5. **Communicate and Justify** Is it possible for a figure to have rotational symmetry and no reflectional symmetry? Explain or give examples.

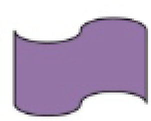
Do You KNOW HOW?

Find the number of lines of symmetry for each figure.

6. 

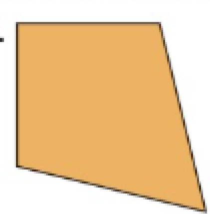
7. 

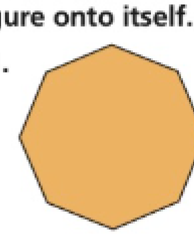
Describe the rotational symmetry of each figure. State whether each has point symmetry.

8. 

9. 

Identify the types of symmetry of each figure. For each figure with reflectional symmetry, identify the lines of symmetry. For each figure with rotational symmetry, identify the angles of rotation that map the figure onto itself.

10. 

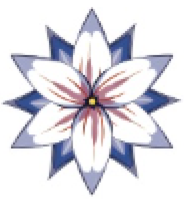
11. 

PRACTICE & PROBLEM SOLVING

UNDERSTAND

12. **Communicate and Justify** Is it possible for a figure to have reflectional symmetry and no rotational symmetry? Explain or give examples.

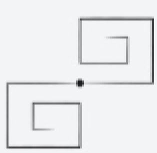
13. **Analyze and Persevere** Explain how you would find the angles of rotational symmetry for the figure shown.



14. **Mathematical Connections** A figure that has 180° rotational symmetry also has point symmetry. Write a conditional to relate those facts. Then, write the converse, inverse, and contrapositive.

15. **Analyze and Persevere** If a figure has 90° rotational symmetry, what other symmetries must it have?

16. **Error Analysis** Yumiko's work is shown below. What error did she make?



The figure is not symmetrical in any way. ✗

17. **Higher Order Thinking** Three types of rigid motion are translations, rotations, and reflections.

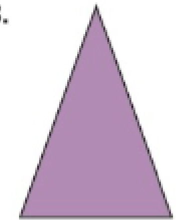
A frieze pattern is a linear pattern that repeats, and it has translational symmetry. An example is shown below.

Find and name some occurrences of frieze patterns in the real world.

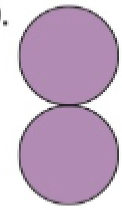
PRACTICE

For Exercises 18 and 19, find all transformations that can be used to map each figure onto itself. SEE EXAMPLE 1

18.



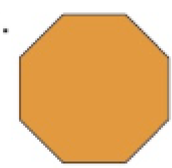
19.



20. How many lines of symmetry does a regular five-pointed star have? SEE EXAMPLE 2

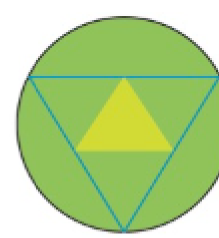
For Exercises 21 and 22, describe the rotational symmetries of each figure. SEE EXAMPLE 3

21.

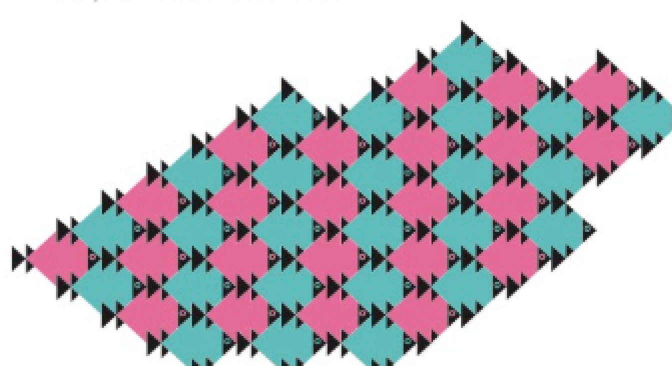


22.

23. What types of symmetry does the figure have? Explain. SEE EXAMPLE 4



24. Does the design have translational symmetry? Explain. SEE EXAMPLE 5

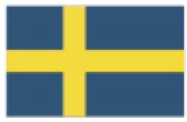


PRACTICE & PROBLEM SOLVING

APPLY

25. Choose Efficient Methods How would you decide which flags show reflection symmetry? Rotational symmetry? No symmetry?

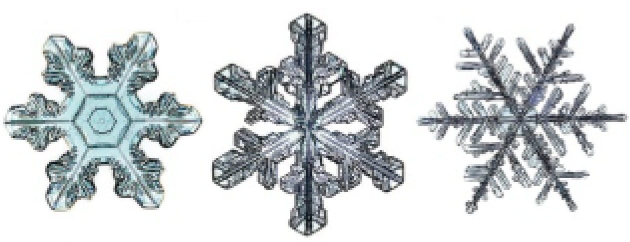

Canada


Sweden


Jamaica

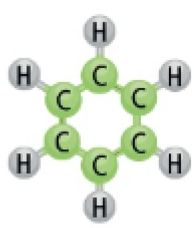

Saint Kitts and Nevis

26. Use Patterns and Structure Make observations about the structure of each snowflake, and describe the types of symmetry that a snowflake can have.

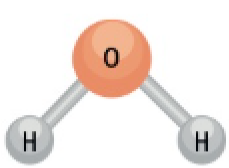


27. Analyze and Persevere Describe the symmetries of each molecule shown.

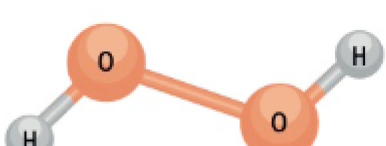
a. benzene



b. water

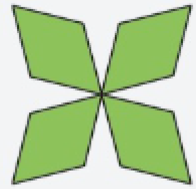


c. hydrogen peroxide



ASSESSMENT PRACTICE

28. Which types of symmetry does the figure display? Select all that apply. GR.2.3



☐ **A.** reflectional symmetry across a vertical line

☐ **B.** reflectional symmetry across a horizontal line

☐ **C.** 120° rotational symmetry

☐ **D.** 180° rotational symmetry

☐ **E.** 60° rotational symmetry

29. SAT/ACT Which letter can be mapped onto itself by a 180° rotation about its center?

Ⓐ E

Ⓑ Y

Ⓒ A

Ⓓ N

30. Performance Task A client wants a graphic designer to create an emblem that has rotational symmetry of 90° and 180°. The client needs the colors of the emblem to be red, yellow, and blue. The emblem should also include the first letter of the company name, X.

Part A The designer begins his design with a polygon. What polygons can he use?

Part B If a figure has rotational symmetry of 90° and 180°, what type of reflectional symmetry does the figure have?

Part C Create two possible designs for the client.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.2.5–Given a geometric figure and a sequence of transformations, draw the transformed figure on a coordinate plane.

MA.912.MTR.7.1

The Perplexing Polygon

Look around and you will see shapes and patterns everywhere you look. The tiles on a floor are often all the same shape and fit together to form a pattern. The petals on a flower often make a repeating pattern around the center of the flower. When you look at snowflakes under a microscope, you'll notice that they are made up of repeating three-dimensional crystals. Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC

3

Topic Review

? TOPIC ESSENTIAL QUESTION

1. What are properties of the four types of rigid motion?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A clockwise ________ of 90° about the origin maps a point in quadrant 1 to quadrant 4.
3. Reflections, translations, rotations, and glide reflections are the four types of ________.
4. A line that a figure is reflected across so that it maps onto itself is called a(n) ________.
5. A ________ is the composition of a reflection and a translation.
6. The set of points that a transformation acts on is called the ________.
7. The result of a transformation is called the ________.

- glide reflection
- image
- line of reflection
- line of symmetry
- preimage
- rigid motion
- rotation
- translation

Concepts & Skills Review

LESSON 3-1 Reflections

Quick Review

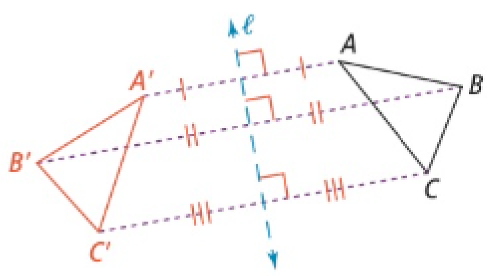
A **rigid motion** is a transformation that preserves length and angle measure.

A reflection is a transformation that reflects a point across a line of reflection m such that the image of a point A on m is A, and for a point B not on m, line m is the perpendicular bisector $\overline{BB'}$.

A reflection is a rigid motion.

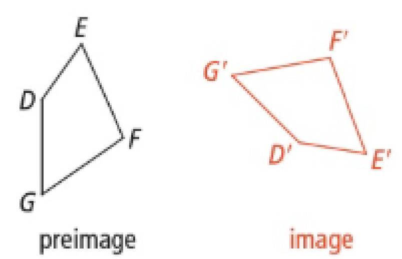
Example

What is the reflection of $\triangle ABC$ across ℓ?



Practice & Problem Solving

8. Does the transformation appear to be a rigid motion?



For Exercises 9 and 10, the vertices of $\triangle HJK$ are $H(-3, 2)$, $J(-1, -3)$, and $K(4, 3)$. What are the coordinates of the vertices of $\triangle H'J'K'$ for each reflection?

9. $r_{y\text{-axis}}$

10. $r_{x\text{-axis}}$

11. **Choose Efficient Methods** Given the coordinates of two points and the equation of a line, how can you check that one point is the image of the other point reflected across the line?

LESSON 3-2 Translations

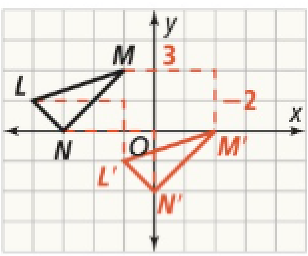
Quick Review

A translation is a transformation that maps all points the same distance and in the same direction, so that for any two points A and B, $AA' = BB'$.

A translation is a rigid motion. Any translation can be expressed as a composition of two reflections across two parallel lines.

Example

What is the graph of $T_{\langle 3, -2 \rangle}(\triangle LMN)$?



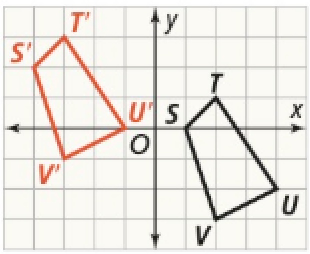
Practice & Problem Solving

For Exercises 12 and 13, the vertices of $\triangle PQR$ are $P(-4, 3)$, $Q(-2, 3)$, and $R(1, -3)$. What are the coordinates of the vertices of $\triangle P'Q'R'$ for each translation?

12. $T_{\langle -3, 2 \rangle}$

13. $T_{\langle 4, -5 \rangle}$

14. What is the translation shown?



15. Analyze and Persevere Given reflections r_h and r_k, where h is a line with equation $x = -3$ and k is a line with equation $x = 2$, how can you determine the distance of the translation resulting from the composition $r_k \circ r_h$?

LESSON 3-3 Rotations

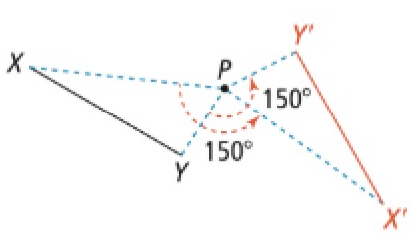
Quick Review

A rotation is a transformation that rotates a point about the center of rotation P by the angle of rotation $x°$ such that the image of P is P, $PA = PA'$, and $m\angle APA' = x$.

A rotation is a rigid motion. Any rotation can be expressed as a composition of reflections across two intersecting lines.

Example

What is the 150° rotation of $\overline{XY}$ about P?



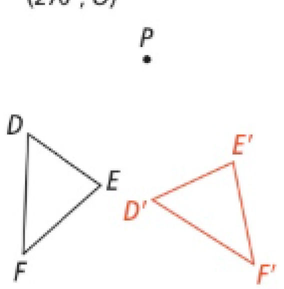
Practice & Problem Solving

For Exercise 16 and 17, the vertices of $\triangle ABC$ are $A(2, -2)$, $B(-3, -2)$, and $C(-1, 3)$. What are the coordinates of the vertices of $\triangle A'B'C'$ for each rotation?

16. $R_{(90°, O)}$

17. $R_{(270°, O)}$

18. Draw two lines of reflection so the composition of the reflections across the lines is equivalent to the rotation about point P shown.



19. Use Patterns and Structure If two lines intersect at a right angle at point P, what rotation is equivalent to the composition of the reflections across the two lines?

LESSON 3-4 Classification of Rigid Motions

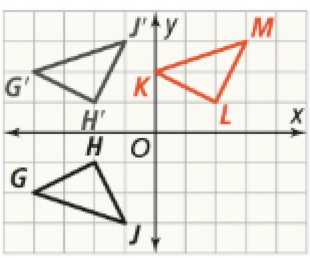
Quick Review

A **glide reflection** is the composition of a reflection followed by a translation.

Any rigid motion is either a translation, reflection, rotation, or glide reflection. As a result, any rigid motion can be expressed as a combination of reflections.

Example

What is a glide reflection that maps $\triangle GHJ$ to $\triangle KLM$?



First reflect $\triangle GHJ$ across the x-axis to get $\triangle G'H'J'$. Then translate $\triangle G'H'J'$ by 4 units to the right to get $\triangle KLM$. The glide reflection is $T_{\langle 4, 0\rangle} \circ r_{x\text{-axis}}$.

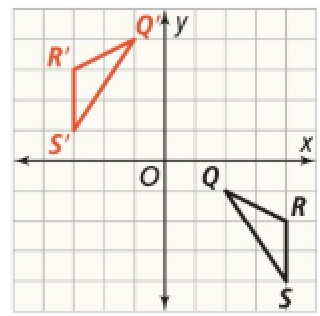
Practice & Problem Solving

For Exercises 20 and 21, the vertices of $\triangle LMN$ are $L(-2, 4)$, $M(1, 2)$, and $N(-3, -5)$. Suppose j is a line with equation $x = 3$ and k is a line with equation $y = -2$. what are the coordinates of the vertices of $\triangle L'M'N'$ for each glide reflection?

20. $T_{\langle -2, 4\rangle} \circ r_j$

21. $T_{\langle 2, -3\rangle} \circ r_k$

22. What is a glide reflection for the transformation shown?



23. Analyze and Persevere Is there more than one way to describe a glide reflection as a composition of a reflection and then a translation? Explain.

LESSON 3-5 Symmetry

Quick Review

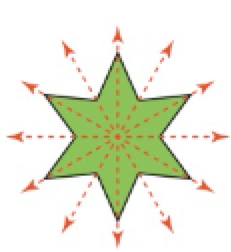
A figure has **symmetry** if a rigid motion can map the figure to itself. If the rigid motion is a reflection, then the symmetry is **reflectional symmetry**. If the rigid motion is a rotation, then the symmetry is **rotational symmetry**.

When the angle of rotation is 180°, the rotational symmetry is called **point symmetry**.

Example

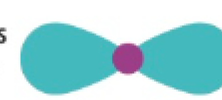
How many lines of symmetry does the figure have?

You can draw a line of reflection through each vertex and the center of the star. There are six lines of symmetry.



Practice & Problem Solving

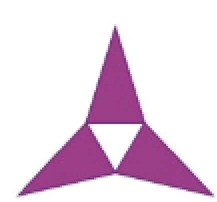
24. Describe the transformations that can be used to map the figure onto itself.



For Exercises 25 and 26, describe all the symmetries of each figure. If the figure has reflectional symmetry, identify all the lines of symmetry. If the figure has rotational symmetry, give the angles of rotation.

25.

26.



27. Use Patterns and Structure Suppose a figure has at least two lines of symmetry. Explain why the figure must have rotational symmetry.

TOPIC
4

Triangle Congruence

TOPIC ESSENTIAL QUESTION

What relationships between sides and angles of triangles can be used to prove triangles congruent?

Topic Overview

enVision® STEM Project
Design a Bridge

4-1 Congruence
GR.2.3, GR.2.6, MTR.5.1, MTR.1.1, MTR.4.1

4-2 Isosceles and Equilateral Triangles
GR.1.3, MTR.4.1, MTR.3.1, MTR.7.1

Mathematical Modeling in 3 Acts
Check It Out!
GR.2.6, GR.1.2, MTR.7.1

4-3 Proving and Applying the SAS and SSS Congruence Criteria
GR.1.2, GR.2.7, MTR.6.1, MTR.1.1, MTR.4.1

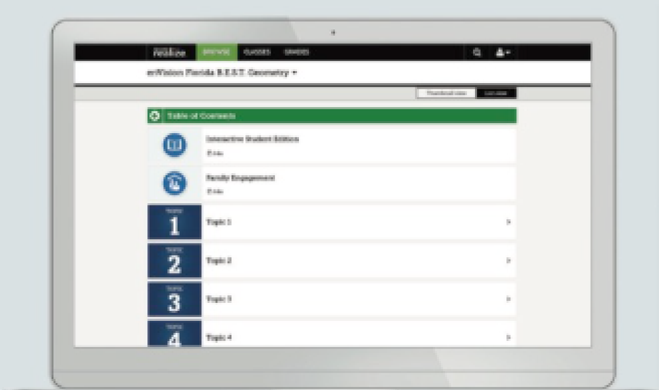
4-4 Proving and Applying the ASA and AAS Congruence Criteria
GR.1.2, GR.1.6, GR.2.7, MTR.5.1, MTR.3.1, MTR.1.1

4-5 Congruence in Right Triangles
GR.1.2, GR.1.6, MTR.5.1, MTR.3.1, MTR.4.1

4-6 Congruence in Overlapping Triangles
GR.1.6, GR.1.2, MTR.1.1, MTR.2.1, MTR.6.1

Topic Vocabulary

- congruence transformation
- congruent



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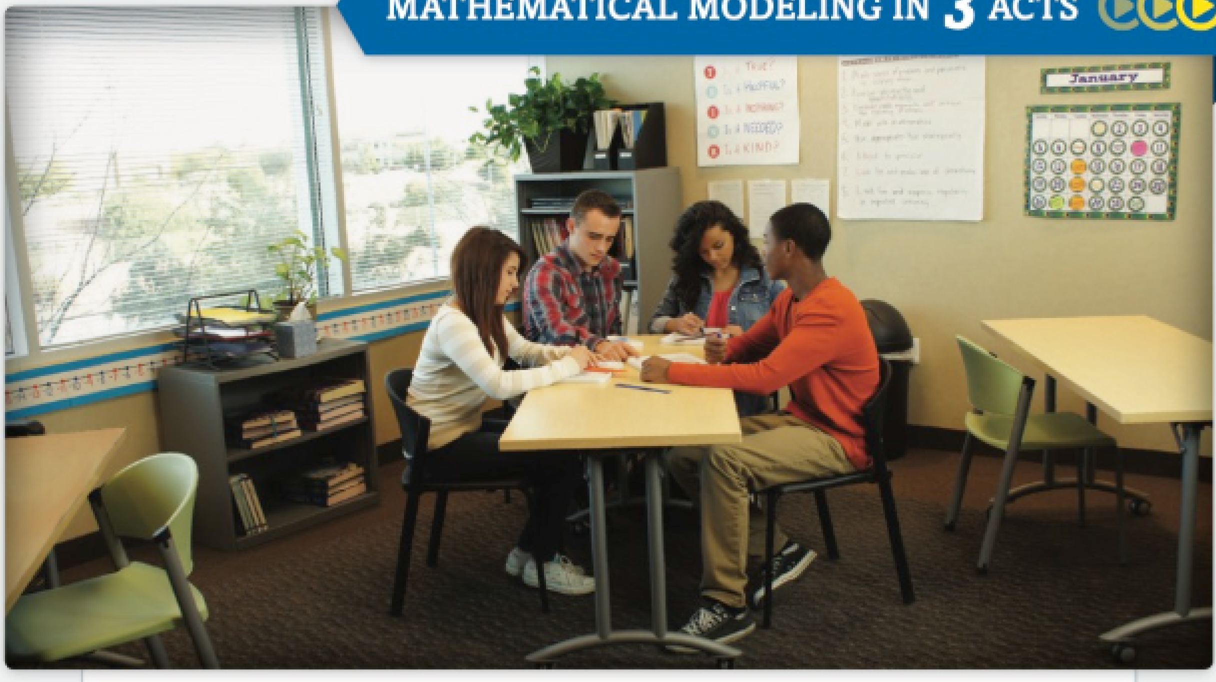
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Check It Out!

Maybe you've played this game before: you draw a picture. Then you try to get a classmate to draw the same picture by giving step-by-step directions but without showing your drawings.

Try it with a classmate. Draw a map of a room in your house or a place in your town. Then give directions to a classmate to draw the map that you drew. How similar are they? Think about your results during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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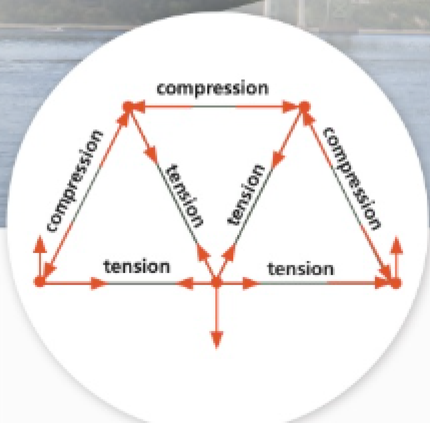
Did You Know?

The **Tacoma Narrows Bridge**, in Washington, collapsed in high winds a few months after it opened in 1940. The bridge was rebuilt in 1950 using a truss for stabilization.



A bridge works by **balancing compression (pressing inward) and tension (pressing outward)**, distributing the load onto the bridge supports.

The design of a truss is based on the strength of a triangle. It distributes a load from a narrow point to a wider base.

BRIDGE LOAD =

Weight of **bridge**

\+

Weight of **people**

\+

Weight of **vehicles**

\+

Weight of **precipitation**

Your Task: Design a Bridge

You and your classmates will analyze different truss bridge designs and how congruent triangles are used in each construction. What type of truss would you use in a bridge design, and why?

4-1 Congruence

I CAN… use a composition of rigid motions to show that two objects are congruent.

VOCABULARY

- congruence transformation
- congruent

MA.912.GR.2.3–Identify a sequence of transformations that will map a given figure onto itself or onto another congruent or similar figure. Also GR.2.6

MA.K12.MTR.5.1, MTR.1.1, MTR.4.1

EXPLORE & REASON

Some corporate logos are distinctive because they make use of repeated shapes.

A designer creates two versions of a new logo for the Bolt Company. Version 1 uses the original image shown at the right and a reflection of it. Version 2 uses reduced copies of the original image.

A. Make a sketch of each version.

B. **Analyze and Persevere** The owner of the company says, "I like your designs, but it is important that the transformed image be the same size and shape as the original image." What would you do to comply with the owner's requirements?

C. What transformations can you apply to the original image that would produce logos acceptable to the owner? Explain.

ESSENTIAL QUESTION What is the relationship between rigid motions and congruence?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Understand Congruence

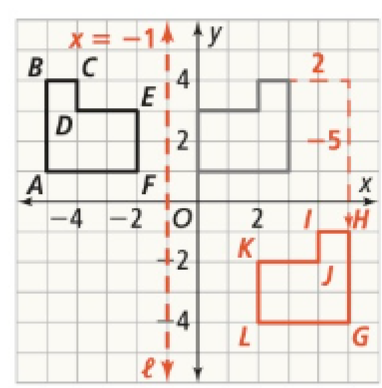
Suppose there is a rigid motion that maps one figure to another. Why does that show that the two figures are congruent?

Figure *ABCDEF* has the following lengths and angle measures, and *GHIJKL* is the image of *ABCDEF* after the rigid motion $T_{\langle 2, -5\rangle} \circ R_{\ell}$.

$AB = 3 \quad BC = 1 \quad CD = 1$

$DE = 2 \quad EF = 2 \quad FA = 3$

All angles in figure *ABCDEF* are right angles.



Because rigid motions preserve length and angle measure, *GHIJKL* has the following lengths and angle measures.

$GH = 3 \quad HI = 1 \quad IJ = 1$

$JK = 2 \quad KL = 2 \quad LG = 3$

All angles in figure *GHIJKL* are right angles.

Since rigid motions preserve measures of corresponding sides and angles, the two figures are *congruent*.

STUDY TIP
Recall that angles with the same measure are congruent, and segments with the same length are congruent.

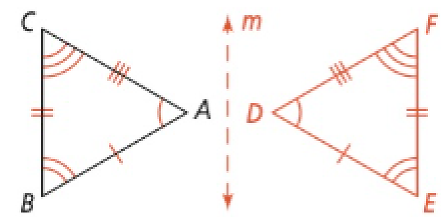
Try It! 1. A 90° rotation about the origin maps △*PQR* to △*LMN*. Are the triangles congruent? Explain.

CONCEPT Congruence

Figures that have the same size and shape are said to be *congruent*. Two figures are **congruent** if there is a rigid motion that maps one figure to the other.

A rigid motion is sometimes called a **congruence transformation** because it maps a figure to a congruent figure.

Use the ≅ symbol to show that two figures are congruent. Since $r_m(\triangle ABC) = \triangle DEF$, $\triangle ABC \cong \triangle DEF$.

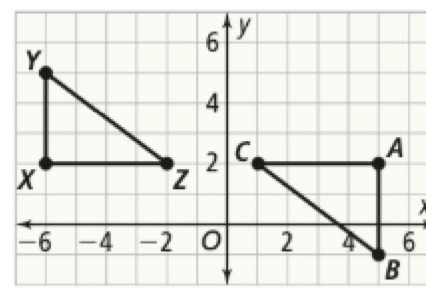


EXAMPLE 2 Verify Congruence

Given $\triangle XYZ \cong \triangle ABC$, what composition of rigid motions maps $\triangle XYZ$ to $\triangle ABC$?

First rotate $\triangle XYZ$ 180° about point Z. Next translate the image three units to the right.

$$\left(T_{\langle 3,\,0\rangle} \circ R_{(180°,\,Z)}\right)(\triangle XYZ) = \triangle ABC$$

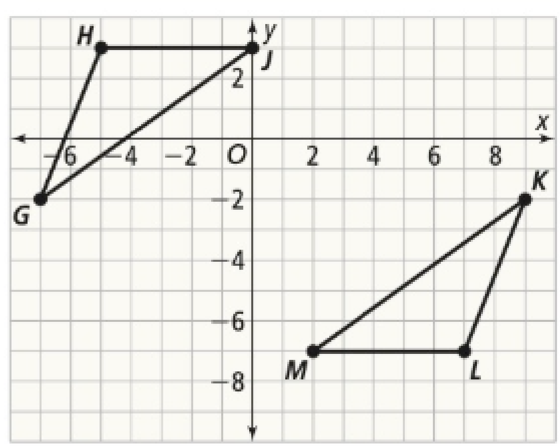


The composition $T_{\langle 3,\,0\rangle} \circ R_{(180°,\,Z)}$ maps $\triangle XYZ$ to $\triangle ABC$.

USE PATTERNS AND STRUCTURE
Think of other solutions. Can you identify another composition of rigid motions that maps one triangle to the other?

Try It!

2. Use the graph shown.

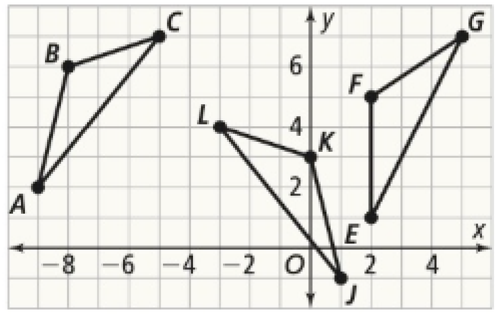


a. Given $\triangle GHJ \cong \triangle KLM$, what is one composition of rigid motions that maps $\triangle GHJ$ to $\triangle KLM$?

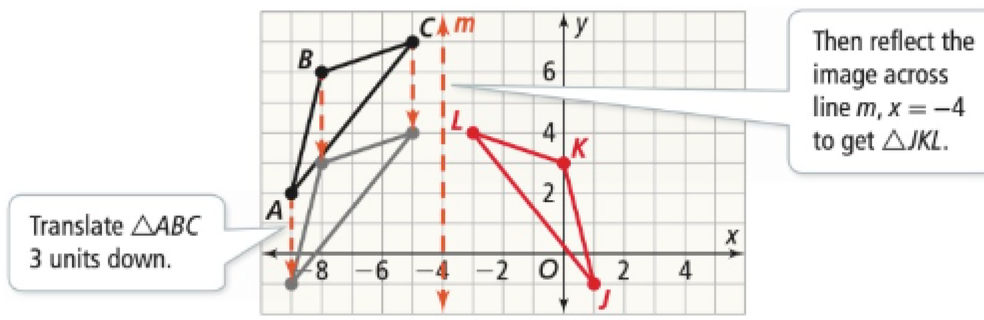
b. What is another composition of rigid motions that maps $\triangle GHJ$ to $\triangle KLM$?

EXAMPLE 3 Identify Congruent Figures

Given $\triangle ABC$, $\triangle EFG$, and $\triangle JKL$, which triangles are congruent?



Find a composition of rigid motions to map $\triangle ABC$ to $\triangle JKL$.

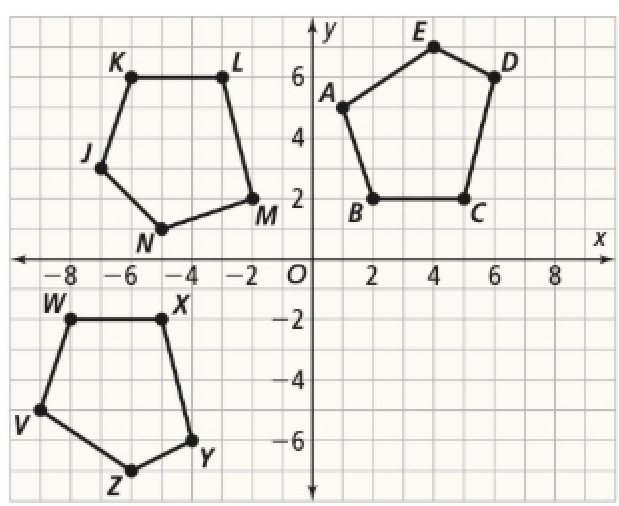


The composition of rigid motions $r_m \circ T_{\langle 0, -3 \rangle}$ maps $\triangle ABC$ to $\triangle JKL$. Therefore, $\triangle ABC \cong \triangle JKL$.

Observe that AC and EG are the longest side in each corresponding triangle, and $AC \neq EG$, so there is no single rigid motion or composition of rigid motions that maps $\triangle ABC$ to $\triangle EFG$. Therefore, they are not congruent.

STUDY TIP
To show that two figures are not congruent, you only need to find one corresponding pair of sides or angles that do not have the same measure.

Try It! **3.** Use the graph shown.



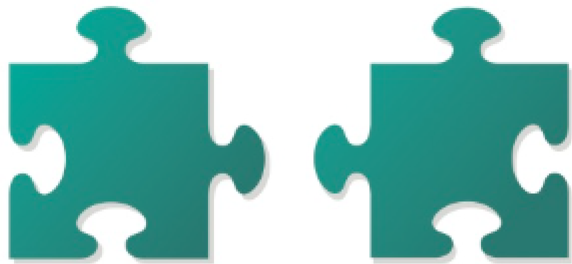
a. Are $ABCDE$ and $JKLMN$ congruent? If so, describe a composition of rigid motions that maps $ABCDE$ to $JKLMN$. If not, explain.

b. Are $ABCDE$ and $VWXYZ$ congruent? If so, describe a composition of rigid motions that maps $ABCDE$ to $VWXYZ$. If not, explain.

EXAMPLE 4 Determine Congruence

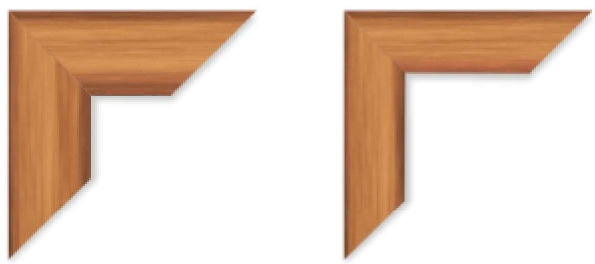
Which pairs of objects are congruent? If a pair of objects is congruent, describe a composition of rigid motions that maps one to the other.

A.



The puzzle pieces are congruent. A reflection across a vertical line maps one puzzle piece to the other.

B.



The frame corners are not congruent. The diagonal segment at the corner of the left frame is longer than the diagonal segment at the corner of the right frame, so the two frame corners are not the same size.

COMMON ERROR

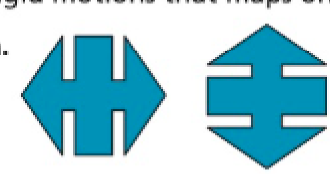
Remember that the composition of rigid motions must map *all* points from the preimage to the image for objects to be congruent.

C.

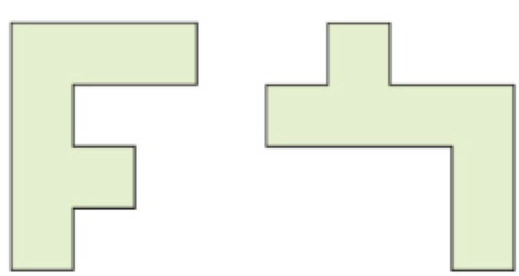
The puzzles are congruent. Translate the preimage puzzle on the left, rotate the figure 90° clockwise, and then reflect over a vertical line.

Try It! **4.** Is the pair of objects congruent? If so, describe a composition of rigid motions that maps one object onto the other.

a.



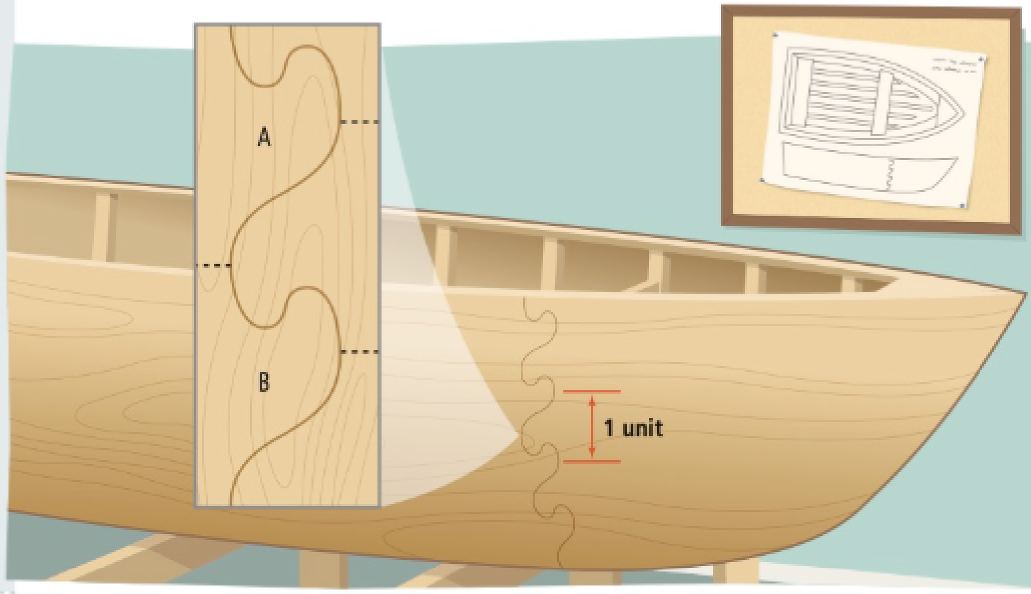
b.



APPLICATION

EXAMPLE 5 Apply Congruence

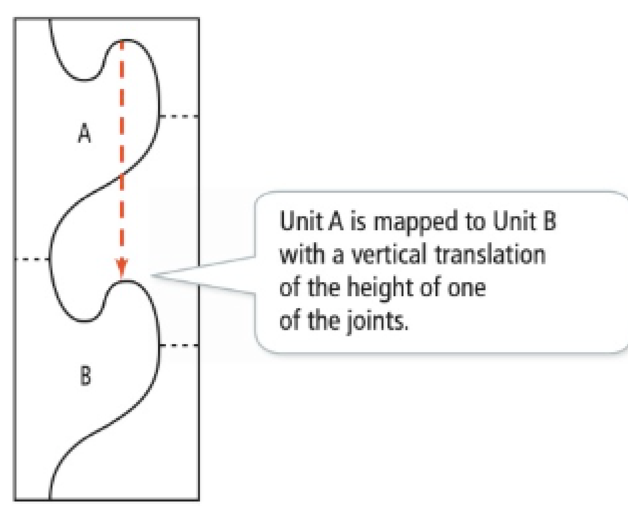
A boat builder plans to connect two pieces of wood by using a puzzle joint as shown. For a successful joint, each unit must be congruent.



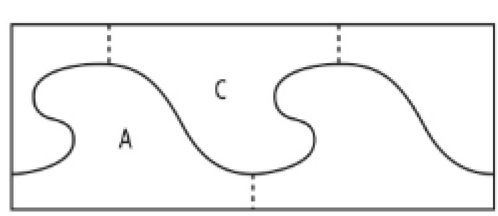
STUDY TIP
A composition of rigid motions can be one rigid motion as well as a composition of more than one rigid motion.

Given Unit A, what composition of rigid motions maps Unit A to Unit B?

Unit B is a translation of Unit A. Unit B has the same size, shape, and orientation as Unit A. The only difference is that it is farther down.



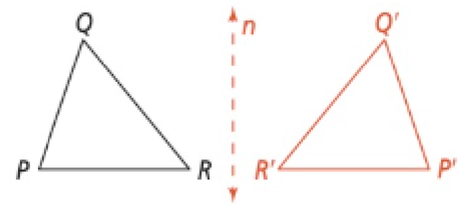
Try It! 5. Is Unit C congruent to Unit A? If so, describe the sequence of rigid motions that maps Unit A to Unit C.



CONCEPT SUMMARY Congruent Figures

WORDS If two figures are congruent, a composition of rigid motions maps one figure to another.

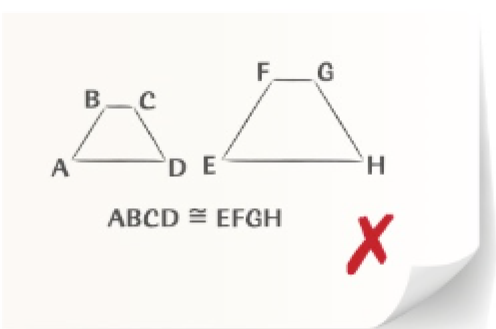
DIAGRAM Since $r_n(\triangle PQR) = \triangle P'Q'R'$, $\triangle PQR \cong \triangle P'Q'R'$.



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What is the relationship between rigid motions and congruence?

2. **Error Analysis** Taylor says *ABCD* and *EFGH* are congruent because he can map *ABCD* to *EFGH* by multiplying each side length by 1.5 and translating the result to coincide with *EFGH*. What is Taylor's error?



3. **Vocabulary** Why is a rigid motion also called a congruence transformation?

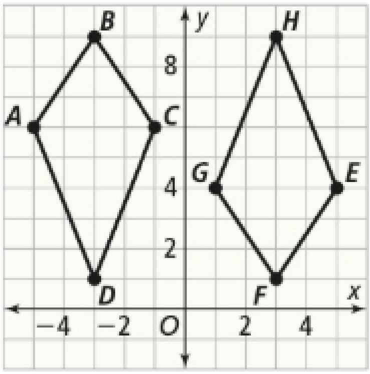
4. **Generalize** For any two line segments that are congruent, what must be true about the lengths of the segments?

5. **Communicate and Justify** A composition of rigid motions maps one figure to another figure. Is each intermediate image in the composition congruent to the original and final figures? Explain.

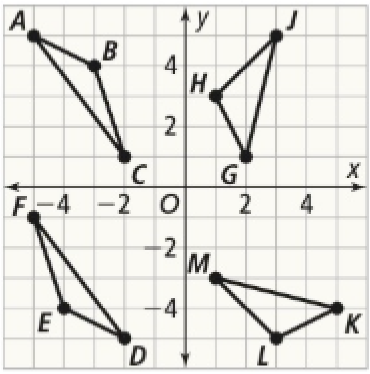
6. **Communicate and Justify** Describe how you can find a rigid motion or composition of rigid motions to map a segment to a congruent segment and an angle to a congruent angle.

Do You KNOW HOW?

7. Given $ABCD \cong EFGH$, what rigid motion, or composition of rigid motions, maps *ABCD* to *EFGH*?



8. Which triangles are congruent?



9. Are Figure A and Figure B congruent? If so, describe a composition of rigid motions that maps Figure A to Figure B. If not, explain.

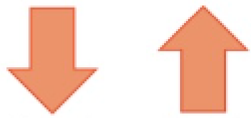
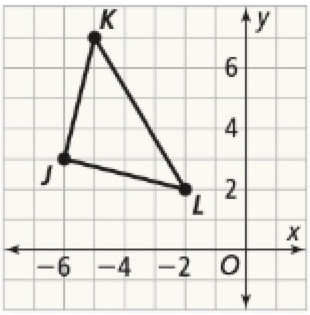


Figure A Figure B

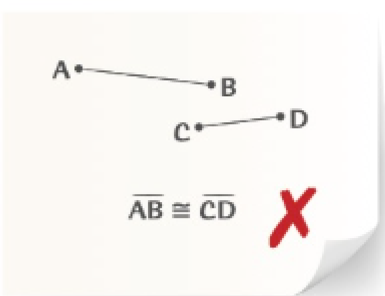
PRACTICE & PROBLEM SOLVING

UNDERSTAND

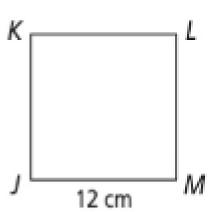
10. **Choose Effective Methods** If $\triangle JKL \cong \triangle RST$, give the coordinates for possible vertices of $\triangle RST$. Justify your answer by describing a composition of rigid motions that maps $\triangle JKL$ to $\triangle RST$.



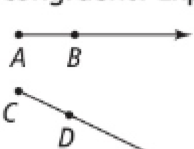
11. **Error Analysis** Yuki says that if all lines are congruent, then all line segments must be congruent. Is Yuki correct? Explain.



12. **Mathematical Connections** Given square *JKLM* and $(T_{\langle -6, 4\rangle} \circ T_{\langle 1, 5\rangle})(JKLM) = RSTU$, what is the area of *RSTU*?

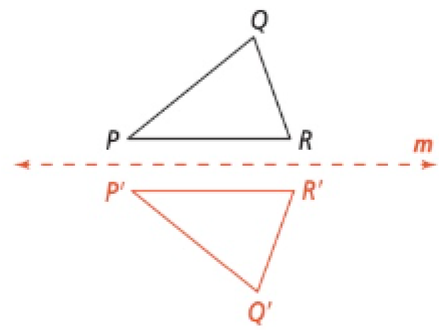


13. **Higher Order Thinking** Are $\overrightarrow{AB}$ and $\overrightarrow{CD}$ congruent? If so, describe a composition of rigid motions that maps any ray to any other ray. If not, explain. Are any two rays congruent? Explain.

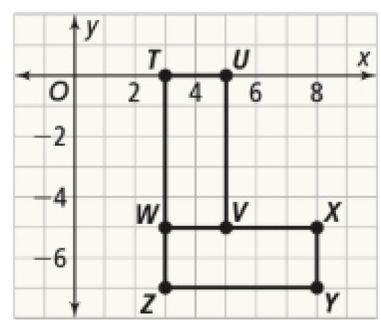


PRACTICE

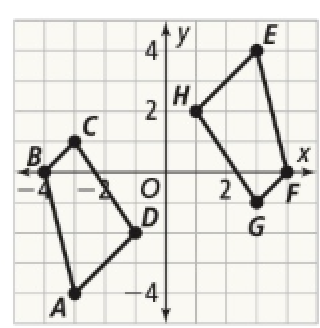
14. Given $r_m(\triangle PQR) = \triangle P'Q'R'$, do $\triangle P'Q'R'$ and $\triangle PQR$ have equal perimeters? Explain.
SEE EXAMPLE 1



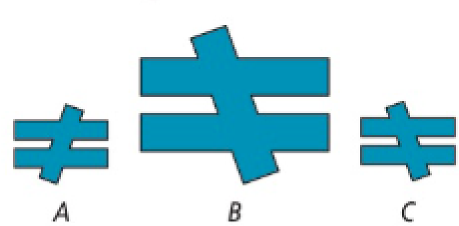
15. Given $WXYZ \cong WTUV$, describe a composition of rigid motions that maps *WXYZ* to *WTUV*.
SEE EXAMPLE 2



16. Are *ABCD* and *EFGH* congruent? If so, describe a composition of rigid motions that maps *ABCD* to *EFGH*. If not, explain.
SEE EXAMPLE 3



17. Which objects are congruent? For any congruent objects, describe a composition of rigid motions that maps the preimage to the image. SEE EXAMPLES 4 AND 5



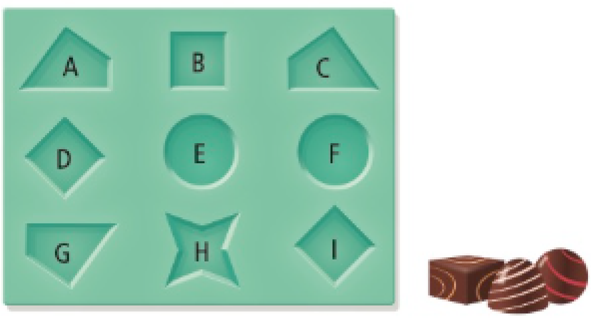
PRACTICE & PROBLEM SOLVING

APPLY

18. **Analyze and Persevere** Using a 3D printer, Emery makes the chocolate mold shown by copying different shapes.

a. Which of the designs in the mold appear to be congruent?

b. Describe a composition of rigid motions that maps the congruent shapes.



19. **Apply Math Models** Are the illustrations of the shoes in the advertisement congruent? If so, describe a composition of rigid motions that maps the left shoe to the right shoe.

20. **Use Patterns and Structure** Describe a rigid motion or a composition of rigid motions that can be used to make sure that each slice of quiche is the same size and shape as the first slice.

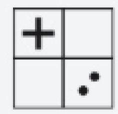
ASSESSMENT PRACTICE

21. The transformation $T_{\langle 3,\,8\rangle} \circ R_{(90°,\,A)}$ maps $\triangle ABC$ to $\triangle DEF$. GR.2.3

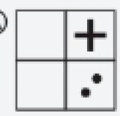
Is $\triangle ABC$ congruent to $\triangle DEF$?

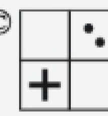
What kind of transformation is $T_{\langle 3,\,8\rangle} \circ R_{(90°,\,A)}$?

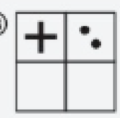
22. **SAT/ACT** A board game token is shown.

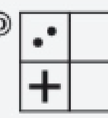


Which is congruent to the token?

Ⓐ 

Ⓒ 

Ⓑ 

Ⓓ 

23. **Performance Task** The Aztec fabric pattern shown is based on the original image.

Part A Identify any images in the pattern that appear to be congruent to the original image.

Part B Describe a composition of rigid motions that maps the original image to each congruent image in the pattern.

Part C For any images in the pattern that are not congruent to the original image, explain how you know they are not congruent.

4-2 Isosceles and Equilateral Triangles

I CAN… apply theorems about isosceles and equilateral triangles to solve problems.

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles.

MA.K12.MTR.4.1, MTR.3.1, MTR.7.1

EXPLORE & REASON

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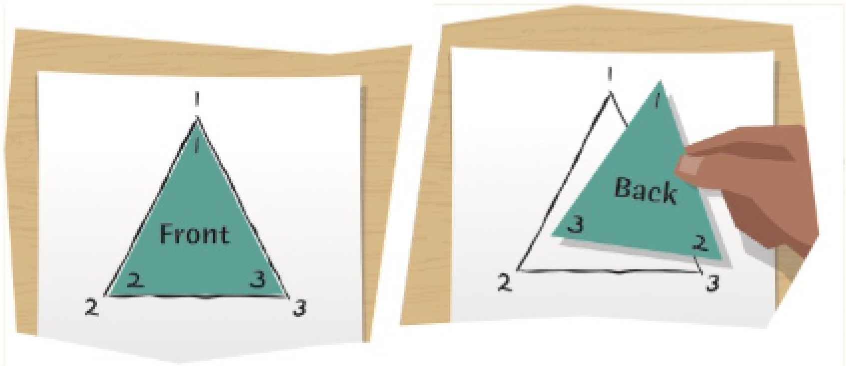
Cut out a triangle with two sides of equal length from a sheet of paper and label its angles 1, 2, and 3. Trace the outline of your triangle on another sheet of paper and label the angles.



A. In how many different ways can you flip, slide, or turn the triangle so that it fits exactly on the outline?

B. **Use Patterns and Structures** How do the angles and sides of the outline correspond to the angles and sides of the triangle?

C. How would your answer to Part A change if all three sides of the triangle were of equal length?

ESSENTIAL QUESTION How are the side lengths and angle measures related in isosceles triangles and in equilateral triangles?

CONCEPTUAL UNDERSTANDING

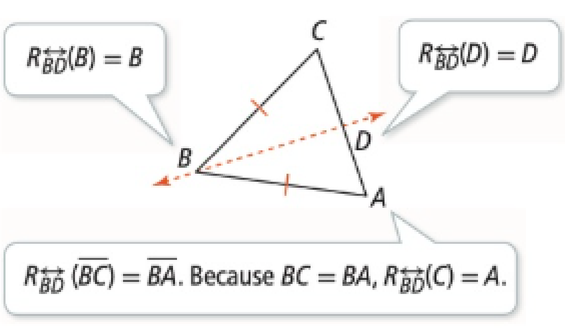
EXAMPLE 1 Understand Angles of Isosceles Triangles

How are the base angles of an isosceles triangle related?

Draw isosceles triangle ABC.

Because $\overline{BC} \cong \overline{BA}$, there is a rigid motion that maps $\overline{BC}$ onto $\overline{BA}$. Draw the angle bisector $\overleftrightarrow{BD}$ of $\angle ABC$, so that $m\angle DBC = m\angle DBA$.

STUDY TIP
The rigid motion that maps $\overline{BC}$ to $\overline{BA}$ must map point B to itself. A reflection across a line that contains point B maps point B to itself.



Since the reflection maps $\triangle BCD$ to $\triangle BAD$, $m\angle BCA = m\angle BAC$. Therefore, $\angle BCA \cong \angle BAC$.

Try It! 1. Copy isosceles $\triangle ABC$. Reflect the triangle across line BC to create the image $\triangle C'B'A'$. What rigid motion maps $\triangle C'B'A'$ onto $\triangle ABC$? Can you use this to show that $\angle A \cong \angle C$? Explain.

THEOREM 4-1 Isosceles Triangle Theorem and the Converse

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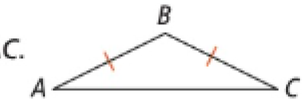
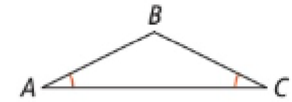
If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

If... $\overline{AB} \cong \overline{BC}$,
Then... $\angle ACB \cong \angle BAC$.



If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

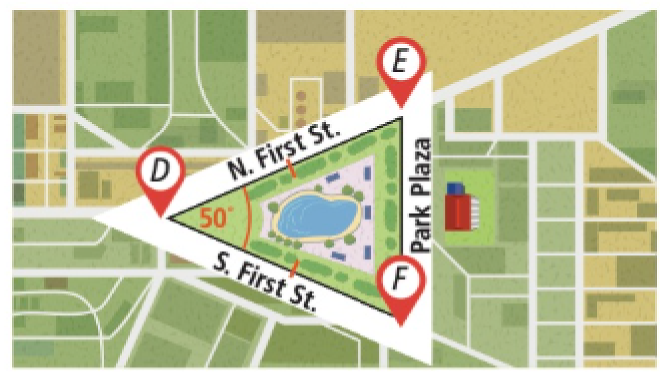
If... $\angle ACB \cong \angle BAC$,
Then... $\overline{AB} \cong \overline{BC}$.



PROOF: SEE EXERCISE 17.

APPLICATION

EXAMPLE 2 Use the Isosceles Triangle Theorem



An architect is designing a community park between N. First St. and S. First St. The pathways on either side of the pool will be equal in length and will provide effective access and circulation around the pool. To protect the landscaping and to minimize erosion, the architect will place a triangular section of triangular cobblestones at the corners along Park Plaza. What angle measure should the architect specify for the corners in her design?

The park is in the shape of an isosceles triangle. Find $m\angle F$ and $m\angle E$.

$$m\angle D + m\angle E + m\angle F = 180$$

$$50 + m\angle F + m\angle F = 180$$

$$m\angle F = 65$$

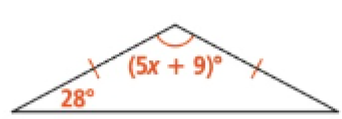
The base angles of the isosceles triangle are congruent by the Isosceles Triangle Theorem, so $m\angle E = m\angle F$.

The landscape architect should specify that the angles at the corners measure 65°.

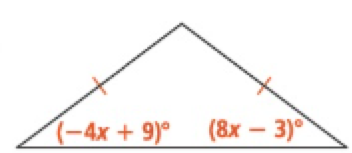
STUDY TIP
Confirm your solution by using your original equation. In this example, the three angles of $\triangle ABC$ are 65°, 65°, and 50°, and the sum is 180°.

Try It! **2.** What is the value of x?

a.

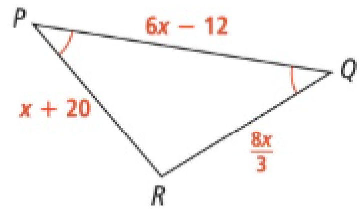


b.



EXAMPLE 3 Use the Converse of the Isosceles Triangle Theorem

What are the lengths of all three sides of the triangle?



COMMON ERROR

Be careful not to set the expressions for the length of a leg and the length of the base equal to each other. Remember that the congruent legs are opposite the congruent base angles.

Step 1 Find the value of x.

$$x + 20 = \frac{8x}{3}$$
$$3x + 60 = 8x$$
$$12 = x$$

Because $\angle P \cong \angle Q$, the sides opposite $\angle P$ and $\angle Q$ are also congruent.

Step 2 Substitute 12 for x to determine the side lengths.

$PR = x + 20 = 12 + 20 = 32$

$PQ = 6x - 12 = 6(12) - 12 = 60$

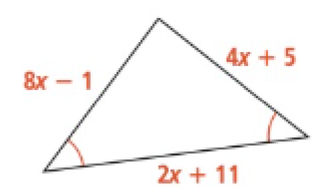
$QR = \frac{8x}{3} = \frac{8(12)}{3} = 32$

Try It!

3. Use the figure shown.

a. What is the value of x?

b. What are the lengths of all three sides of the triangle?



THEOREM 4-2

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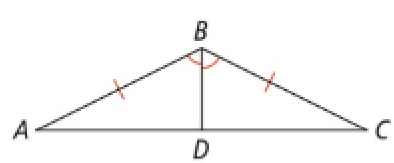
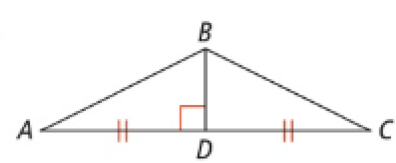
If a line or line segment bisects the vertex angle of an isosceles triangle, then it is also the perpendicular bisector of the opposite side.

If...



$\overline{AB} \cong \overline{BC}$ and $m\angle ABD = m\angle CBD$

Then...



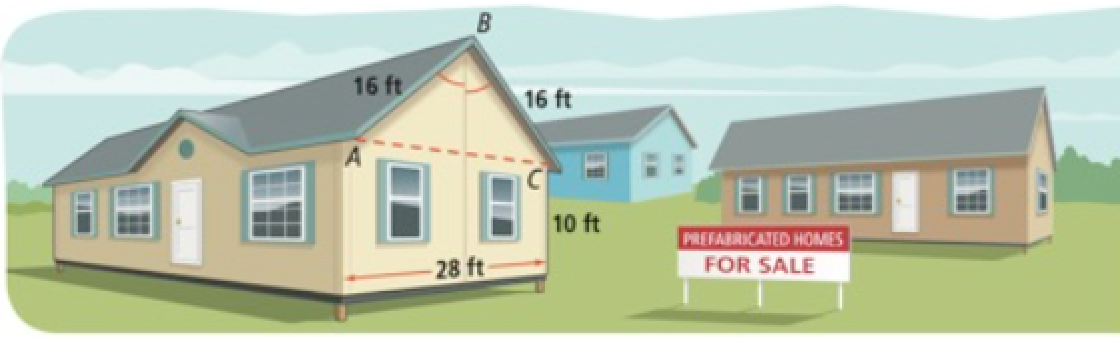
$\overline{BD} \perp \overline{AC}$ and $\overline{AD} \cong \overline{DC}$

PROOF: SEE EXERCISE 13.

APPLICATION

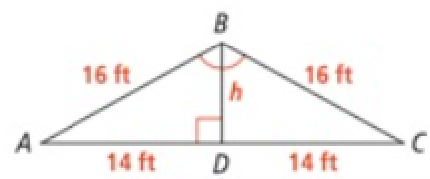
EXAMPLE 4 Use Perpendicular Bisectors to Solve Problems

A prefabricated house is delivered to a foundation in two symmetric halves that are assembled on-site. Along the planned route to the site, the truck must pass under a bridge that has a clearance height of 17 feet. Should the trucker plan a different route for delivering the house? Explain.



COMMUNICATE AND JUSTIFY
Consider the information the figure provides about $\triangle ABC$. What other information given in the figure is needed?

Draw a diagram to represent the roof.



$\overline{BD}$ is the perpendicular bisector of $\overline{AC}$ so $AD = CD = 14$ ft.

Use the Pythagorean Theorem to find the height h.

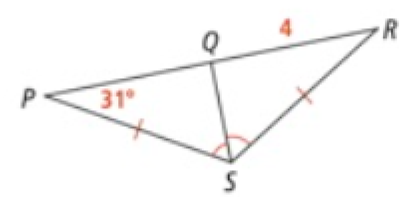
$$h^2 + 14^2 = 16^2$$
$$h^2 = 60$$
$$h = \sqrt{60} \approx 7.7$$

The house is approximately 17.7 feet tall, so the total height that the truck must clear is greater than 17.7 ft. The trucker should plan a different route to the site.

Try It! 4. Use the figure shown.

a. What is $m\angle RSQ$?

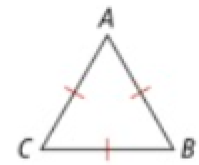
b. What is PR?



CONCEPT Equilateral Triangles

An equilateral triangle is equiangular.

An equiangular triangle is equilateral.

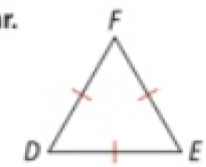


If... $\angle A \cong \angle B \cong \angle C$, **then...** $\overline{AB} \cong \overline{BC} \cong \overline{AC}$.

If... $\overline{AB} \cong \overline{BC} \cong \overline{AC}$, **then...** $\angle A \cong \angle B \cong \angle C$.

EXAMPLE 5 Prove that Equilateral Triangles are Equiangular

A. Prove that equilateral triangles are equiangular.

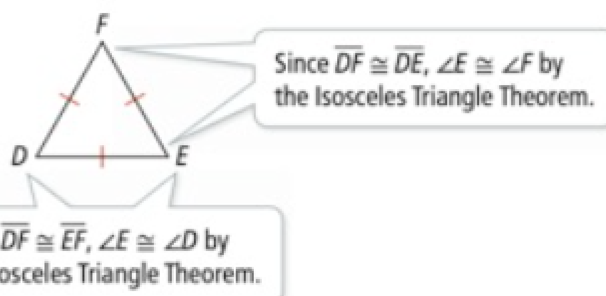


Given: $\overline{DE} \cong \overline{EF} \cong \overline{DF}$

Prove: $\angle D \cong \angle E \cong \angle F$

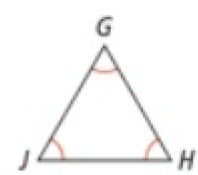
Plan: Use the fact that an equilateral triangle is also an isosceles triangle to show that all three angles are congruent.

Proof:



Since $\angle D \cong \angle E$ and $\angle E \cong \angle F$, $\angle D \cong \angle E \cong \angle F$.

B. Prove that equiangular triangles are equilateral.



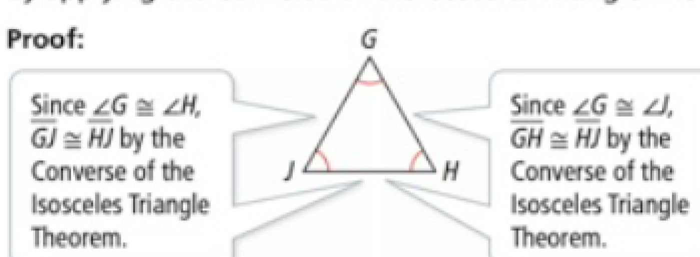
Given: $\angle G \cong \angle H \cong \angle J$

Prove: $\overline{GH} \cong \overline{HJ} \cong \overline{GJ}$

STUDY TIP
When planning a proof, consider proofs that may be parallel in structure.

Plan: Use a strategy similar to the one in part A by applying the Converse of the Isosceles Triangle Theorem.

Proof:

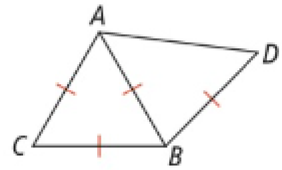


Since $\overline{GJ} \cong \overline{HJ}$ and $\overline{GH} \cong \overline{HJ}$, $\overline{GH} \cong \overline{HJ} \cong \overline{GJ}$.

Try It! 5. What rotation can be used to show the angles of an equilateral triangle are congruent?

EXAMPLE 6 Find Angle Measures in Isosceles and Equilateral Triangles

A. If $m\angle CBD = 130$, what is $m\angle BAD$?



Step 1 **Find $m\angle ABD$.**

$$m\angle ABC + m\angle ABD = m\angle CBD$$
$$60 + m\angle ABD = 130$$
$$m\angle ABD = 70$$

Since $\triangle ABC$ is equilateral, $m\angle ABC = 60$.

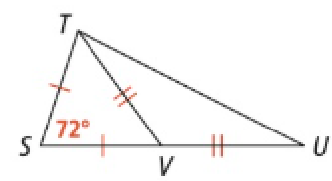
Step 2 **Use the Isosceles Triangle Theorem to find $m\angle BAD$.**

$$m\angle ABD + m\angle BDA + m\angle BAD = 180$$
$$70 + m\angle BAD + m\angle BAD = 180$$
$$m\angle BAD = 55$$

Since $\triangle ABD$ is isosceles, $m\angle BDA = m\angle BAD$.

COMMON ERROR
You may think you have solved the problem after finding one angle measure. Make sure you provide the measure of the angle asked for in the question.

B. What is $m\angle U$?



Step 1 Find $m\angle SVT$. Write an equation using the Triangle Angle-Sum Theorem.

$$m\angle S + m\angle STV + m\angle SVT = 180$$
$$72 + m\angle SVT + m\angle SVT = 180$$
$$2m\angle SVT = 108$$
$$m\angle SVT = 54$$

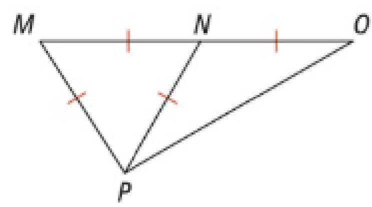
Since $\triangle STV$ is isosceles, $m\angle STV = m\angle SVT$.

Step 2 Find $m\angle U$. Write an equation using the Triangle Exterior Angle Theorem.

$$m\angle SVT = m\angle VTU + m\angle U$$
$$54 = m\angle U + m\angle U$$
$$54 = 2m\angle U$$
$$m\angle U = 27$$

Since $\triangle VTU$ is isosceles, $m\angle VTU = m\angle U$.

Try It! **6.** Find each angle measure in the figure.

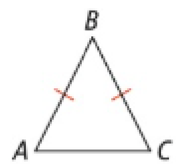


a. $m\angle PNO$ **b.** $m\angle NOP$

CONCEPT SUMMARY Isosceles and Equilateral Triangles

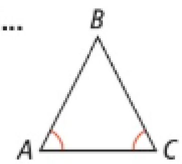
ISOSCELES TRIANGLES

If...



$\overline{AB} \cong \overline{BC}$

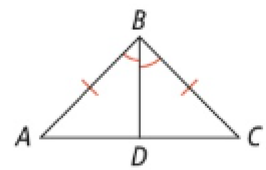
Then...



$\angle ACB \cong \angle BAC$

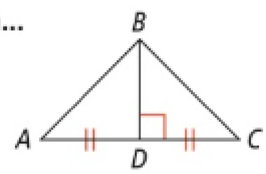
PERPENDICULAR BISECTOR

If...



$\overline{AB} \cong \overline{BC}$ and $m\angle ABD = m\angle CBD$

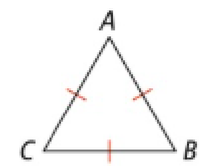
Then...



$\overline{BD} \perp \overline{AC}$ and $AD = DC$

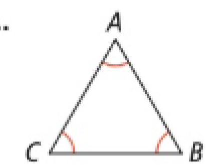
EQUILATERAL TRIANGLES

If...



$\overline{AB} \cong \overline{BC} \cong \overline{AC}$

Then...

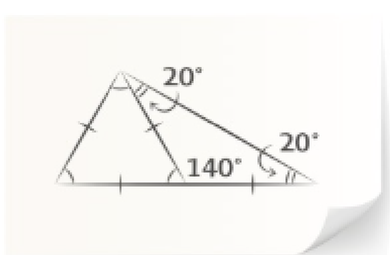


$\angle A \cong \angle B \cong \angle C$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the side lengths and angle measures related in isosceles triangles and in equilateral triangles?

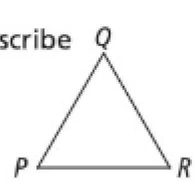
2. **Error Analysis** Nate drew the following diagram to represent an equilateral triangle and an isosceles triangle. What mistake did Nate make?



3. **Vocabulary** How can you distinguish the base of an isosceles triangle from a leg?

4. **Communicate and Justify** Is it possible for the vertex of an isosceles triangle to be a right angle? Explain why or why not, and state the angle measures of the triangle, if possible.

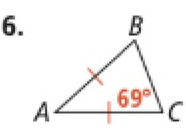
5. **Choose Efficient Methods** Describe five rigid motions that map equilateral triangle $\triangle PQR$ onto itself.



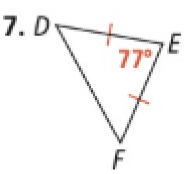
Do You KNOW HOW?

For Exercises 6 and 7, find the unknown angle measures.

6.

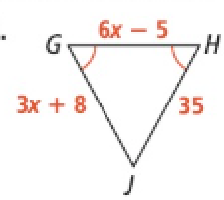


7.

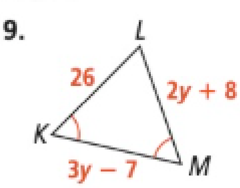


For Exercises 8 and 9, find the lengths of all three sides of the triangle.

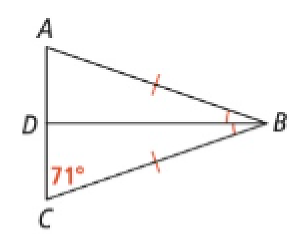
8.



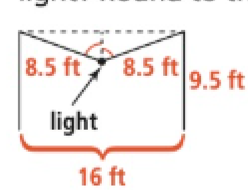
9.



10. What is $m\angle ABD$ in the figure shown?



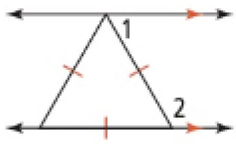
11. A light is suspended between two poles as shown. How far above the ground is the light? Round to the nearest tenth of a foot.



PRACTICE & PROBLEM SOLVING

UNDERSTAND

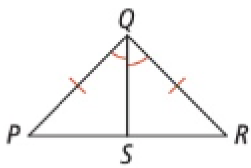
12. Mathematical Connections What are the measures of ∠1 and ∠2? Explain.



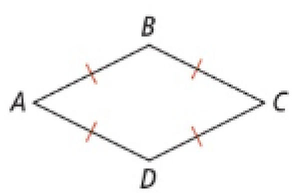
13. Communicate and Justify Use the properties of rigid motions to write a proof of Theorem 4-2.

Given: $\overline{PQ} \cong \overline{QR}$ and $m\angle PQS = m\angle RQS$

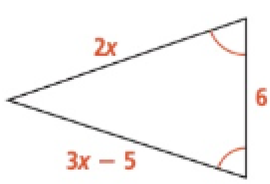
Prove: $\overline{QS} \perp \overline{PR}$ and $PS = SR$



14. Use Patterns and Structure Prove that $\angle BAD \cong \angle BCD$ and $\angle ABC \cong \angle CDA$.



15. Error Analysis Amaya is asked to find the side lengths of the triangle shown. What is her error?



From the top leg and the base, $2x = 6$, so $x = 3$. Substitute x into the expression for the bottom leg's length to get $3(3) - 5 = 4$. ✗

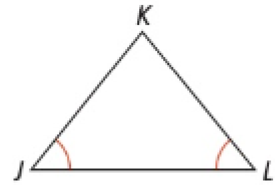
16. Higher Order Thinking Deondra draws points at (1, 5) and (1, −1) on a coordinate plane. Each point will be a vertex of an isosceles right triangle. What are two possible points in the second quadrant that she can specify as a vertex of her triangle? Explain.

PRACTICE

17. Use rigid motions to write a proof of the Converse of the Isosceles Triangle Theorem.
SEE EXAMPLE 1

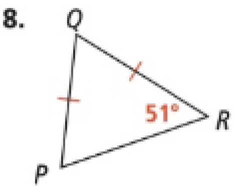
Given: $\angle J \cong \angle L$

Prove: $JK \cong KL$

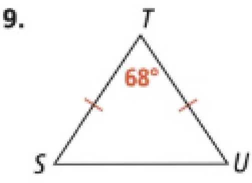


Find the unknown angle measures in each triangle.
SEE EXAMPLE 2

18.

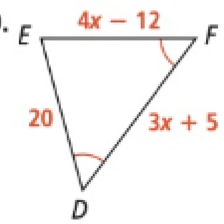


19.

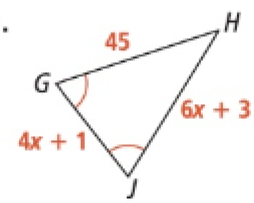


Find the lengths of all three sides of each triangle.
SEE EXAMPLE 3

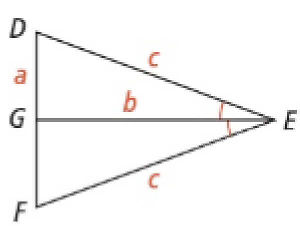
20.



21.



Use the figure shown for Exercises 22 and 23.
SEE EXAMPLE 4



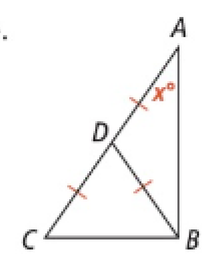
22. What is $m\angle DEG$ if $m\angle DFE = 70$?

23. What is the value of b if $a = 8$ and $c = 24$?

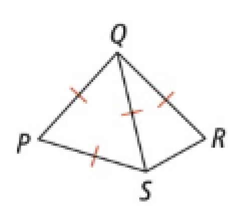
24. Prove that ∠ABC is a right angle.
SEE EXAMPLE 5

Given: $\overline{AD} \cong \overline{BD} \cong \overline{CD}$

Prove: $m\angle ABC = 90$



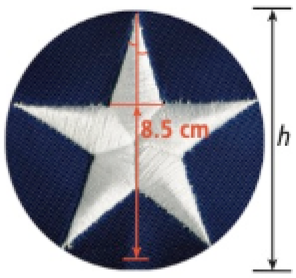
25. Given $m\angle PSR = 134$, what is the measure of ∠SQR?
SEE EXAMPLE 6



PRACTICE & PROBLEM SOLVING

APPLY

26. Analyze and Persevere Each of the five points on a star produced for a flag is an isosceles triangle with leg length 6 cm and base length 4.2 cm. What is the total height h of each star? Round to the nearest tenth of a centimeter.



27. Communicate and Justify The front of the tent below has the shape of an equilateral triangle.

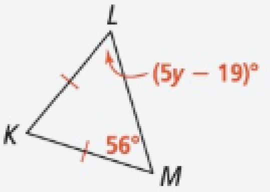
a. What is the side length of the triangle? Round to the nearest tenth of a foot.

b. Explain the method you use to calculate the length.

28. Apply Math Models For a crane to lift the beam shown below, the beam and the two support cables must form an isosceles triangle with height h. If the distance between the cables along the beam is 18 ft and the height h is 8 ft, what is the total length of the two cables? Round to the nearest tenth of a foot.

ASSESSMENT PRACTICE

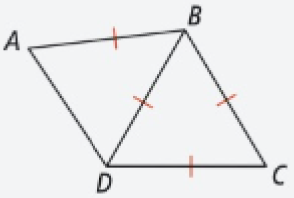
29. Consider the following triangle. GR.1.3



a. Write an equation you can solve to find the value of y.

b. What is $m\angle K$?

30. SAT/ACT Given $m\angle ABC = 114$, what is $m\angle BAD$?



Ⓐ 54

Ⓑ 63

Ⓒ 60

Ⓓ 72

31. Performance Task Emaan designs the birdhouse shown below.

Part A What is the total height of the birdhouse? Show your work.

Part B If Emaan decides to change the design by increasing each side of the roof from 12.5 cm to 15.2 cm, what will be the new height of the birdhouse? All other labeled dimensions on the birdhouse will remain unchanged.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.2.6–Apply rigid transformations to map one figure onto another to justify that the two figures are congruent. Also GR.1.2
MA.K12.MTR.7.1

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Check It Out!

Maybe you've played this game before: you draw a picture. Then you try to get a classmate to draw the same picture by giving step-by-step directions but without showing your drawings.

Try it with a classmate. Draw a map of a room in your house or a place in your town. Then give directions to a classmate to draw the map that you drew. How similar are they? Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know in order to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

4-3 Proving and Applying the SAS and SSS Congruence Criteria

I CAN… use SAS and SSS to determine whether triangles are congruent.

MA.912.GR.1.2–Prove triangle congruence or similarity using Side-Side-Side, Side-Angle-Side, Angle-Side-Angle, Angle-Angle-Side, Angle-Angle and Hypotenuse-Leg. Also GR.2.7

MA.K12.MTR.6.1, MTR.1.1, MTR.4.1

CONCEPTUAL UNDERSTANDING

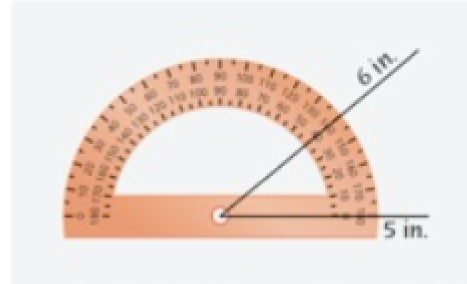
EXPLORE & REASON

Powered By desmos

Make five triangles that have a 5-inch side, a 6-inch side, and one 40° angle.

A. How many unique triangles can you make?

B. **Commiunicate and Justify** How are the unique triangles different from each other?

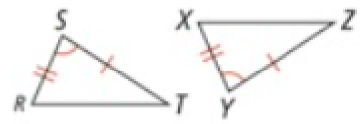


ESSENTIAL QUESTION

How are SAS and SSS used to show that two triangles are congruent?

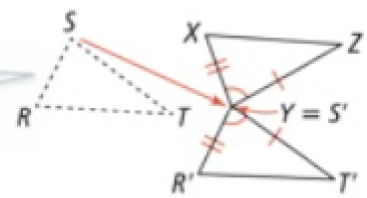
EXAMPLE 1 Explore the Side-Angle-Side (SAS) Congruence Criterion

Given two triangles with two pairs of sides congruent and the included angles congruent, verify that the triangles are congruent.

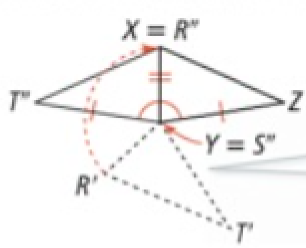


To prove that the triangles are congruent, show that a rigid motion maps $\triangle RST$ to $\triangle XYZ$.

Translate $\triangle RST$ so point S maps to point Y.



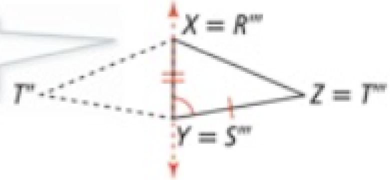
Since the translation maps S to Y, $Y = S'$.



Rotate $\triangle R'S'T'$ about point Y, so $\overrightarrow{S''R''}$ coincides with $\overrightarrow{YX}$.

Since rotation preserves length, $R''S'' = RS = XY$, so $X = R''$.

Then reflect $\triangle R''S''T''$ across $\overleftrightarrow{XY}$.



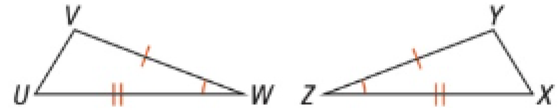
Since reflection preserves angle measure and length, $m\angle R'''S'''T''' = m\angle RST = m\angle XYZ$, so $\overrightarrow{S'''T'''}$ coincides with $\overrightarrow{YZ}$. Additionally, $S'''T''' = ST = YZ$, so Z coincides with T'''.

Because the vertices of $\triangle R'''S'''T'''$ coincide with the vertices of $\triangle XYZ$, there exists a rigid motion that maps $\triangle RST$ to $\triangle XYZ$. By the definition of congruence, $\triangle RST \cong \triangle XYZ$.

CONTINUED ON THE NEXT PAGE

GENERALIZE
Consider the relationship between corresponding parts of any pair of congruent triangles. How do the corresponding sides and angles compare to each other?

Try It! 1. What rigid motion or composition of rigid motions shows that $\triangle UVW$ maps to $\triangle XYZ$?

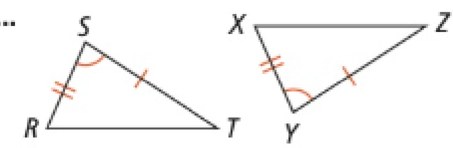


THEOREM 4-3 Side-Angle-Side (SAS) Congruence Criterion

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.

PROOF: SEE EXAMPLE 1.

If...



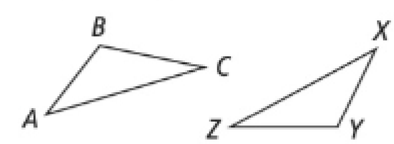
Then... $\triangle RST \cong \triangle XYZ$

THEOREM 4-4 Corresponding Parts of Congruent Triangles are Congruent (CPCTC)

If two triangles are congruent, then each pair of corresponding sides is congruent and each pair of corresponding angles is congruent.

PROOF: SEE EXERCISE 13.

If... $\triangle ABC \cong \triangle XYZ$

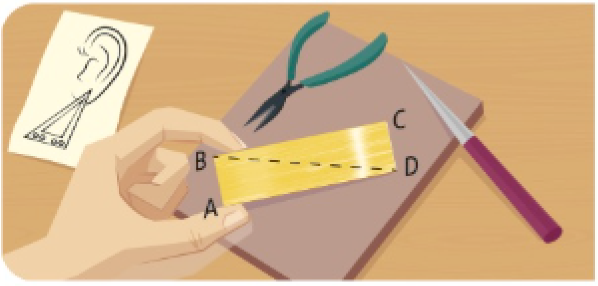


Then... $\overline{AB} \cong \overline{XY}$, $\overline{BC} \cong \overline{YZ}$, $\overline{AC} \cong \overline{XZ}$, $\angle A \cong \angle X$, $\angle B \cong \angle Y$, and $\angle C \cong \angle Z$.

APPLICATION

EXAMPLE 2 Apply the SAS Congruence Criterion

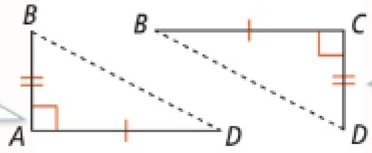
Allie cuts two triangles from a rectangular piece of metal along the dashed line to make earrings. How can Allie show that the earrings are the same size and shape?



Draw diagrams to represent the earrings.

LEARN TOGETHER
How can you share your ideas and communicate your thinking with others?

Because the sheet is a rectangle, $m\angle A = m\angle C = 90$, so $\angle A \cong \angle C$.



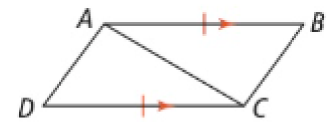
Opposite sides of a rectangle have the same length, so $\overline{AD} \cong \overline{BC}$ and $\overline{AB} \cong \overline{CD}$.

By SAS, $\triangle ABD \cong \triangle CDB$.

By CPCTC, all the corresponding sides and angles of the earrings are congruent, so the earrings are the same size and shape.

CONTINUED ON THE NEXT PAGE

Try It! 2. Given that $\overline{AB} \parallel \overline{CD}$ and $\overline{AB} \cong \overline{CD}$, how can you show that $\angle B \cong \angle D$?

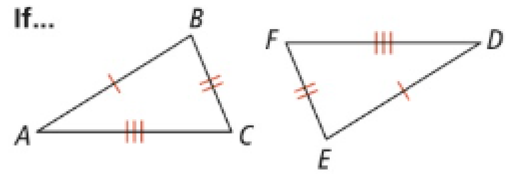


THEOREM 4-5 Side-Side-Side (SSS) Congruence Criterion

If three sides of one triangle are congruent to three sides of another triangle, then the two triangles are congruent.

PROOF: SEE EXAMPLE 3.

If...



Then... $\triangle ABC \cong \triangle DEF$

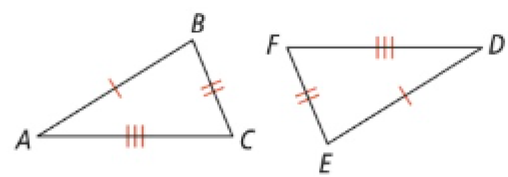
PROOF

EXAMPLE 3 Prove the Side-Side-Side (SSS) Congruence Criterion

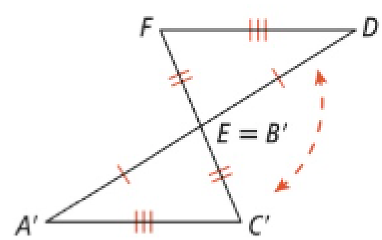
Prove the SSS Congruence Criterion.

Given: $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$

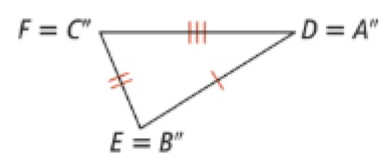
Prove: $\triangle ABC \cong \triangle DEF$



Proof: First, translate $\triangle ABC$ so point B maps to point E. Since the translation maps B to E, $E = B'$.

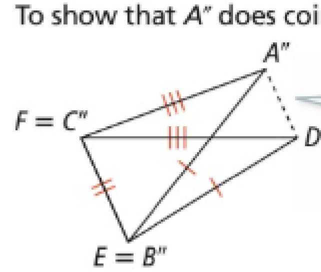


Then rotate $\triangle A'B'C'$ about point E so the image $\overline{B''C''}$ coincides with $\overline{EF}$. It appears that A'' coincides with D'', but this needs to be proven.



STUDY TIP
Use indirect reasoning to show that A'' must coincide with D.

To show that A'' does coincide with D, assume that A'' does not coincide.



Draw $\overline{A''D}$. Then $\overline{A''D}$ is the base of two isosceles triangles, $\triangle FA''D$ and $\triangle EDA''$.

By the Isosceles Triangle Theorem, $\angle FA''D \cong \angle FDA''$ and $\angle EA''D \cong \angle EDA''$. From the diagram, observe that $m\angle FA''D > m\angle EA''D$ and $m\angle FDA'' < m\angle EDA''$.

$m\angle FA''D > m\angle EA''D$

$m\angle FDA'' > m\angle EA''D$

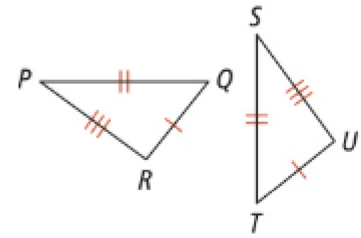
$m\angle FDA'' > m\angle EDA''$

Substitute $m\angle FDA''$ for $m\angle FA''D$, and substitute $m\angle EDA''$ for $m\angle EA''D$.

This contradicts the observation that $\angle FDA'' < \angle EDA''$. Therefore, A'' must coincide with D. Since $D = A''$, $E = B''$, and $F = C''$, there exists a rigid motion that maps $\triangle ABC$ to $\triangle DEF$, so $\triangle ABC \cong \triangle DEF$.

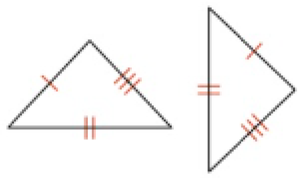
CONTINUED ON THE NEXT PAGE

Try It! 3. Show that there is a rigid motion that maps △*PQR* to △*STU*.
Hint: Be sure to consider a reflection when mapping △*PQR* to △*STU*.

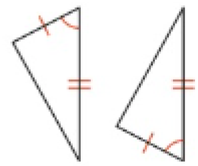


EXAMPLE 4 Determine Congruent Triangles

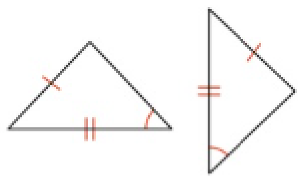
A. Which of the following pairs are congruent by SAS or SSS?



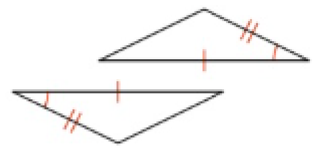
Congruent by SSS



Congruent by SAS



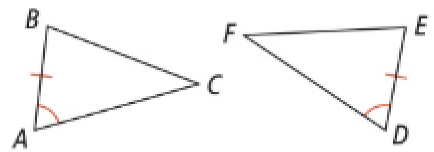
Cannot be determined



Congruent by SAS

COMMON ERROR
It may not be enough to identify two pairs of congruent sides and one pair of congruent angles. Recall that to apply SAS, the congruent angle pair must be the included angles.

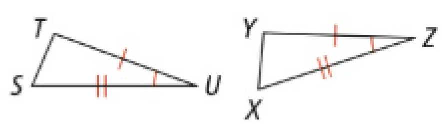
B. What additional information is needed to show △*ABC* ≅ △*DEF* by SAS? By SSS?



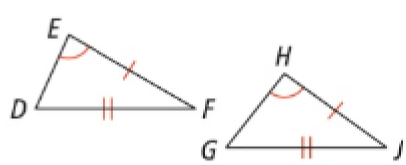
To show △*ABC* ≅ △*DEF* by SAS, you need $\overline{AC} \cong \overline{DF}$.

To show △*ABC* ≅ △*DEF* by SSS, you need $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$.

Try It! 4. a. Is △*STU* congruent to △*XYZ*? Explain.



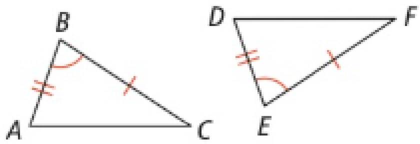
b. Is any additional information needed to show △*DEF* ≅ △*GHJ* by SAS? Explain.



CONCEPT SUMMARY Triangle Congruence Criteria

THEOREM 4-3 Side-Angle-Side (SAS)

If...

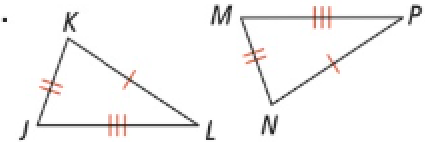


$\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle B \cong \angle E$

Then... $\triangle ABC \cong \triangle DEF$

THEOREM 4-5 Side-Side-Side (SSS)

If...



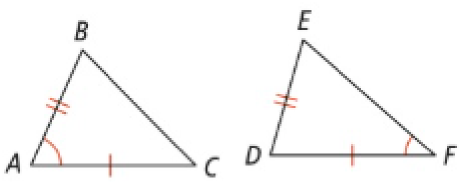
$\overline{JK} \cong \overline{MN}$, $\overline{JL} \cong \overline{MP}$, and $\overline{KL} \cong \overline{NP}$

Then... $\triangle JKL \cong \triangle MNP$

Do You UNDERSTAND?

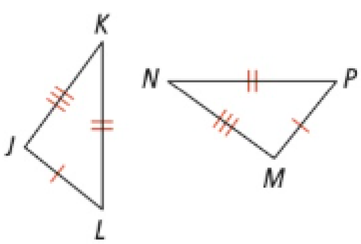
1. **ESSENTIAL QUESTION** How are SAS and SSS used to show that two triangles are congruent?

2. **Error Analysis** Elijah says $\triangle ABC$ and $\triangle DEF$ are congruent by SAS. Explain Elijah's error.

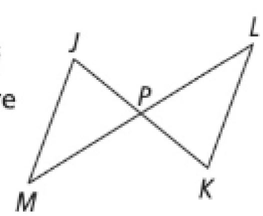


3. **Check for Reasonableness** Suppose $\overline{PR} \cong \overline{ST}$ and $\angle P \cong \angle S$. Ron wants to prove $\triangle PQR \cong \triangle STU$ by SAS. He says that all he needs to do is show $\overline{RQ} \cong \overline{SU}$. Will this work? Explain.

4. **Communicate and Justify** How would you decide what theorem to use to prove $\angle JKL \cong \angle MNP$? Explain.



5. **Analyze and Persevere** Suppose that $\overline{JK}$ and $\overline{LM}$ bisect each other. Is there enough information to show that $\triangle JPM \cong \triangle KPL$? Explain.

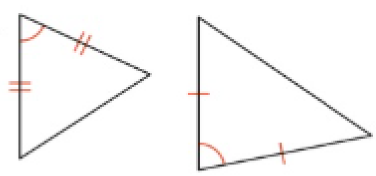


Do You KNOW HOW?

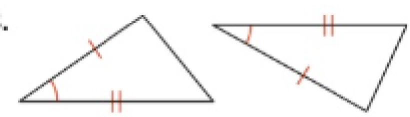
For Exercises 6–8, which pairs of triangles are congruent by SAS? By SSS?

6.

7.

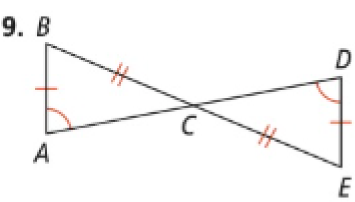


8.

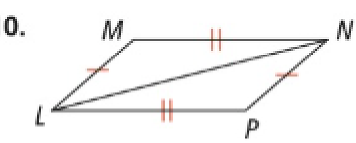


For Exercises 9–11, are the triangles congruent? Explain.

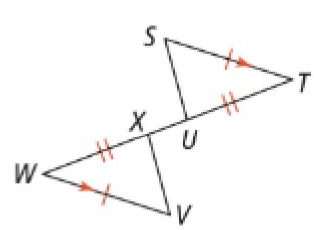
9.



10.



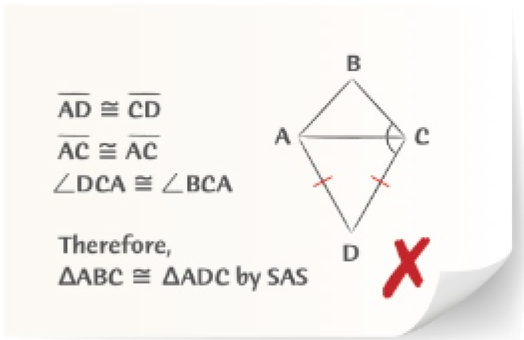
11.



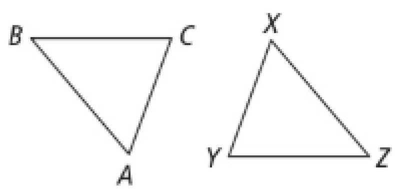
PRACTICE & PROBLEM SOLVING

UNDERSTAND

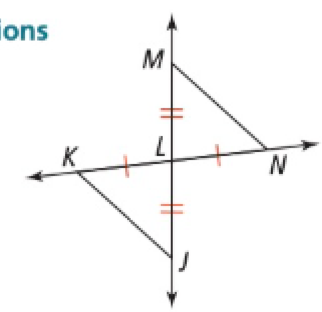
12. Error Analysis Zhang says $\triangle ABC$ is congruent to $\triangle ADC$. Explain the error in Zhang's work.



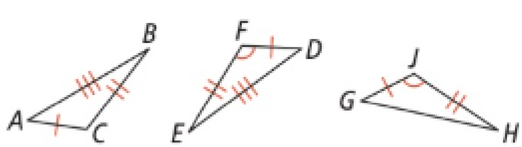
13. Use Patterns and Structure Given $\triangle ABC \cong \triangle XYZ$, use a rigid motion to prove Theorem 4-4, Corresponding Parts of Congruent Triangles are Congruent.



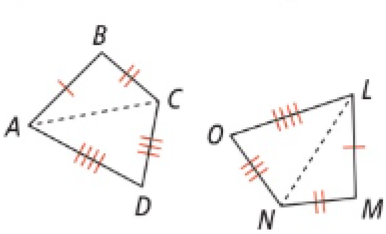
14. Mathematical Connections Is $\triangle JKL$ congruent to $\triangle MNL$? Explain.



15. Analyze and Persevere Why is $\triangle ABC \cong \triangle GHJ$?

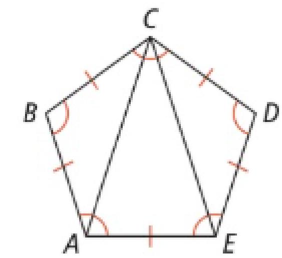


16. Higher Order Thinking Given quadrilaterals $ABCD$ and $LMNO$, and $\overline{AC} \cong \overline{LN}$, how can you show that the corresponding angles of the quadrilaterals are congruent?

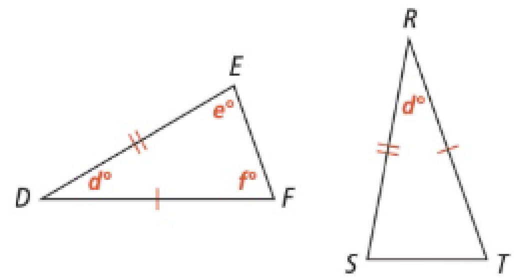


PRACTICE

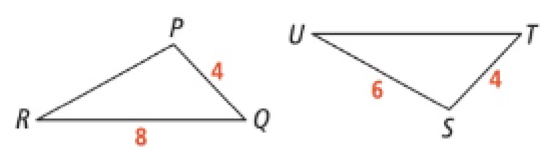
17. Prove $\triangle ACE$ is an isosceles triangle.
SEE EXAMPLES 1 AND 2



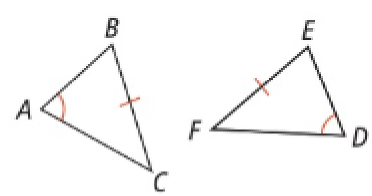
18. What is $m\angle RTS$? Justify your answer.
SEE EXAMPLES 1 AND 2



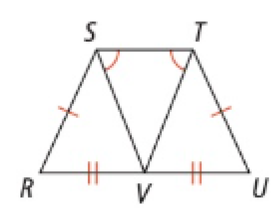
19. What additional information is needed to show that $\triangle PQR \cong \triangle STU$ by SSS? SEE EXAMPLE 3



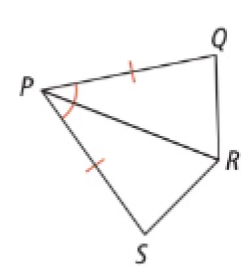
20. What additional information is needed to show that $\triangle ABC \cong \triangle DEF$? SEE EXAMPLE 3



21. Is $\triangle RSV \cong \triangle UTV$? Explain.
SEE EXAMPLE 4



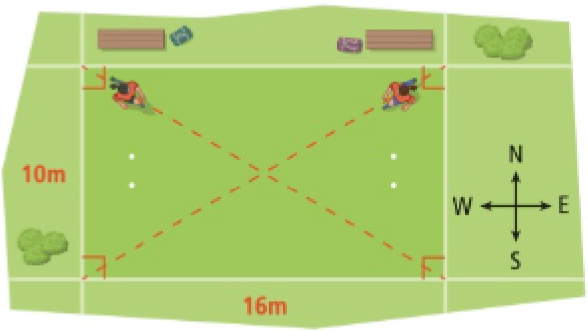
22. Is $\triangle PQR \cong \triangle PSR$? Explain.
SEE EXAMPLE 4



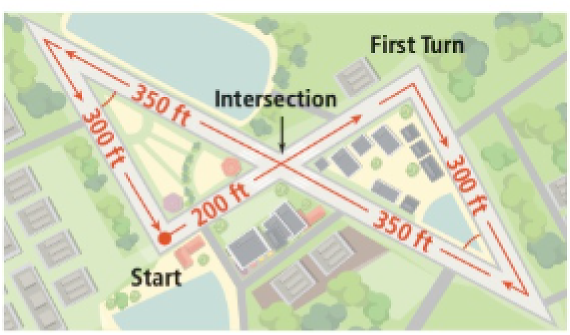
PRACTICE & PROBLEM SOLVING

APPLY

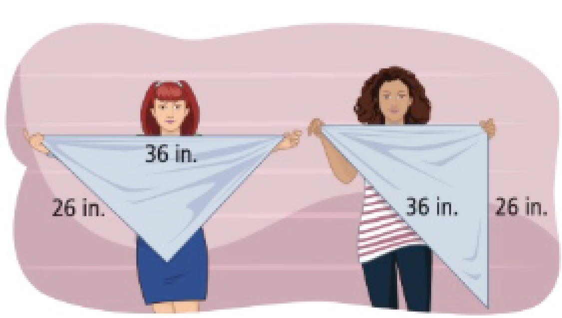
23. Check for Reasonableness Kathryn runs from the northwest corner to the southeast corner of a rugby field and Mia runs from the northeast corner to the southwest corner. Mia says she ran farther. Is she correct? Explain.



24. Choose Efficient Methods Following the route shown, what is the total distance traveled by the architectural tour if it ends where it started? What properties and theorems did you use to find the distance?

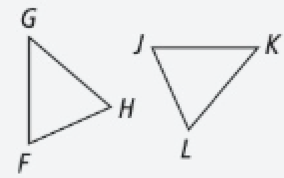


25. Analyze and Persevere Justice and Leah both made a triangular scarf. Do the scarves have the same size and shape? What do you notice about the information that is given?



ASSESSMENT PRACTICE

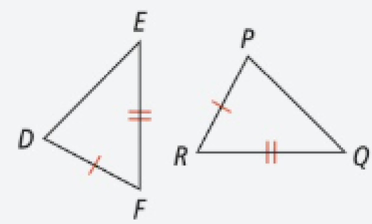
26. Select all the sets of congruent parts that are sufficient to conclude that $\triangle FGH \cong \triangle JKL$?



GR.1.2

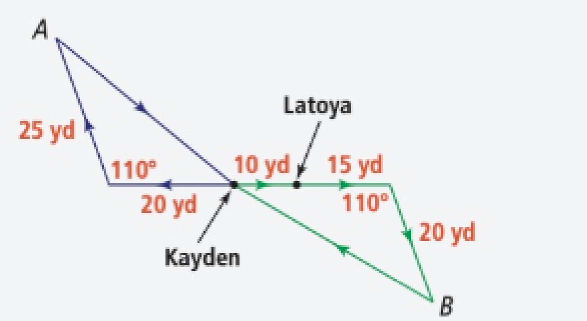
- ☐ **A.** $\overline{FG} \cong \overline{JK}$, $\overline{GH} \cong \overline{KL}$, $\overline{FH} \cong \overline{JL}$
- ☐ **B.** $\overline{FG} \cong \overline{JK}$, $\overline{FH} \cong \overline{JL}$, $\angle FHG \cong \angle JLK$
- ☐ **C.** $\overline{GH} \cong \overline{KL}$, $\overline{FG} \cong \overline{JK}$, $\angle FGH \cong \angle JKL$
- ☐ **D.** $\overline{GH} \cong \overline{KL}$, $\overline{FH} \cong \overline{JL}$, $\angle FHG \cong \angle JLK$
- ☐ **E.** $\overline{GH} \cong \overline{KL}$, $\overline{FH} \cong \overline{JL}$, $\angle FGH \cong \angle JKL$

27. SAT/ACT Consider $\triangle DEF$ and $\triangle PQR$. Which additional piece of information would allow you to conclude that $\triangle DEF \cong \triangle PQR$?



Ⓐ $\angle D \cong \angle P$
Ⓑ $\angle E \cong \angle Q$
Ⓒ $\angle D \cong \angle Q$
Ⓓ $\angle F \cong \angle R$

28. Performance Task In a marching band show, Kayden and Latoya start 10 yards apart. Kayden marches the path in blue and Latoya marches the path in green.



Part A Are the triangles formed by the paths congruent? Explain.

Part B Are the angle measures that Kayden and Latoya turn at points A and B the same? Explain.

4-4 Proving and Applying the ASA and AAS Congruence Criteria

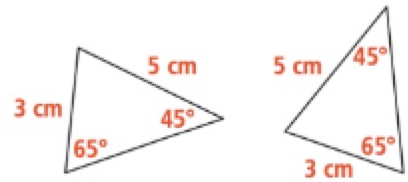
I CAN… determine congruent triangles by comparing two angles and one side.

MA.912.GR.1.2–Prove triangle congruence or similarity using Side-Side-Side, Side-Angle-Side, Angle-Side-Angle, Angle-Angle-Side, Angle-Angle and Hypotenuse-Leg. Also GR.1.6, GR.2.7

MA.K12.MTR.5.1, MTR.3.1, MTR.1.1

EXPLORE & REASON

Are these triangles congruent?



A. Analyze and Persevere Assume the triangles are *not* congruent. What contradictions can you find to contradict your assumption? Explain.

B. Is it sufficient to say that the triangles are congruent because of the contradictions you found? Explain.

ESSENTIAL QUESTION

How are ASA and AAS used to show that triangles are congruent?

CONCEPTUAL UNDERSTANDING

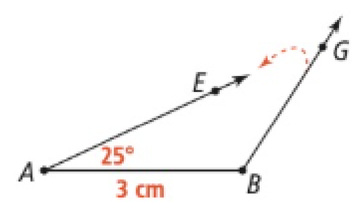
EXAMPLE 1 Explore the ASA Congruence Criterion

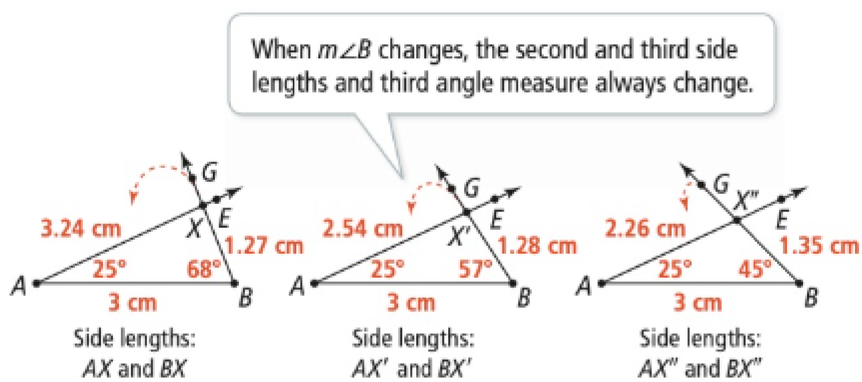
Powered By desmos

How many possible triangles can you determine when given two angles and the included side of a triangle?

Consider $\overline{AB}$, $\overrightarrow{AE}$, and $\overrightarrow{BG}$, where $\overline{AB}$ and $\overrightarrow{AE}$ form a 25° angle.

Let $\overrightarrow{BG}$ rotate counterclockwise about point *B* to form a 68° angle, a 57° angle, and a 45° angle. The rays will intersect to form some triangles.

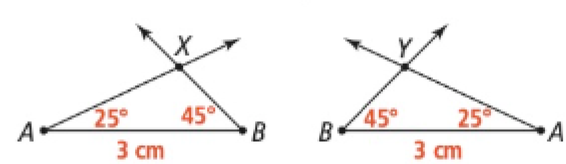




USE PATTERNS AND STURCTURE
Think about how change in one part of a figure affects the rest of the figure. Does the result follow a pattern?

Notice that once the 25° angle, and the side of length 3 cm are set, there is exactly one way to complete the triangle with a 68° angle, a 57° angle, or a 25° angle. So for each unique combination of *AB*, $m\angle A$, and $m\angle B$, there is a unique triangle.

Try It! **1.** What is the relationship between $\triangle AXB$ and $\triangle AYB$?

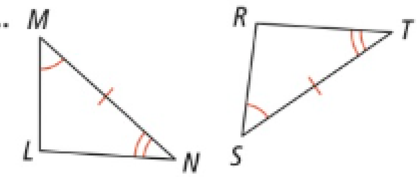


THEOREM 4-6 Angle-Side-Angle (ASA) Congruence Criterion

If two angles of one triangle and the included side are congruent to two angles and the included side of another triangle, then the two triangles are congruent.

PROOF: SEE EXAMPLE 2.

If...

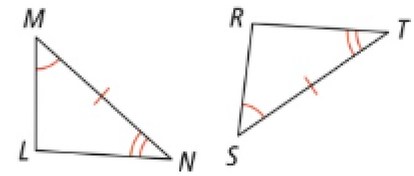


Then... $\triangle MLN \cong \triangle SRT$

PROOF

EXAMPLE 2 Prove the Angle-Side-Angle (ASA) Congruence Criterion

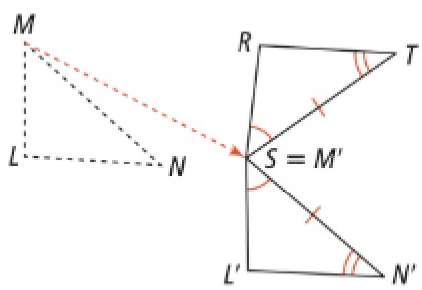
Given two triangles with two pairs of angles congruent and the included sides congruent, prove the triangles are congruent.



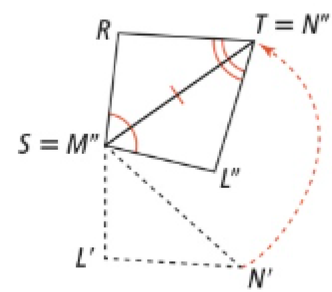
Given: $\angle LMN \cong \angle RST$, $\angle LNM \cong \angle RTS$, $\overline{MN} \cong \overline{ST}$

Prove: $\triangle MLN \cong \triangle SRT$

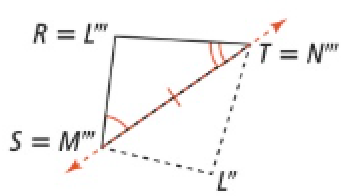
Translate $\triangle MLN$ to map M to S.



Rotate $\triangle M'L'N'$ about point S to map $\overline{M'N'}$ to $\overline{ST}$.



Reflect $\triangle M''L''N''$ across $\overleftrightarrow{ST}$.

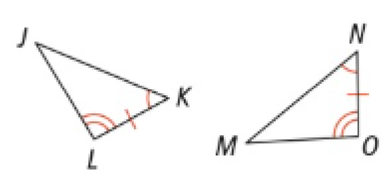


STUDY TIP
When all vertices coincide, there exists a rigid motion that maps one triangle to the other. By definition, the triangles are congruent.

Because angle measures are preserved with reflections, $\overrightarrow{M'''L'''}$ coincides with $\overrightarrow{SR}$ and $\overrightarrow{N'''L'''}$ coincides with $\overrightarrow{TR}$. The intersection of $\overrightarrow{SR}$ and $\overrightarrow{TR}$ is at both R and L'''. Since the intersection of two rays is unique, R and L''' coincide.

Therefore, $\triangle MLN \cong \triangle SRT$.

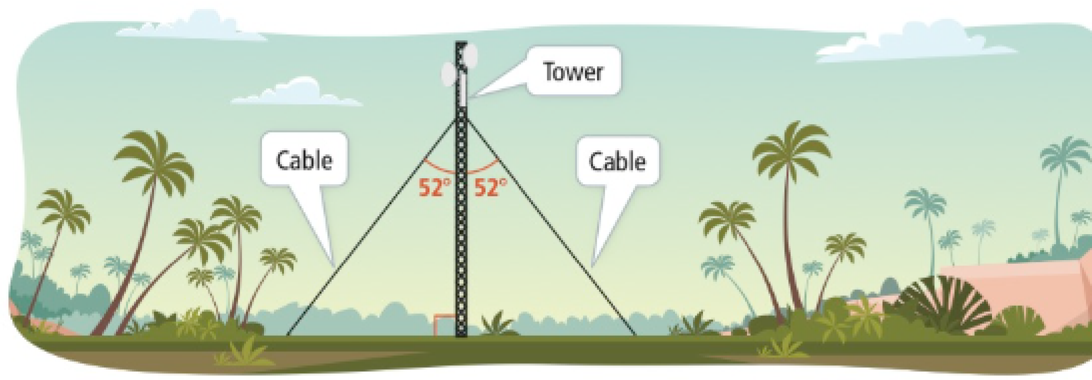
Try It! 2. Describe a series of transformations that shows $\triangle JKL \cong \triangle MNO$.



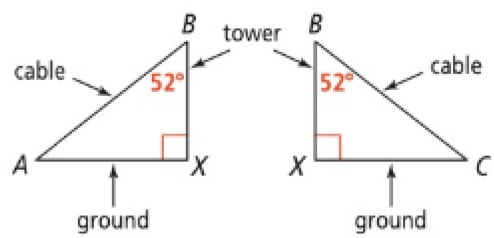
APPLICATION

EXAMPLE 3 Apply the Angle-Side-Angle (ASA) Congruence Criterion

A technician installs cables from a cell phone tower to the ground. To pass inspection, both cables must be the same length. Does this installation meet the cable-length requirement? Explain.



Formulate The ground, tower, and cables form two triangles, $\triangle ABX$ and $\triangle CBX$.



Compute All right angles are congruent, so $\angle AXB \cong \angle CXB$.
By the Reflexive Property of Congruence, $\overline{BX} \cong \overline{BX}$.
Also, $m\angle ABX = 52°$ and $m\angle CBX = 52°$, so $\angle ABX \cong \angle CBX$.

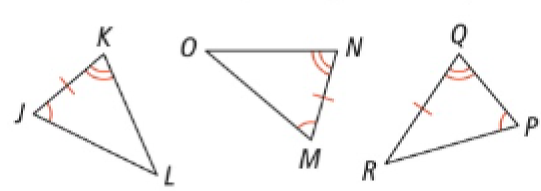
Two angles and the included side of $\triangle ABX$ are congruent to two angles and the included side of $\triangle CBX$. Therefore, $\triangle ABX \cong \triangle CBX$ by ASA.

By CPCTC, $\overline{AB} \cong \overline{CB}$. So, both cables are the same length.

Interpret The cable lengths do meet the inspection requirement.

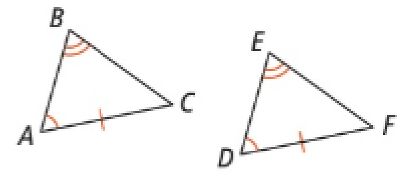
Try It! **3. a.** Are $\triangle JKL$ and $\triangle MNO$ congruent? Explain.

b. Are $\triangle JKL$ and $\triangle PQR$ congruent? Explain.



EXAMPLE 4 Investigate the Angle-Angle-Side (AAS) Congruence Criterion

Given $\triangle ABC$, is the triangle determined by $\angle A$, $\angle B$, and the non-included side $\overline{AC}$ unique?



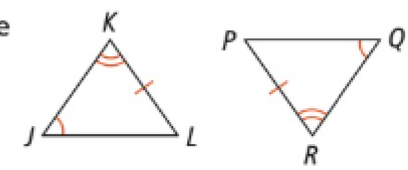
Assume that $\triangle ABC$ is not unique. Then there must exist $\triangle DEF$ such that $\triangle ABC \ncong \triangle DEF$, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\overline{AC} \cong \overline{DF}$.

By the Triangle Angle-Sum Theorem, $m\angle C = 180 - m\angle A - m\angle B$ and $m\angle F = 180 - m\angle D - m\angle E$. Since $\angle A \cong \angle D$ and $\angle B \cong \angle E$, $m\angle C = 180 - m\angle D - m\angle E = m\angle F$, so $\angle C \cong \angle F$.

Therefore, by ASA, $\triangle ABC \cong \triangle DEF$, and the assumption is false. A unique triangle is determined by $\angle A$ and $\angle B$ and the non-included side $\overline{AC}$.

HAVE A GROWTH MINDSET
What other strategies can you try when you get stuck?

Try It! 4. Using the figures shown, describe a sequence of rigid motions that maps $\triangle JKL$ to $\triangle QRP$.

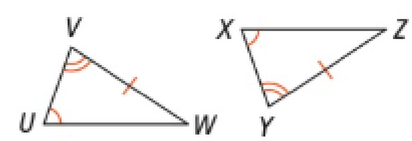


THEOREM 4-7 Angle-Angle-Side (AAS) Congruence Criterion

If two angles and a nonincluded side of one triangle are congruent to two angles and a nonincluded side of another triangle, then the two triangles are congruent.

PROOF: SEE EXERCISE 16.

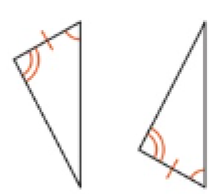
If...



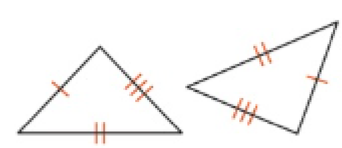
Then... $\triangle UVW \cong \triangle XYZ$

EXAMPLE 5 Use Triangle Congruence Criteria

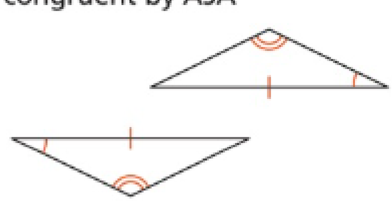
A. State whether each pair of triangles is congruent by SAS, SSS, ASA, or AAS, or if the congruence cannot be determined.



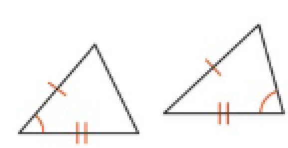
congruent by ASA



congruent by SSS



congruent by AAS



cannot be determined

COMMON ERROR
Be careful not to just assume that triangles are congruent when given two pairs of congruent angles and one pair of congruent sides. The congruent sides must be corresponding sides.

CONTINUED ON THE NEXT PAGE

EXAMPLE 5 CONTINUED

B. Prove that $\overline{FH} \cong \overline{JL}$.

Given: $\overline{GH} \cong \overline{KL}$, $\angle GFH \cong \angle KJL$, and $\angle FGH \cong \angle JKL$

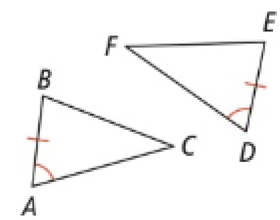
Prove: $\overline{FH} \cong \overline{JL}$

Statements	Reasons
1) $\angle F \cong \angle J$, $\angle G \cong \angle K$, $\overline{GH} \cong \overline{KL}$	1) Given
2) $\triangle FGH \cong \triangle JKL$	2) AAS
3) $\overline{FH} \cong \overline{JL}$	3) CPCTC

When triangle congruence applies, you can conclude the remaining sides and angles are congruent by CPCTC.

Try It! **5. a.** What additional information is needed to show $\triangle ABC \cong \triangle DEF$ by ASA?

b. What additional information is needed to show $\triangle ABC \cong \triangle DEF$ by AAS?



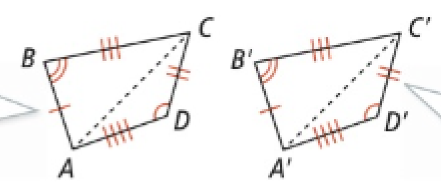
EXAMPLE 6 Determine Congruent Polygons

USE PATTERNS STRUCTURE
Consider the diagonals of a polygon. Can any polygon be divided into a figure composed of triangles?

All sides and angles of *ABCD* are congruent to the corresponding sides and angles of *A′B′C′D′*. Is *ABCD* congruent to *A′B′C′D′*?

Each polygon can be divided into two triangles by the diagonals shown.

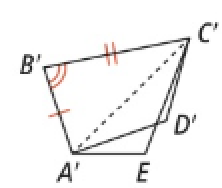
$\triangle ABC \cong \triangle A'B'C'$ by SAS.



$\triangle ADC \cong \triangle A'D'C'$ by SAS.

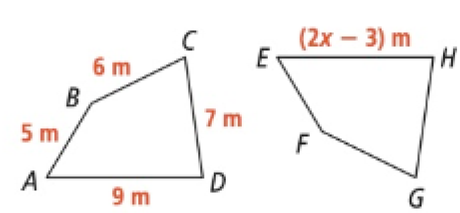
Since $\triangle ABC \cong \triangle A'B'C'$, there is a rigid motion that maps $\triangle ABC$ to $\triangle A'B'C'$. Consider this rigid motion applied to $\triangle ADC$. Side $\overline{AC}$ maps to $\overline{A'C'}$ since $\overline{AC}$ is shared by both $\triangle ABC$ and $\triangle ADC$.

Now suppose that this rigid motion maps D to some point other than D'. Call that point E. Since $\triangle A'D'C'$ and $\triangle A'EC'$ are congruent to $\triangle ADC$, $\triangle A'D'C' \cong \triangle A'EC'$. Since $\angle C'A'E \cong \angle C'A'D'$, E lies on $\overleftrightarrow{A'D'}$. By a similar argument, E lies on $\overleftrightarrow{C'D'}$. So E must be the point D'.



Since the rigid motion that maps $\triangle ABC$ to $\triangle A'B'C'$ also maps $ABCD$ to $A'B'C'D'$, $ABCD$ is congruent to $A'B'C'D'$.

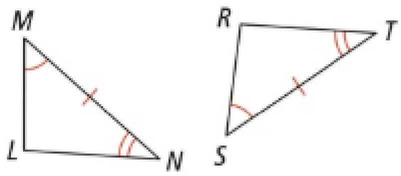
Try It! **6.** Given $ABCD \cong EFGH$, what is the value of x?



CONCEPT SUMMARY Triangle Congruence

THEOREM 4-6 Angle-Side-Angle (ASA)

If...

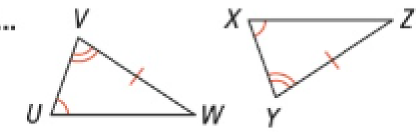


$\overline{MN} \cong \overline{ST}$, $\angle M \cong \angle S$, and $\angle N \cong \angle T$

Then... $\triangle MLN \cong \triangle SRT$

THEOREM 4-7 Angle-Angle-Side (AAS)

If...



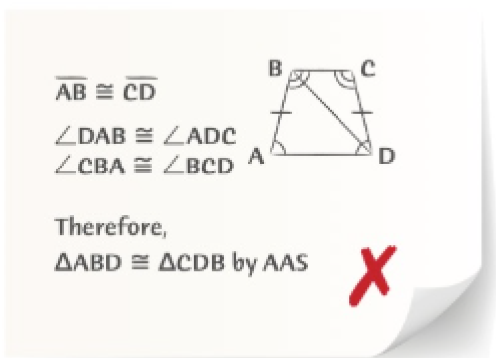
$\overline{VW} \cong \overline{YZ}$, $\angle U \cong \angle X$, and $\angle V \cong \angle Y$

Then... $\triangle UVW \cong \triangle XYZ$

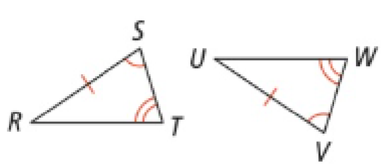
Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are ASA and AAS used to show that triangles are congruent?

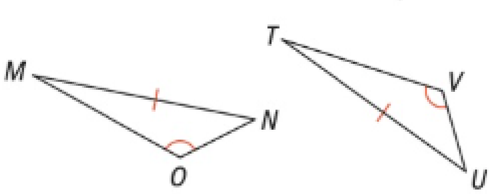
2. **Error Analysis** Why is Terrell's conclusion incorrect?



3. **Use Patterns and Structure** How can you tell which property of triangle congruence shows $\triangle RST \cong \triangle UVW$?



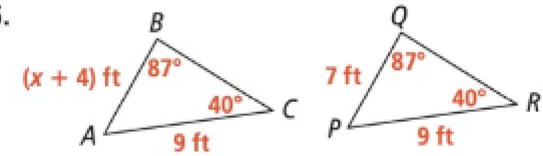
4. **Analyze and Persevere** Is there a congruence relationship that is sufficient to show that $\triangle MNO \cong \triangle TUV$? Explain.



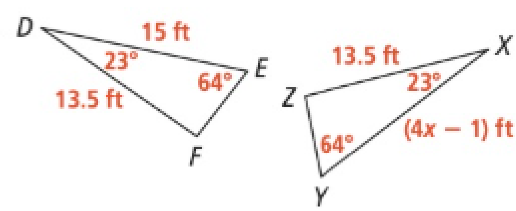
Do You KNOW HOW?

For Exercises 5 and 6, find the value of *x*.

5.

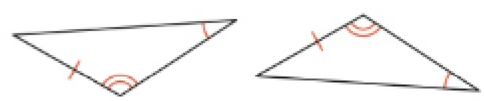


6.

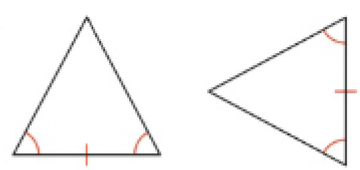


For Exercises 7 and 8, state whether the triangles are congruent and by which theorem.

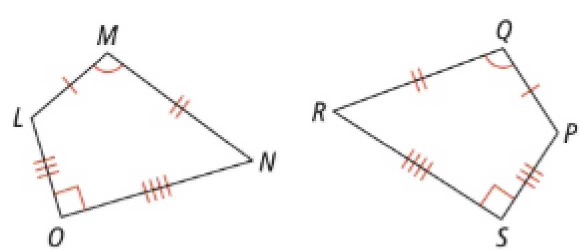
7.



8.



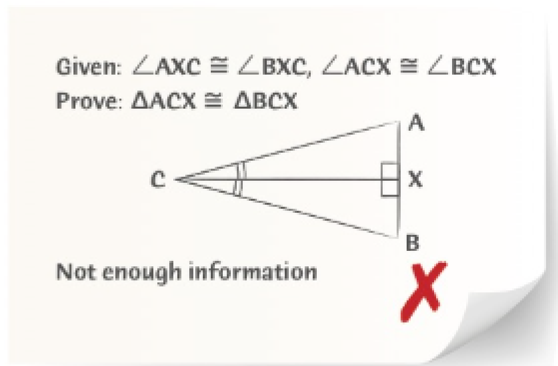
9. Why is $LMNO \cong PQRS$?



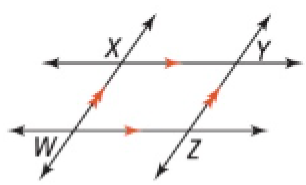
PRACTICE & PROBLEM SOLVING

UNDERSTAND

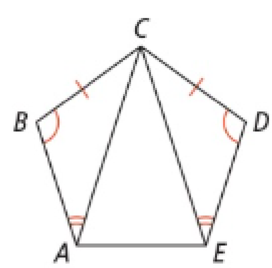
10. Error Analysis Stacy says there is not enough information to prove $\triangle ACX \cong \triangle BCX$. Explain why Stacy's statement is incorrect.



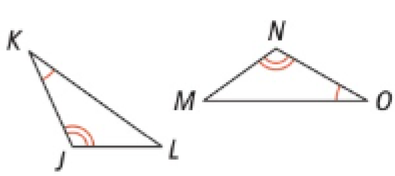
11. Mathematical Connections Given $\overleftrightarrow{WZ} \parallel \overleftrightarrow{XY}$ and $\overleftrightarrow{WX} \parallel \overleftrightarrow{ZY}$, write a two-column proof to show $\overline{WX} \cong \overline{YZ}$.



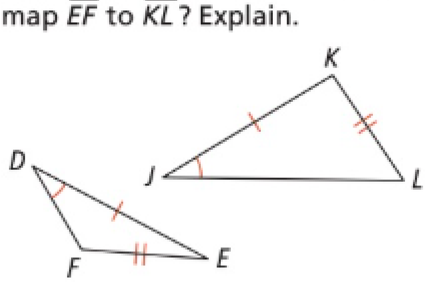
12. Use Patterns and Structure Given the figure shown, write a two-column proof to prove $\angle CAE \cong \angle CEA$.



13. Choose Efficient Methods How might you decide what additional piece of information you need to prove $\triangle JKL \cong \triangle NOM$?

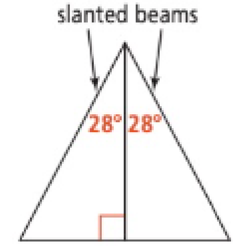


14. Higher Order Thinking Describe a composition of rigid motions that maps $\overline{DE}$ to $\overline{JK}$ and $\angle D$ to $\angle J$. Does this same composition map $\overline{EF}$ to $\overline{KL}$? Explain.



PRACTICE

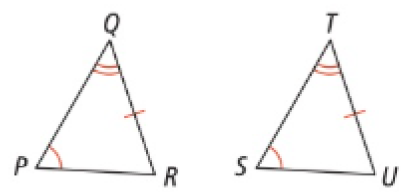
15. Carpenters build a set of triangular roof supports, each with the measurements shown. How can the carpenters be sure all the slanted beams are the same length? SEE EXAMPLES 1–3



16. Prove the Angle-Angle-Side Congruence Criterion. SEE EXAMPLE 4

Given: $\angle P \cong \angle S$, $\angle Q \cong \angle T$, $\overline{QR} \cong \overline{TU}$

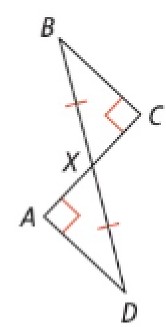
Prove: $\triangle PQR \cong \triangle STU$



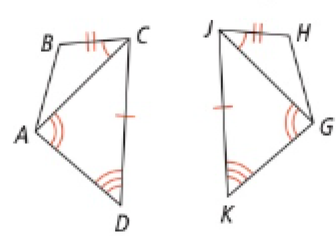
17. Write a proof. SEE EXAMPLE 5

Given: $\angle A \cong \angle C$, $\overline{BX} \cong \overline{DX}$

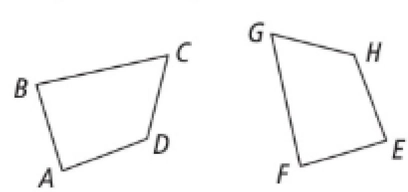
Prove: $\overline{AX} \cong \overline{CX}$



18. Is $ABCD \cong GHJK$? Explain. SEE EXAMPLE 6



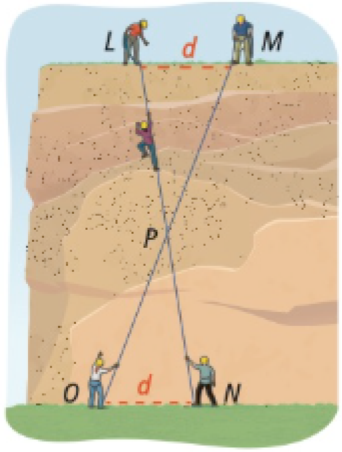
19. If $ABCD \cong EFGH$, are all corresponding parts congruent? Explain. SEE EXAMPLE 6



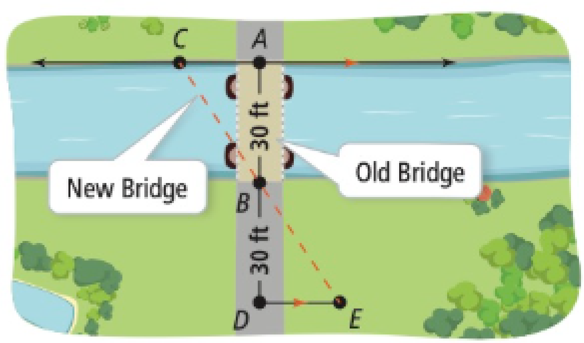
PRACTICE & PROBLEM SOLVING

APPLY

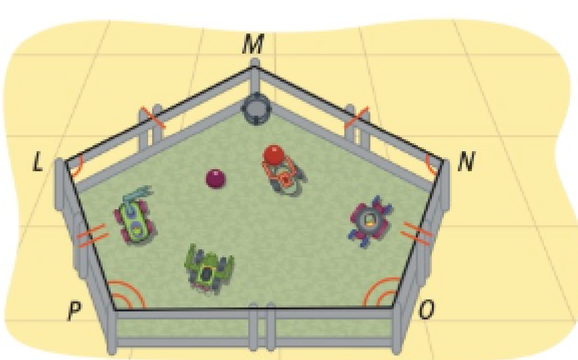
20. **Use Patterns and Structure** Climbers want to determine the halfway point up a vertical cliff. If the top and bottom are parallel, why is point *P*, where the ropes intersect, halfway up the cliff?



21. **Choose Efficient Methods** Keisha, Dwayne, and Lonzell are planning for a new bridge to replace the old bridge. The new bridge will start at point *B*, where Dwayne is standing, and end at point *C*, where Keisha is standing. Lonzell walks to point *D* and then walks parallel to the river until he reaches point *E*, where he sees Dwayne and Keisha are aligned. Why is the distance from *E* to *B* the length of the new bridge?

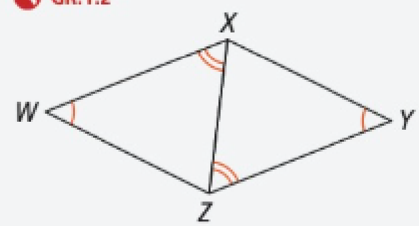


22. **Communicate and Justify** The Robotics Club wants to divide their robot battle arena into two congruent arenas for a tournament. Paxton says that if they build a wall perpendicular to and bisecting $\overline{PO}$ from *M*, then the arenas will be congruent. Is Paxton correct? Explain.

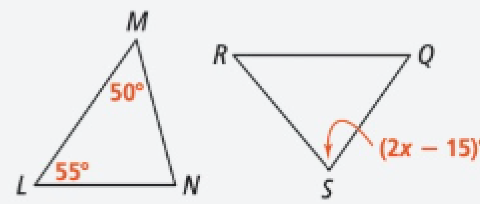


ASSESSMENT PRACTICE

23. Given the figure shown, list all the pairs of congruent sides and pairs of congruent angles.
GR.1.2

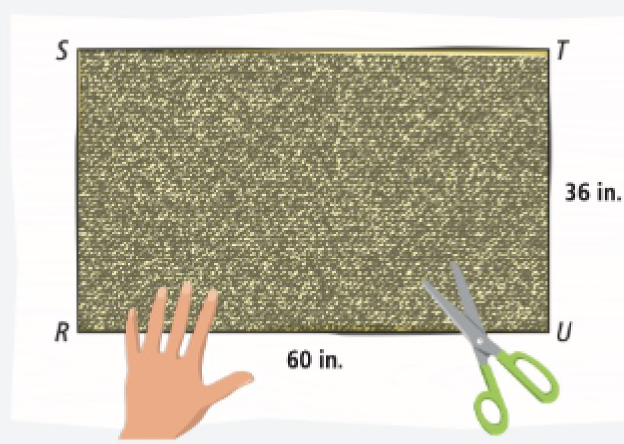


24. **SAT/ACT** Given $\triangle LMN \cong \triangle QRS$, what is the value of *x*?



Ⓐ 30 Ⓑ 35 Ⓒ 45 Ⓓ 60

25. **Performance Task** Gregory wants to make four congruent triangular flags using as much of the rectangular canvas shown as possible.



Part A Draw and label a diagram to show how Gregory should cut the fabric.

Part B Explain why the flags are congruent.

Part C Is there another way Gregory can cut the fabric to make 4 congruent triangular flags using the same amount of fabric? Explain.

4-5 Congruence in Right Triangles

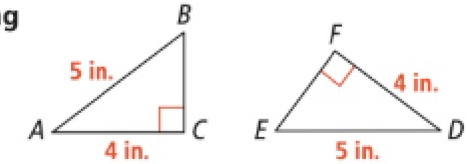
I CAN… identify congruent right triangles.

MA.912.GR.1.2–Prove triangle congruence or similarity using Side-Side-Side, Side-Angle-Side, Angle-Side-Angle, Angle-Angle-Side, Angle-Angle and Hypotenuse-Leg. Also GR.1.6

MA.K12.MTR.5.1, MTR.3.1, MTR.4.1

CRITIQUE & EXPLAIN

Seth and Jae wrote the following explanations of why the two triangles are congruent.



Seth

There are two pairs of congruent sides, $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$, and a pair of congruent right angles, $\angle C \cong \angle F$. So $\triangle ABC \cong \triangle DEF$ by SSA.

Jae

The lengths of $\overline{BC}$ and $\overline{EF}$ are 3 in., since these are 3-4-5 right triangles. There are three pairs of congruent sides, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\overline{BC} \cong \overline{EF}$. So $\triangle ABC \cong \triangle DEF$ by SSS.

A. Do you think either student is correct? Explain.

B. **Generalize** Describe when you can state that two right triangles are congruent if you are only given two pairs of congruent sides and a right angle in each triangle.

ESSENTIAL QUESTION What minimum criteria are needed to show that right triangles are congruent?

CONCEPTUAL UNDERSTANDING

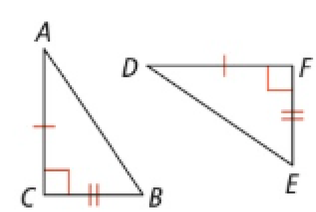
EXAMPLE 1 Investigate Right Triangle Congruence

When any two pairs of corresponding sides are congruent, can you show that two right triangles $\triangle ABC$ and $\triangle DEF$ are congruent? Explain.

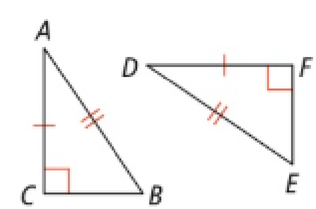
Given that right triangles have one pair of congruent corresponding angles with right angles, look to see what else is congruent.

STUDY TIP
To visualize congruent corresponding parts, draw copies of $\triangle ABC$ and $\triangle DEF$. Then mark the triangles to show the congruent relationships.

- If both pairs of corresponding legs are congruent, use SAS with $\overline{AC} \cong \overline{DF}$, $\angle C \cong \angle F$, and $\overline{BC} \cong \overline{EF}$.



- If one pair of corresponding legs is congruent along with the hypotenuses, apply the Pythagorean Theorem to show that the other pair of corresponding legs is also congruent.



$$BC = \sqrt{AB^2 - AC^2} = \sqrt{DE^2 - DF^2} = EF$$

The right triangles are congruent by SSS.

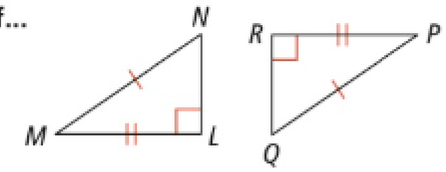
Try It! 1. Can you show that two right triangles are congruent when any one pair of corresponding acute angles is congruent and any one pair of corresponding legs is congruent? Explain.

THEOREM 4-8 Hypotenuse-Leg (HL) Theorem

If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent.

PROOF: SEE EXERCISE 9.

If...



Then... $\triangle MNL \cong \triangle PQR$

APPLICATION

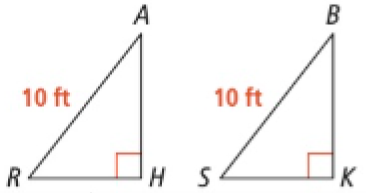
EXAMPLE 2 Use the Hypotenuse-Leg (HL) Theorem

Ashton is washing windows using a 10-foot ladder. For the first window, the ladder reaches the window when he places the base of the ladder at the rose bush. How can he determine where to place the ladder to be sure it reaches the last window?

The ground, the ladder, and the side of the house form a right triangle, $\triangle RAH$. When Ashton moves the ladder, there will be another right triangle, $\triangle SBK$.

COMMON ERROR
Remember that the triangles must be right triangles in order to use the HL Theorem. Be sure the situation, like this one, describes right triangles.

Ashton wants $AH = BK$. If the two triangles are congruent, then he knows $\overline{AH} \cong \overline{BK}$ by CPCTC.

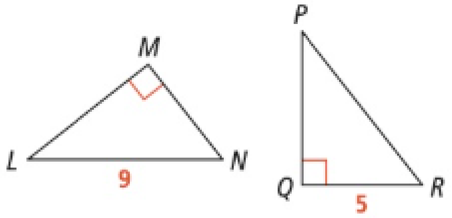


To make sure the triangles are congruent, Ashton can place the base of the ladder so that $RH = SK$.

If $RH = SK$, then $\triangle RAH \cong \triangle SBK$ by the HL Theorem.

Thus, by placing the base of the ladder the same distance away from the house as the rose bush, the ladder will reach the last window.

Try It! **2.** What information is needed in order to apply the Hypotenuse-Leg (HL) Theorem?



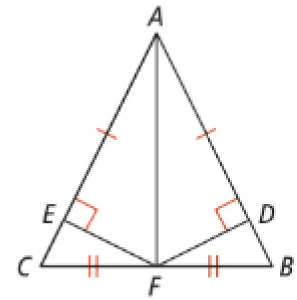
PROOF

EXAMPLE 3 Write a Proof Using the Hypotenuse-Leg (HL) Theorem

Write a proof to show that a triangle is isosceles.

Given: $\overline{FD} \perp \overline{AB}$, $\overline{FE} \perp \overline{AC}$, $\overline{AE} \cong \overline{AD}$, $\overline{FC} \cong \overline{FB}$

Prove: $\triangle ABC$ is isosceles.



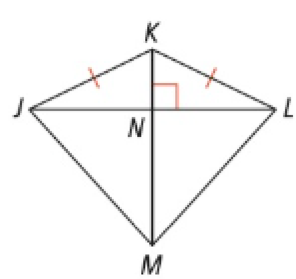
Statement	Reason
1) $\overline{FD} \perp \overline{AB}$, $\overline{FE} \perp \overline{AC}$	**1)** Given
2) $m\angle FDA = m\angle FDB = 90$, $m\angle FEA = m\angle FEC = 90$	**2)** Def. of $\perp$
3) $\triangle FEA$, $\triangle FDA$, $\triangle FEC$, and $\triangle FDB$ are rt. triangles	**3)** Def. of rt. triangle
4) $\overline{AE} \cong \overline{AD}$	**4)** Given
5) $\overline{AF} \cong \overline{AF}$	**5)** Refl. Prop. of Congruence
6) $\triangle FEA \cong \triangle FDA$	**6)** HL Theorem
7) $\overline{EF} \cong \overline{DF}$	**7)** CPCTC
8) $\overline{FC} \cong \overline{FB}$	**8)** Given
9) $\triangle FDB \cong \triangle FEC$	**9)** HL Theorem
10) $\angle ECF \cong \angle DBF$	**10)** CPCTC
11) $\overline{AC} \cong \overline{AB}$	**11)** Converse of Isosc. Triangle Thm.
12) $\triangle ABC$ is isosceles.	**12)** Def. of isosc. triangle

USE PATTERNS AND STRUCTURE
Consider what properties can be used to identify congruent parts in two triangles. What property shows that a common side is congruent to itself?

Try It! **3.** Write a proof to show that two triangles are congruent.

Given: $\overline{JL} \perp \overline{KM}$, $\overline{JK} \cong \overline{LK}$

Prove: $\triangle JKM \cong \triangle LKM$

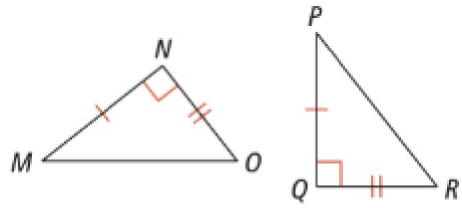


CONCEPT SUMMARY Congruence of Right Triangles

Triangle congruence theorems apply to right triangles.

THEOREM 4-3 Side-Angle-Side (SAS)

If...

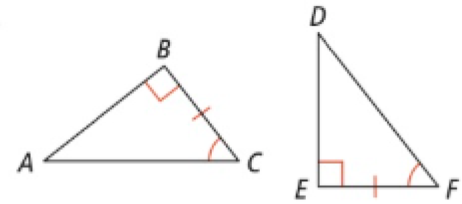


$\overline{MN} \cong \overline{PQ}$ and $\overline{NO} \cong \overline{QR}$

Then... $\triangle MNO \cong \triangle PQR$

THEOREM 4-6 Angle-Side-Angle (ASA)

If...

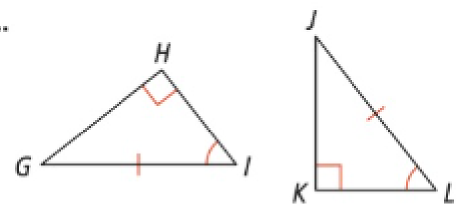


$\overline{BC} \cong \overline{EF}$ and $\angle C \cong \angle F$

Then... $\triangle ABC \cong \triangle DEF$

THEOREM 4-7 Angle-Angle-Side (AAS)

If...

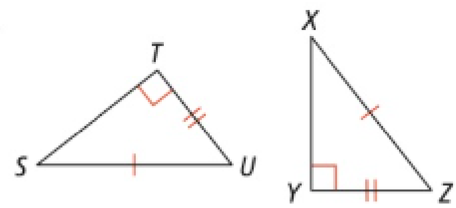


$\overline{GI} \cong \overline{JL}$ and $\angle I \cong \angle L$

Then... $\triangle GHI \cong \triangle JKL$

THEOREM 4-8 Hypotenuse-Leg (HL) Theorem

If...

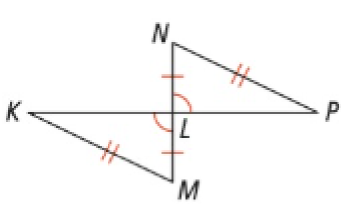


Then... $\triangle STU \cong \triangle XYZ$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What minimum criteria are needed to show that right triangles are congruent?

2. **Error Analysis** Yama stated that $\triangle KLM \cong \triangle PLN$ by the HL Theorem. What mistake did Yama make?



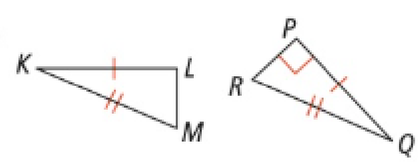
3. **Use Patterns and Structure** What are the three conditions that two triangles must meet in order to apply the HL Theorem?

4. **Analyze and Persevere** The HL Theorem is a side-side-angle theorem for right triangles. Why does it prove congruence for two right triangles but not prove congruence for two acute triangles or for two obtuse triangles?

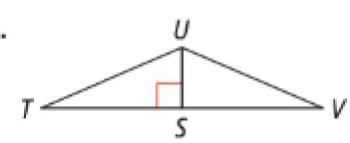
Do You KNOW HOW?

What information is needed to prove the triangles are congruent using the Hypotenuse-Leg (HL) Theorem?

5.

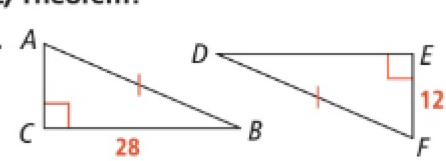


6.

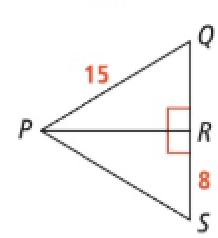


What information would be sufficient to show the two triangles are congruent by the Hypotenuse-Leg (HL) Theorem?

7.



8.



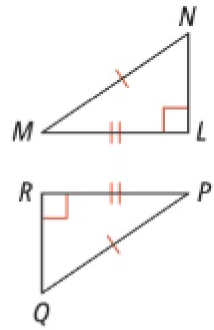
PRACTICE & PROBLEM SOLVING

UNDERSTAND

9. Use Patterns and Structure Follow the steps to prove the HL Theorem.

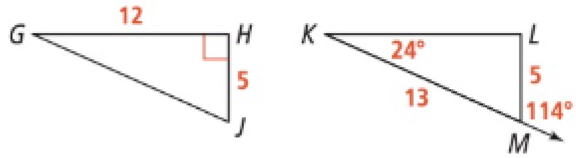
Given: Right triangles $\triangle MNL$ and $\triangle PQR$, $\overline{MN} \cong \overline{PQ}$, $\overline{ML} \cong \overline{PR}$

Prove: $\triangle MNL \cong \triangle PQR$



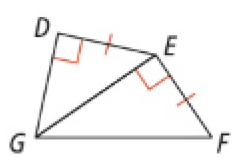
- Show that there is a rigid motion that maps L to R and M to P so that N' and Q are on opposite sides of $\overleftrightarrow{PR}$.
- Then show that $\triangle PQN'$ is isosceles.
- Show that $\triangle M'N'L' \cong \triangle PQR$, so $\triangle MNL \cong \triangle PQR$.

10. Mathematical Connections Consider the figures.



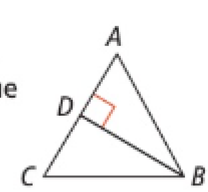
Describe the steps you would have to take before you could use the HL Theorem to prove $\triangle GHJ \cong \triangle KLM$.

11. Error Analysis Mohamed wrote the paragraph proof to show that $\triangle DEG \cong \triangle EFG$. What mistake did he make?



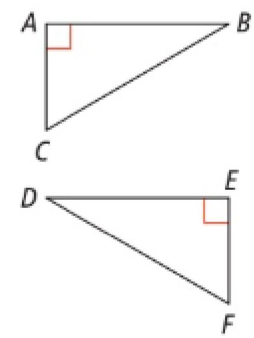
> ΔDEG and ΔEFG are right triangles. The figure shows $\overline{DE} \cong \overline{EF}$, AND $\overline{EG} \cong \overline{EG}$ by the Reflexive Property. Therefore, by the HL theorem, ΔDEG ≅ ΔEFG. ✗

12. Higher Order Thinking Suppose $\triangle ABC$ is an equilateral triangle. Use the HL Theorem to explain why any segment perpendicular to a side from the opposite vertex produces two congruent triangles. Would the same be true if $\triangle ABC$ were an isosceles triangle that was not equilateral? Explain.



PRACTICE

For Exercises 13–16, you are given a theorem and a congruence statement. What additional information is needed to prove that the triangles are congruent? SEE EXAMPLE 1



13. By using ASA, given $\overline{AC} \cong \overline{EF}$

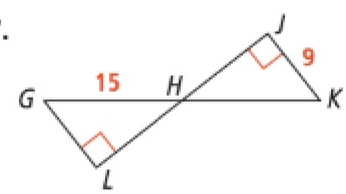
14. By using AAS, given $\angle B \cong \angle D$

15. By using SAS, given $\overline{AB} \cong \overline{DE}$

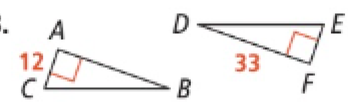
16. By using HL, given $\overline{CB} \cong \overline{DF}$

For Exercises 17–18, what information would be sufficient to show that the triangles are congruent by the HL Theorem? SEE EXAMPLE 2

17.



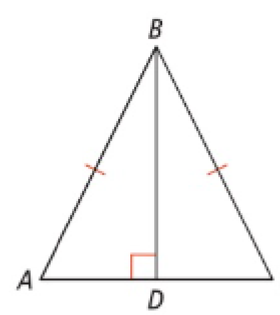
18.



For Exercises 19–20, write a proof using the HL Theorem to show that the triangles are congruent. SEE EXAMPLE 3

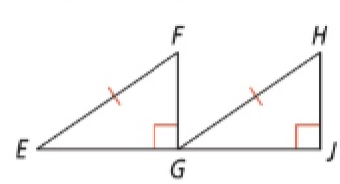
19. Given: $\overline{AB} \cong \overline{CB}$, $\overline{AC} \perp \overline{DB}$

Prove: $\triangle ABD \cong \triangle CBD$



20. Given: $\overline{EF} \cong \overline{GH}$, G is the midpoint of $\overline{EJ}$

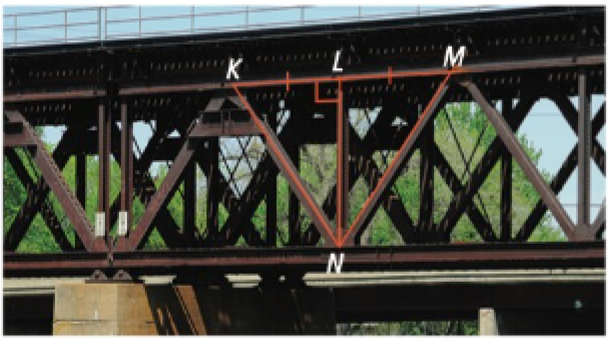
Prove: $\triangle EFG \cong \triangle GHJ$



PRACTICE & PROBLEM SOLVING

APPLY

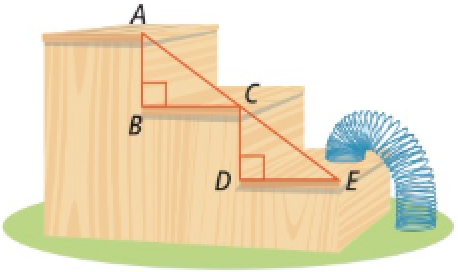
21. Analyze and Persevere Part of a truss bridge consists of crossbeam $\overline{KM}$ and a perpendicular beam $\overline{LN}$. What beams could an engineer measure in order to show $\triangle KLN \cong \triangle MLN$ using the HL Theorem?



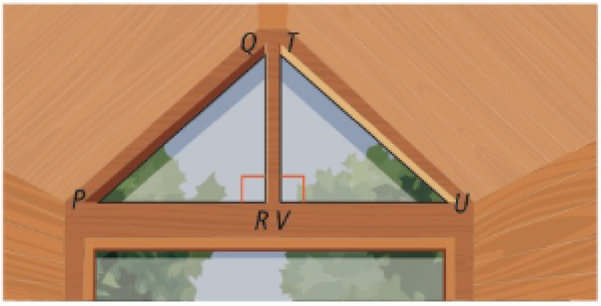
22. Choose Efficient Methods Raul wants to verify that the steps built by a carpenter are uniform by checking that $\triangle ABC \cong \triangle CDE$. The carpenter assures him they are because $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DE}$.

a. Explain why Raul cannot use the HL Theorem to prove $\triangle ABC \cong \triangle CDE$.

b. Is there another theorem that Raul can apply to prove $\triangle ABC \cong \triangle CDE$? If so, state the theorem. If not, explain why not.

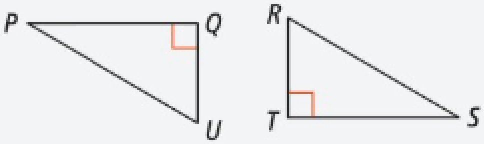


23. Communicate and Justify What are the fewest measurements that a homeowner could make to be certain that the front windows shown below are congruent?



ASSESSMENT PRACTICE

24. For each congruence statement, state the theorem that can be used to prove that the two triangles are congruent. GR.1.2



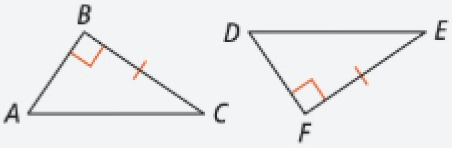
a. $\overline{PQ} \cong \overline{ST}$ and $\overline{QU} \cong \overline{TR}$

b. $\overline{PU} \cong \overline{SR}$ and $\overline{QU} \cong \overline{TR}$

c. $\overline{QU} \cong \overline{TR}$ and $\angle U \cong \angle R$

d. $\overline{QU} \cong \overline{TR}$ and $\angle P \cong \angle S$

25. SAT/ACT Which statement proves the triangles are congruent using the HL Theorem?



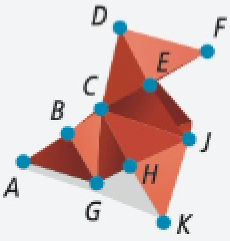
Ⓐ $\angle A \cong \angle D$

Ⓑ $\overline{AB} \cong \overline{DF}$

Ⓒ $\angle B \cong \angle F$

Ⓓ $\overline{AC} \cong \overline{DE}$

26. Performance Task Holly makes the origami figure shown. Assume that every angle that appears to be a right angle is a right angle.



Part A What can Holly measure so that she can use the HL Theorem to prove that $\triangle ABG \cong \triangle CBG$?

Part B Holly measures to find that $HK = DE$ and $HJ = EC$. Is it possible for her to apply the HL Theorem to prove that $\triangle JHK \cong \triangle CED$? Explain.

Part C Choose two other triangles on the figure. Describe what Holly could measure to prove the triangles are congruent by using the HL Theorem.

4-6 Congruence in Overlapping Triangles

I CAN… use triangle congruence to solve problems with overlapping triangles.

MA.912.GR.1.6–Solve mathematical and real-world problems involving congruence or similarity in two-dimensional figures. Also GR.1.2
MA.K12.MTR.1.1, MTR.2.1, MTR.6.1

EXPLORE & REASON

Powered By desmos

Look at the painting shown.

A. How many triangles can you find?

B. **Analyze and Persevere** What strategy did you use to count the triangles? How well did your strategy work?

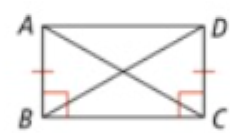
ESSENTIAL QUESTION

Which theorems can be used to prove that two overlapping triangles are congruent?

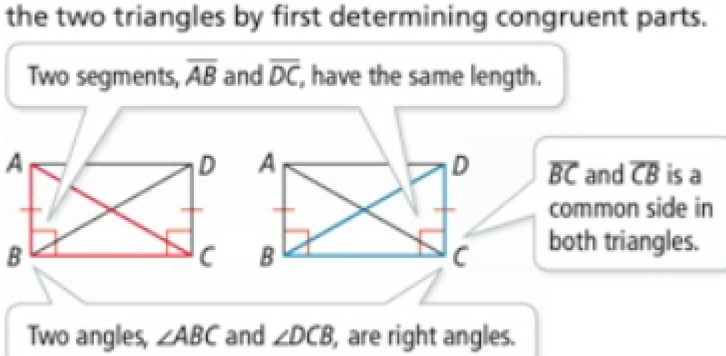
CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Identify Corresponding Parts in Triangles

Figure *ABCD* is a rectangle with diagonals $\overline{AC}$ and $\overline{BD}$. Why is it important to identify corresponding parts of overlapping triangles?



Consider $\triangle ABC$ and $\triangle DCB$. Identify the corresponding sides and angles in the two triangles by first determining congruent parts.



Use what you know about congruent parts to identify the corresponding vertices. Then use the corresponding vertices to identify the corresponding angles and sides.

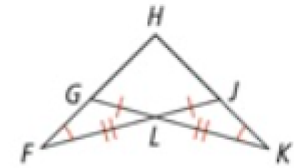
Corresponding angles:	Corresponding sides:
$\angle ACB$ and $\angle DBC$	$\overline{AC}$ and $\overline{DB}$
$\angle CBA$ and $\angle BCD$	$\overline{CB}$ and $\overline{BC}$
$\angle BAC$ and $\angle CDB$	$\overline{BA}$ and $\overline{CD}$

Once you identify the corresponding angles and sides, you can determine if the triangles are congruent.

COMMON ERROR
Be careful to name corresponding segments correctly. While $\overline{BC}$ is congruent to itself, $\overline{BC}$ corresponds to $\overline{CB}$ in the two triangles.

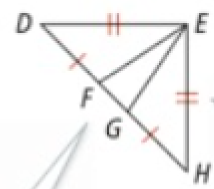
CONTINUED ON THE NEXT PAGE

Try It! 1. What are the corresponding sides and angles in $\triangle FHJ$ and $\triangle KHG$?



EXAMPLE 2 Use Common Parts of Triangles

Is $\angle EGD \cong \angle EFH$?

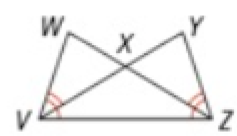


Since $\overline{DE} \cong \overline{HE}$, $\triangle DEH$ is isosceles, and therefore $\angle D \cong \angle H$.

Since $\overline{FG} \cong \overline{FG}$ and $\overline{DF} \cong \overline{HG}$, $\overline{DG} \cong \overline{HF}$ by the Segment Addition Postulate.

Because $\overline{DE} \cong \overline{HE}$, $\angle D \cong \angle H$, and $\overline{DG} \cong \overline{HF}$, $\triangle EDG \cong \triangle EHF$ by SAS. By CPCTC, $\angle EGD \cong \angle EFH$.

Try It! 2. Are $\overline{VW}$ and $\overline{ZY}$ congruent? Explain.

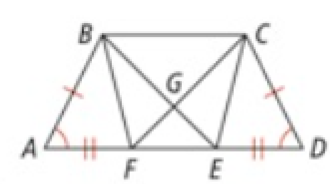


EXAMPLE 3 Prove That Two Triangles Are Congruent

Write a proof to show that $\triangle BFE$ is congruent to $\triangle CEF$.

Given: $\overline{AB} \cong \overline{DC}$, $\overline{AF} \cong \overline{DE}$, and $\angle A \cong \angle D$

Prove: $\triangle BFE \cong \triangle CEF$

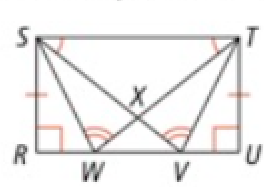


Proof: Given that $\overline{AF} \cong \overline{DE}$ and $\overline{FE} \cong \overline{FE}$, $\overline{AE} \cong \overline{DF}$ by the Segment Addition Postulate. Since $\overline{AB} \cong \overline{DC}$ and $\angle A \cong \angle D$, $\triangle ABE \cong \triangle DCF$ by SAS.

This means that by CPCTC, $\overline{BE} \cong \overline{CF}$ and $\angle BEA \cong \angle CFD$. Therefore, by SAS, $\triangle BFE \cong \triangle CEF$.

ANALYZE AND PERSEVERE
There are often multiple ways to complete a proof. How could you use SSS triangle congruence in this proof?

Try It! 3. Write a proof to show that $\triangle SRV \cong \triangle TUW$.



APPLICATION

EXAMPLE 4 Separate Overlapping Triangles

A city runs three triangular bus routes to various attractions. How can you draw a separate triangle for each route? Are any of the routes the same length?

Use the map and the list of locations for each route to help you draw the triangles. Add length and angle information to your diagrams.

STUDY TIP
When drawing each triangle, you might find it helpful to cover the parts of the diagram that you are not using.

By HL, the triangles representing the green route and the red route are congruent. Therefore, the green route and the red route are the same length.

Try It! 4. A new route will stop at the History Museum, Water Park, Zoo, Science Museum, and Theater. Draw a triangle to represent the new route. Include any length or angle information that is given in the diagram.

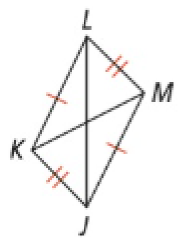
CONCEPT SUMMARY Congruence in Overlapping Triangles

All congruence criteria can be applied to overlapping triangles.

THEOREM 4-4

Side-Side-Side (SSS)

If...

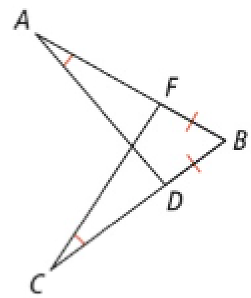


Then... $\triangle KLM \cong \triangle MJK$ and $\triangle LMJ \cong \triangle JKL$

THEOREM 4-6

Angle-Angle-Side (AAS)

If...

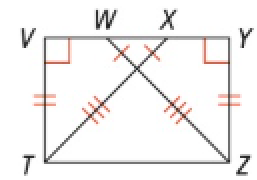


Then... $\triangle ABD \cong \triangle CBF$

THEOREM 4-7

Hypotenuse-Leg (HL) Theorem

If...

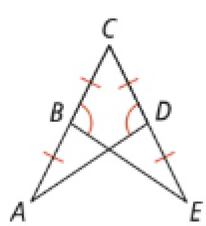


Then... $\triangle VXT \cong \triangle YWZ$

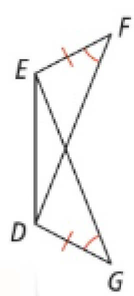
Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** Which theorems can be used to prove two overlapping triangles are congruent?

2. **Choose Efficient Methods** How could you prove that $\triangle ACD \cong \triangle ECB$?

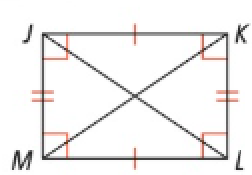


3. **Error Analysis** Nicholas wrote a proof to show that $\triangle EFD \cong \triangle DGE$. Explain Nicholas's error. Is it possible to prove the triangles congruent? Explain.



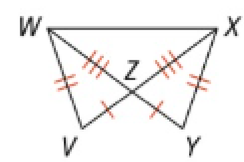
Since $\overline{EF} \cong \overline{DG}$, $\angle F \cong \angle G$, and $\overline{ED} \cong \overline{ED}$, by SAS, $\Delta EFD \cong \Delta DGE$.

4. **Analyze and Persevere** Quadrilateral *JKLM* is a rectangle. Which triangles are congruent to $\triangle JKL$? Explain.

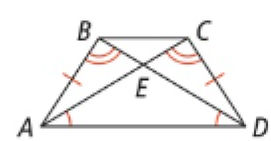


Do You KNOW HOW?

5. What are the corresponding sides and angles in $\triangle WXV$ and $\triangle XWY$?



In Exercises 6–9, name a side or angle congruent to each given side or angle.

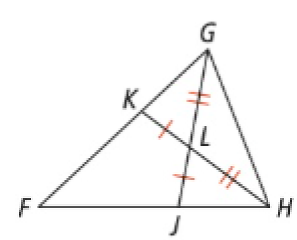


6. $\angle CDA$

7. $\overline{DB}$

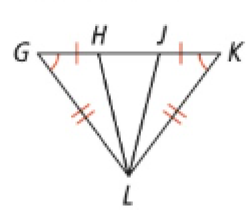
8. $\angle FGH$

9. $\overline{HJ}$

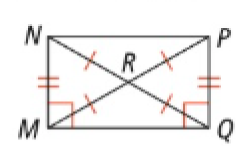


For Exercises 10 and 11, name a theorem that can be used to prove that each pair of triangles is congruent.

10. $\triangle GJL$ and $\triangle KHL$



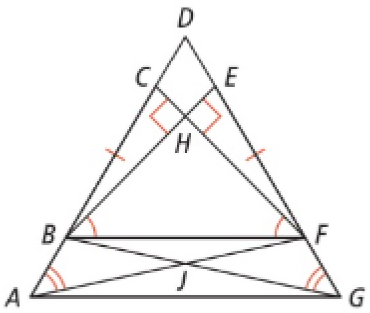
11. $\triangle NQM$ and $\triangle PMQ$



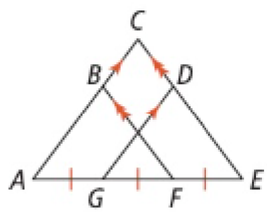
PRACTICE & PROBLEM SOLVING

UNDERSTAND

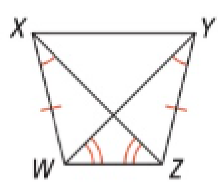
12. Communicate and Justify Write a proof to show that $\overline{AF} \cong \overline{GB}$.



13. Mathematical Connections Explain why $\triangle ABF \cong \triangle GDE$.

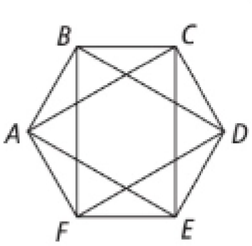


14. Error Analysis Dyani wrote a proof to show that $\angle XWY \cong \angle YZX$. What is her error?



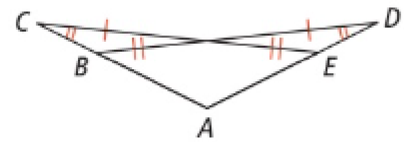
Since $\angle WXZ \cong \angle ZYW$, $\angle XZW \cong \angle YWX$, and $\overline{XW} \cong \overline{YZ}$, by AAS, $\Delta XWZ \cong \Delta YWZ$. Therefore, by CPCTC, $\angle XWY \cong \angle YZX$.

15. Higher Order Thinking Hexagon *ABCDEF* is a regular hexagon with all sides and angles congruent. List all sets of congruent triangles with vertices that are also vertices of the hexagon, and list all sets of congruent quadrilaterals with vertices that are also vertices of the hexagon.



PRACTICE

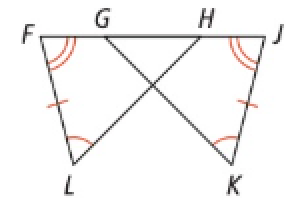
16. What are the corresponding parts of $\triangle CAE$ and $\triangle DAB$? SEE EXAMPLE 1



For Exercises 17–20, identify which side or angle is congruent to each given part. SEE EXAMPLE 2

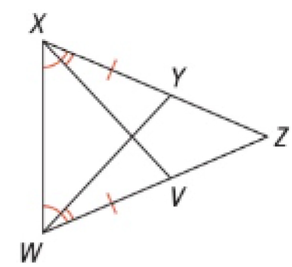
17. $\angle JGK$

18. $\overline{HL}$

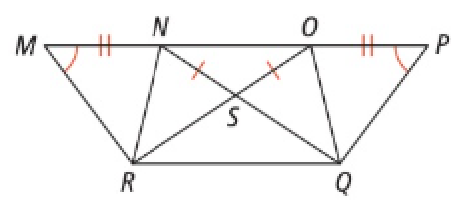


19. $\angle WYZ$

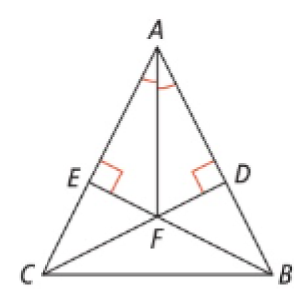
20. $\overline{XV}$



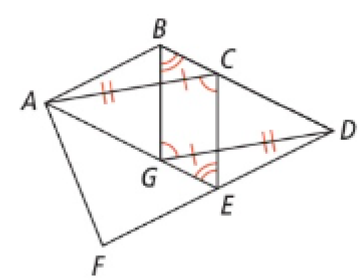
21. Write a proof to show triangles $\triangle MRO$ and $\triangle PQN$ are congruent. SEE EXAMPLE 3



22. Write a proof to show that $\triangle BCE \cong \triangle CBD$. SEE EXAMPLE 3



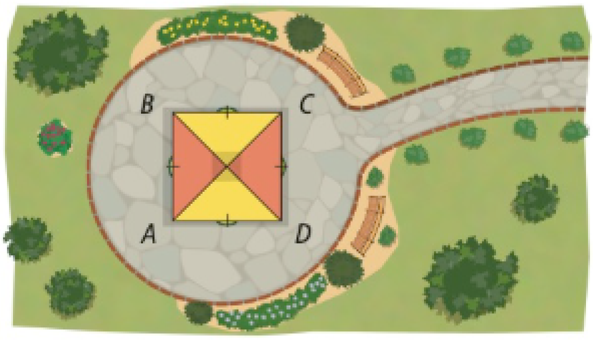
23. Draw separate diagrams showing $\triangle AEC$ and $\triangle DBG$. SEE EXAMPLE 4



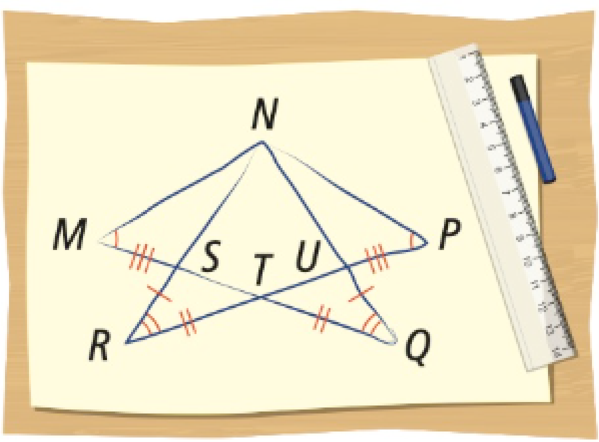
PRACTICE & PROBLEM SOLVING

APPLY

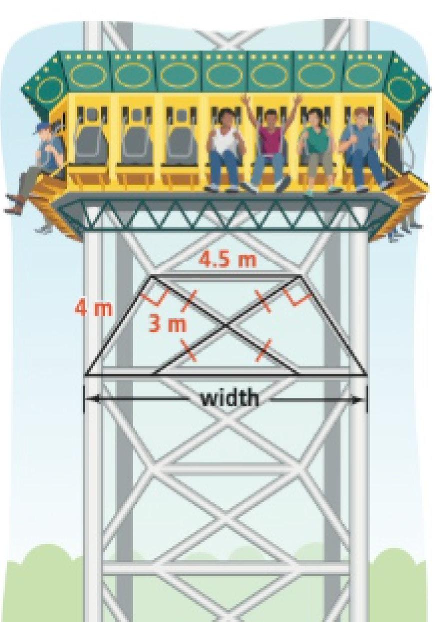
24. Communicate and Justify Parker wants to place red trim along the seams, $\overline{AC}$ and $\overline{BD}$, of a patio umbrella. He assumes they are the same length. Is he correct? Explain.



25. Apply Math Models A student is checking whether the design she drew is symmetric. Can she determine whether $\overline{MN}$ and $\overline{PN}$ are the same length? Explain.

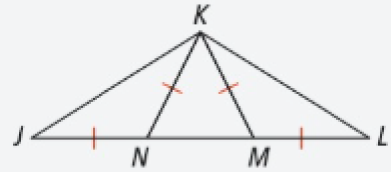


26. Represent and Connect The support for a drop tower ride is shown in the diagram. What is the width of the support? Round to the nearest hundredth.



ASSESSMENT PRACTICE

27. Which statement is false? GR.1.6



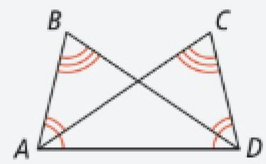
Ⓐ $\overline{KN} \cong \overline{KL}$

Ⓑ $\triangle KMJ \cong \triangle KNL$

Ⓒ $\angle KJN \cong \angle KLM$

Ⓓ $\overline{MJ} \cong \overline{NL}$

28. SAT/ACT Which theorem could you use to prove $\triangle ABD \cong \triangle DCA$?



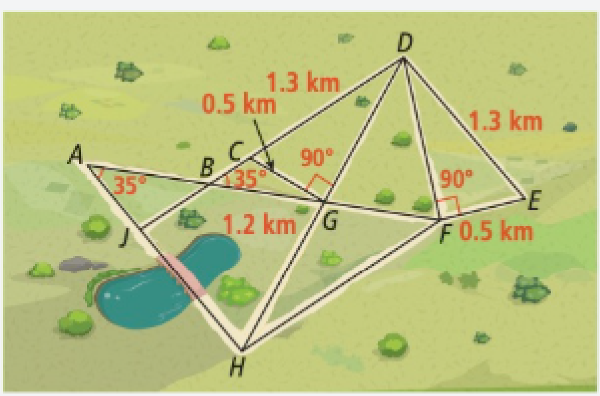
Ⓐ SAS

Ⓑ AAS

Ⓒ SSS

Ⓓ AAA

29. Performance Task The diagram shows running trails at a park.



Part A Lucy ran the triangular route represented by $\triangle BDF$. Kaitlyn starts from point H and wants to run the same distance as Lucy. What triangular route can Kaitlyn run? Explain.

Part B Draw separate triangles to represent the routes the two girls ran. Label as many side lengths and angle measures as you can determine.

Part C Can you determine the distances that the girls ran? Explain.

TOPIC 4

Topic Review

TOPIC ESSENTIAL QUESTION

1. What relationships between sides and angles of triangles can be used to prove triangles congruent?

Vocabulary Review

Choose the correct term to complete each sentence.

- base
- base angle
- congruence transformation
- congruent
- leg
- vertex

2. Figures that have the same size and shape are said to be ________.
3. The side of an isosceles triangle that is opposite the vertex is called the ________.
4. A rigid motion is sometimes called a ________ because it maps a figure to a figure with the same shape and size.
5. The legs of an isosceles triangle form an angle called the ________.

Concepts & Skills Review

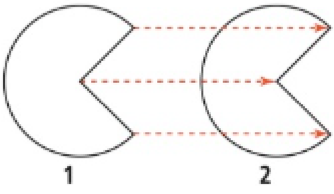
LESSON 4-1 Congruence

Quick Review

Two figures are **congruent** if there is a rigid motion, or sequence of rigid motions, that maps one figure to the other.

Example

Figure 1 is translated right to form Figure 2. Are the figures congruent?

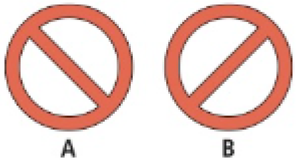


Yes, a translation is a congruence transformation.

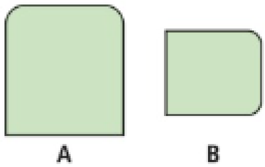
Practice & Problem Solving

For Exercises 6 and 7, determine if Figure A and Figure B are congruent. If so, describe the sequence of rigid motions that maps Figure A to Figure B. If not, explain.

6.



7.



8. **Analyze and Persevere** If a figure is reflected across the same line twice, is the resulting image congruent to that figure? Explain.

LESSON 4-2 Isosceles and Equilateral Triangles

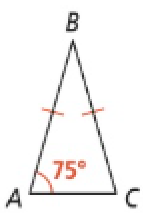
Quick Review

Two sides of a triangle are congruent if and only if the angles opposite those sides are also congruent.

A line that bisects the **vertex** angle of an isosceles triangle is the perpendicular bisector of the opposite side.

Example

What is the measure of ∠C?

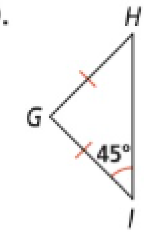


The sides opposite ∠A and ∠C are congruent, so ∠A ≅ ∠C; m∠A is 75°, so m∠C is 75°.

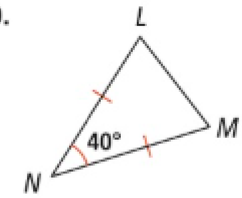
Practice & Problem Solving

Find the unknown angle measures for each triangle.

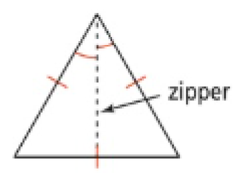
9.



10.



11. Use Patterns and Structure A zipper bisects the vertex of the front of the tent, which is in the shape of an equilateral triangle, forming two triangles. What are the angle measures of the resulting triangles?



LESSON 4-3 Proving and Applying the SAS and SSS Congruence Criteria

Quick Review

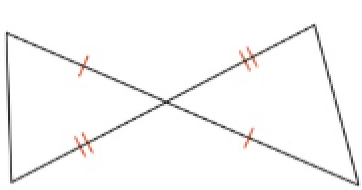
Two triangles are congruent if two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle (SAS).

Two triangles are congruent if three sides of one triangle are congruent to three sides of the other triangle (SSS).

Corresponding parts of congruent triangles are congruent (CPCTC).

Example

Show that the triangles are congruent.

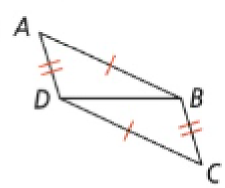


They are congruent by SAS, because they have two pairs of congruent sides and the included angles are vertical angles and so are congruent.

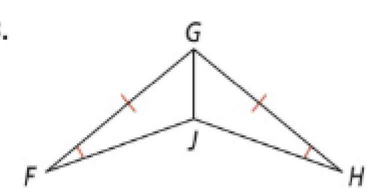
Practice & Problem Solving

Which pairs of triangles are congruent by SAS or SSS? Explain.

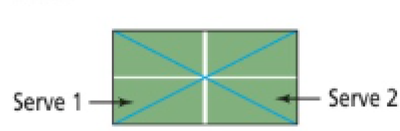
12.



13.



14. Analyze and Persevere The paths of a ping-pong ball during two separate serves intersect at the center of the table. How many pairs of congruent triangles can you find?



LESSON 4-4 Proving and Applying the ASA and AAS Congruence Criteria

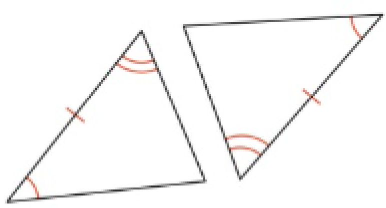
Quick Review

Two triangles are congruent if two angles of one triangle and the included side are congruent to two angles and the included side of the other triangle (ASA).

Two triangles are congruent if two angles and a nonincluded side of one triangle are congruent to two angles and a nonincluded side of the other triangle (AAS).

Example

Show that the triangles are congruent.

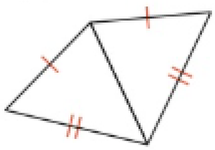


They are congruent by ASA because they have two pairs of congruent angles that have a congruent side between the angles.

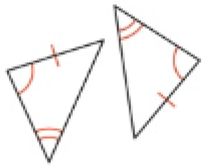
Practice & Problem Solving

State whether each pair of triangles is congruent by SAS, SSS, ASA, AAS, or the congruence cannot be determined. Justify your answer.

15.

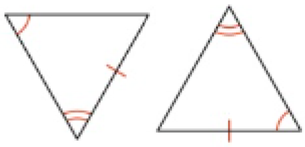


16.



17. **Generalize** Are two triangles with two pairs of congruent angles and one pair of congruent sides always congruent? Explain.

18. **Analyze and Persevere** Are the triangles congruent? If they are, by which congruence criterion? Explain.



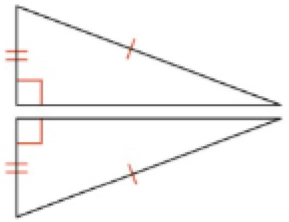
LESSON 4-5 Congruence in Right Triangles

Quick Review

If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and leg of another triangle, then the triangles are congruent (HL).

Example

Show that the triangles are congruent.

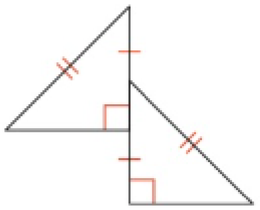


They are congruent by HL because the triangles are right triangles, and the hypotenuses and one pair of legs are congruent.

Practice & Problem Solving

Prove that the pair of triangles is congruent.

19.



20. **Communicate Precisely** Triangle *ABC* is an isosceles triangle with the vertex bisected by a line segment. Draw the triangle and prove that the resulting triangles are congruent.

LESSON 4-6 Congruence in Overlapping Triangles

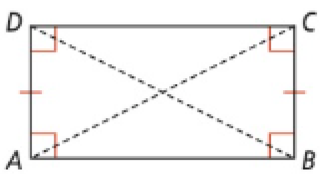
Quick Review

If two triangles are overlapping, all congruence criteria—AAS, ASA, HL, SAS, and SSS—can still be applied.

To identify congruent overlapping triangles, first identify the parts of each triangle. Then test if any of the congruence criteria hold.

Example

Given rectangle *ABCD*, how can you prove that $\angle ADB \cong \angle BCA$?

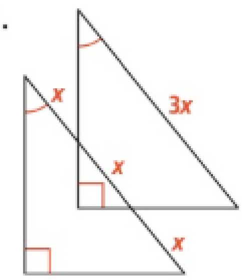


Opposite sides of a rectangle are congruent, so $\overline{AD} \cong \overline{BC}$. All right angles are congruent, so $\angle DAB \cong \angle CBA$. By the Reflexive Property, $\overline{AB} \cong \overline{BA}$. Thus, $\triangle ADB \cong \triangle BCA$ by SAS, and $\angle ADB \cong \angle BCA$ by CPCTC.

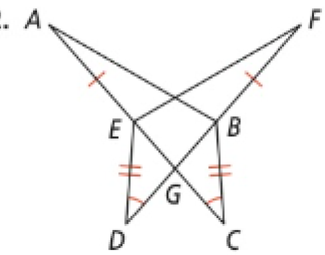
Practice & Problem Solving

For Exercise 21, prove that the pair of triangles is congruent. For Exercise 22, which pairs are of triangles are congruent? Explain.

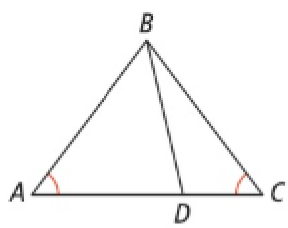
21.



22.



23. **Use Patterns and Structure** Describe where to place point *E* along $\overline{AC}$ such that $\triangle ABD \cong \triangle CBE$. Then explain why the triangles are congruent.



TOPIC 5

Relationships in Triangles

TOPIC ESSENTIAL QUESTION

How are the sides, segments, and angles of triangles related?

Topic Overview

enVision® STEM Project
Find the Center of Mass

5-1 Perpendicular and Angle Bisectors
GR.1.1, GR.5.2, MTR.4.1, MTR.5.1, MTR.6.1

5-2 Bisectors in Triangles
GR.1.3, GR.5.3, GR.6.3, MTR.1.1, MTR.2.1, MTR.3.1

Mathematical Modeling in 3 Acts:
Making It Fair
GR.1.3, MTR.7.1

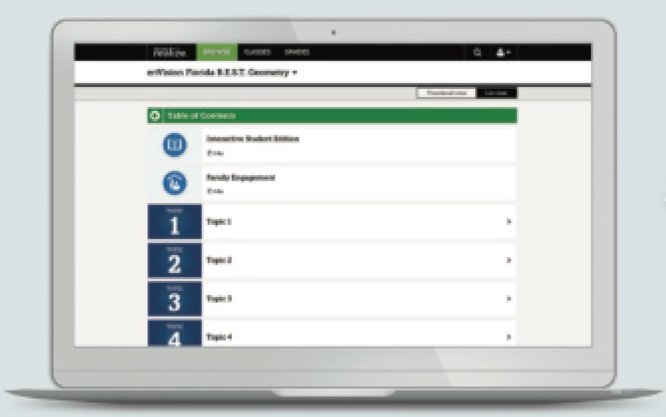
5-3 Medians and Altitudes
GR.1.3, GR.3.3, MTR.1.1, MTR.4.1, MTR.2.1

5-4 Inequalities in One Triangle
GR.1.3, LT.4.8, MTR.5.1, MTR.1.1, MTR.7.1

5-5 Inequalities in Two Triangles
GR.1.3, LT.4.8, MTR.4.1, MTR.5.1, MTR.7.1

Topic Vocabulary

- altitude
- centroid
- circumcenter
- circumscribed
- concurrent
- equidistant
- incenter
- inscribed
- median
- orthocenter
- point of concurrency



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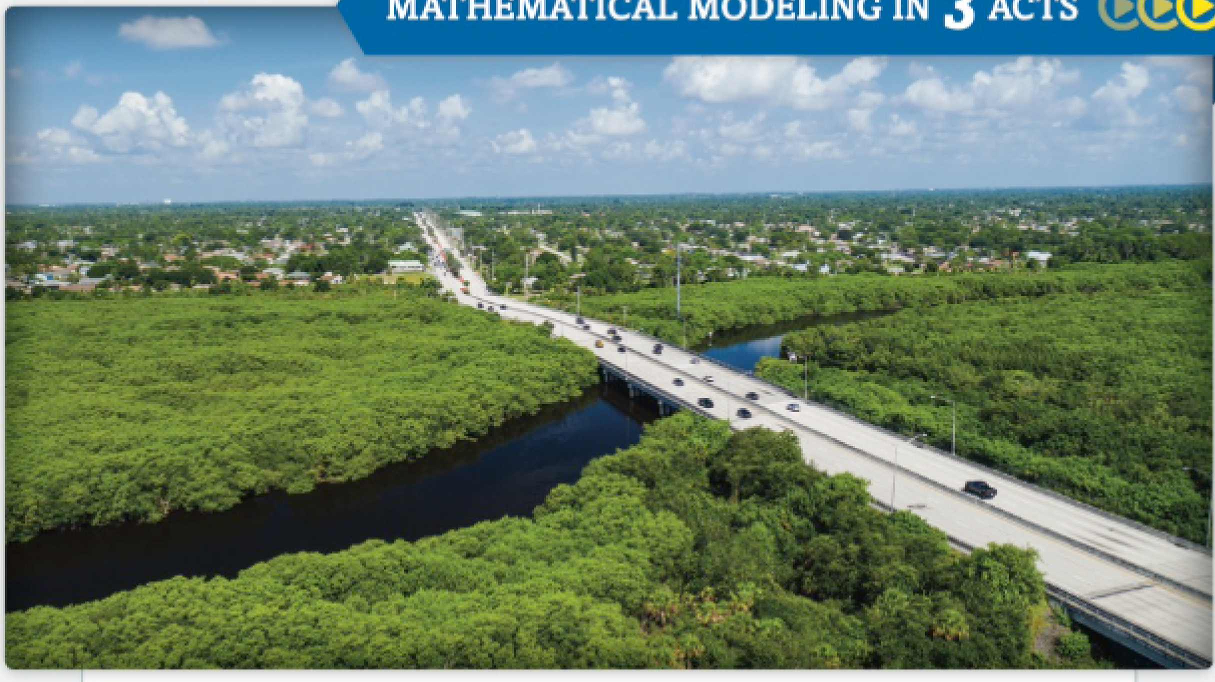
ACTIVITIES Complete ***Explore & Reason, Model & Discuss***, and ***Critique & Explain*** activities. Interact with ***Examples*** and ***Try Its.***

ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.



Making it Fair

In rural areas, county planners often work with local officials from a number of small towns to establish a regional medical center to serve all of the nearby communities.

County planners might also establish regional medical evacuation centers to transport patients with serious trauma to larger medical centers. The locations of these regional centers are carefully planned. Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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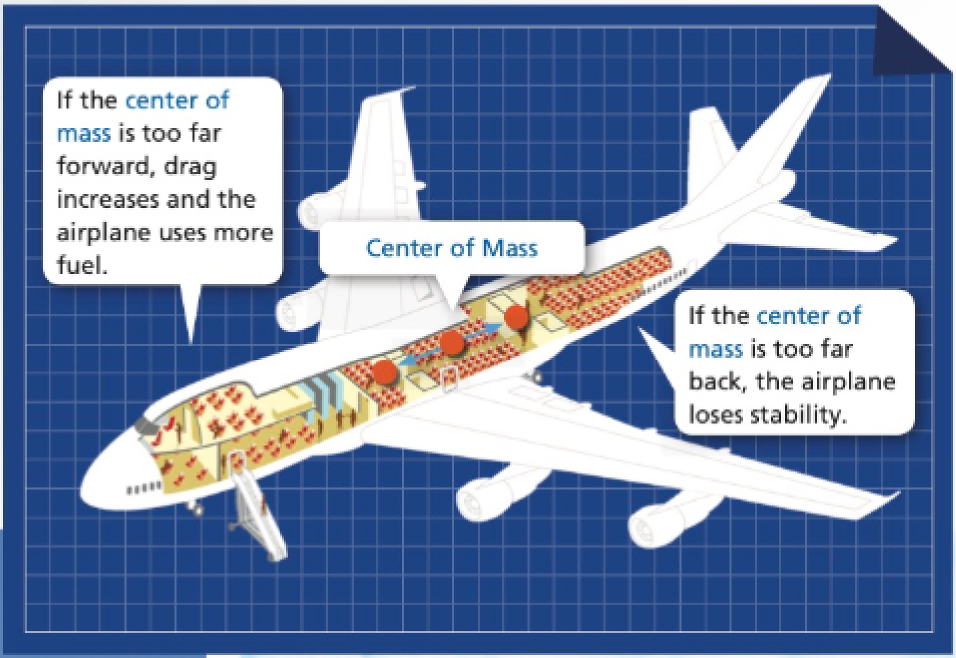
Did You Know?

An object's **center of mass** is the single point at which its mass is evenly dispersed and the object is in balance.

When an object is moving within Earth's gravitational field, the object's center of mass is sometimes called its *center of gravity*.



Athletes adjust their **center of mass** to maximize performance.

Your Task: Find the Center of Mass

In this project, you and your classmates will find the center of mass for a triangular object using mathematics. You will also find the center of mass for an irregular object through experimentation.

5-1 Perpendicular and Angle Bisectors

I CAN… use perpendicular and angle bisectors to solve problems.

VOCABULARY

- equidistant

MA.912.GR.1.1–Prove relationships and theorems about lines and angles. Solve mathematical and real-world problems involving postulates, relationships and theorems of lines and angles. Also GR.5.2

MA.K12.MTR.4.1, MTR.5.1, MTR.6.1

MODEL & DISCUSS

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A new high school will be built for Brighton and Springfield. The location of the school must be the same distance from each middle school. The distance between the two middle schools is 18 miles.

A. Trace the points for the schools on a piece of paper. Locate a new point that is 12 mi from each school. Compare your point with the points of other students. Is there more than one location for the new high school? Explain.

B. **Communicate and Justify** Can you find locations for the new high school that are the same distance from each middle school for any given distance? Explain.

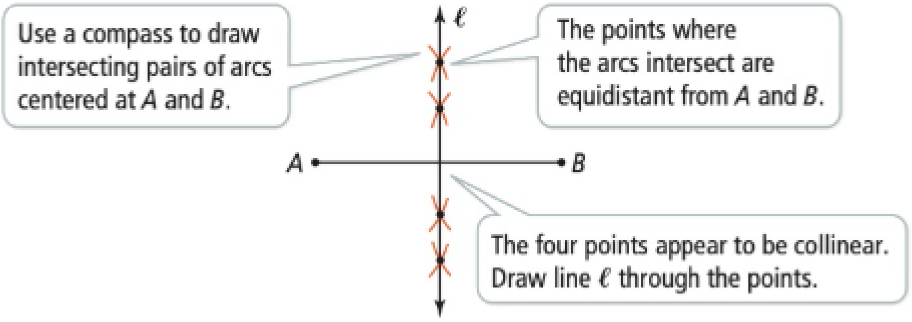
ESSENTIAL QUESTION What is the relationship between a segment and the points on its perpendicular bisector? Between an angle and the points on its bisector?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Find Equidistant Points

How can you find points that are equidistant from the endpoints of $\overline{AB}$? What do you notice about these points and their relationship with $\overline{AB}$?

A point that is the same distance from two points is **equidistant** from the points.



The points that are equidistant from A and B appear to lie on line ℓ. Line ℓ appears to be perpendicular to and bisect $\overline{AB}$ in the same plane as $\overline{AB}$. You can use a ruler and a protractor to support this hypothesis.

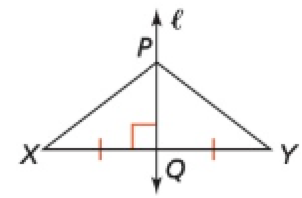
COMMON ERROR
Be sure not to change the compass setting when drawing each pair of intersecting arcs from each endpoint.

Try It! 1. Draw a pair of fixed points, and find points that are equidistant from the two fixed points. Draw a line through the set of equidistant points. Repeat this process for several pairs of fixed points. What conjecture can you make about points that are the same distance from a given pair of points?

THEOREM 5-1 Perpendicular Bisector Theorem

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If...



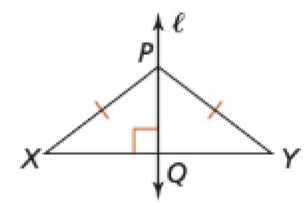
PROOF: SEE EXAMPLE 2.

Then... $PX = PY$

THEOREM 5-2 Converse of the Perpendicular Bisector Theorem

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

If...



PROOF: SEE EXAMPLE 2 TRY IT.

Then... $XQ = YQ$

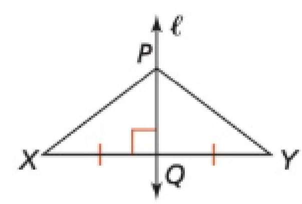
PROOF

EXAMPLE 2 Prove the Perpendicular Bisector Theorem

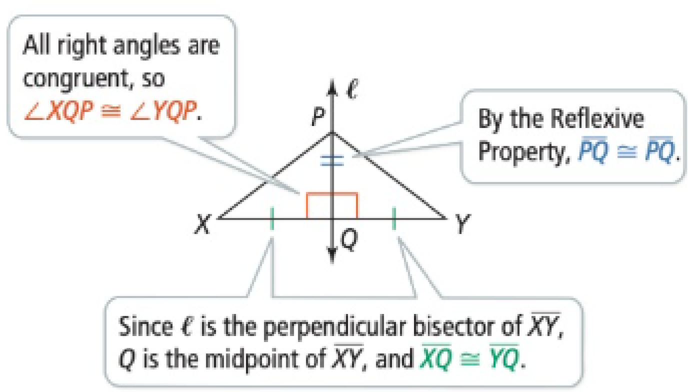
Prove the Perpendicular Bisector Theorem.

Given: ℓ is the perpendicular bisector of $\overline{XY}$.

Prove: $PX = PY$



Proof:



STUDY TIP

Remember that if a line is a perpendicular bisector of a segment, you can conclude two things: the line is perpendicular to the segment, and it bisects the segment.

By SAS, $\triangle XQP \cong \triangle YQP$. Therefore, $\overline{PX} \cong \overline{PY}$ by CPCTC, so $PX = PY$.

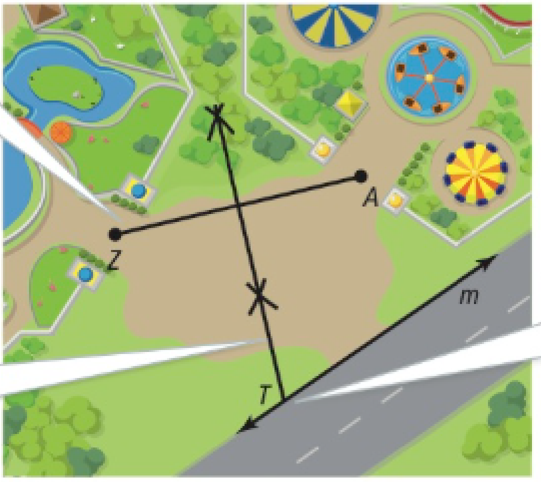
Try It! **2.** Prove the Converse of the Perpendicular Bisector Theorem.

APPLICATION

EXAMPLE 3 Use a Perpendicular Bisector

Mr. Lee wants to park his ice cream cart on Main Street so that he is equidistant from the entrances of the amusement park and the zoo. Where should Mr. Lee park? How can he determine where to park?

Mr. Lee can use the perpendicular bisector of the segment that connects the two entrances to find the location.



Step 1 Label the entrances of the amusement park and zoo as points *A* and *Z*, and draw line *m* for Main Street.

Step 2 Draw $\overline{AZ}$, and construct the perpendicular bisector.

Step 3 Mark point *T* where the perpendicular bisector and line *m* intersect.

Mr. Lee should park his cart at point *T*, because it is equidistant from both entrances.

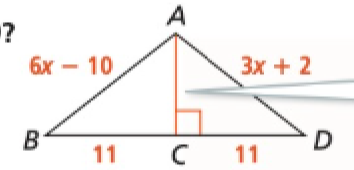
LEARN TOGETHER
How can you respectfully disagree and manage your emotions?

Try It! **3.** The entrances are 40 feet apart. Mr. Lee decides to move his cart off the street to the area between the Main Street and the entrances to the amusement park and the zoo. How can you find where Mr. Lee should park if he must be 30 feet from both entrances?

EXAMPLE 4 Apply the Perpendicular Bisector Theorem

STUDY TIP
Look for relationships in the diagram to help you solve a problem. For example, a right angle marker tells you that two line segments are perpendicular.

What is the value of AD?



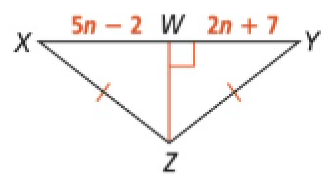
By definition, $\overline{AC}$ is the perpendicular bisector of $\overline{BD}$.

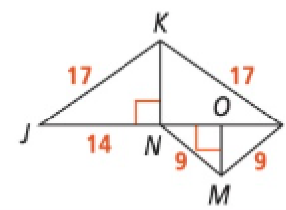
$$AB = AD$$
$$6x - 10 = 3x + 2$$
$$3x = 12$$
$$x = 4$$
$$AD = 3(4) + 2$$
$$AD = 14$$

The lengths are equal by the Perpendicular Bisector Theorem.

Evaluate the expression for AD.

Try It! **4. a.** What is the value of WY? **b.** What is the value of OL?





EXAMPLE 5 Find Equidistant Points from the Sides of an Angle

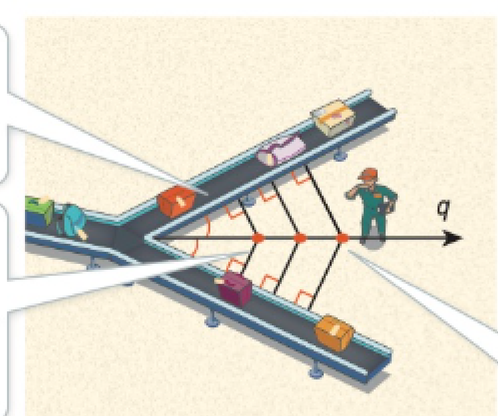
An airport baggage inspector needs to stand equidistant from two conveyor belts. How can the inspector determine where he should stand?

Use pairs of corresponding points on each conveyor belt that are the same distance away from the vertex of the angle. To be equidistant from the conveyor belts, a point must have the same distance from corresponding points.

CHECK FOR REASONABLENESS
Think about the tools you can use to make sure that segments are perpendicular. What tool would you use?

Draw the lines perpendicular from each pair of corresponding points.

The distance between a point and a line is the length of the segment perpendicular from the line to the point.



The points of intersection are equidistant from each belt and appear to be collinear.

Ray q appears to be the angle bisector. You can use a protractor to support this.

The inspector can determine where to stand by choosing a point on the angle bisector.

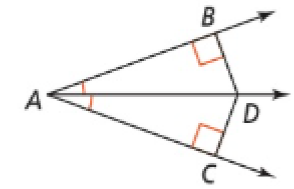
Try It! **5.** Consider two triangles that result from drawing perpendicular segments from where the inspector stands to the conveyor belts. How are the triangles related? Explain.

THEOREM 5-3 Angle Bisector Theorem

If a point is on the bisector of an angle, then it is equidistant from the two sides of the angle.

PROOF: SEE EXERCISE 9.

If...



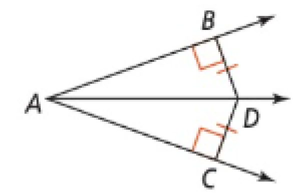
Then... $BD = CD$

THEOREM 5-4 Converse of the Angle Bisector Theorem

If a point is equidistant from two sides of an angle, then it is on the angle bisector.

PROOF: SEE EXERCISE 10.

If...

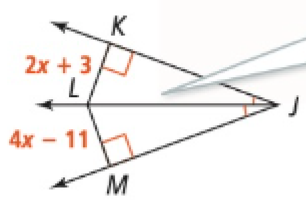


Then... $m\angle BAD = m\angle CAD$

EXAMPLE 6 Apply the Angle Bisector Theorem

STUDY TIP
To apply the Angle Bisector Theorem, be sure a diagram reflects the necessary conditions—angles are marked as congruent and right angles are marked to indicate that segments are perpendicular to the sides.

What is the value of *KL*?



$\overrightarrow{JL}$ is the angle bisector of $\angle KJM$ since $m\angle KJL = m\angle MJL$.

$KL = ML$

$2x + 3 = 4x - 11$

$2x = 14$

$x = 7$

$KL = 2(7) + 3$

$KL = 17$

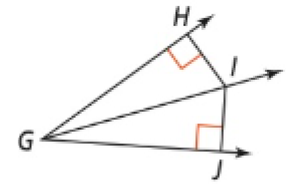
The lengths are equal by the Angle Bisector Theorem.

Evaluate the expression for *KL*.

Try It! **6.** Use the figure shown.

a. If $HI = 7$, $IJ = 7$, and $m\angle HGI = 25$, what is $m\angle IGJ$?

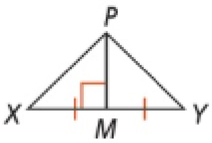
b. If $m\angle HGJ = 57$, $m\angle IGJ = 28.5$, and $HI = 12.2$, what is the value of IJ?



CONCEPT SUMMARY Perpendicular and Angle Bisectors

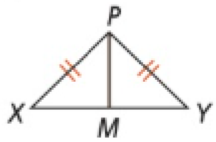
THEOREM 5-1 Perpendicular Bisector Theorem

If...



$XM = YM$ and $\overline{PM} \perp \overline{XY}$

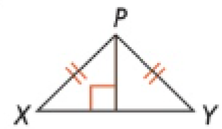
Then...



$PX = PY$

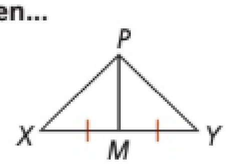
THEOREM 5-2 Converse of Perpendicular Bisector Theorem

If...



$PX = PY$

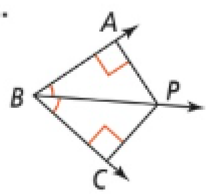
Then...



$XM = YM$

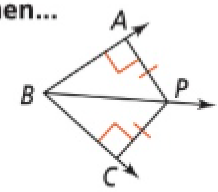
THEOREM 5-3 Angle Bisector Theorem

If...



$\angle ABP \cong \angle CBP$

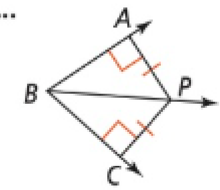
Then...



$AP = CP$

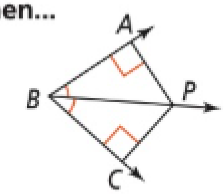
THEOREM 5-4 Converse of Angle Bisector Theorem

If...



$AP = CP$

Then...



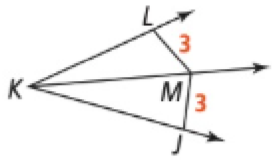
$\angle ABP \cong \angle CBP$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What is the relationship between a segment and the points on its perpendicular bisector? Between an angle and the points on its bisector?

2. **Vocabulary** How can you determine if a point is *equidistant* from the sides of an angle?

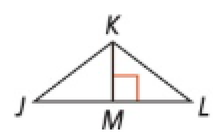
3. **Error Analysis** River says that $\overrightarrow{KM}$ is the bisector of $\angle LKJ$ because $LM = MJ$. Explain the error in River's reasoning.



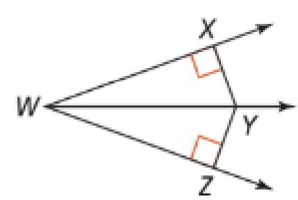
4. **Communicate and Justify** You know that $\overline{AB}$ is the perpendicular bisector of $\overline{XY}$, and $\overline{XY}$ is the perpendicular bisector of $\overline{AB}$. What can you conclude about the side lengths of quadrilateral $AXBY$? Explain.

Do You KNOW HOW?

5. If $JL = 14$, $KL = 10$, and $ML = 7$, what is JK?



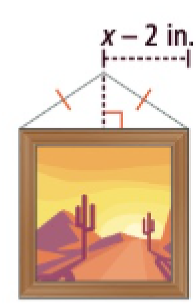
Use the figure shown for Exercises 6 and 7.



6. If $\angle XWY \cong \angle ZWY$ and $XY = 4$, what is YZ?

7. If $XY = ZY$ and $m\angle ZWY = 18$, what is $m\angle XWZ$?

8. What is an algebraic expression for the area of the square picture and frame?



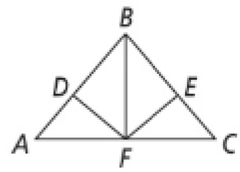
PRACTICE & PROBLEM SOLVING

UNDERSTAND

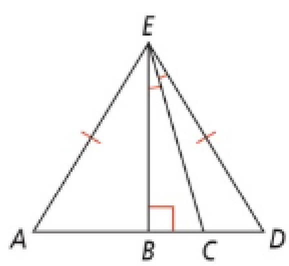
9. **Communicate and Justify** Write a two-column proof for the Angle Bisector Theorem.

10. **Communicate and Justify** Write a paragraph proof for the Converse of the Angle Bisector Theorem.

11. **Use Patterns and Strcture** In the diagram below, $AB = BC$, $DF = EF$, and $m\angle BDF = m\angle BEF = 90°$. Is $\triangle ADF \cong \triangle CEF$? Justify your answer.



12. **Error Analysis** A student analyzed the diagram and incorrectly concluded that $AB = 2BC$. Explain the student's error.

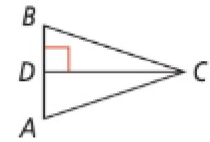


$\overline{EB}$ is the perpendicular bisector of $\overline{AD}$, so $AB = BD$.
$\angle BEC \cong \angle DEC$, so $BC = CD$.
$BC + CD = BD = AB$, and
$BC + CD = BC + BC = 2BC$,
so $AB = 2BC$.

13. **Higher Order Thinking** Describe the process of constructing the bisector of an angle. Draw a diagram and explain how this construction can be related to the Angle Bisector Theorem.

PRACTICE

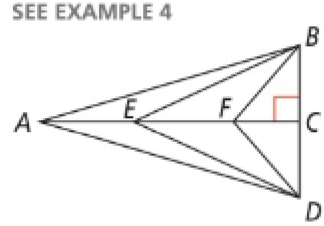
Use the figure shown for Exercises 14 and 15.
SEE EXAMPLES 1–3



14. If $AD = 3$, $AC = 8$, and $BD = 3$, what is the perimeter of $\triangle ABC$?

15. If $BC = 10$, $AB = 7$, and the perimeter of $\triangle ABC$ is 27, what is the value of BD?

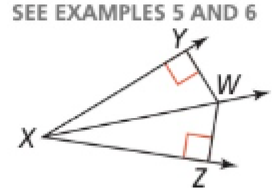
Use the figure shown for Exercises 16 and 17.
SEE EXAMPLE 4



16. If $AD = 21$, $BF = 8$, and $DF = 8$, what is the value of AB?

17. If $EB = 6.2$, $CD = 3.3$, and $ED = 6.2$, what is the value of BD?

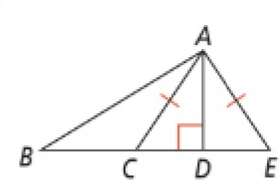
Use the figure shown for Exercises 18 and 19.
SEE EXAMPLES 5 AND 6



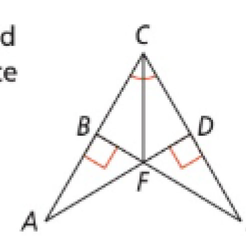
18. If $m\angle YXW = 21$, $YW = 5$, and $WZ = 5$, what is $m\angle ZXY$?

19. If $m\angle YXZ = 38$, $m\angle WXZ = 19$, and $WZ = 8.1$, what is the value of YW?

20. If $CD = 4$ and the perimeter of $\triangle ABC$ is 23, what is the perimeter of $\triangle ABE$?



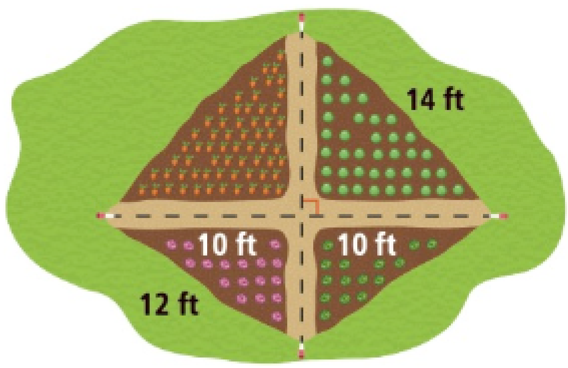
21. Given that $\angle ACF \cong \angle ECF$ and $m\angle ABF = m\angle EDF = 90$, write a two-column proof to show that $\triangle ABF \cong \triangle EDF$.



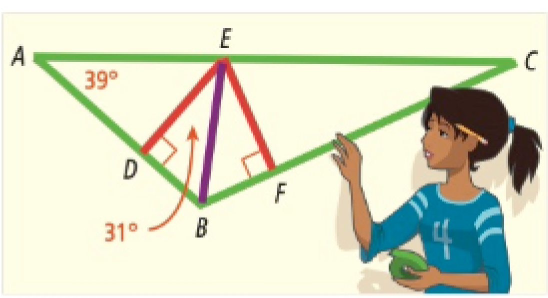
PRACTICE & PROBLEM SOLVING

APPLY

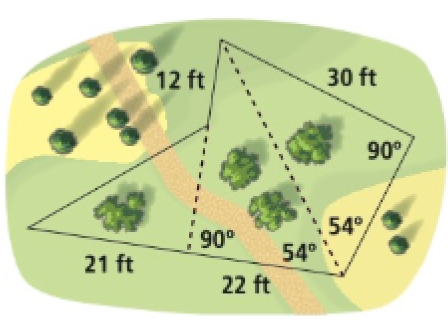
22. Analyze and Persevere A gardener wants to replace the fence along the perimeter of her garden. How much new fencing will be required?



23. Use Patterns and Structure An artist uses colored tape to divide sections of a mural. She places the tape so that $DB = BF$. She needs to cut a piece of paper to cover $\triangle EFC$ while she works on other sections. What angles should she cut so she only covers the triangle?



24. Mathematical Connections A surveyor took some measurements of a piece of land. The owner needs to know the area of the land to determine the value. What is the area of the piece of land?



ASSESSMENT PRACTICE

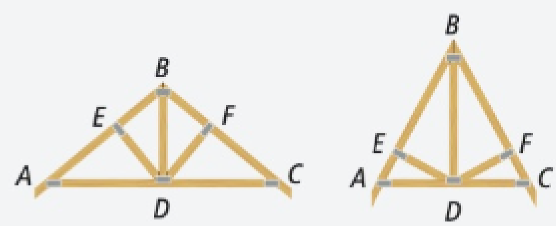
25. $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{XY}$. Point P is the midpoint of $\overline{XY}$. Select all the always true statements. GR.1.1

- ☐ **A.** $AP = XP$
- ☐ **B.** $AB = XY$
- ☐ **C.** $AP = BP$
- ☐ **D.** $XB = YB$
- ☐ **E.** $AY = XB$
- ☐ **F.** $XP = YP$

26. SAT/ACT Points G, J, and K are not collinear, and $GJ = GK$. If P is a point on $\overline{JK}$, which of the following conditions is sufficient to prove that $\overleftrightarrow{GP}$ is the perpendicular bisector of $\overline{JK}$?

Ⓐ $JG = PG$
Ⓑ $m\angle GPJ = 90$
Ⓒ $\angle GJK \cong \angle GKJ$
Ⓓ $PK = PG$

27. Performance Task A manufacturer makes roofing trusses in a variety of sizes. All of the trusses have three supports, as shown, with $\overline{ED} \perp \overline{AB}$ and $\overline{FD} \perp \overline{BC}$.



Part A One builder needs $\angle ABD$ and $\angle CBD$ to be congruent for a project. You need to check that a truss meets the builder's requirement. The only tools you have are a measuring tape and a steel square, which is a carpentry tool for measuring right angles. How can you use these tools to verify the angles are congruent?

Part B In addition to the requirement of the first builder, another builder also needs $\overline{AB}$ and $\overline{BC}$ to be congruent as well as $\overline{AD}$ and $\overline{DC}$. Using the same tools, how can you efficiently verify that all three pairs are congruent? Explain.

5-2 Bisectors in Triangles

I CAN… use triangle bisectors to solve problems.

VOCABULARY

- circumcenter
- circumscribed
- concurrent
- incenter
- inscribed
- point of concurrency

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles. **Also GR.5.3, GR.6.3**

MA.K12.MTR.1.1, MTR.2.1, MTR.3.1

MODEL & DISCUSS

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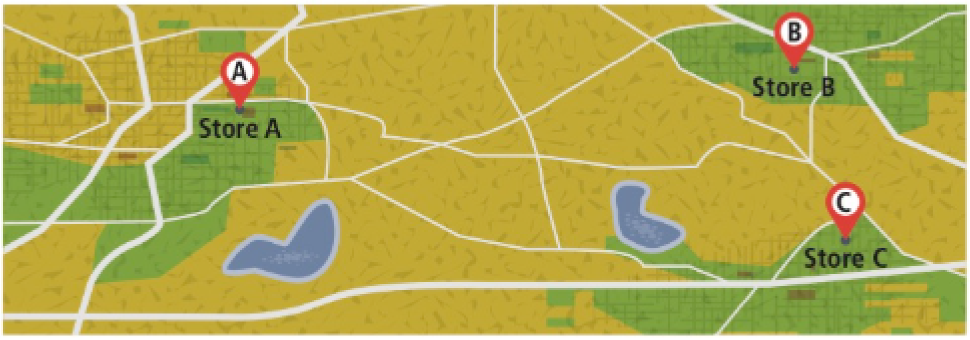
A sporting goods company has three stores in three different towns. They want to build a distribution center so that the distance from each store to the distribution center is as close to equal as possible.



A. Points *A, B,* and *C* represent the locations of the three stores. Trace the points on a piece of paper. Locate a point *D* that appears to be the same distance from *A, B,* and *C* by sight only.

B. **Communicate and Justify** Measure the length from points *A, B,* and *C* to point *D* on your diagram. Are the lengths equal? If not, can you find a better location for point *D*? Explain.

C. What do you think is the quickest way to find the best point *D* in similar situations?

ESSENTIAL QUESTION

What are the properties of the perpendicular bisectors in a triangle? What are the properties of the angle bisectors in a triangle?

The Concurrency of the Perpendicular Bisectors Theorem explains the relationship between the perpendicular bisectors of a triangle.

THEOREM 5-5 Concurrency of Perpendicular Bisectors

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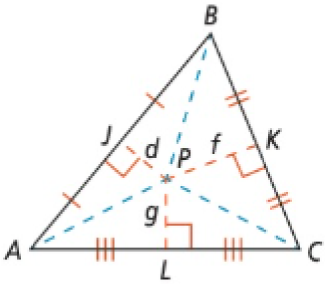
The perpendicular bisectors of the sides of a triangle are concurrent at a point equidistant from the vertices.

PROOF: SEE EXAMPLE 1.

If...

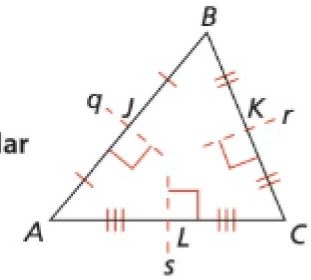


Then... d, f, and g intersect at P and $PA = PB = PC$

PROOF

EXAMPLE 1 Prove Theorem 5-5

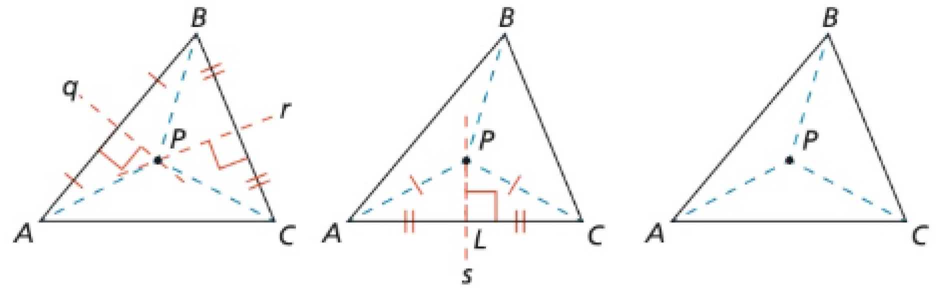
When three or more lines intersect at one point, the lines are concurrent. The point where the lines intersect is called the point of concurrency. How do you prove the Concurrency of Perpendicular Bisectors Theorem?



Given: $\triangle ABC$ with midpoints J, K, and L, and perpendicular bisectors q, r, and s.

Prove: Lines q, r, and s are concurrent at a point that is equidistant from A, B, and C.

COMMON ERROR
Make sure that you prove everything required to complete the proof. For this proof, you need to show both that the lines are concurrent and that they are equidistant from the vertices.



Let P be the point of intersection of q and r. By the Perpendicular Bisector Theorem, $PA = PB$, and $PB = PC$. Therefore $PA = PC$.

By the Converse of the Perpendicular Bisector Theorem, P also lies on the perpendicular bisector of $\overline{AC}$.

P is the point of concurrency of q, r, and s. Since $PA = PB = PC$, point P is equidistant from A, B, and C.

Try It! **1.** Verify the Concurrency of Perpendicular Bisectors Theorem on acute, right, and obtuse triangles using a straightedge and compass or geometry software.

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Investigate Circumscribed Circles

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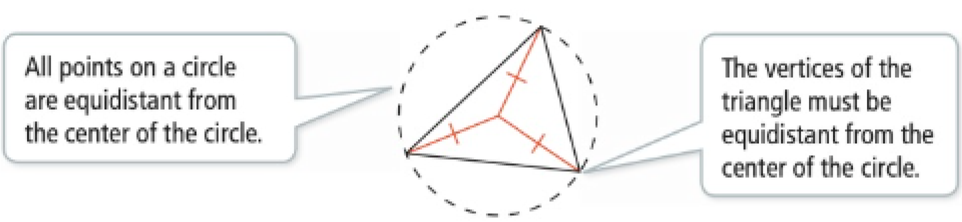
How can you construct a circle that contains the three vertices of a given triangle?

The circle that contains all three vertices of a triangle is the circumscribed circle of the triangle.



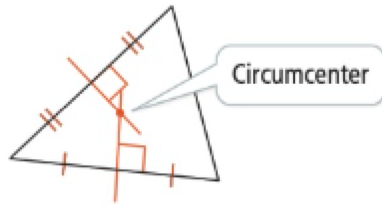
By the Concurrency of Perpendicular Bisectors Theorem, the perpendicular bisectors intersect at a point that is equidistant from the vertices.

CONTINUED ON THE NEXT PAGE

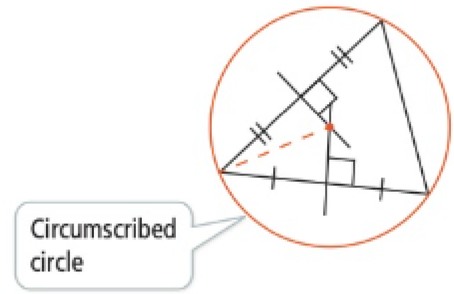
EXAMPLE 2 CONTINUED

The point of concurrency of the perpendicular bisectors of a triangle is the **circumcenter**, so the circumcenter is the center of the circumscribed circle of the triangle.

COMMUNICATE AND JUSTIFY
When you are learning a new concept, think about how to explain it in your own words. How would you explain the circumcenter of a triangle to another student?



First, construct two perpendicular bisectors to find the circumcenter.



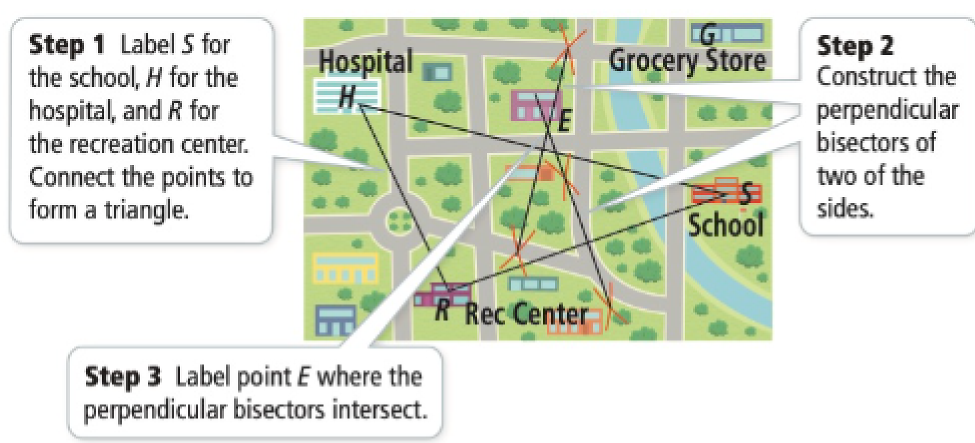
Then construct the circle centered on the circumcenter passing through any vertex.

Try It! **2.** What conjecture can you make about the location of the circumcenter for acute, right, and obtuse triangles?

APPLICATION

EXAMPLE 3 Use a Circumcenter

A city manager wants to place a new emergency siren so that it is the same distance from the school, hospital, and recreation center. Where should the emergency siren be placed?



The city manager should place the emergency siren at point E, because it is equidistant to the three locations.

Try It! **3.** If the city manager decided to place the siren so that it is the same distance from the hospital, school, and grocery store, how can she find the location?

Another kind of special segment in a triangle is the angle bisector. Like the perpendicular bisectors of a triangle, the angle bisectors of a triangle also have a point of concurrency.

THEOREM 5-6 Concurrency of Angle Bisectors

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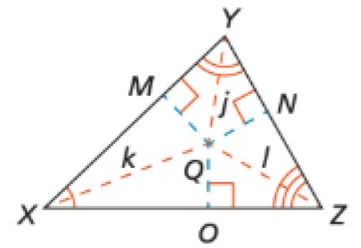
The angle bisectors of the angles of a triangle are concurrent at a point equidistant from the sides of the triangle.

If...



PROOF: SEE EXERCISE 10.

Then... j, k, and l intersect at Q and $QM = QN = QO$

EXAMPLE 4 Investigate Inscribed Circles

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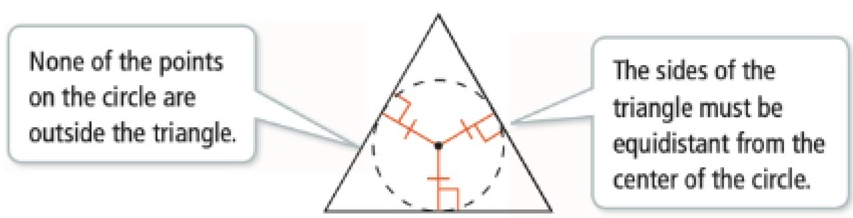
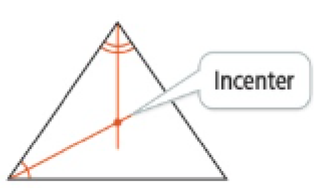
How can you construct a circle that intersects each side of a given triangle in exactly one point?

The circle that intersects each side of a triangle at exactly one point and has no points outside of the triangle is the **inscribed** circle of the triangle.

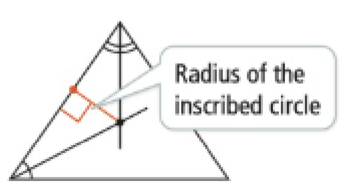


HAVE A GROWTH MINDSET
In what ways can you be inquisitive and open to learning new things?

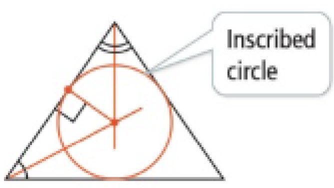
By the Concurrency of Angle Bisectors Theorem, the angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle. The point of concurrency of the angle bisectors of a triangle is the **incenter**, so the incenter is the center of the inscribed circle of the triangle.



First, construct two angle bisectors to find the incenter.



Next, construct a perpendicular segment from the incenter to any side.



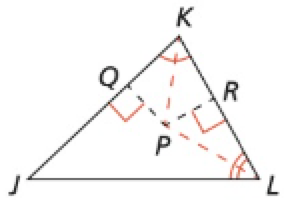
Finally, construct the circle centered on the incenter and passing through the point of intersection of the perpendicular segment and the side.

Try It! 4. Do you think the incenter of a triangle can ever be located on a side of the triangle? Explain.

EXAMPLE 5 Identify and Use the Incenter of a Triangle

If $QP = 3(x + 1)$ and $RP = 5x - 3$, what is the radius of the inscribed circle of $\triangle JKL$?

Since $\overline{KP}$ and $\overline{LP}$ are angle bisectors of $\triangle JKL$, P is the incenter of $\triangle JKL$. Therefore, $QP = RP$.



STUDY TIP
Remember that $QP = RP$, so evaluating either expression for $x = 3$ gives the value of the radius. Select the expression that is easier to evaluate.

Step 1 Solve for x.

$$QP = RP$$
$$3(x + 1) = 5x - 3$$
$$3x + 3 = 5x - 3$$
$$6 = 2x$$
$$x = 3$$

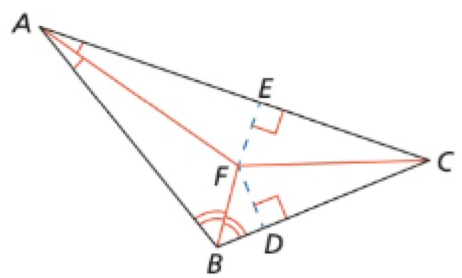
Step 2 Find the radius.

$$RP = 5x - 3$$
$$= 5(3) - 3$$
$$= 12$$

The radius of the incircle is equal to RP.

The radius of the inscribed circle is 12.

Try It! **5.** Use the figure shown.



a. If $m\angle BAF = 15$ and $m\angle CBF = 52$, what is $m\angle ACF$?

b. If $EF = 3y - 5$ and $DF = 2y + 4$, what is the distance from F to $\overline{AB}$?

CONCEPT SUMMARY Bisectors in Triangles

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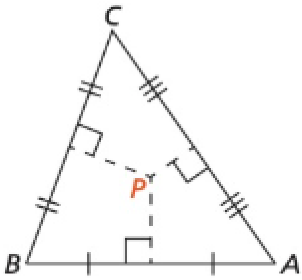
PROPERTIES Perpendicular Bisectors

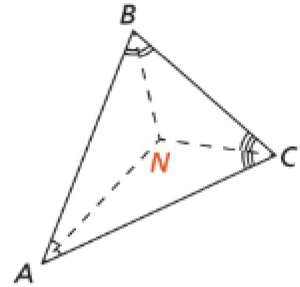
Perpendicular bisectors intersect at the circumcenter.



The circumcenter is equidistant from the vertices.

PROPERTIES Angle Bisectors

Angle bisectors intersect at the incenter.

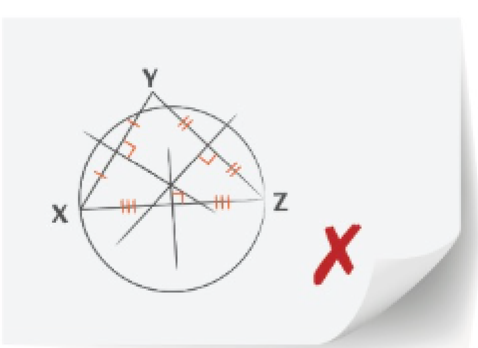


The incenter is equidistant from the sides.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are the properties of the perpendicular bisectors in a triangle? What are the properties of the angle bisectors in a triangle?

2. **Error Analysis** Terrence constructed the circumscribed circle for $\triangle XYZ$. Explain Terrence's error.

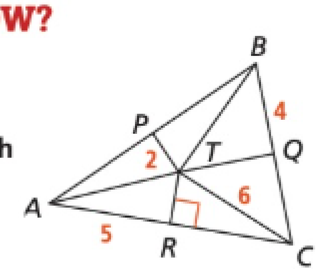


3. **Vocabulary** What parts of the triangle is the *circumcenter* equidistant from? What parts of the triangle is the *incenter* equidistant from?

4. **Analyze and Persevere** Is it possible for the circumscribed circle and the inscribed circle of a triangle to be the same? Explain your reasoning.

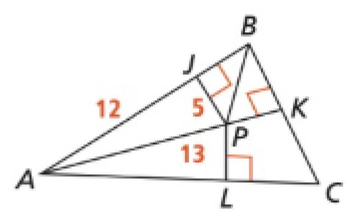
Do You KNOW HOW?

The perpendicular bisectors of $\triangle ABC$ are $\overline{PT}$, $\overline{QT}$, and $\overline{RT}$. Find each value.



5. AT 6. RC

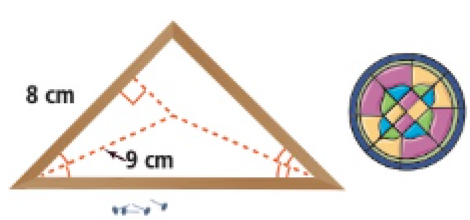
Two of the angle bisectors of $\triangle ABC$ are $\overline{AP}$ and $\overline{BP}$. Find each value.



7. PK

8. Perimeter of $\triangle APL$

9. An artist will place a circular piece of stained glass inside the triangular frame so that the glass touches each side of the frame. What is the diameter of the stained glass? Round to the nearest tenth.



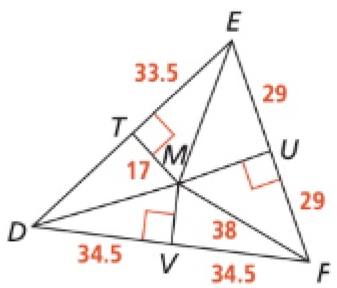
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Use Patterns and Structure** Write a Proof of Theorem 5-6: In $\triangle ABC$, let the angle bisectors of $\angle A$ and $\angle B$ intersect at point P. Show that P is equidistant from each side of $\triangle ABC$, and that $\overline{CP}$ bisects $\angle C$.

11. **Higher Order Thinking** A right triangle has vertices $X(0, 0)$, $Y(0, 2a)$, $Z(2b, 0)$. What is the circumcenter of the triangle? Make a conjecture about the diameter of a circle that is circumscribed about a right triangle.

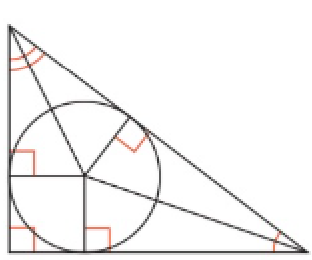
12. **Error Analysis** What is the error that a student made in finding the perimeter of $\triangle DTM$? Correct the error.



$DT = 34.5$, $TM = 17$, $DM = 34.5$.
The perimeter of ΔDTM is
$34.5 + 17 + 34.5 = 86$. ✗

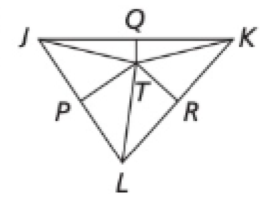
13. **Mathematical Connections** A triangle with incenter P has side lengths x, y, and z. The distance from P to each side is a. Write an expression for the area of the triangle. Use the distributive property to factor your expression.

14. **Choose Efficient Methods** In a right triangle with side lengths of 3, 4, and 5, what is the radius of the inscribed circle? Show your work. (Hint: Let r be the radius. Label the lengths of each segment formed by the perpendiculars to the sides.)

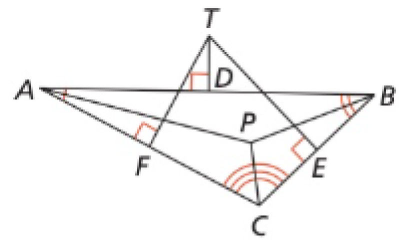


PRACTICE

15. The perpendicular bisectors of $\triangle JKL$ are $\overline{PT}$, $\overline{QT}$, and $\overline{RT}$. Name three isosceles triangles. SEE EXAMPLE 1



Use the diagram below for Exercises 16–18. Points D, E, and F are the midpoints of the sides of $\triangle ABC$. SEE EXAMPLES 2 AND 4

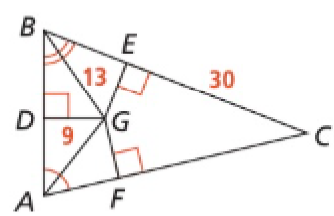


16. Which point is the center of a circle that contains A, B, and C?

17. Which point is the center of a circle that intersects each side of $\triangle ABC$ at exactly one point?

18. The perpendicular bisector of $\overline{AB}$ is m and the perpendicular bisector of $\overline{BC}$ is n. Lines m and n intersect at T. If $TA = 8.2$, what is TC? SEE EXAMPLE 3

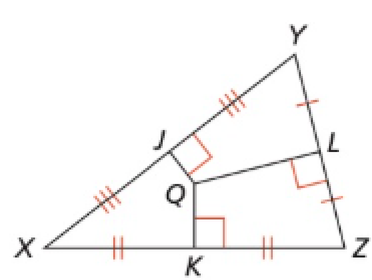
Find the values. SEE EXAMPLE 5



19. EG

20. GF

If $XY = 24$, $XZ = 22$, and $JQ = 5$, find the values. Round to the nearest tenth.



21. The radius of the circumscribed circle of $\triangle XYZ$

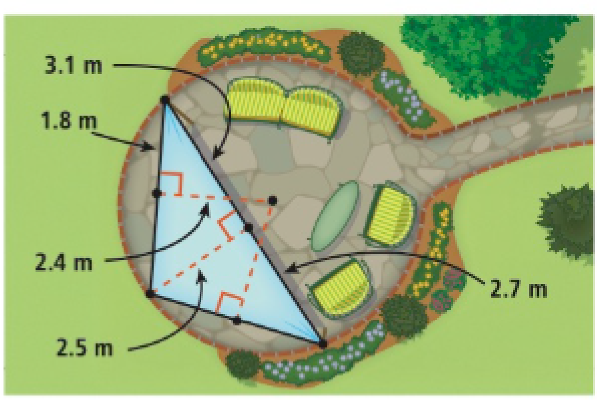
22. QK

PRACTICE & PROBLEM SOLVING

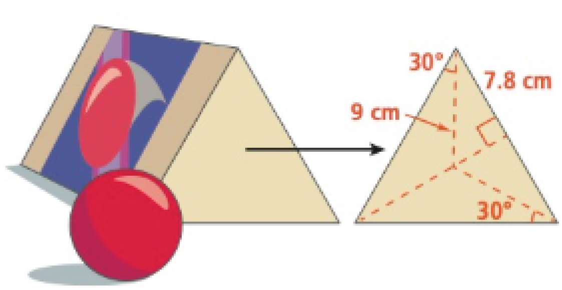
APPLY

23. Represent and Connect A maintenance crew wants to build a shed at a location that is the same distance from each path. Where should the shed be located? Justify your answer with a diagram.

24. Apply Math Models What is the area of the patio **not** covered by the sunshade? Round to the nearest tenth, and explain how you found your answer.



25. Analyze and Persevere A ball manufacturer wants to stack three balls, each with an 8-centimeter diameter, in a box that is an equilateral triangular prism. The diagram shows the dimensions of the bases. Will the balls fit in the box? Explain how you know.



ASSESSMENT PRACTICE

26. In $\triangle ABC$, $\overline{AB}$ has midpoint M, and ℓ is the perpendicular bisector of $\overline{AB}$ and the angle bisector of $\angle ACB$. Select all the statements that must be true. GR.1.1

☐ **A.** The radius of the inscribed circle of $\triangle ABC$ is AM.

☐ **B.** $AC = CB$

☐ **C.** Both the circumcenter and incenter of $\triangle ABC$ are on ℓ.

☐ **D.** The circumcenter of $\triangle ABC$ is inside the triangle.

☐ **E.** The radius of the circumscribed circle of $\triangle ABC$ is AM.

27. SAT/ACT Circle O intersects $\overline{AB}$ only at F, $\overline{BC}$ only at G, and $\overline{AC}$ only at H. Which equation is true?

Ⓐ $AH = AC$

Ⓑ $m\angle OFB = 90$

Ⓒ $OB = OC$

Ⓓ $OF = OC$

Ⓔ $\angle BAO \cong \angle ABO$

28. Performance Task Edison High School is designing a new triangular pennant. The school mascot will be inside a circle, and the circle must touch each side of the pennant. The circle should fill as much of the pennant as possible.

Part A Using a straightedge and compass, draw at least 4 different types of triangles for the pennant. Construct an inscribed circle in each triangle.

Part B Make a table about your pennants. Include side lengths, type of triangle, circle radius and area, triangle area, and ratio of circle area to triangle area.

Part C What type of triangle do you recommend that they use? Justify your answer.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.1.3–Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

MA.K12.MTR.7.1

Making It Fair

In rural areas, county planners often work with local officials from a number of small towns to establish a regional medical center to serve all of the nearby communities.

County planners might also establish regional medical evacuation centers to transport patients with serious trauma to larger medical centers. The locations of these regional centers are carefully planned. Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

5-3 Medians and Altitudes

I CAN… find the points of concurrency for the medians of a triangle and the altitudes of a triangle.

VOCABULARY

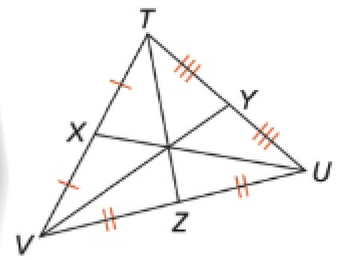
- altitude
- centroid
- median
- orthocenter

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles. **Also GR.3.3**
MA.K12.MTR.1.1, MTR.4.1, MTR.2.1

CRITIQUE & EXPLAIN

Aisha wrote the following explanation of the relationships in the triangle.

I can see that $\angle TVY \cong \angle YVU$, $\angle VUX \cong \angle XUT$, and $\angle UTZ \cong \angle ZTV$ because $\overline{TZ}$, $\overline{VY}$, and $\overline{UX}$ bisect the sides opposite each vertex. By the Concurrency of Angle Bisectors Theorem, $\overline{VY}$, $\overline{UX}$, and $\overline{TZ}$ are concurrent.



A. Why is Aisha's explanation not correct?

B. Communicate and Justify What can you do in the future to avoid Aisha's mistake?

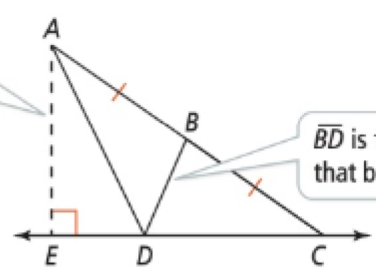
ESSENTIAL QUESTION

What are the properties of the medians in a triangle? What are the properties of the altitudes in a triangle?

EXAMPLE 1 Identify Special Segments in Triangles

What are the *altitude* and *median* that are shown in $\triangle ADC$?

An **altitude** is a perpendicular segment from a vertex of a triangle to the line containing the side opposite the vertex. A **median** of a triangle is a segment that has endpoints at a vertex and the midpoint of the side opposite the vertex.



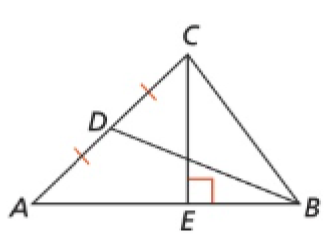
STUDY TIP
Recall that if congruence marks are not on a diagram, you cannot assume congruence.

Try It!

1. Use the figure shown.

a. What are the altitude and median that are shown in $\triangle ABC$?

b. Copy the triangle and draw the other altitudes and medians of the triangle.

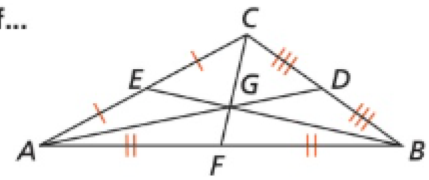


THEOREM 5-7 Concurrency of Medians

The medians of a triangle are concurrent at a point that is two-thirds the distance from each vertex to the midpoint of the opposite side.

PROOF: SEE LESSON 9-2.

If...

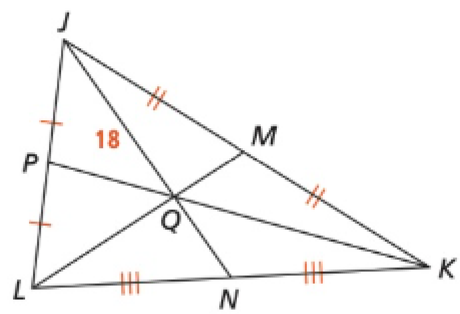


Then...

$AG = \frac{2}{3}AD \quad BG = \frac{2}{3}BE \quad CG = \frac{2}{3}CF$

EXAMPLE 2 Find the Length of a Median

What is the length of $\overline{JN}$ in the figure?



The medians of $\triangle JKL$ are $\overline{JN}$, $\overline{KP}$, and $\overline{LM}$. Point Q is the point of concurrency of the medians. The point of concurrency of the medians of a triangle is called the **centroid**.

COMMON ERROR
Be careful not to confuse which part is $\frac{1}{3}$ of the length of the median and which part is $\frac{2}{3}$ of the length.

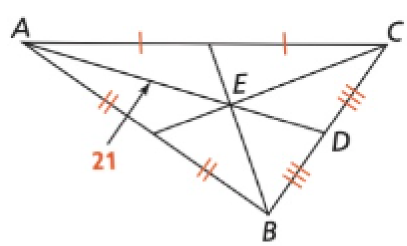
$\frac{2}{3}JN = JQ$ — Use the Concurrency of Medians Theorem.

$\frac{2}{3}JN = 18$

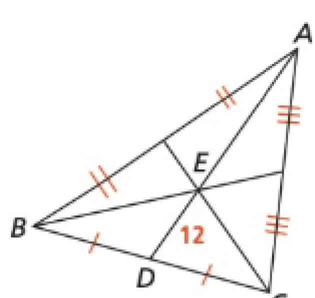
$JN = 27$

Try It! **2.** Find AD for each triangle.

a.



b.

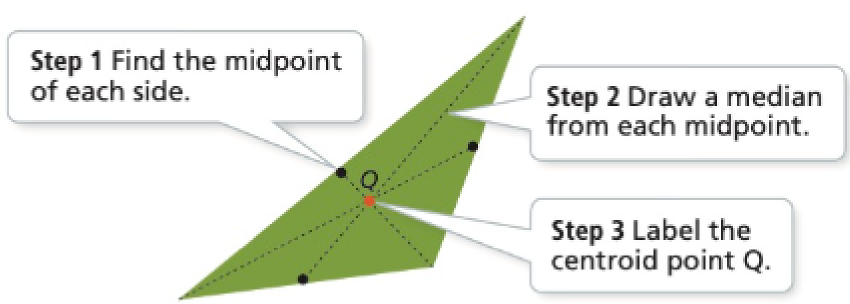


EXAMPLE 3 Locate the Centroid

An artist wants to balance a triangular piece of wood at a single point so that the triangle is parallel to the ground. Where should he balance the triangle?

A triangle is balanced at its *center of gravity,* which is at the centroid of the triangle.

ANALYZE AND PERSEVERE
Think about what tools you can use to find the midpoint. What tool would you use?

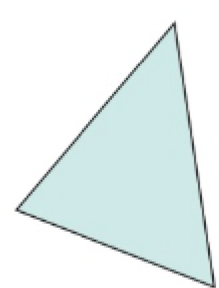


The artist should balance the triangle at point *Q*.

Try It! **3.** Copy the triangle shown.

a. Use the medians of the triangle to locate its centroid.

b. Use a ruler to verify the centroid is two-thirds the distance from each vertex to the midpoint of the opposite side.

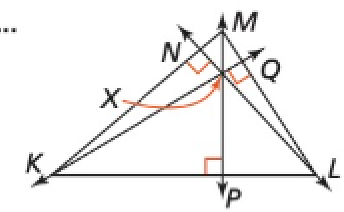


THEOREM 5-8 Concurrency of Altitudes

The lines that contain the altitudes of a triangle are concurrent.

PROOF: SEE LESSON 9-2.

If...



Then... $\overleftrightarrow{KQ}$, $\overleftrightarrow{LN}$, and $\overleftrightarrow{MP}$ are concurrent at X.

CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Locate the Orthocenter

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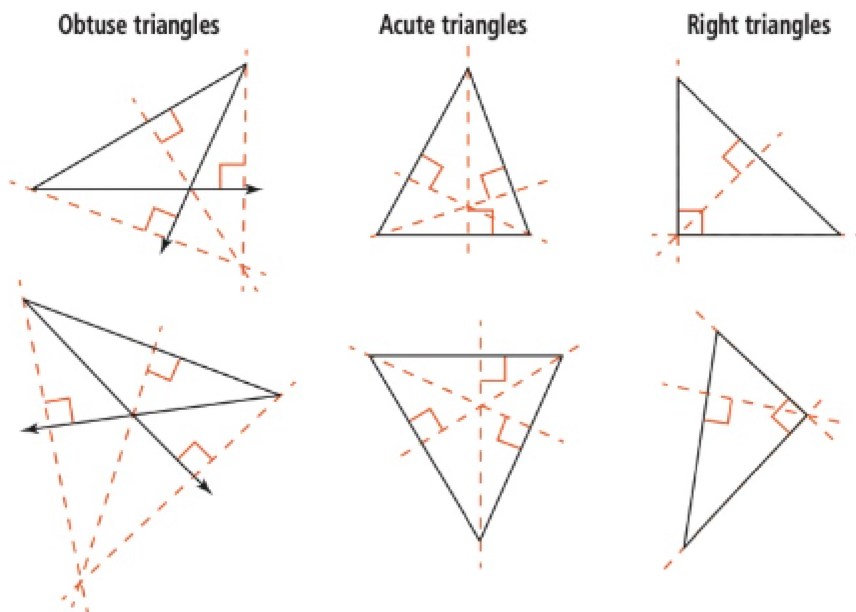
The orthocenter is the point of concurrency of the lines containing the altitudes of a triangle. How does the type of triangle (obtuse, acute, right) relate to the location of its orthocenter?

Draw at least two of each type of triangle. Describe any relationship you notice between the type of triangle and the location of its orthocenter.



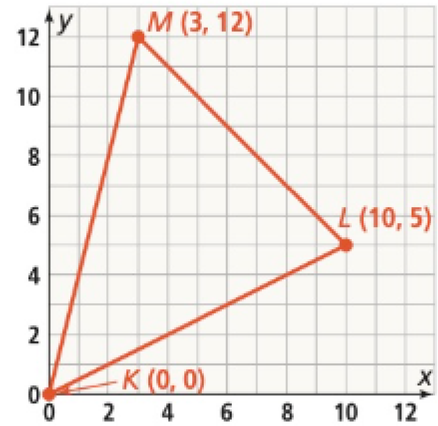
The orthocenter is outside an obtuse triangle and inside an acute triangle. For a right triangle, the orthocenter is at the vertex of the right angle.

STUDY TIP
Recall that when you make a conjecture by observing a few examples, you are not actually proving the conjecture.

Try It! **4.** What is the relationship between an isosceles triangle and the location of its orthocenter? Explain your answer.

EXAMPLE 5 Find the Orthocenter of a Triangle

Orthocenters can be found using constructions or coordinate geometry. Where is the orthocenter of $\triangle KLM$?



Since the orthocenter is the point of concurrency of the altitudes, find the equations for two altitudes, and solve for the point of intersection.

Step 1 Find the slopes of two sides of the triangle.

$$\text{slope of } \overline{KL} = \frac{5-0}{10-0} = \frac{1}{2}$$

$$\text{slope of } \overline{LM} = \frac{12-5}{3-10} = -1$$

Step 2 Use the point slope form, $y - y_1 = m(x - x_1)$, to write the equations of the altitudes perpendicular to $\overline{KL}$ and $\overline{LM}$.

STUDY TIP
Remember that since an altitude is perpendicular to a side of the triangle, you must find the reciprocal and reverse the sign of the slope of a side to find the slope of an altitude.

Equation of the altitude perpendicular to $\overline{KL}$:

$$y - 12 = -2(x - 3)$$
$$y = -2x + 6 + 12$$
$$y = -2x + 18$$

Point M is the vertex opposite $\overline{KL}$, and the slope of a line perpendicular to $\overline{KL}$ is -2.

Equation of the altitude perpendicular to $\overline{LM}$:

$$y - 0 = 1(x - 0)$$
$$y = x$$

Point K is the vertex opposite $\overline{LM}$, and the slope of a line perpendicular to $\overline{LM}$ is 1.

Step 3 Solve the system of equations to determine the coordinates of the point of intersection.

$$y = -2x + 18$$
$$y = -2(y) + 18$$
$$y + 2y = 18$$
$$3y = 18$$
$$y = 6$$

Since $y = x$, substitute y in the equation $y = -2x + 18$. Then solve for y.

Since $y = x$, $x = 6$.

Since all three altitudes intersect at the orthocenter, the intersection of two altitudes is sufficient to determine the orthocenter. The orthocenter of $\triangle KLM$ is (6, 6).

Try It! **5.** Find the orthocenter of a triangle with vertices at each of the following sets of coordinates.

a. (0, 0), (10, 4), (8, 9) **b.** (0, 0), (6, 3), (8, 9)

CONCEPT SUMMARY Medians and Altitudes

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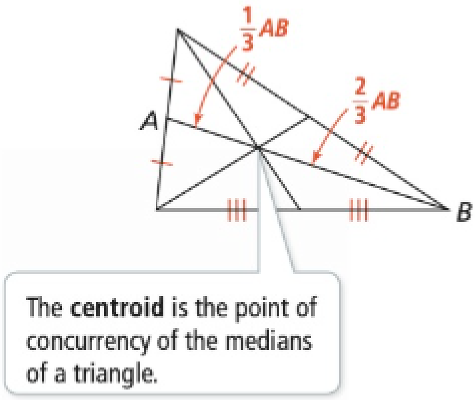
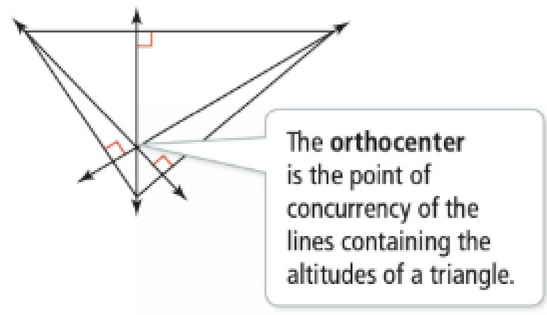
MEDIANS



ALTITUDES

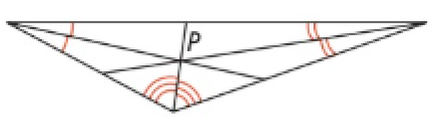


Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are the properties of the medians in a triangle? What are the properties of the altitudes in a triangle?

2. **Vocabulary** The prefix *ortho-* means "upright" or "right." How can this meaning help you remember which segments of a triangle have a point of concurrency at the orthocenter?

3. **Error Analysis** A student labeled P as the centroid of the triangle. What error did the student make? Explain.



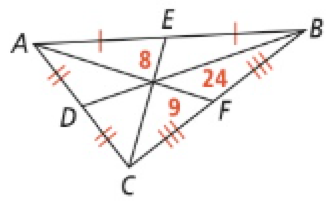
4. **Communicate and Justify** Why is an orthocenter sometimes outside a triangle but a centroid is always inside?

5. **Use Patterns and Structure** Consider the three types of triangles: acute, obtuse, and right. What is the relationship between the type of triangle and the location of the orthocenter? Does the type of triangle tell you anything about the location of the centroid?

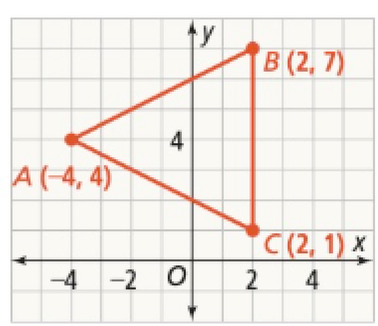
6. **Generalize** For any right triangle, where is the orthocenter located?

Do You KNOW HOW?

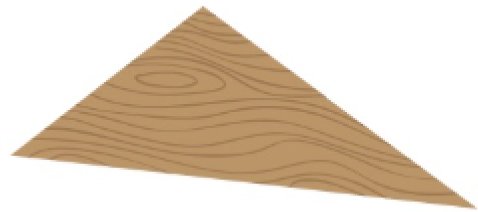
7. Find the length of each of the medians of the triangle.



8. Where is the orthocenter of $\triangle ABC$?



9. A crane operator needs to lift a large triangular piece of plywood. Copy the triangle and use its medians to locate the centroid.



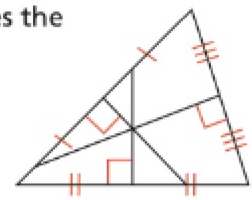
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. Choose Efficient Methods Describe the process for finding the orthocenter of a triangle that is on a coordinate plane.

11. Use Patterns and Structure Given the midpoints of a triangle, which two points of concurrency can you locate? Which point of concurrency can you locate if you only know the angle bisectors? Which two points of concurrency can you locate by only drawing perpendicular segments?

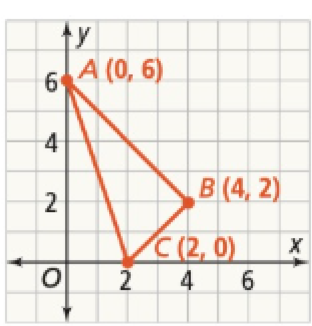
12. Error Analysis A student uses the following explanation to identify the triangle's point of concurrency. Explain the student's error.



> A perpendicular segment bisects each side of the triangle. According to the Concurrency of Altitudes Theorem, the segments are concurrent. The point of concurrency is the orthocenter. ✗

13. Communicate and Justify Draw several different types of triangles and compare the locations of the centroid and the circumcenter of each triangle. What conjecture can you make about the type of triangle that has a common centroid and circumcenter? Explain.

14. Mathematical Connections Where is the centroid of $\triangle ABC$?

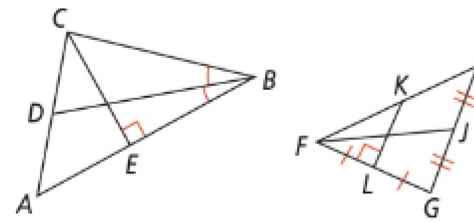


- Locate the midpoints of any two sides.
- Find the equations of two medians using the vertex and the opposite midpoint.
- Solve the system of the two equations to find the coordinates of the centroid.

How can you verify that the coordinates you found are correct?

PRACTICE

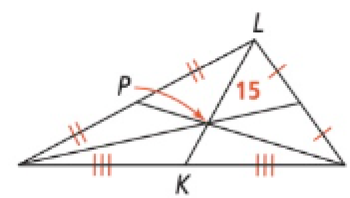
15. Identify whether each segment is an altitude, an angle bisector, a median, or a perpendicular bisector. SEE EXAMPLE 1



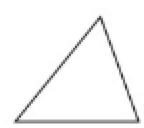
a. $\overline{BD}$ **b.** $\overline{FJ}$

c. $\overline{CE}$ **d.** $\overline{KL}$

16. What is the value of $\overline{KL}$? SEE EXAMPLE 2

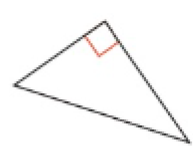


17. Copy the triangle and use its medians to locate the centroid. SEE EXAMPLE 3

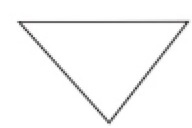


18. State whether the orthocenter of each triangle is inside the triangle, outside the triangle, or on the triangle. Explain your reasoning. SEE EXAMPLE 4

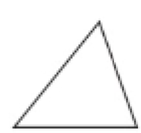
a.



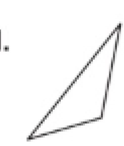
b.



c.



d.



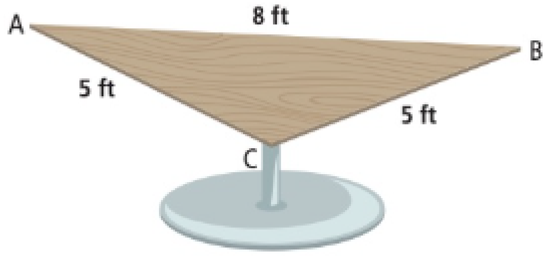
19. Find the coordinates of the orthocenter of a triangle with vertices at each set of points on a coordinate plane. SEE EXAMPLE 5

a. (0, 0), (8, 4), (4, 22)

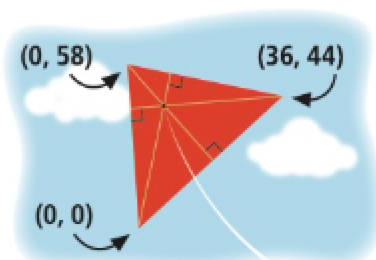
b. (3, 1), (10, 8), (5, 13)

APPLY

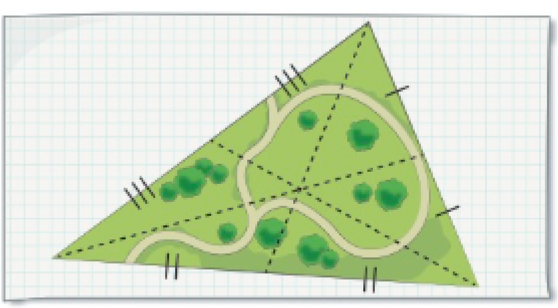
20. **Represent and Connect** A large triangular-shaped table is supported by a single pole at the center of gravity. How far is vertex C from the center of gravity?



21. **Analyze and Persevere** To support a triangular kite, Hana attaches thin strips of wood from each vertex perpendicular to the opposite edge. She then attaches the kite's string at the point of concurrency. To calculate the point of concurrency, she determines the coordinates of each vertex on a coordinate plane. What are the coordinates where the wood strips cross? Round your answer to the nearest hundredth.

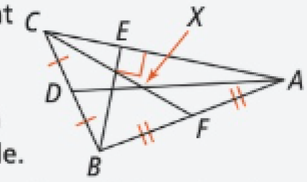


22. **Higher Order Thinking** A designer wants to place a fountain at the intersection of the shortest paths from each side to the opposite vertex. What mistake is made on her model? At what point of concurrency should the fountain be located?



ASSESSMENT PRACTICE

23. Select the true statement about $\triangle ABC$.
GR.1.3



Ⓐ The segment $\overline{AD}$ is an altitude of the triangle.
Ⓑ The segment $\overline{BE}$ is a median of the triangle.
Ⓒ The point X is the orthocenter of the triangle.
Ⓓ The point X is the centroid of the triangle.

24. **SAT/ACT** A triangle with vertices at (3, 4) and (9, 17) has a centroid at (8, 16). What are the coordinates of the third vertex?

Ⓐ (10, 4)
Ⓑ (10, 7)
Ⓒ (12, 14)
Ⓓ (12, 27)

25. **Performance Task** Steve is designing a mobile with triangular pieces of wood, where each piece attaches to a wire at the center of gravity and hangs parallel to the ground. The side lengths of the triangles will be between 4 cm and 8 cm.

Part A Describe how Steve can find the center of gravity for any triangular piece. Then model this process by finding the center of gravity of a triangle with side lengths 5 cm, 5 cm, and 6 cm.

Part B Is it possible for a triangle attached at the orthocenter to hang so that it is parallel to the ground? If it is possible, describe the triangle. What are possible side lengths for such a triangle? If it is not possible, explain why not.

5-4 Inequalities in One Triangle

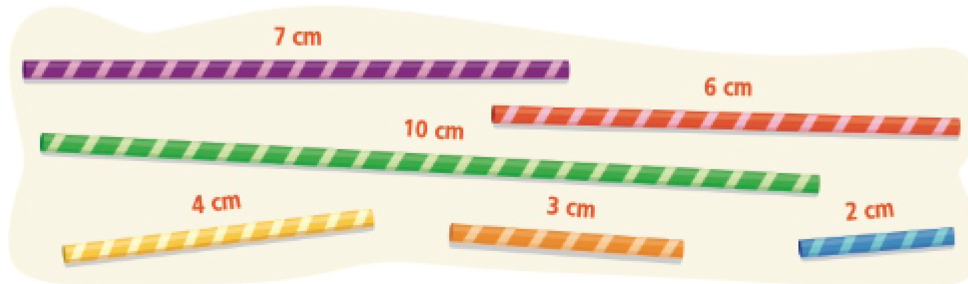
I CAN… use theorems to compare the sides and angles of a triangle.

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles. **Also LT.4.8**

MA.K12.MTR.5.1, MTR.1.1, MTR.7.1

EXPLORE & REASON

Cut several drinking straws to the sizes shown.



A. Take your two shortest straws and your longest straw. Can they form a triangle? Explain.

B. Try different combinations of three straws to form triangles. Which side length combinations work? Which combinations do not work?

C. **Use Patterns and Structure** What do you notice about the relationship between the combined lengths of the two shorter sides and the length of the longest side?

ESSENTIAL QUESTION

What are some relationships between the sides and angles of any triangle?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Investigate Side and Angle Relationships

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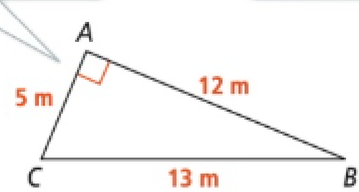
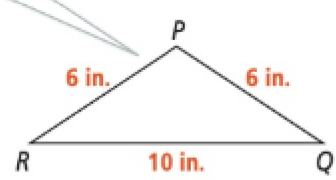
Draw a right triangle and a non-right triangle. How is the largest angle measure of each triangle related to the side lengths?

STUDY TIP
Recall that the non-right angles in a right triangle are acute. This means the right angle is the largest angle.

In the right triangle, $\angle A$ is the largest angle and $\overline{BC}$ is the longest side.



Angle P appears to be obtuse, while $\angle Q$ and $\angle R$ appear to be acute. The largest angle $\angle P$ is across from longest side $\overline{QR}$.



The largest angle appears to be opposite the longest side.

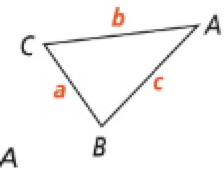
Try It! 1. Which angle measure appears to be the smallest in $\triangle MNP$? How is it related to the side lengths?

THEOREM 5-9

If two sides of a triangle are not congruent, then the larger angle lies opposite the longer side.

PROOF: SEE EXERCISE 13.

If... $b > a$



Then... $m\angle B > m\angle A$

EXAMPLE 2 Use Theorem 5-9

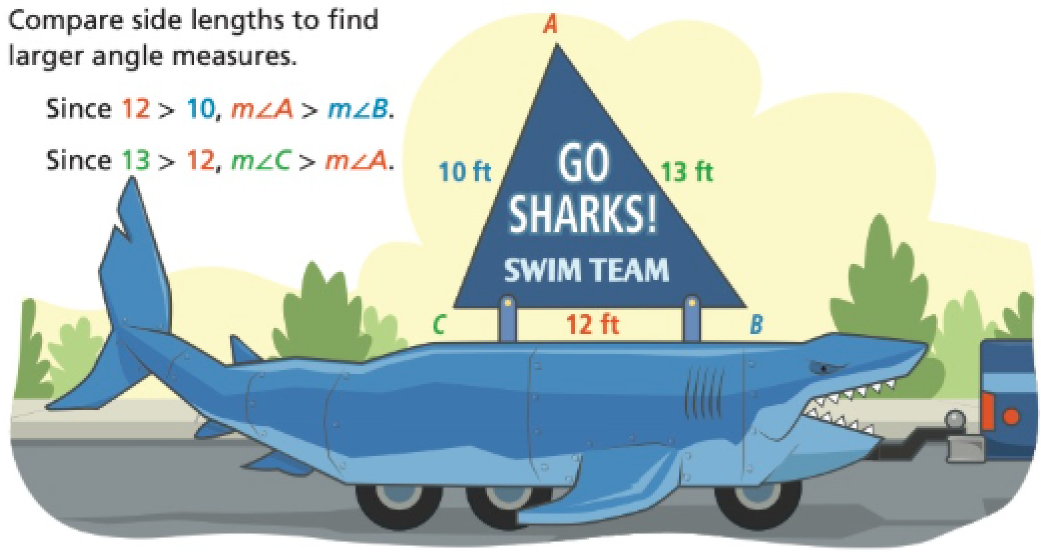
To support a triangular piece of a float, a brace is placed at the largest angle and a guide wire is placed at the smallest angle. Which angle is the largest? Which angle is the smallest?

GENERALIZE
Think about the relationships between pairs of numbers. How can you use the relationships between pairs of numbers to order all the numbers?

Compare side lengths to find larger angle measures.

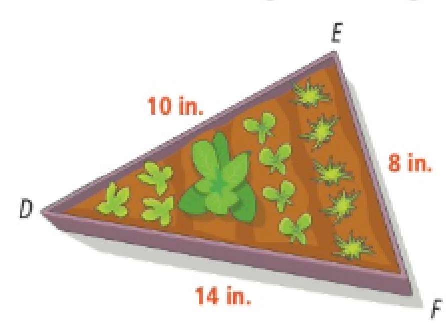
Since $12 > 10$, $m\angle A > m\angle B$.

Since $13 > 12$, $m\angle C > m\angle A$.



Combining the inequalities of the angle measures, $m\angle C > m\angle A > m\angle B$. Thus, the largest angle is $\angle C$ and the smallest angle is $\angle B$.

Try It! **2.** Lucas sketched a diagram for a garden box.



a. Which angle is the largest?

b. Which angle is the smallest?

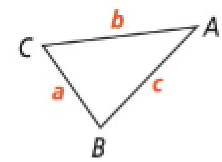
THEOREM 5-10 Converse of Theorem 5-9

If two angles of a triangle are not congruent, then the longer side lies opposite the larger angle.

PROOF: SEE EXAMPLE 3.

If... $m\angle B > m\angle A$

Then... $b > a$

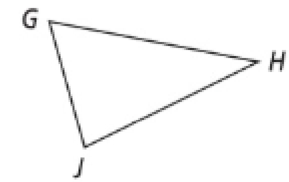


PROOF

EXAMPLE 3 Prove Theorem 5-10

STUDY TIP
Recall that in an indirect proof, you assume the opposite of what you are trying to prove and then show that this leads to a contradiction.

Use indirect reasoning to prove Theorem 5–10; assume that $GH \leq HJ$. This means that $GH = HJ$ or $GH < HJ$.



Given: $m\angle J > m\angle G$

Prove: $GH > HJ$

First show that assuming $GH = HJ$ leads to a contradiction of the given condition that $m\angle J > m\angle G$.

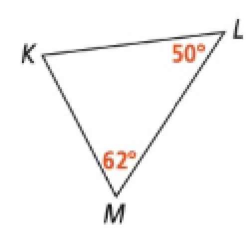
$GH = HJ$ (Assumption) → $\overline{GH} \cong \overline{HJ}$ (Def. of congruent) → $\triangle GHJ$ is isosceles. (Def. of isosceles) → $\angle J \cong \angle G$ (Isosceles Triangle Thm.)

By the definition of congruence $m\angle J = m\angle G$, which contradicts $m\angle J > m\angle G$. So the assumption is false.

Try It! 3. To complete the proof of Theorem 5-10, show that assuming $GH < HJ$ leads to a contradiction of the given condition that $m\angle J > m\angle G$.

EXAMPLE 4 Use Theorem 5-10

Which side of $\triangle KLM$ is the longest?



By Theorem 5-10, the longest side of the triangle is across from the largest angle. Find the unknown angle measure.

STUDY TIP
When needed information is missing from a diagram, you may need to apply properties and theorems to find the needed information.

$$m\angle K + m\angle L + m\angle M = 180$$
$$m\angle K + 50 + 62 = 180$$
$$m\angle K = 68$$

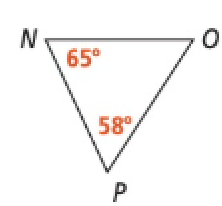
Apply the Triangle Angle-Sum Theorem.

The largest angle in the triangle is $\angle K$, so the longest side is the side opposite $\angle K$. The longest side is $\overline{LM}$.

Try It! 4. Identify the sides of $\triangle NOP$.

a. Which side is the longest?

b. Which side is the shortest?

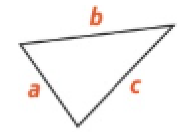


THEOREM 5-11 Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

PROOF: SEE EXERCISE 14.

If...



Then... $a + b > c$

$a + c > b$

$b + c > a$

EXAMPLE 5 Use the Triangle Inequality Theorem

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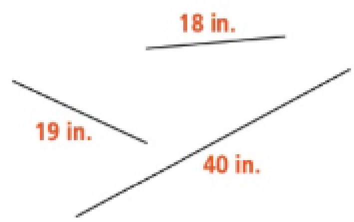
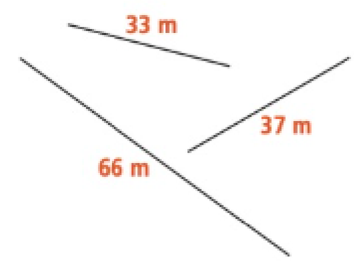
A. Which of the following sets of segments could be the sides of a triangle?



Set 1

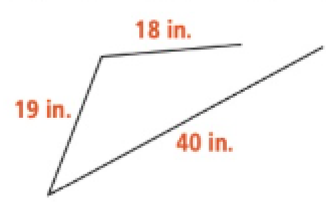


Set 2

Determine if the sum of the two shorter side lengths is longer than the longest side length.

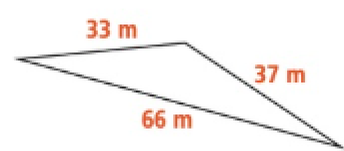
$18 + 19 = 37$

Since $37 < 40$, the segments in Set 1 cannot form a triangle.



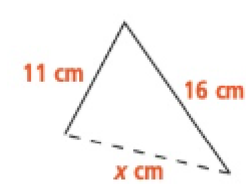
$33 + 37 = 70$

Since $70 > 66$, the segments in Set 2 can form a triangle.



B. A triangle has sides that measure 11 cm and 16 cm. What are the possible lengths of the third side?

Apply the Triangle Inequality Theorem.



$x + 11 > 16$ $\quad$ $x + 16 > 11$ $\quad$ $11 + 16 > x$

- If $x + 11 > 16$, then $x > 5$. So, x is greater than 5.
- The inequality $x + 16 > 11$ is true for all positive values of x, so this inequality only tells you that $x > 0$.
- If $11 + 16 > x$, then $27 > x$, so x is less than 27.

Therefore, $5 < x < 27$.

The third side of the triangle could be between 5 cm and 27 cm long.

COMMON ERROR

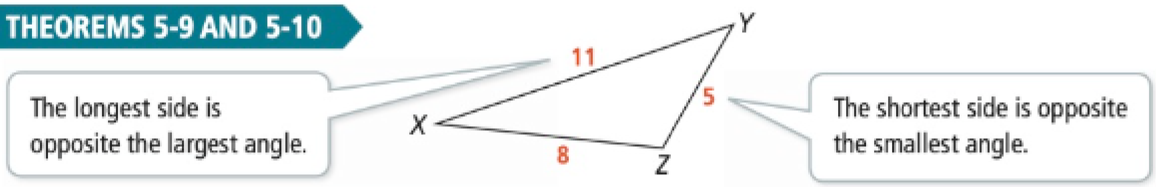
You may compare any two sides to a third side, but you must compare the shorter two sides to the longest side to determine whether a triangle is possible.

Try It! **5. a.** Could a triangle have side lengths 16 m, 39 m, and 28 m?

b. A triangle has side lengths that are 30 in. and 50 in. What are the possible lengths of the third side?

CONCEPT SUMMARY Inequalities in One Triangle

THEOREMS 5-9 AND 5-10



$ZY < XZ < XY$
$m\angle X < m\angle Y < m\angle Z$

THEOREM 5-11 Triangle Inequality Theorem

The sum of the lengths of any two sides is greater than the length of the third side.

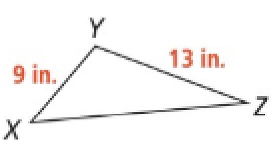
$5 + 8 > 11$
$5 + 11 > 8$
$8 + 11 > 5$

Do You UNDERSTAND?

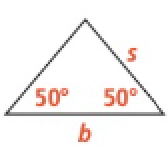
1. **ESSENTIAL QUESTION** What are some relationships between the sides and angles of any triangle?

2. **Use Patterns and Structure** If a triangle has three different side lengths, what does that tell you about the measures of its angles?

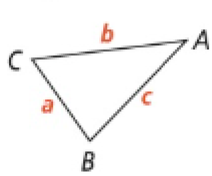
3. **Error Analysis** Richard says that $\angle X$ must be the largest angle in $\triangle XYZ$. Explain his error.



4. **Use Patterns and Structure** An isosceles triangle has base angles that each measure 50. How could you determine whether *b* or *s* is greater?

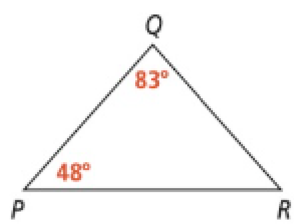


5. **Generalize** In $\triangle ABC$, $a < c < b$. List the angles in order from smallest to largest.



Do You KNOW HOW?

Identify the sides of $\triangle PQR$.



6. Which side is the longest?

7. Which side is the shortest?

Determine whether each set of lengths could form a triangle.

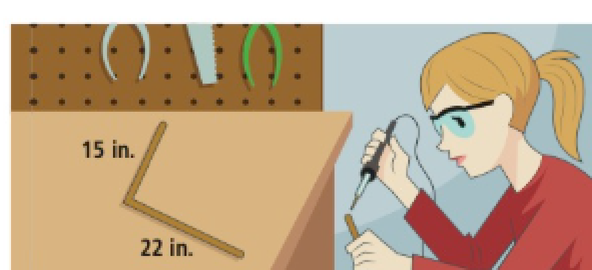
8. 5, 2, and 3

9. 55, 76, and 112

10. 102, 95, and 157

11. 17, 17, and 35

12. Kelsey is welding 3 metal rods to make a triangle. If the lengths of two of the rods are 15 in. and 22 in., what are the possible lengths of a third rod?



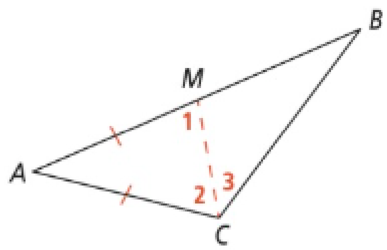
PRACTICE & PROBLEM SOLVING

UNDERSTAND

13. Communicate and Justify Fill in the missing reasons in the proof of Theorem 5-9. (Hint: The Comparison Property of Inequality states that if $a = b + c$ and $c > 0$, then $a > b$.)

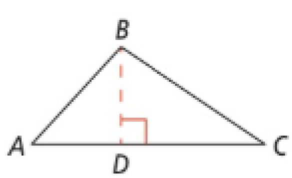
Given: $AB > AC$, $\overline{AC} \cong \overline{AM}$

Prove: $m\angle ACB > m\angle B$



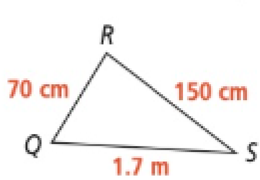
Statements	Reasons
1) $\overline{AC} \cong \overline{AM}$	1) Given
2) $m\angle 1 = m\angle 2$	2) Isosc. Triangle Thm.
3) $m\angle ACB = m\angle 2 + m\angle 3$	3)
4) $m\angle ACB > m\angle 2$	4)
5) $m\angle ACB > m\angle 1$	5)
6) $m\angle 1 = m\angle B + m\angle 3$	6) Ext. Angles Thm.
7) $m\angle 1 > m\angle B$	7)
8) $m\angle ACB > m\angle B$	8)

14. Communicate and Justify Write a paragraph proof for Theorem 5-11. Use the figure shown and prove that $AB + CB > AC$.



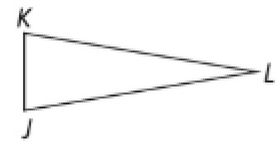
15. Error Analysis A student said that a triangle with side lengths of 3 ft and 4 ft could have a third side with a length of 7 ft. Explain why the student is incorrect. What is a correct statement about the third side of the triangle?

16. Error Analysis Tia says that $\angle Q$ must be the largest angle in $\triangle QRS$ because $150 > 70 > 1.7$. Explain Tia's error.

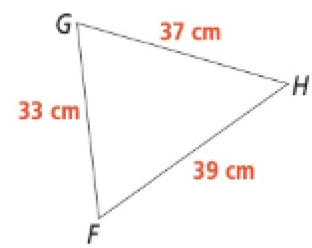


PRACTICE

17. Which angle measure appears to be the smallest in $\triangle JKL$? What can you conclude about the side opposite that angle? SEE EXAMPLE 1.



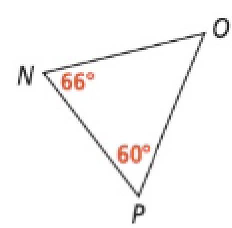
Identify the angles of $\triangle FGH$. SEE EXAMPLE 2.



18. Which angle is the smallest?

19. Which angle is the largest?

Identify the sides of $\triangle NOP$. SEE EXAMPLES 3 AND 4.



20. Which side is the longest?

21. Which side is the shortest?

Determine whether the side lengths could form a triangle. SEE EXAMPLE 5.

22. 13, 15, 9

23. 8, 15, 7

24. 35, 20, 11

25. 65, 32, 40

Given two sides of a triangle, determine the range of possible lengths of the third side. SEE EXAMPLE 5.

26. 10 in. and 12 in.

27. 5 ft and 10 ft

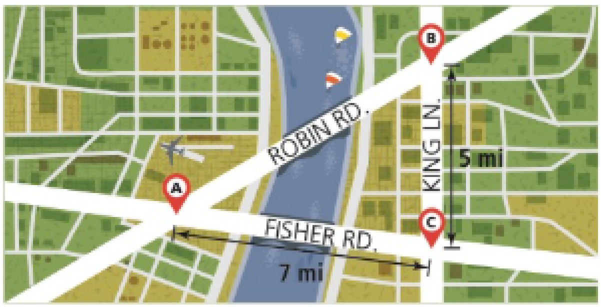
28. 200 m and 300 m

29. 90 km and 150 km

PRACTICE & PROBLEM SOLVING

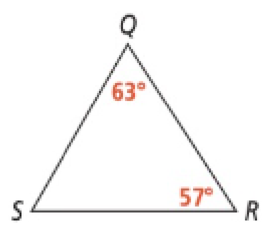
APPLY

30. Analyze and Persevere It took Ines 2 hours to bicycle the perimeter shown at a constant speed of 10 miles per hour. Which two roads form the largest angle?

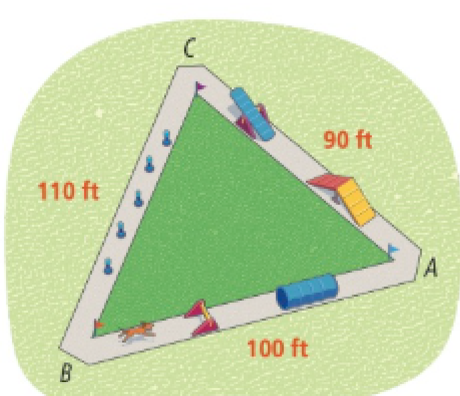


31. Apply Math Models A jewelry designer plans to make a triangular pendant out of gold wire. The wire costs $31.65 per centimeter. What is the possible range of costs for the wire?

32. Use Patterns and Structure A stage manager must use tape to outline a triangular platform on the set. Order the sides of the platform from longest to shortest.



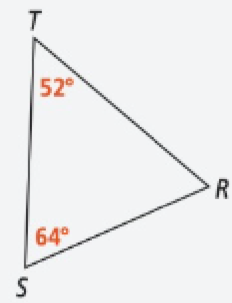
33. Analyze and Persevere A dog running an agility course has difficulty making turns. The sharper the angle, the more difficult the turn. Which corner is most difficult for her to turn?



ASSESSMENT PRACTICE

34. The lengths of two sides of a triangle are 13 and 20. What is the range of values for the length x of the third side? GR.1.3

35. SAT/ACT Look at $\triangle RST$.



Which statement is false?

Ⓐ $TS = TR$

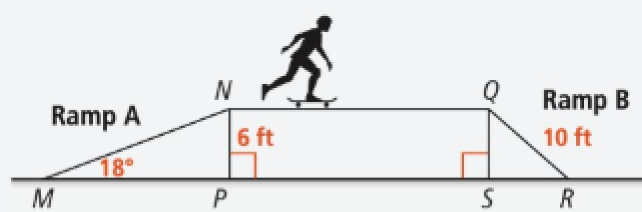
Ⓑ $m\angle STR < m\angle TRS$

Ⓒ $TR > SR$

Ⓓ $TS < SR$

Ⓔ $TS + TR > SR$

36. Performance Task Teo designed a skateboard ramp.



Part A List the sides of ramp A in order from shortest to longest.

Part B List the angles of ramp B from smallest to largest, and explain how you know.

Part C Ramp B cannot be steeper than 45°. Is it possible to build ramp B so that $\overline{SR}$ is shorter than 6 ft? Explain.

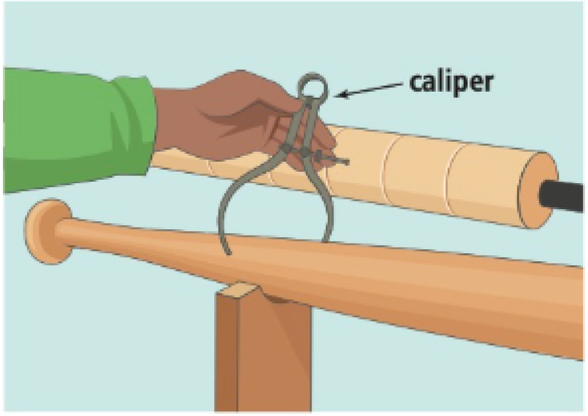
5-5 Inequalities in Two Triangles

I CAN… compare a pair of sides of two triangles when the remaining pairs of sides are congruent.

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles. **Also LT.4.8**
MA.K12.MTR.4.1, MTR.5.1, MTR.7.1

EXPLORE & REASON

A woodworker uses a caliper to measure the widths of a bat to help him determine the widths for a new bat. The woodworker places the open tips of the caliper on the bat. The distance between the tips is a width of the bat.



A. Suppose a caliper opens to an angle of 25° for one width of a bat and opens to an angle of 35° for another. What can you conclude about the widths of the bat?

B. **Use Patterns and Structure** Next, suppose you use a caliper to measure the width of a narrow part of a bat and a wider part of the bat. What can you predict about the angle to which the caliper opens each time?

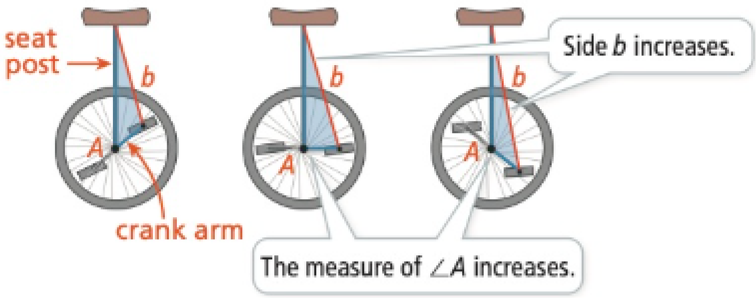
ESSENTIAL QUESTION When two triangles have two pairs of congruent sides, how are the third pair of sides and the pair of angles opposite the third pair of sides related?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Investigate Side Lengths in Triangles

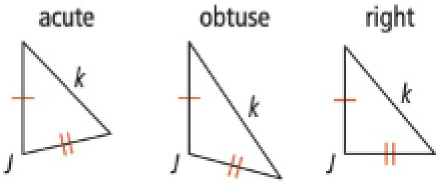
As a rider pedals a unicycle, how do the measure of ∠A and length b change? What does this suggest about the change in the triangle?

USE PATTERNS AND STRUCTURE
Consider how multiple diagrams are used to show the relationship between moving parts. What changes and what remains the same between diagrams?



If two sides of a triangle stay the same, but the measure of the angle between them increases, the length of the third side also increases.

Try It! **1.** Compare the measure of ∠J and side length k for the triangles.

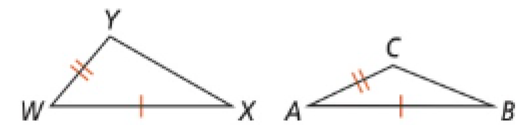


THEOREM 5-12 Hinge Theorem

If two sides of one triangle are congruent to two sides of another triangle, and the included angles are not congruent, then the longer third side is opposite the larger included angle.

PROOF: SEE EXERCISE 9.

If... $m\angle YWX > m\angle CAB$

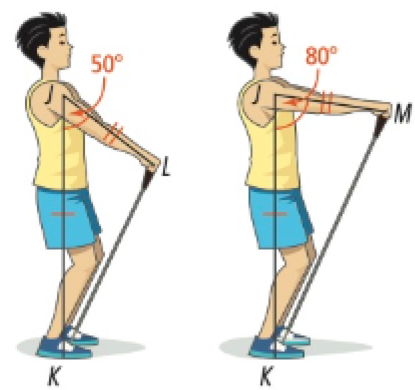


Then... $XY > BC$

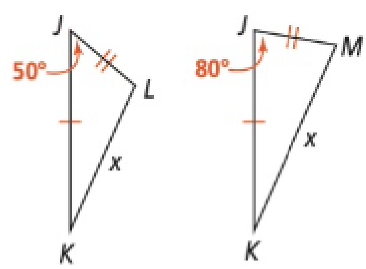
APPLICATION

EXAMPLE 2 Apply the Hinge Theorem

The tension in the exercise band varies proportionally with the stretch distance. The tension *T* is described by the function $T(x) = kx$, where *k* is a constant that depends on the elasticity of the band and *x* is the stretch distance. Which position shown in the figures has a greater tension in the band?



Formulate Model each figure with a triangle.



Compute $T(LK)$ is the tension when the angle is 50°, and $T(MK)$ is the tension when the angle is 80°. Since $m\angle MJK > m\angle LJK$, apply the Hinge Theorem.

$$MK > LK$$

$$\frac{T(MK)}{k} > \frac{T(LK)}{k}$$

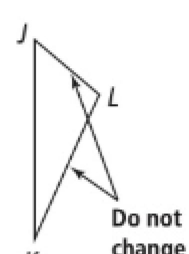
$$T(MK) > T(LK)$$

Since the tension T is equal to the product of k and the stretch distance, we can substitute $\frac{T}{k}$ for each distance.

Interpret A larger angle corresponds to a larger distance from the man's hands to his feet. The larger distance corresponds to a higher tension.

The tension is greater when the man pulls higher on the tension band.

Try It! **2.** The man keeps his arms extended and the length of the tension band the same. If he wants to make the measure of $\angle L$ smaller, how would $\overline{JK}$ change?

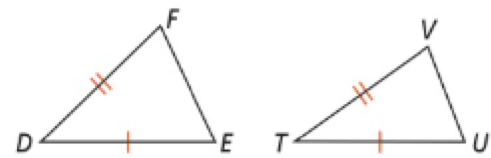


THEOREM 5-13 Converse of the Hinge Theorem

If two sides of one triangle are congruent to two sides of another triangle, and the third sides are not congruent, then the larger included angle is opposite the longer third side.

PROOF: SEE EXAMPLE 3.

If... $EF > UV$



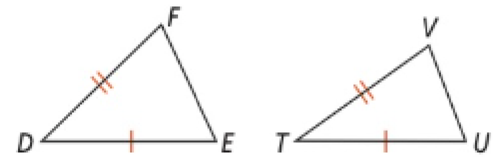
Then... $m\angle D > m\angle T$

EXAMPLE 3 Prove the Converse of the Hinge Theorem

Use indirect reasoning to prove the Converse of the Hinge Theorem.

Given: $\overline{DF} \cong \overline{TV}$; $\overline{DE} \cong \overline{TU}$; $EF > UV$

Prove: $m\angle FDE > m\angle VTU$



Assume that $m\angle FDE$ is not greater than $m\angle VTU$, that is, that $m\angle FDE = m\angle VTU$, or $m\angle FDE < m\angle VTU$.

Assuming that $m\angle FDE = m\angle VTU$, $\angle FDE \cong \angle VTU$. Applying SAS, $\triangle DEF \cong \triangle TUV$, so by CPCTC, $\overline{EF} \cong \overline{UV}$ and $EF = UV$. But, this contradicts $EF > UV$.

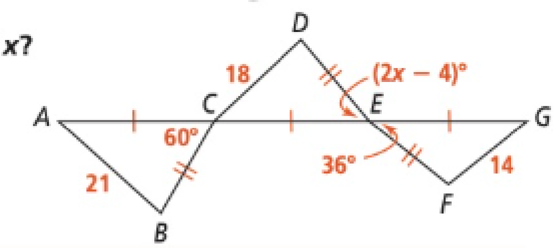
STUDY TIP

If you get stuck when writing a proof, make a list of things you know and what you want to prove.

Try It! **3.** To complete the proof of the Hinge Theorem, show that assuming $m\angle FDE < m\angle VTU$ leads to a contradiction of the given statement, $EF > UV$.

EXAMPLE 4 Apply the Converse of the Hinge Theorem

What are the possible values of *x*?



Since $FG < CD$ and $CD < AB$, apply the Converse of the Hinge Theorem.

$$m\angle FEG < m\angle CED < m\angle ACB$$
$$36 < 2x - 4 < 60$$
$$40 < 2x < 64$$
$$20 < x < 32$$

Use the Converse of the Hinge Theorem.

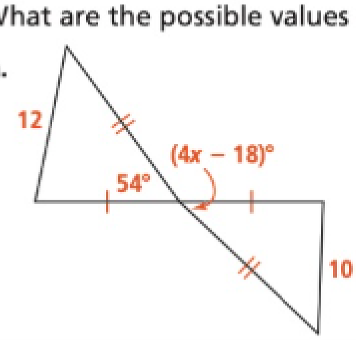
The possible values for *x* are between 20 and 32.

COMMON ERROR

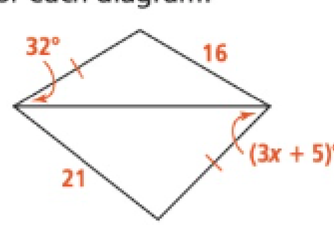
Be careful to use the correct inequality sign when comparing triangles. After you write the inequality, check a second time to be sure it indicates that the larger angle is opposite the longer side.

Try It! **4.** What are the possible values of *x* for each diagram?

a.



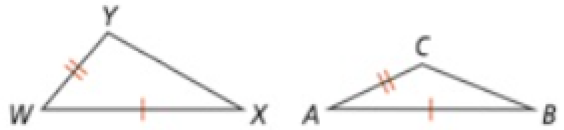
b.



CONCEPT SUMMARY Inequalities in Two Triangles

THEOREM 5-12 Hinge Theorem

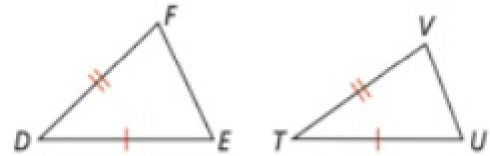
If... $\overline{WX} \cong \overline{AB}$, $\overline{WY} \cong \overline{AC}$, and $m\angle W > m\angle A$



Then... $XY > BC$

THEOREM 5-13 Converse of the Hinge Theorem

If... $\overline{DF} \cong \overline{TV}$, $\overline{DE} \cong \overline{TU}$, and $EF > UV$

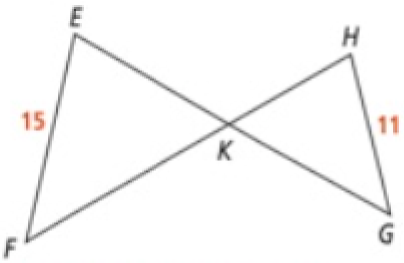


Then... $m\angle D > m\angle T$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** When two triangles have two pairs of congruent sides, how are the third pair of sides and the pair of angles opposite the third pair of sides related?

2. **Error Analysis** Venetta applies the Converse of the Hinge Theorem to conclude that $m\angle EKF > m\angle HKG$ for the triangles shown. Is Venetta correct? Explain your answer.

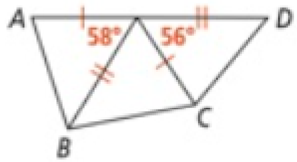


3. **Analyze and Persevere** Why must the angles described in the Hinge Theorem be between the congruent pairs of sides?

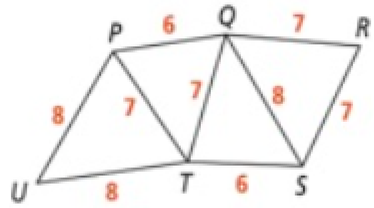
4. **Communicate and Justify** The Hinge Theorem is also known as the Side-Angle-Side Inequality Theorem or SAS Inequality Theorem. How are the requirements for applying the Hinge Theorem similar to the requirements for applying SAS? How are the requirements different?

Do You KNOW HOW?

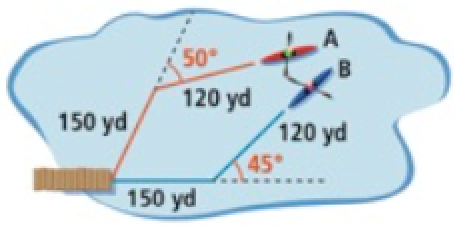
5. Order AB, BC, and CD from least to greatest.



6. Order the measures of $\angle PTU$, $\angle SQT$, and $\angle QSR$ from least to greatest.



7. Kayak A and kayak B leave a dock as shown. Which kayak is closer to the dock?



PRACTICE & PROBLEM SOLVING

UNDERSTAND

8. **Error Analysis** Tonya has the scissors shown.

Tonya writes the following description of how she will use the Hinge Theorem with the scissors.

> If you open the right pair of scissors to an angle of 30° and open the left pair of scissors to an angle of 45°, then by the Hinge Theorem, the distance between the blade tips of the left pair of scissors will be larger. ✗

What is the mistake in her use of the Hinge Theorem?

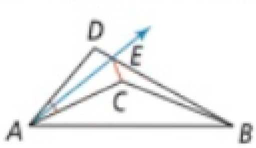
9. **Communicate and Justify** Write a paragraph proof of the Hinge Theorem.

Given: $\overline{WX} \cong \overline{AB}$, $\overline{WY} \cong \overline{AC}$, $m\angle W > m\angle A$

Prove: $XY > BC$

Use the following outline.

- Find a point D outside $\triangle ABC$ so $\overline{AD} \cong \overline{WY}$ and $\angle DAB \cong \angle YWX$.
- Show that $\triangle WXY \cong \triangle ABD$.
- Construct the angle bisector of $\angle CAD$. Let point E be the point where the angle bisector intersects $\overline{BD}$.

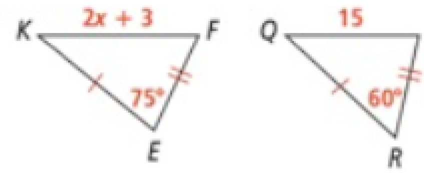


- Show that $\triangle ACE \cong \triangle ADE$ so $\overline{CE} \cong \overline{DE}$.
- Show that $DB = CE + EB$.
- Use the Triangle Inequality Theorem on $\triangle BCE$.

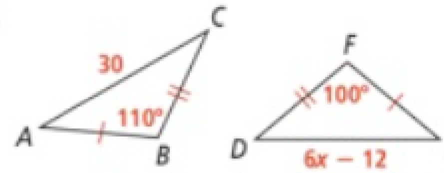
PRACTICE

10. Write an inequality describing the range of x for each pair of triangles. SEE EXAMPLES 1 AND 2.

a.

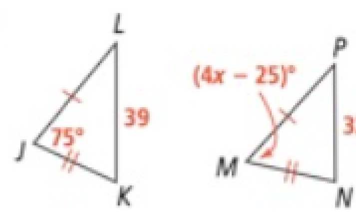


b.

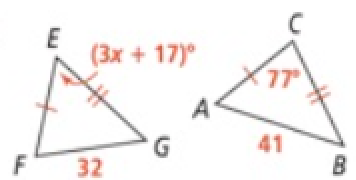


11. Write an inequality describing the possible values of x for each pair of triangles. SEE EXAMPLES 3 AND 4.

a.

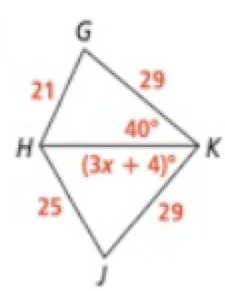


b.

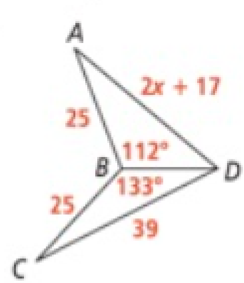


12. Write an inequality describing the possible values of x for each diagram.

a.



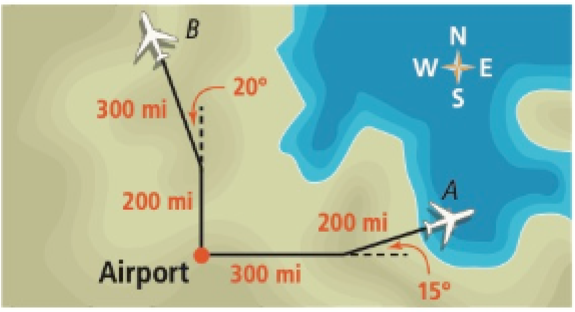
b.



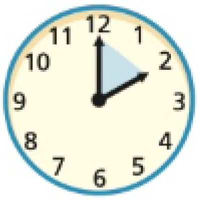
PRACTICE & PROBLEM SOLVING

APPLY

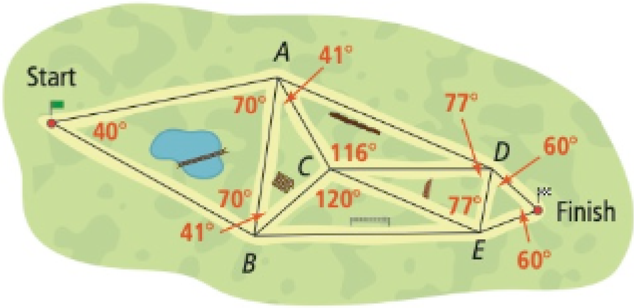
13. **Communicate and Justify** Airplane A flies 300 miles due east of an airport and then flies 200 miles at 15° north of east. Airplane B flies 200 miles due north and then flies 300 miles at 20° west of north. Which airplane is closer to the airport? Explain how you know.



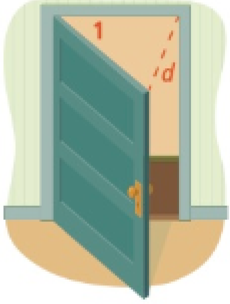
14. **Apply Math Models** According to the Hinge Theorem, is the distance between the tips of the hands greater at 4:00 or at 7:00? Explain how the distance changes throughout a day.



15. **Mathematical Connections** Determine the shortest path from start to finish on the obstacle course.

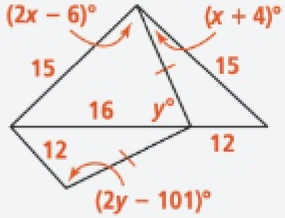


16. **Higher Order Thinking** When $m\angle 1 = 75$, $d = 43$ in., and when $m\angle 1 = 100$, $d = 54$ in. Neil wants to know how wide a sofa he can buy if he can open the door at most 85°. Using the Hinge Theorem or the Converse of the Hinge Theorem, can you determine the exact value of d when $m\angle 1 = 85$? If you can, explain the method. If not, explain what you can determine about the distance.



ASSESSMENT PRACTICE

17. Select all of the inequalities that you can conclude from the diagram. GR.1.3



☐ **A.** $x < 10$

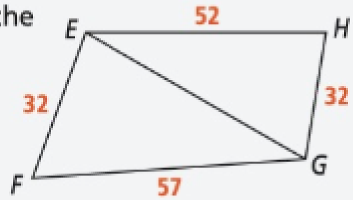
☐ **B.** $x > 10$

☐ **C.** $y < 101$

☐ **D.** $y > 101$

☐ **E.** $x < 18$

18. **SAT/ACT** Which of the following can you conclude from the diagram?



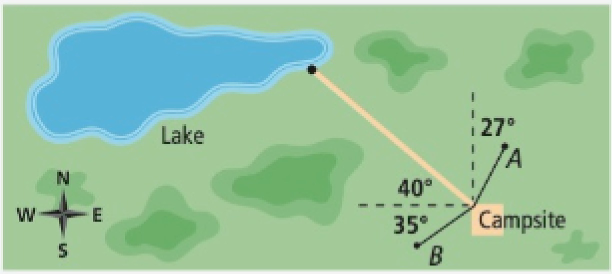
Ⓐ $m\angle EFG = m\angle GHE$

Ⓑ $m\angle FGE = m\angle HEG$

Ⓒ $m\angle GEF > m\angle EGH$

Ⓓ $m\angle FGE > m\angle EGH$

19. **Performance Task** Abby, Danielle, and Jacy walk from their campsite to get to the lake. The lake is located 3 miles away in the direction of 40° north of west.



Part A Abby walks along a straight path in the direction of 27° east of north for 1 mile to point A. Using the Hinge Theorem, if Danielle walks along a straight path in the direction of 35° south of west for 1 mile to point B, who is closer to the lake?

Part B Jacy also walks for 1 mile from the campsite along a different straight path than Abby. Her straight-line distance to the lake is shorter than Abby's distance. What directions could Jacy have taken?

TOPIC 5

Topic Review

TOPIC ESSENTIAL QUESTION

1. How are the sides, segments, and angles of triangles related?

Vocabulary Review

Choose the correct term to complete each sentence.

2. The ________ is the point of concurrency of the angle bisectors of a triangle.
3. Three or more lines that intersect at one point are ________.
4. The point of concurrency of the altitudes of a triangle is the ________.
5. A perpendicular segment from a vertex to the line containing the side opposite the vertex is a(n) ________ of a triangle.
6. A point that is the same distance from two points is ________ from the points.
7. A(n) ________ of a triangle has endpoints at a vertex and at the midpoint of the side opposite the vertex.

- altitude
- centroid
- circumcenter
- concurrent
- equidistant
- incenter
- median
- orthocenter

Concepts & Skills Review

LESSON 5-1 Perpendicular and Angle Bisectors

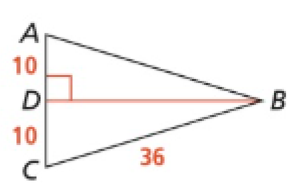
Quick Review

Perpendicular bisectors and angle bisectors are related to the segments or angles they bisect:

- Any point on the perpendicular bisector of a segment is **equidistant** from the endpoints of the segment.
- Any point on the bisector of an angle is equidistant from the two sides of the angle.

Example

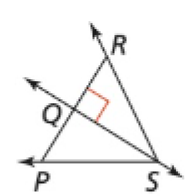
Find *AB*.



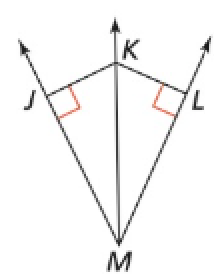
DB is the perpendicular bisector of $\overline{AC}$, so $AB = CB$. $AB = 36$.

Practice & Problem Solving

Use the given values to find each unknown.



8. If $PS = 36$, $PQ = 3x + 5$, $QR = 6x - 10$, and $RS = 36$, then $PR = ■$.
9. If $PS = 4x + 8$, $PQ = 29$, $RS = 5x - 3$, and $QR = 29$, then $PS = ■$.



10. If $JM = 12$, $LM = 12$, and $m\angle JMK = 25$, then $m\angle KML = ■$.
11. If $m\angle JML = 49$, $m\angle JMK = 24.5$, and $JK = 17$, then $KL = ■$.
12. Use Patterns and Structure A point on a perpendicular bisector is 7 cm from each endpoint of the bisected segment and 5 cm from the point of intersection. To the nearest tenth, what is the length of the segment?

LESSON 5-2 Bisectors in Triangles

Quick Review

The perpendicular bisectors of a triangle are concurrent at the point equidistant from the vertices of the triangle. This point is called the **circumcenter**.

The angle bisectors of a triangle are concurrent at a point equidistant from the sides of the triangle. This point is called the **incenter**.

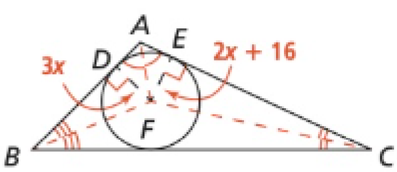
Example

For $\triangle ABC$, what is the radius of the inscribed circle?

$DF = EF$

$3x = 2x + 16$

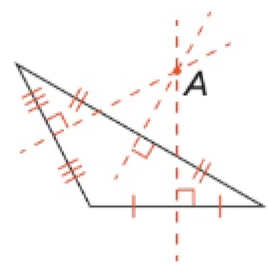
$x = 16$

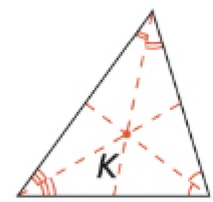


The radius is $DF = 3x = 3(16) = 48$.

Practice & Problem Solving

Identify each point of concurrency.



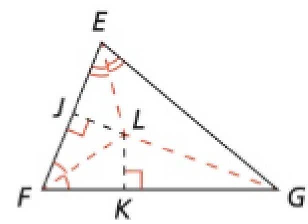


13. incenter **14.** circumcenter

Use the diagram to find each unknown quantity.

15. If $m\angle GFL = 34$, and $m\angle GEL = 36$, what is $m\angle FGL$?

16. If $JL = 5$, what is the measure of KL?



17. **Communicate and Justify** The circumcenter of a triangle is on one side of the triangle. Explain how to find the area of the circumscribed circle of the triangle given the lengths of the sides.

LESSON 5-3 Medians and Altitudes

Quick Review

A **median** of a triangle is a line segment from the midpoint of one side to the opposite vertex. The medians of a triangle are concurrent at the **centroid**. The distance from a vertex to the centroid is two-thirds the length of the median from that vertex.

An **altitude** of a triangle is a line segment perpendicular to one side and ending at the opposite vertex. The lines containing the altitudes of a triangle are concurrent at the **orthocenter**.

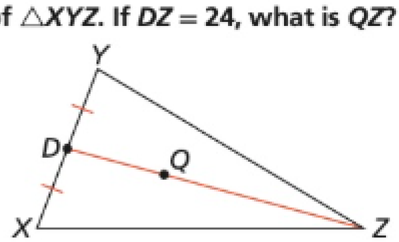
Example

Q is the centroid of $\triangle XYZ$. If $DZ = 24$, what is QZ?

$QZ = \frac{2}{3} DZ$

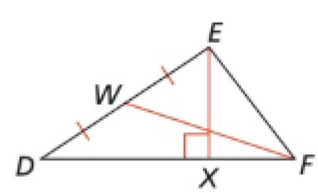
$QZ = \frac{2}{3}(24)$

$QZ = 16$



Practice & Problem Solving

Identify each segment type.



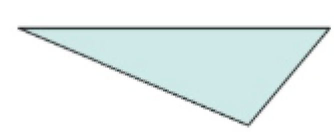
18. median **19.** altitude

Find the orthocenter of each triangle with the given set of vertices.

20. (2, 0), (2, 12), (8, 6) **21.** (7, 8), (9, 6), (5, 4)

22. **Analyze and Persevere** The orthocenter of $\triangle ABC$ is point B. What is the measure of $\angle ABC$?

23. A plastic triangle is suspended parallel to the ground by a string attached at the centroid. Copy the triangle and show where the string should be attached.



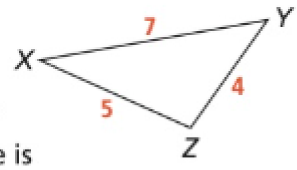
LESSON 5-4 Inequalities in One Triangle

Quick Review

In a triangle, if two sides are not congruent, then the larger angle is opposite the longer side. If two angles are not congruent, then the longer side is opposite the larger angle.

$XY > XZ > YZ$

$m\angle Z > m\angle Y > m\angle X$



The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

$4 + 5 > 7 \quad 4 + 7 > 5 \quad 5 + 7 > 4$

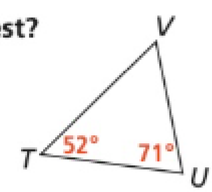
Example

Which side of $\triangle TUV$ is the longest?

$m\angle T + m\angle U + m\angle V = 180$

$52 + 71 + m\angle V = 180$

$m\angle V = 57$



The largest angle is $\angle U$, so the longest side is $\overline{VT}$.

Practice & Problem Solving

Determine if the lengths can form a triangle.

24. 14, 32, 18

25. 14, 25, 29

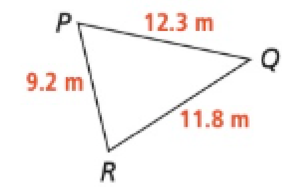
26. 37, 22, 56

27. 87, 35, 41

Use the figure for Exercises 28 and 29.

28. Which angle has the least measure?

29. Which angle has the greatest measure?



30. Use Patterns and Structure Why must the sum of two sides of a triangle be greater than the third side?

31. Choose Efficient Methods Two sides of a triangular garden are 6.4 m and 8.2 m. The gardener buys fencing for $29.25 per meter. What is the range of total cost of the fencing?

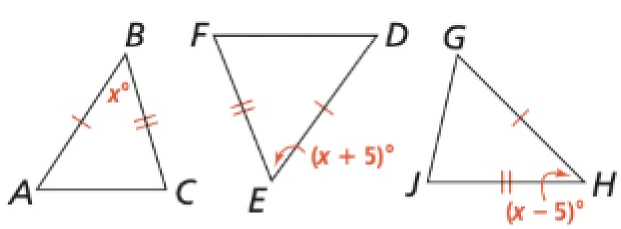
LESSON 5-5 Inequalities in Two Triangles

Quick Review

The Hinge Theorem states that if two triangles have two congruent sides, and the included angles are not congruent, then the longer third side is opposite the larger included angle.

Example

Order AC, DF, and GJ from greatest to least.



The triangles have two congruent sides.

$\overline{AB} \cong \overline{DE} \cong \overline{GH}$ and $\overline{BC} \cong \overline{EF} \cong \overline{HJ}$

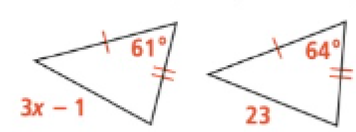
The included angles are not congruent.

$x + 5 > x > x - 5$

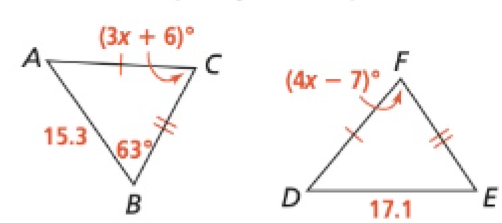
Therefore, $DF > AC > GJ$.

Practice & Problem Solving

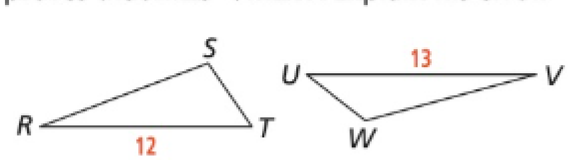
32. Write an inequality for the possible values of x.



33. Write an inequality for the possible values of x.



34. Communicate and Justify Cameron incorrectly says the Converse of the Hinge Theorem proves that $m\angle S < m\angle W$. Explain his error.



TOPIC
6 Quadrilaterals

TOPIC ESSENTIAL QUESTION

How are properties of parallelograms used to solve problems and to classify quadrilaterals?

Topic Overview

enVision® STEM Project:
Design a Quadrilateral Lift

6-1 Kites and Trapezoids
GR.1.5, MTR.5.1, MTR.6.1, MTR.7.1

6-2 Properties of Parallelograms
GR.1.4, MTR.2.1, MTR.4.1, MTR.5.1

6-3 Proving a Quadrilateral Is a Parallelogram
GR.1.4, MTR.1.1, MTR.2.1, MTR.3.1

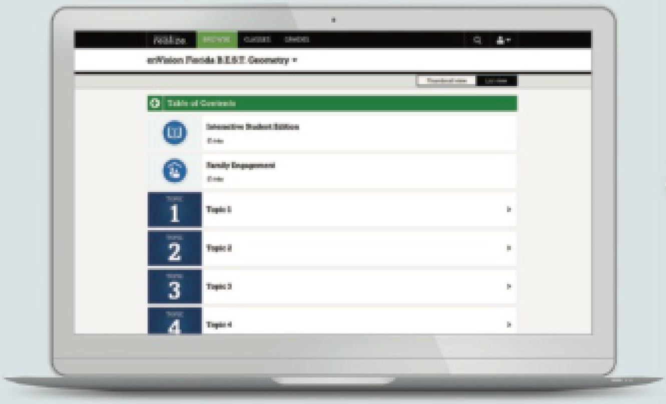
6-4 Properties of Special Parallelograms
GR.1.4, MTR.5.1, MTR.1.1, MTR.6.1

6-5 Conditions of Special Parallelograms
GR.1.4, MTR.4.1, MTR.2.1, MTR.7.1

Mathematical Modeling in 3 Acts:
Picture This
GR.1.4, GR.1.5, MTR.7.1

Topic Vocabulary

- isosceles trapezoid
- midsegment of a trapezoid



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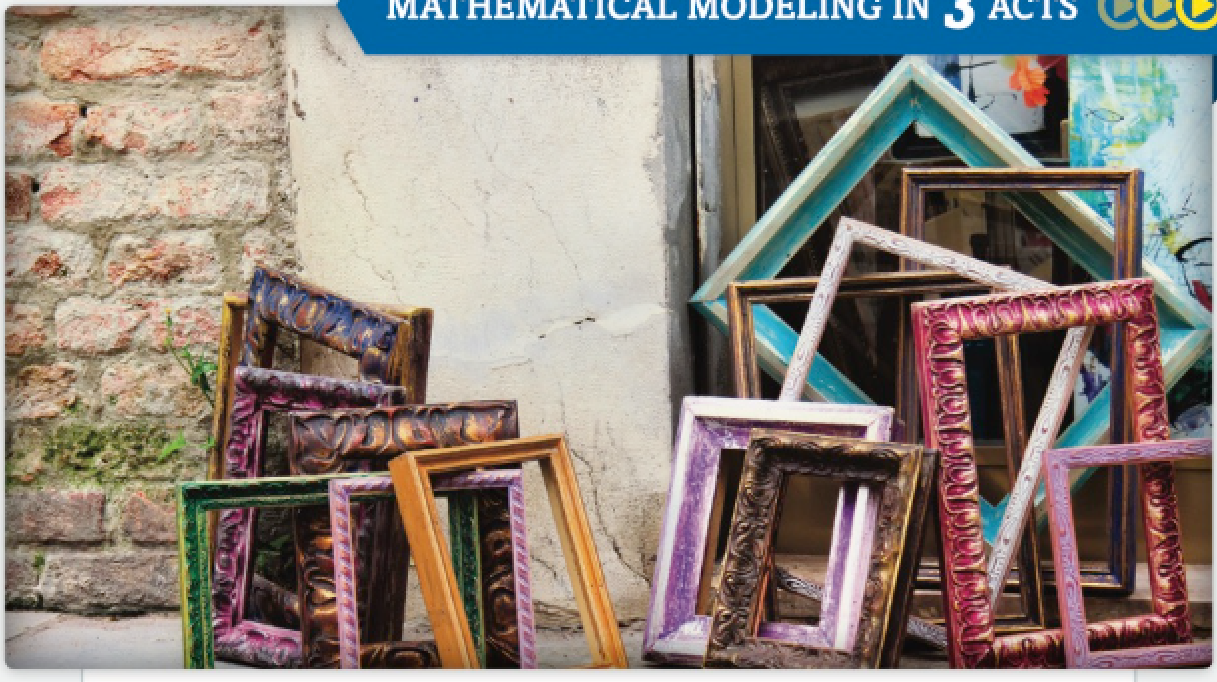
PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Picture This

There are many sizes and designs of picture frames to enhance a piece of art. While frames are often rectangular, there is no limit to the shape of a picture frame. From art galleries to personal homes, many people carefully arrange and display art, choosing shapes and styles that harmonize with and balance the surroundings.

How might you use the geometric properties of picture frames to organize a collection of pictures? Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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Did You Know?

The rhinoceros beetle of Central and South America can lift up to **850 times** its own weight. That's equivalent to a 150-pound human lifting some 120,000 pounds, or **30 cars**.

Cargo bay

The world's largest cargo plane can carry more than **half a million pounds**. The cargo bay is 142 feet long, which is longer than the length of the first airplane *flight* by the Wright brothers, in 1903.

Your Task: Design a Quadrilateral Lift

A 50,000-pound bus needs to be lifted 6 feet off the ground for engine repairs. You and your classmates will analyze quadrilaterals and design a hydraulic lift for a mechanic to use for those repairs.

6-1 Kites and Trapezoids

I CAN… use triangle congruence to understand kites and trapezoids.

VOCABULARY

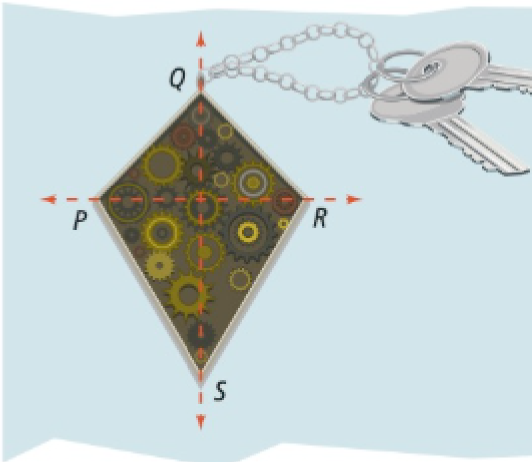
- isosceles trapezoid
- midsegment of a trapezoid

MA.912.GR.1.5–Prove relationships and theorems about trapezoids. Solve mathematical and real-world problems involving postulates, relationships and theorems of trapezoids.
MA.K12.MTR.5.1, MTR.6.1, MTR.7.1

CRITIQUE & EXPLAIN

Manuel draws a diagram of kite *PQRS* with $\overleftrightarrow{QS}$ as the line of symmetry over a design of a kite-shaped key fob. He makes a list of conclusions based on the diagram.

- $\overline{PR} \perp \overline{QS}$
- $\overline{QP} \cong \overline{QR}$
- $\overline{SP} \cong \overline{SR}$
- $\overline{PR}$ bisects $\overline{QS}$.
- ΔPQR is an equilateral triangle.
- ΔPSR is an isosceles triangle.



A. Which of Manuel's conclusions do you agree with? Which do you disagree with? Explain.

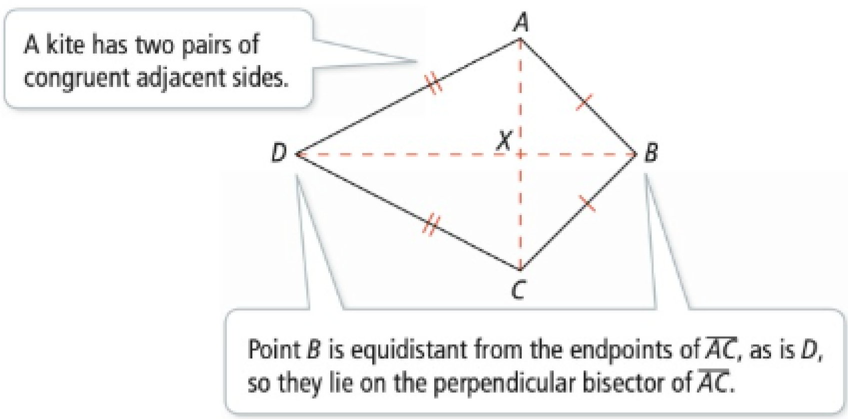
B. **Use Patterns and Structure** What other conclusions are supported by the diagram?

ESSENTIAL QUESTION How are diagonals and angle measures related in kites and trapezoids?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Investigate the Diagonals of a Kite

How are the diagonals of a kite related?



STUDY TIP
Remember that you must show that both *B* and *D* are on the perpendicular bisector in order to show that one diagonal is the perpendicular bisector of the other. It is not sufficient to show that only one is on the perpendicular bisector.

The diagonals of a kite are perpendicular to each other. At least one diagonal bisects the other.

Try It! **1. a.** What is the measure of $\angle AXB$?

b. If $AX = 3.8$, what is AC?

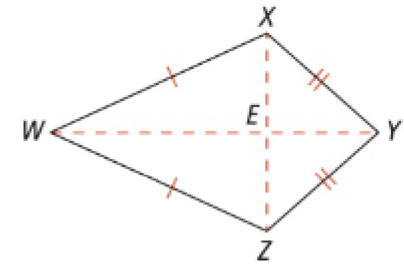
c. If $BD = 10$, does $BX = 5$? Explain.

THEOREM 6-1

Powered By desmos

The diagonals of a kite are perpendicular.

If...

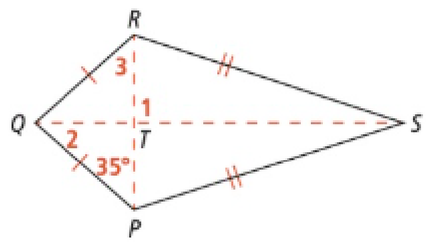


Then... $\overline{WY} \perp \overline{XZ}$

PROOF: SEE EXERCISE 12.

EXAMPLE 2 Use the Diagonals of a Kite

Quadrilateral *PQRS* is a kite with diagonals $\overline{QS}$ and $\overline{PR}$.



A. What is $m\angle 1$?

The diagonals of a kite are perpendicular, so $m\angle 1 = 90$.

B. What is $m\angle 2$?

The sum of the angles of $\triangle PQT$ is 180.

$$m\angle 2 + 35 + 90 = 180$$

$$m\angle 2 = 55$$

C. What is $m\angle 3$?

Since $\triangle PQR$ is an isosceles triangle, $\angle 3 \cong \angle QPT$.

So, $m\angle 3 = 35$.

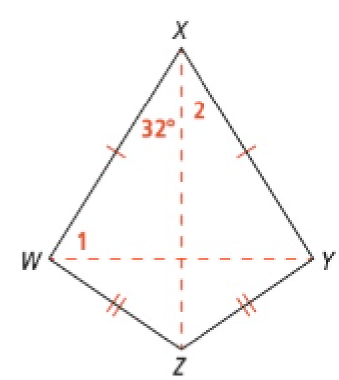
COMMON ERROR

You may incorrectly assume angles are congruent just from their appearance. Always check that you can prove congruence first.

Try It! **2.** Quadrilateral *WXYZ* is a kite.

a. What is $m\angle 1$?

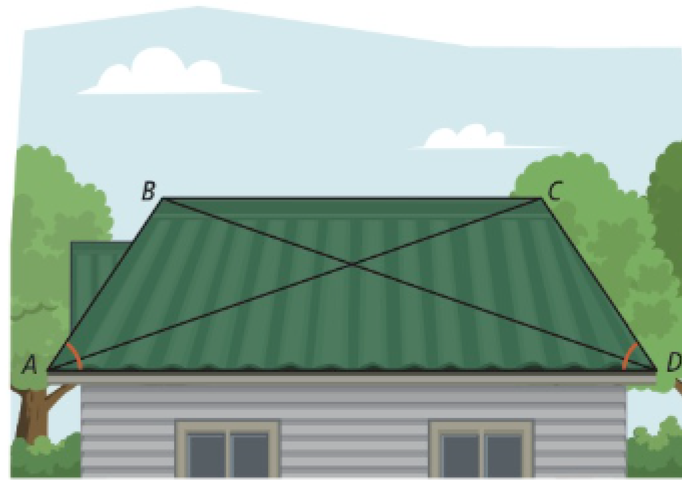
b. What is $m\angle 2$?



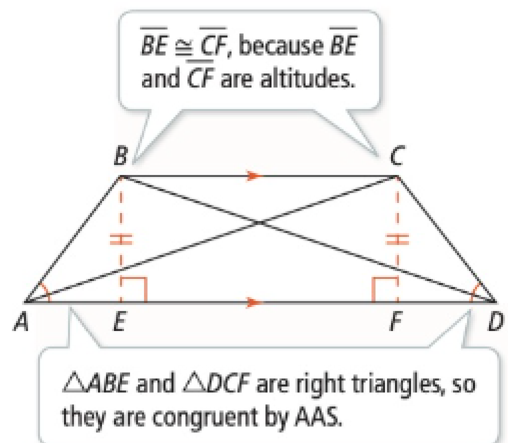
APPLICATION

EXAMPLE 3 Explore Parts of an Isosceles Trapezoid

Kiyo is designing a trapezoid-shaped roof. In order for the roof to be symmetric, the overlapping triangles △*DAB* and △*ADC* must be congruent. Will the roof be symmetric?



Step 1 Show △*ABE* ≅ △*DCF*.



HAVE A GROWTH MINDSET
When it takes time to learn something new, how do you stick with it?

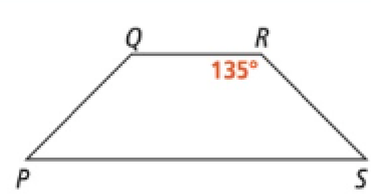
Step 2 Show △*DAB* ≅ △*ADC*.

By CPCTC, $\overline{AB} \cong \overline{CD}$. By the Reflexive Property of Congruency, $\overline{AD} \cong \overline{DA}$. So, △*DAB* ≅ △*ADC* by SAS.

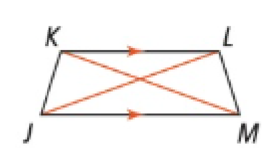
The overlapping triangles are congruent, so the roof is symmetric.

ABCD is an example of an **isosceles trapezoid**, which is a trapezoid with congruent base angles.

Try It! **3. a.** Given isosceles trapezoid *PQRS*, what are $m\angle P$, $m\angle Q$, and $m\angle S$?



b. In isosceles trapezoid *JKLM*, what can you conclude about diagonals $\overline{JL}$ and $\overline{KM}$? Explain.

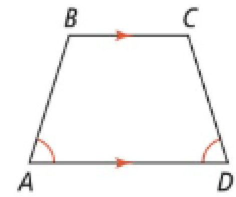


THEOREM 6-2

In an isosceles trapezoid, the legs are congruent.

PROOF: SEE LESSON 6-2, EXERCISE 17.

If...



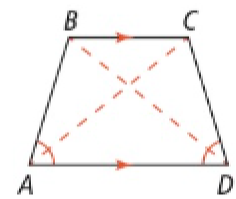
Then... $\overline{AB} \cong \overline{CD}$

THEOREM 6-3

The diagonals of an isosceles trapezoid are congruent.

PROOF: SEE EXERCISE 18.

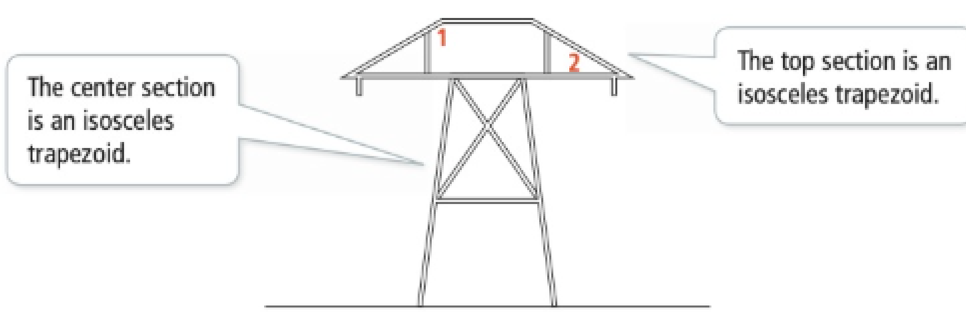
If...



Then... $\overline{AC} \cong \overline{DB}$

EXAMPLE 4 Solve Problems Involving Isosceles Trapezoids

All horizontal beams of the high-voltage transmission tower are parallel to the ground.



A. If $m\angle 1 = 138$, what is $m\angle 2$?

The sum of the interior angle measures of a quadrilateral is 360.

$$m\angle 1 + m\angle 1 + m\angle 2 + m\angle 2 = 360$$
$$138 + 138 + 2(m\angle 2) = 360$$
$$276 + 2(m\angle 2) = 360$$
$$2(m\angle 2) = 84$$
$$m\angle 2 = 42$$

The base angles are congruent.

The measure of $\angle 2$ is 42.

CONTINUED ON THE NEXT PAGE

ANALYZE AND PERSEVERE
How are the measures of $\angle 1$ and $\angle 2$ related? Is this true for all opposite angles in an isosceles trapezoid?

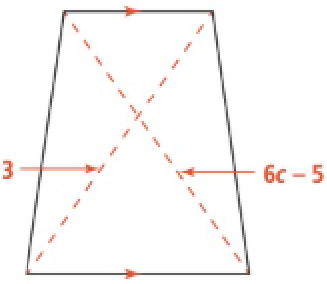
GENERALIZE

Why might this strategy work for isosceles trapezoids but not for trapezoids with noncongruent legs?

EXAMPLE 4 CONTINUED

B. One cross support in the center of the tower measures $4c + 3$, and the other measures $6c - 5$. What is the length of each cross support?

The cross supports are diagonals of an isosceles trapezoid, so they are congruent.



Step 1 Find the value of *c*.

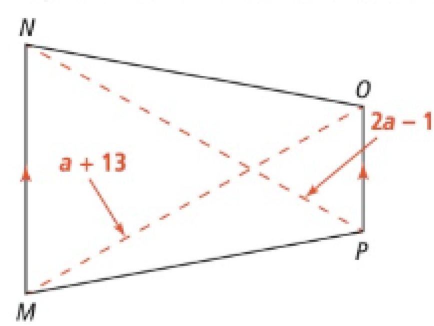
$$4c + 3 = 6c - 5$$
$$8 = 2c$$
$$4 = c$$

Step 2 Find the lengths of the diagonals.

$$4c + 3 = 4(4) + 3$$
$$= 19$$
$$6c - 5 = 6(4) - 5$$
$$= 19$$

The length of each cross support measures 19 ft.

Try It! 4. Given isosceles trapezoid *MNOP* where the given expressions represent the measures of the diagonals, what is the value of *a*?



THEOREM 6-4 Trapezoid Midsegment Theorem

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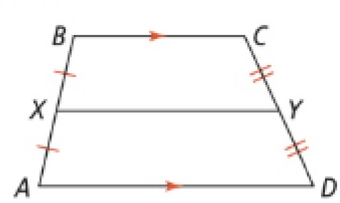
In a trapezoid, the segment containing the midpoints of the two legs is parallel to the bases, and its length is half the sum of the lengths of the bases.

PROOF: SEE LESSON 9-2.

If...

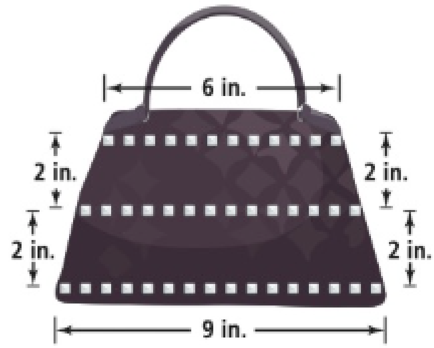


Then... $\overline{XY} \parallel \overline{AD}$, $\overline{XY} \parallel \overline{BC}$, and $XY = \frac{1}{2}(AD + BC)$

APPLICATION

EXAMPLE 5 Apply the Trapezoid Midsegment Theorem

Paxton makes trapezoidal handbags for her friends. She stiches decorative trim along the top, middle, and bottom on both sides of the handbags. How much trim does she need for three handbags? Explain.



Formulate ◀ The top and bottom sides of the handbag are the bases of a trapezoid. The left and right sides are the legs. Since the middle segment divides both legs in half, it is the midsegment of the trapezoid. The **midsegment of a trapezoid** is the segment that connects the midpoints of the legs.

Let x represent the length of the midsegment in inches.

Compute ◀ **Step 1** Find the value of x.

$$x = \frac{1}{2}(6 + 9)$$

$$x = 7.5$$

Apply the Trapezoid Midsegment Theorem with the base lengths 6 and 9.

The length of the midsegment is 7.5 in.

Step 2 Find the amount of trim that she needs.

First, find the amount for one side.

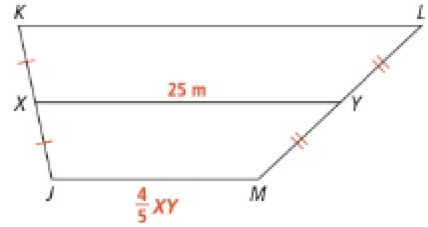
$$6 + 9 + 7.5 = 22.5$$

Then, multiply by 2 for the number of sides per handbag and by 3 for the number of handbags.

$$22.5 \cdot 2 \cdot 3 = 135$$

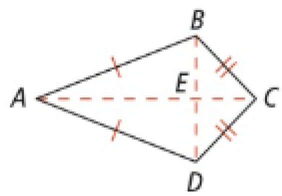
Interpret ◀ Paxton needs 135 inches of trim.

Try It! **5.** Given trapezoid *JKLM*, what is *KL*?



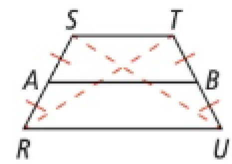
CONCEPT SUMMARY Kites and Trapezoids

	Kites	Trapezoids
WORDS	A kite is a quadrilateral with two pairs of adjacent sides congruent. At least one diagonal is a perpendicular bisector of the other.	A trapezoid is a quadrilateral with at least one pair of parallel sides. The length of the midsegment is the average of the lengths of the two bases. A trapezoid with congruent base angles is an isosceles trapezoid that has congruent legs and congruent diagonals.
DIAGRAMS	Quadrilateral *ABCD* is a kite.	Quadrilateral *RSTU* is an isosceles trapezoid.



$\overline{AC} \perp \overline{BD}$

$BE = ED$



$\overline{SU} \cong \overline{TR}$

$AB = \frac{1}{2}(ST + RU)$

$\overline{AB} \parallel \overline{ST} \parallel \overline{RU}$

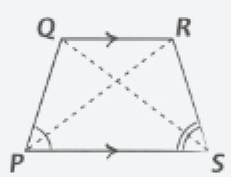
$m\angle RST = m\angle UTS$

$m\angle SRU = m\angle TUR$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are diagonals and angle measures related in kites and trapezoids?

2. **Error Analysis** What is Reagan's error?



By Theorem 6-3, $\overline{PR} \cong \overline{QS}$

3. **Vocabulary** If $\overline{XY}$ is the midsegment of a trapezoid, what must be true about point *X* and point *Y*?

4. **Check for Reasonableness** Emaan says every kite is composed of 4 right triangles. Is he correct? Explain.

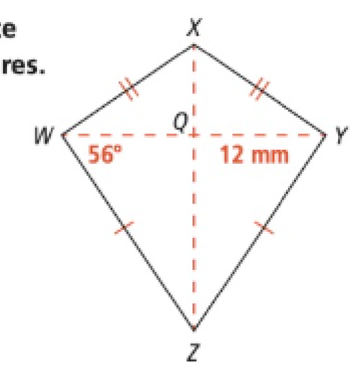
Do You KNOW HOW?

For Exercises 5–7, use kite *WXYZ* to find the measures.

5. $m\angle XQY$

6. $m\angle YZQ$

7. *WY*

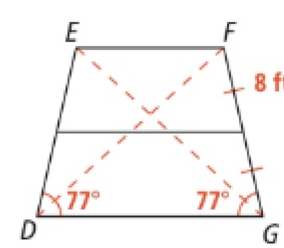


For Exercises 8–10, use trapezoid *DEFG* with $EG = 21$ ft and $m\angle DGF = 77$ to find each measure.

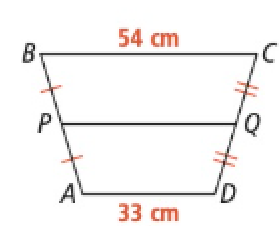
8. *ED*

9. *DF*

10. $m\angle DEF$



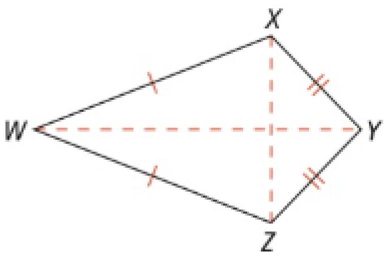
11. What is the length of $\overline{PQ}$?



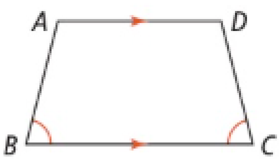
PRACTICE & PROBLEM SOLVING

UNDERSTAND

12. Use Patterns and Structure Write a two-column proof to show that the diagonals of a kite are perpendicular.

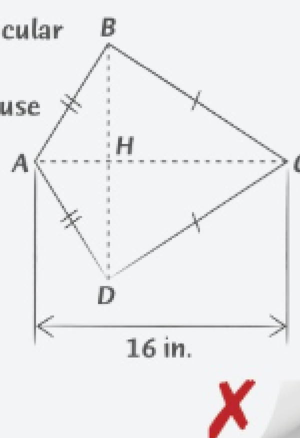


13. Quadrilateral *ABCD* is an isosceles trapezoid. Prove that the opposite angles of *ABCD* are supplementary.

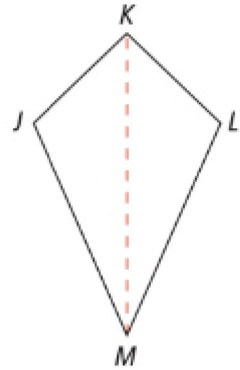


14. Error Analysis What is Emery's error?

$\overline{BD}$ is the perpendicular bisector of $\overline{AC}$, so $HC = 8$ in. because $AC = 16$ in.

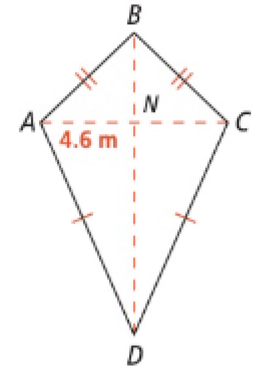


15. Higher Order Thinking
Given kite *JKLM* with diagonal $\overline{KM}$, $JK < JM$, and $KL < LM$, prove that $\angle JMK$ is congruent to $\angle LMK$.

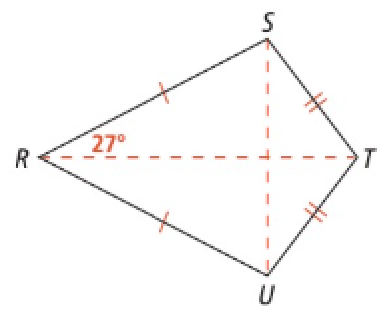


PRACTICE

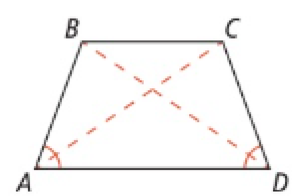
16. Given kite *ABCD*, in which $AN = 4.6$ m, what is *AC*?
SEE EXAMPLE 1



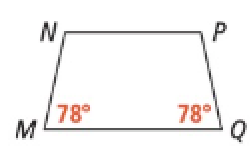
17. Given kite *RSTU*, what is $m\angle RUS$? SEE EXAMPLE 2



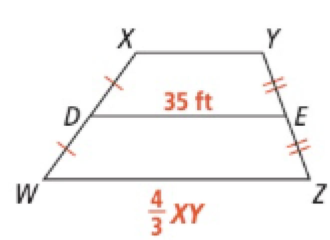
18. Write a two-column proof to show that the diagonals of an isosceles trapezoid are congruent. SEE EXAMPLES 3 AND 4



19. Given trapezoid *MNPQ*, what is $m\angle MNP$?
SEE EXAMPLE 4



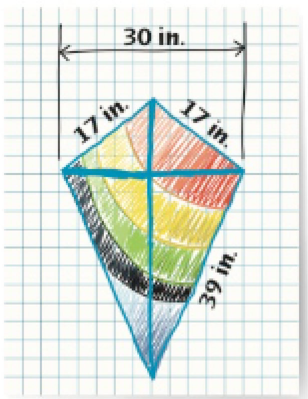
20. Given trapezoid *WXYZ*, what is *XY*?
SEE EXAMPLE 5



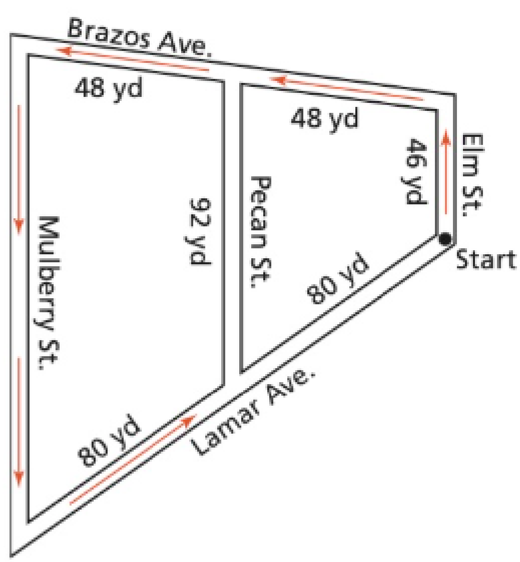
PRACTICE & PROBLEM SOLVING

APPLY

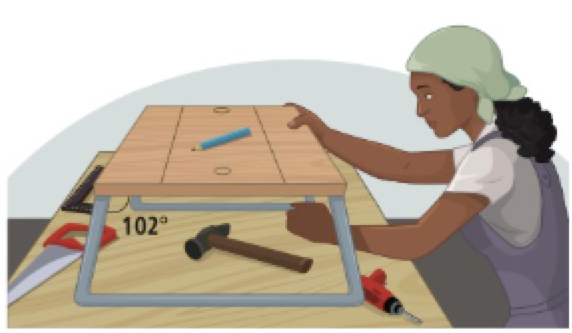
21. Represent and Connect Gregory plans to make a kite like the one shown. He has 1,700 square inches of plastic sheeting. Does Gregory have enough plastic to make the kite? Explain.



22. Apply Math Models Coach Murphy uses the map to plan a 2-mile run for the track team. How many times will the team run the route shown?



23. Use Patterns and Structure Abby builds a bench with the seat parallel to the ground. She bends pipe to make the leg and seat supports in the shape of isosceles trapezoids. At what angles should she bend the pipe? Explain.

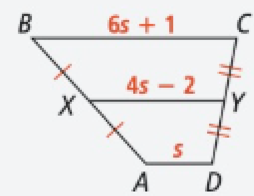


ASSESSMENT PRACTICE

24. Choose the word or phrase that completes the sentence: The diagonals of an isosceles trapezoid are always ______. GR.1.5

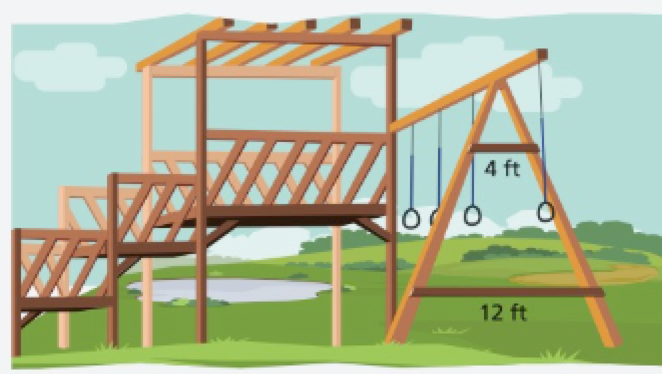
Ⓐ parallel
Ⓑ perpendicular
Ⓒ congruent
Ⓓ both bisectors of each other

25. SAT/ACT Given trapezoid *ABCD*, what is the length of $\overline{XY}$?



Ⓐ $3\frac{3}{5}$ Ⓑ $4\frac{2}{3}$ Ⓒ 5 Ⓓ 11 Ⓔ 18

26. Performance Task Cindy is a member of a volunteer group that built the play structure shown.



Part A Cindy wants to add three more boards evenly spaced between and parallel to the bottom and top boards of the triangular frame. Based on the lengths of the top and bottom boards shown, what will be the lengths of each of the three additional boards? Explain.

Part B How can Cindy determine where to nail the new boards into the existing frame?

Part C What other measurements should Cindy find to be certain that the boards will fit exactly onto the triangular frame?

6-2 Properties of Parallelograms

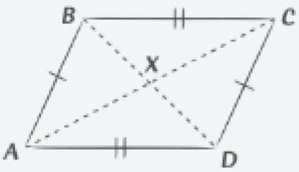
I CAN… use the properties of parallel lines, diagonals, and triangles to investigate parallelograms.

MA.912.GR.1.4–Prove relationships and theorems about parallelograms. Solve mathematical and real-world problems involving postulates, relationships and theorems of parallelograms.
MA.K12.MTR.2.1, MTR.4.1, MTR.5.1

CRITIQUE & EXPLAIN

Kennedy lists all the pairs of congruent triangles she finds in quadrilateral *ABCD*.

$\overline{AD} \parallel \overline{BC}$ $\quad \overline{AB} \parallel \overline{CD}$
$\angle ABC \cong \angle CDA$
$\angle DAB \cong \angle BCD$



Congruent triangles:
$\triangle DAB \cong \triangle BCD$ by SAS
$\triangle ABC \cong \triangle CDA$ by SAS

A. Is Kennedy's justification for triangle congruence correct for each pair?

B. **Use Patterns and Structure** Did Kennedy overlook any pairs of congruent triangles? If not, explain how you know. If so, name them and explain how you know they are congruent.

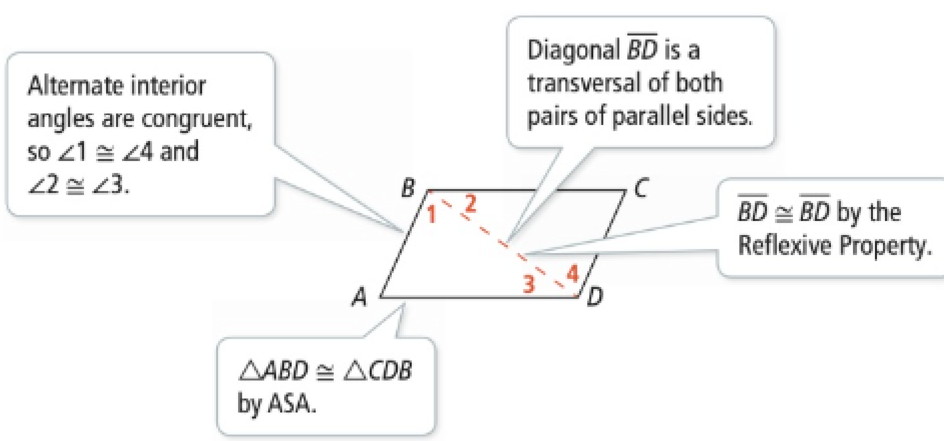
ESSENTIAL QUESTION What are the relationships of the sides, the angles, and the diagonals of a parallelogram?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Explore Opposite Sides of Parallelograms

How do the lengths of the opposite sides of a parallelogram compare to each other?

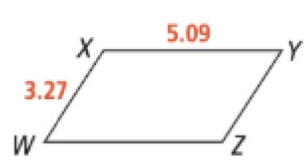
Quadrilateral *ABCD* is a parallelogram.



USE PATTERNS AND STRUCTURE
Can you use the same strategy to show other relationships in a parallelogram?

By CPCTC, $\overline{AD} \cong \overline{CB}$ and $\overline{AB} \cong \overline{CD}$, so the lengths of the opposite sides are congruent to each other.

Try It! **1.** Given parallelogram *WXYZ*, what is *YZ*?



THEOREM 6-5

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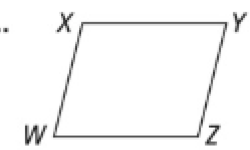
If a quadrilateral is a parallelogram, then its opposite sides are congruent.

PROOF: SEE EXERCISE 13.

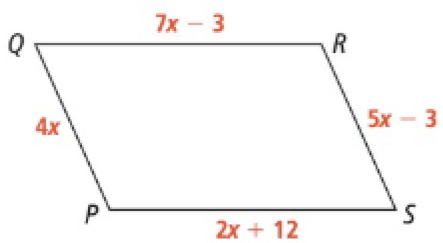
If... $\overline{WX} \parallel \overline{ZY}$, $\overline{WZ} \parallel \overline{XY}$



Then... $\overline{WX} \cong \overline{YZ}$, $\overline{WZ} \cong \overline{XY}$

EXAMPLE 2 Use Opposite Sides of a Parallelogram

Quadrilateral *PQRS* is a parallelogram.



STUDY TIP
Remember there is often more than one way to solve a problem. You could also find the value of x by solving the equation $4x = 5x - 3$, since $\overline{QP}$ and $\overline{RS}$ are also opposite sides of the parallelogram.

A. What is the value of *x*?

$7x - 3 = 2x + 12$

$5x = 15$

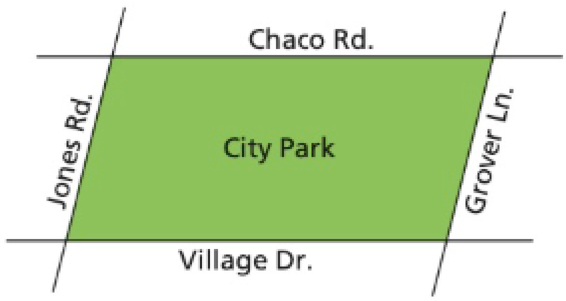
$x = 3$

$\overline{QR} \cong \overline{PS}$ because they are opposite sides of a parallelogram.

B. What is the length of each side of *PQRS*?

$PQ = 4x$	$QR = 7x - 3$	$RS = 5x - 3$	$PS = 2x + 12$
$= 4(3)$	$= 7(3) - 3$	$= 5(3) - 3$	$= 2(3) + 12$
$= 12$	$= 21 - 3$	$= 15 - 3$	$= 6 + 12$
	$= 18$	$= 12$	$= 18$

Try It! 2. The 600-meter fence around City Park forms a parallelogram. The fence along Chaco Road is twice as long as the fence along Grover Lane. What is the length of the fence along Jones Road?

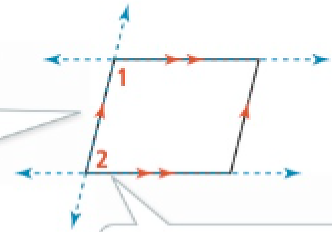


EXAMPLE 3 Explore Angle Measures in Parallelograms

A. How are consecutive angles in a parallelogram related?

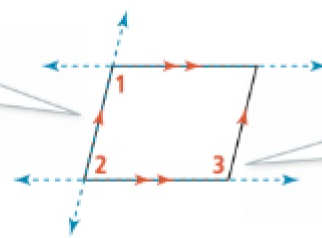
You can use what you know about angle relationships formed when parallel lines are cut by a transversal.

STUDY TIP
You can visualize a parallelogram as parallel lines intersected by transversals that are also parallel. This may help you determine how the angles are related.



Consecutive angles ∠1 and ∠2 are supplementary.

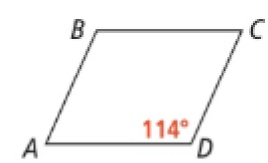
B. How are opposite angles in a parallelogram related?



In the figure, ∠1 and ∠3 are opposite angles. Both angles are supplementary to ∠2, so opposite angles in a parallelogram are congruent.

Try It! **3. a.** Given parallelogram *ABCD*, what are $m\angle A$ and $m\angle C$?

b. What is $m\angle B$?



THEOREM 6-6

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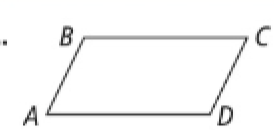
If a quadrilateral is a parallelogram, then its consecutive angles are supplementary.

PROOF: SEE EXERCISE 15.

If...



$\overline{AB} \parallel \overline{DC}$
$\overline{AD} \parallel \overline{BC}$

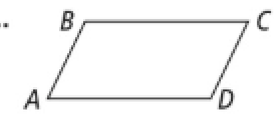
Then... $m\angle A + m\angle B = 180$
$m\angle B + m\angle C = 180$
$m\angle C + m\angle D = 180$
$m\angle D + m\angle A = 180$

THEOREM 6-7

If a quadrilateral is parallelogram, then opposite angles are congruent.

PROOF: SEE EXERCISE 23.

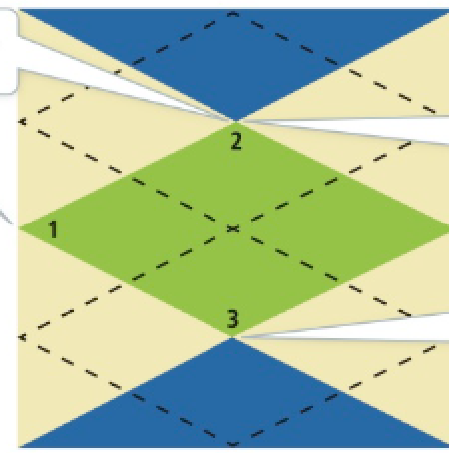
If... $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$



Then... $\angle A \cong \angle C$
$\angle B \cong \angle D$

EXAMPLE 4 Use Angles of a Parallelogram

The green shape in the fabric design is a parallelogram. The measure of $\angle 2$ is twice the measure of $\angle 1$. What are $m\angle 1$, $m\angle 2$, and $m\angle 3$?



COMMON ERROR
You may incorrectly write $m\angle 1 = 2m\angle 2$, but $m\angle 1 = 2m\angle 2$ means that $m\angle 1$ is twice $m\angle 2$.

Find $m\angle 1$.

$$m\angle 1 + m\angle 2 = 180$$
$$m\angle 1 + 2m\angle 1 = 180$$
$$m\angle 1 = 60$$

Find $m\angle 2$.

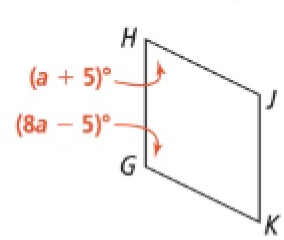
$$m\angle 2 = 2m\angle 1$$
$$m\angle 2 = 2(60)$$
$$m\angle 2 = 120$$

Find $m\angle 3$.

$$m\angle 3 = m\angle 2$$
$$m\angle 3 = 120$$

The measures of $\angle 1$, $\angle 2$, and $\angle 3$ are 60, 120, and 120, respectively.

Try It! **4.** Use the parallelogram shown.



a. Given parallelogram *GHJK*, what is the value of *a*?

b. What are $m\angle G$, $m\angle H$, $m\angle J$, and $m\angle K$?

THEOREM 6-8

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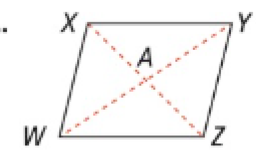
If a quadrilateral is a parallelogram, then its diagonals bisect each other.

PROOF: SEE EXAMPLE 5.

If...



$\overline{WX} \parallel \overline{ZY}$

$\overline{WZ} \parallel \overline{XY}$

Then... $\overline{AW} \cong \overline{AY}$

$\overline{AX} \cong \overline{AZ}$

PROOF

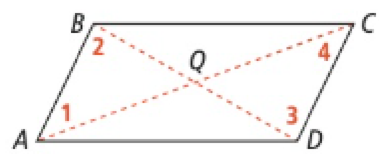
EXAMPLE 5 Explore the Diagonals of a Parallelogram

How are the diagonals of a parallelogram related?

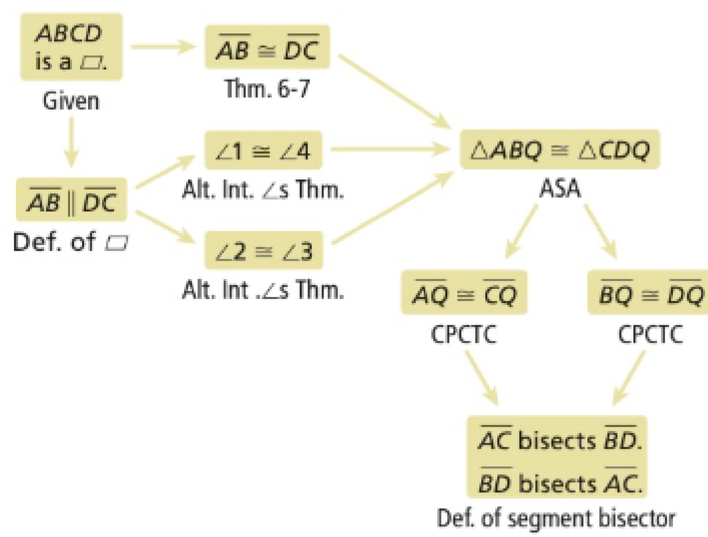
$\overline{AC}$ and $\overline{BD}$ are the diagonals of parallelogram $ABCD$.

Given: $ABCD$ is a parallelogram.

Prove: $\overline{AQ} \cong \overline{CQ}$, $\overline{BQ} \cong \overline{DQ}$



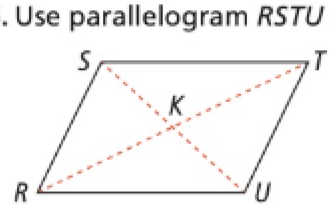
Proof:



STUDY TIP

The given information is usually the best statement to begin a proof.

Try It! **5.** Use parallelogram $RSTU$ with $SU = 35$ and $KT = 19$.



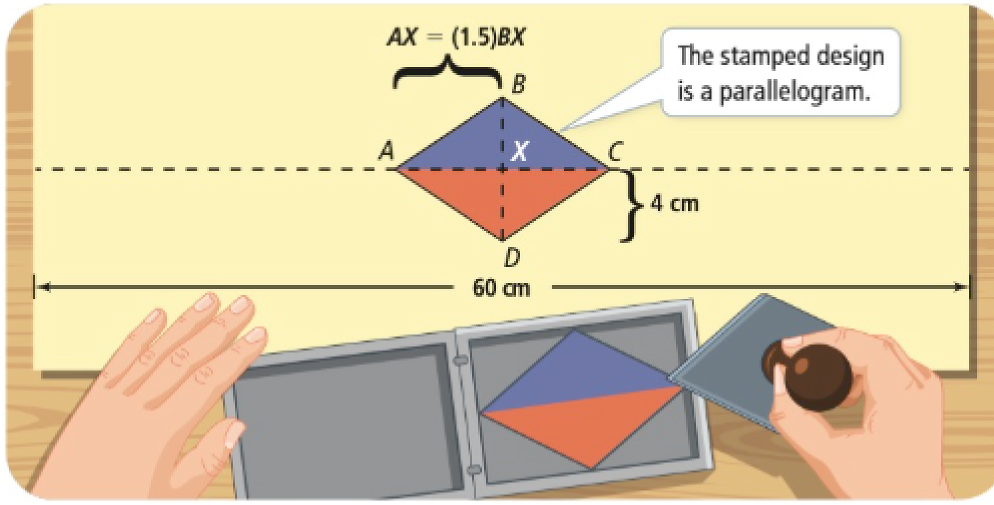
a. What is SK?

b. What is RT?

APPLICATION

EXAMPLE 6 Find Unknown Lengths in a Parallelogram

Corey stamps the orange and purple pattern shown on the front of a poster she is making. How many times will she need to stamp the design to make a row 60 cm wide along the dashed line?



Formulate ◀ By Theorem 6-8, the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other. So $\overline{BX} \cong \overline{DX}$, and $\overline{AX} \cong \overline{CX}$.

Compute ◀ **Step 1** Determine the length of the diagonal, $\overline{AC}$.

$BX = DX$ — Diagonals bisect each other.

$BX = 4$

$AX = 1.5\ BX$ — Given

$AX = 1.5(4)$

$AX = 6$

$AC = 2(AX)$ — Diagonals bisect each other.

$AC = 2(6)$

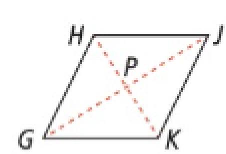
$AC = 12$

Step 2 Find the number of times Corey needs to stamp.

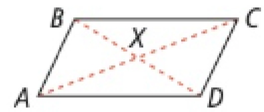
$60 \div 12 = 5$

Interpret ◀ Corey will need to stamp the design 5 times to make a row 60 cm wide.

Try It! **6.** Given parallelogram *GHJK*, if $PK = 4$ and $HK = \frac{2}{3}(GJ)$, what is *GP*?



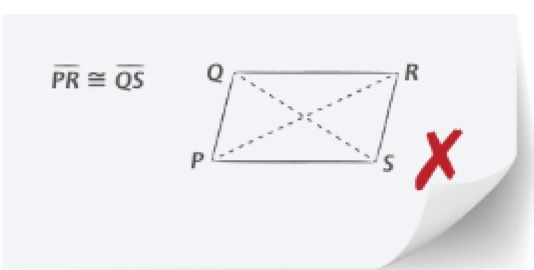
CONCEPT SUMMARY Properties of Parallelograms

	Angles of Parallelograms	Sides and Diagonals of Parallelograms
WORDS	Consecutive angles of a parallelogram are supplementary. Opposite angles of a parallelogram are congruent.	Opposite sides of a parallelogram are congruent. Diagonals of a parallelogram bisect each other.
SYMBOLS	**If...**  $\overline{AD} \parallel \overline{CB}$ $\overline{AB} \parallel \overline{DC}$ **Then...** $m\angle A + m\angle B = 180$ $m\angle B + m\angle C = 180$ $m\angle C + m\angle D = 180$ $m\angle D + m\angle A = 180$ $m\angle A = m\angle C$ $m\angle B = m\angle D$	**If...**  $\overline{AD} \parallel \overline{CB}$ $\overline{AB} \parallel \overline{DC}$ **Then...** $AB = CD$ $AD = BC$ $AX = CX$ $BX = DX$

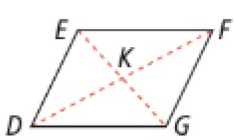
Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What are the relationships of the sides, the angles, and the diagonals of a parallelogram?

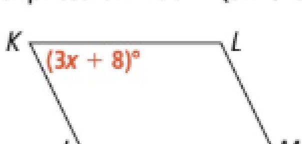
2. **Error Analysis** What is Carla's error?



3. **Analyze and Persevere** If you knew the length of $\overline{DF}$ in parallelogram $DEFG$, how would you find the length of $\overline{DK}$? Explain.



4. **Use Patterns and Structure** Given parallelogram $JKLM$, what could the expression $180 - (3x + 8)$ represent? Explain.

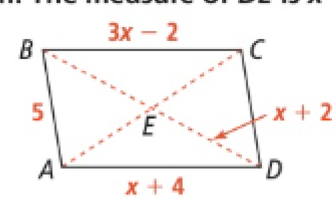


Do You KNOW HOW?

For Exercises 5 and 6, use parallelogram *ABCD* to find each length. The measure of $\overline{DE}$ is $x + 2$.

5. BC

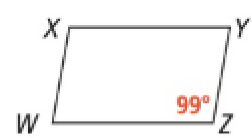
6. BD



For Exercises 7 and 8, use parallelogram *WXYZ* to find each angle measure.

7. $m\angle WXY$

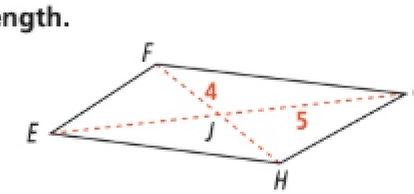
8. $m\angle XYZ$



For Exercises 9 and 10, use parallelogram *EFGH* to find each length.

9. EJ

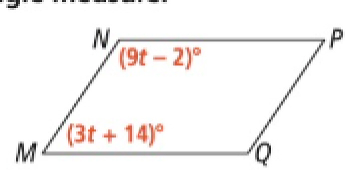
10. FH



For Exercises 11 and 12, use parallelogram *MNPQ* to find each angle measure.

11. $m\angle NPQ$

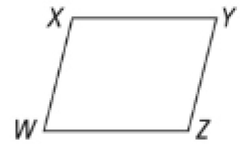
12. $m\angle PQM$



PRACTICE & PROBLEM SOLVING

UNDERSTAND

13. Communicate and Justify Write a proof of Theorem 6-5.

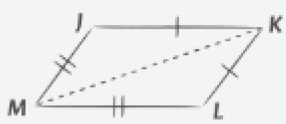


Given: $\overline{WX} \parallel \overline{ZY}$, $\overline{WZ} \parallel \overline{XY}$

Prove: $\overline{WX} \cong \overline{ZY}$, $\overline{WZ} \cong \overline{XY}$

14. Error Analysis In the statements shown, explain the student's error. What shape is the quadrilateral?

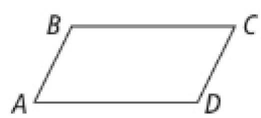
$\overline{JK} \cong \overline{KL}$ and $\overline{LM} \cong \overline{MJ}$.
$\angle MJK \cong \angle KLM$



Therefore, $\triangle MJK \cong \triangle KLM$ by SAS. The triangular halves of *JKLM* are congruent, so *JKLM* must be a parallelogram.

✗

15. Communicate and Justify Write a proof of Theorem 6-6.

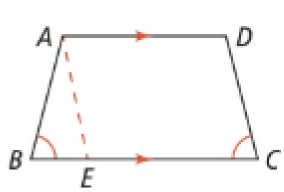


Given: $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$

Prove: $m\angle A + m\angle B = 180$
$m\angle B + m\angle C = 180$
$m\angle C + m\angle D = 180$
$m\angle D + m\angle A = 180$

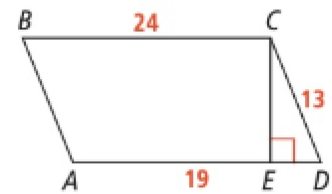
16. Represent and Connect In a parallelogram, opposite sides are congruent, and opposite angles are congruent. If all sides in a parallelogram are congruent, are all angles congruent also? Draw a picture to explain your answer.

17. Prove that the legs of an isosceles trapezoid are congruent. *Hint:* Use the figure below and construct $\overline{AE}$ to be parallel to $\overline{DC}$.

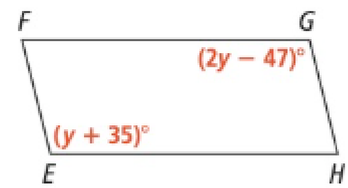


PRACTICE

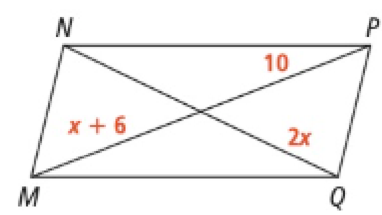
18. What are the values of *AB* and *DE* in parallelogram *ABCD*? SEE EXAMPLES 1 AND 2



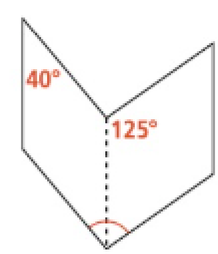
19. Quadrilateral *EFGH* is a parallelogram. What is $m\angle F$? SEE EXAMPLES 3 AND 4



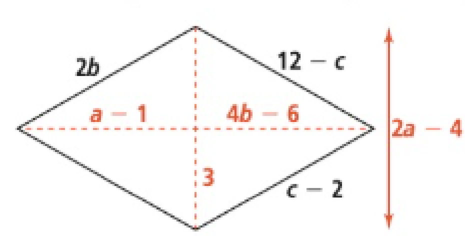
20. Quadrilateral *MNPQ* is a parallelogram. What is *NQ*? SEE EXAMPLES 5 AND 6



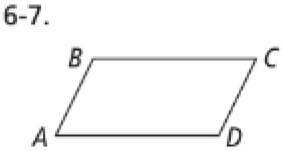
21. The figure below can be divided into two parallelograms. What is the angle measure of the point at the bottom?



22. Find the perimeter of the parallelogram.



23. Write a proof of Theorem 6-7.



Given: $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$

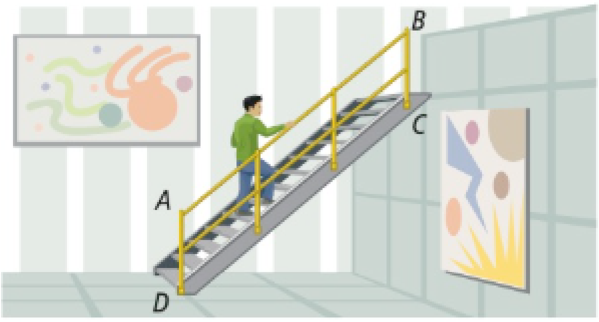
Prove: $\angle A \cong \angle C$, $\angle B \cong \angle D$

PRACTICE & PROBLEM SOLVING

APPLY

24. **Represent and Connect** All four arms of a mechanical jack are the same length, and they form a parallelogram. Turning the crank pulls the arms together, raising the top of the jack. How high is the top of the jack when the crank is 5 inches off the ground? Explain.

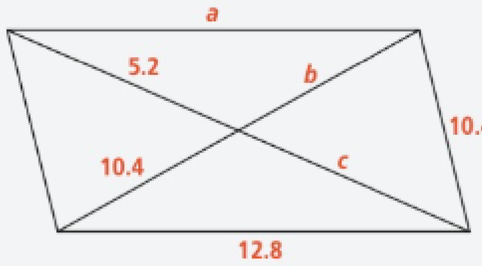
25. **Use Patterns and Structure** The handrails for a steel staircase form a parallelogram *ABCD*. Additional bars are needed one third and two thirds of the way up the stairs. Explain why the additional bars must be the same length as the end bars.



26. **Higher Order Thinking** Reagan designs a pattern consisting of large squares of the same size, small squares of the same size, and some parallelograms. She wants to replicate the pattern using tiles for her bathroom. Are the vertical and horizontal parallelograms congruent? Explain.

ASSESSMENT PRACTICE

27. Find the values of *a*, *b*, and *c* in the parallelogram. GR.1.4



28. **SAT/ACT** In parallelogram *ABCD*, which angle is congruent to ∠*ABC*?

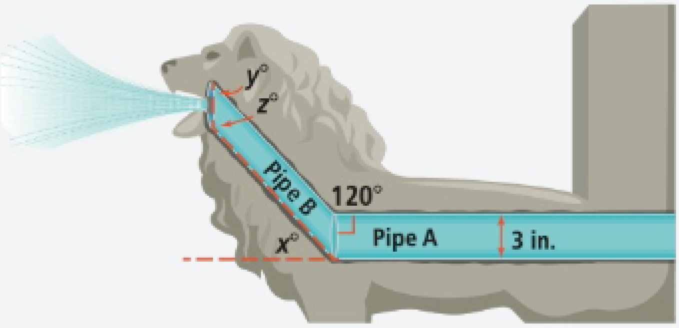
Ⓐ ∠*ABD*

Ⓑ ∠*CDA*

Ⓒ ∠*BCD*

Ⓓ ∠*DAB*

29. **Performance Task** A pipe at an amusement park sprays water onto visitors. A cross section of each pipe has the shape of a parallelogram.



Part A Pipe A makes a 120° angle with Pipe B. What are the interior angles of parallelogram B? What is *x*, the measure of the angle that Pipe B makes with the horizontal? Explain.

Part B Park engineers fasten a circular cap onto the end of Pipe B. In the middle of the cap is a nozzle to turn the spray of water into a mist. If the diameter of Pipe A is 3 inches, what is the diameter of the circular cap? Explain.

Part C What are *y* and *z*, the angle measures that the cap makes with Pipe B? Explain.

6-3 Proving a Quadrilateral Is a Parallelogram

I CAN… use properties of sides, angles, and diagonals to identify a parallelogram.

MA.912.GR.1.4–Prove relationships and theorems about parallelograms. Solve mathematical and real-world problems involving postulates, relationships and theorems of parallelograms.
MA.K12.MTR.1.1, MTR.2.1, MTR.3.1

EXPLORE & REASON

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Sketch the quadrilaterals as described in the table. Include the diagonals.

	Parallel Sides	Congruent Sides
Quadrilateral 1	0 pairs	2 consecutive pairs
Quadrilateral 2	1 pair	exactly 1 nonparallel pair
Quadrilateral 3	2 pairs	2 opposite pairs

A. Measure the angles of each quadrilateral. How are the angle measures in Quadrilateral 1 related to each other? In Quadrilateral 2? In Quadrilateral 3?

B. Measure the diagonals of each quadrilateral. How are the diagonals in Quadrilateral 1 related to each other? In Quadrilateral 2? In Quadrilateral 3?

C. **Communicate and Justify** Compare the relationships among the angles and diagonals of Quadrilateral 3 to those of the other two quadrilaterals. Are there any relationships that make Quadrilateral 3 unique?

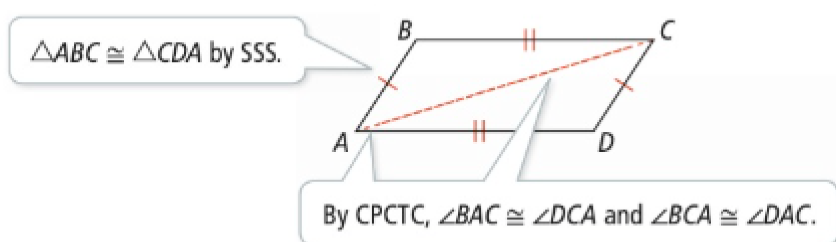
ESSENTIAL QUESTION

Which properties determine whether a quadrilateral is a parallelogram?

EXAMPLE 1 Investigate Sides to Confirm a Parallelogram

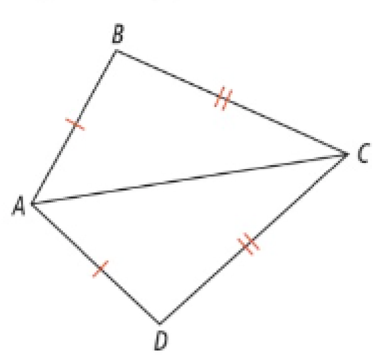
In quadrilateral *ABCD*, $\overline{AC}$ is a diagonal, $\overline{AB} \cong \overline{CD}$, and $\overline{AD} \cong \overline{BC}$. Is *ABCD* a parallelogram? Explain.

STUDY TIP
Recall that any segment is congruent to itself by the Reflexive Property of Congruence.



By the Converse of the Alternate Interior Angles Theorem, $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$. By definition, quadrilateral *ABCD* is a parallelogram.

Try It! **1.** Explain why you cannot conclude that *ABCD* is a parallelogram.

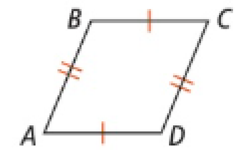


THEOREM 6-9 Converse of Theorem 6-5

If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

PROOF: SEE EXAMPLE 1.

If...



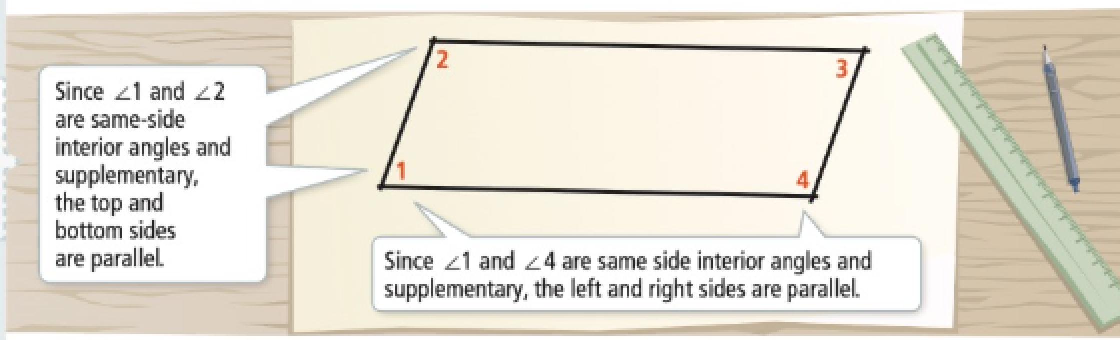
$\overline{AB} \cong \overline{CD}$
$\overline{AD} \cong \overline{BC}$

Then... *ABCD* is a parallelogram.

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Explore Angle Measures to Confirm a Parallelogram

A. Teo sketches a design of a quadrilateral-shaped building. If ∠1 is supplementary to ∠2 and ∠4, is his design a parallelogram?



LEARN TOGETHER
How do you value other perspectives and points of view respectfully?

The quadrilateral has two pairs of parallel sides, so it is a parallelogram. The design is a parallelogram.

B. Teo sketches a second design in which ∠1 is congruent to ∠3, and ∠2 is congruent to ∠4. Is that design a parallelogram?

The sum of interior angles is 360.

$$m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = 360$$
$$m\angle 1 + m\angle 2 + m\angle 1 + m\angle 2 = 360$$
$$2(m\angle 1 + m\angle 2) = 360$$
$$m\angle 1 + m\angle 2 = 180$$

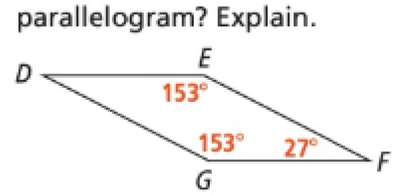
Since ∠1 ≅ ∠3 and ∠2 ≅ ∠4, substitute $m\angle 1$ for $m\angle 3$ and $m\angle 2$ for $m\angle 4$.

Substitute $m\angle 4$ for $m\angle 2$.

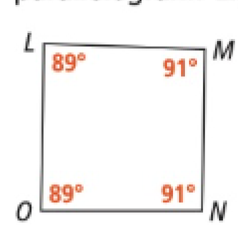
$$m\angle 1 + m\angle 4 = 180$$

Because the edges form a quadrilateral with one angle supplementary to both consecutive angles and from the result in part A, the second design is also a parallelogram.

Try It! **2. a.** Is *DEFG* a parallelogram? Explain.



b. Is *LMNO* a parallelogram? Explain.

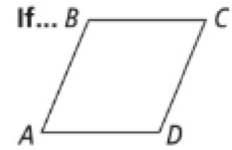


THEOREM 6-10 Converse of Theorem 6-6

If an angle of a quadrilateral is supplementary to both of its consecutive angles, then the quadrilateral is a parallelogram.

PROOF: SEE EXERCISE 12.

If...



$m\angle A + m\angle B = 180$

$m\angle A + m\angle D = 180$

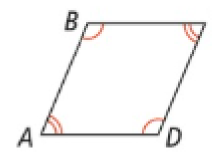
Then... *ABCD* is a parallelogram.

THEOREM 6-11 Converse of Theorem 6-7

If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

PROOF: SEE EXERCISE 14.

If...



$\angle A \cong \angle C$

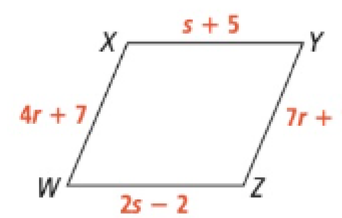
$\angle B \cong \angle D$

Then... *ABCD* is a parallelogram.

EXAMPLE 3 Find Values to Make Parallelograms

A. For what values of *r* and *s* is *WXYZ* a parallelogram?

Quadrilateral *WXYZ* is a parallelogram if both pairs of opposite sides are congruent.



$7r + 1 = 4r + 7$ $\qquad$ $2s - 2 = s + 5$

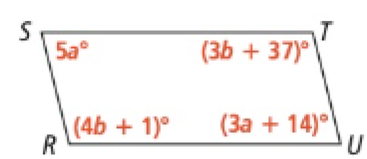
$r = 2$ $\qquad$ $s = 7$

If $r = 2$ and $s = 7$, then *WX* and *ZY* are both 15, and *XY* and *WZ* are both 12. So *WXYZ* is a parallelogram.

GENERALIZE
Think about the properties of a parallelogram. What do you know about a quadrilateral that is a parallelogram?

B. For what values of *a* and *b* is *RSTU* a parallelogram?

Quadrilateral *RSTU* is a parallelogram if both pairs of opposite angles are congruent.



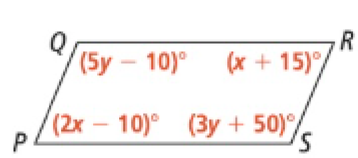
$5a = 3a + 14$ $\qquad$ $4b + 1 = 3b + 37$

$2a = 14$ $\qquad$ $b = 36$

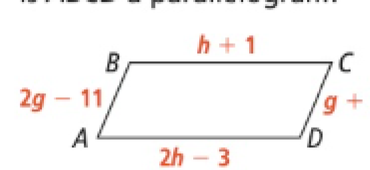
$a = 7$

If $a = 7$ and $b = 36$, then angles *S* and *U* are both 35° and angles *T* and *R* are both 145°. So *RSTU* is a parallelogram.

Try It! **3. a.** If $x = 25$ and $y = 30$, is *PQRS* a parallelogram?



b. If $g = 14$ and $h = 5$, is *ABCD* a parallelogram?

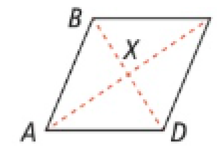


THEOREM 6-12 Converse of Theorem 6-8

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

PROOF: SEE EXAMPLE 4.

If...



$\overline{AX} \cong \overline{CX}$

$\overline{BX} \cong \overline{DX}$

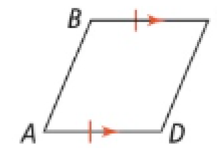
Then... *ABCD* is a parallelogram.

THEOREM 6-13

If one pair of opposite sides of a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram.

PROOF: SEE EXERCISE 20.

If...



$\overline{AD} \cong \overline{BC}$

$\overline{AD} \parallel \overline{BC}$

Then... *ABCD* is a parallelogram.

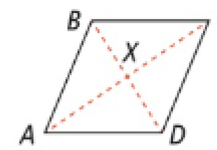
PROOF

EXAMPLE 4 Investigate Diagonals to Confirm a Parallelogram

Given: $\overline{AX} \cong \overline{CX}$ and $\overline{BX} \cong \overline{DX}$

Prove: *ABCD* is a parallelogram

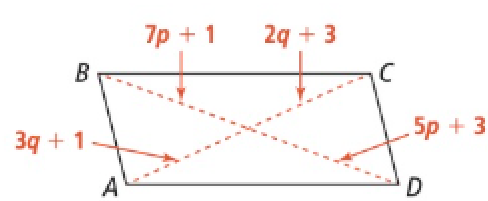
Proof:



Statements	Reasons
1) $\overline{AX} \cong \overline{CX}$ and $\overline{BX} \cong \overline{DX}$	**1)** Given
2) $\angle AXD \cong \angle CXB$ and $\angle AXB \cong \angle CXD$	**2)** Vertical Angles Theorem
3) $\triangle AXD \cong \triangle CXB$ and $\triangle AXB \cong \triangle CXD$	**3)** SAS
4) $\overline{AD} \cong \overline{CB}$ and $\overline{AB} \cong \overline{CD}$	**4)** CPCTC
5) *ABCD* is a parallelogram.	**5)** Theorem 6-9

COMMON ERROR
Remember, noncongruent diagonals may bisect each other, just as congruent diagonals do.

Try It! **4.** For what values of *p* and *q* is *ABCD* a parallelogram?

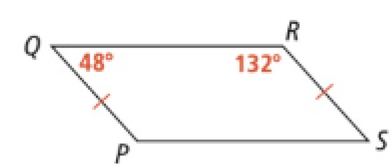


EXAMPLE 5 Identify a Parallelogram

A. Is *PQRS* a parallelogram? Explain.

Same-side interior angles *Q* and *R* are supplementary, so $\overline{QP} \parallel \overline{RS}$.

PQRS is a parallelogram by Theorem 6-13.



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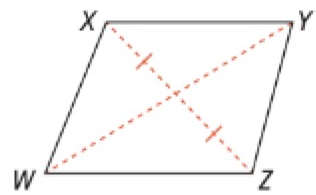
STUDY TIP
After determining that there is not enough information, it is good practice to think about what additional information would be needed to show that the quadrilateral is a parallelogram and why.

EXAMPLE 1 CONTINUED

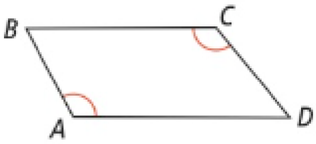
B. Is *WXYZ* a parallelogram? Explain.

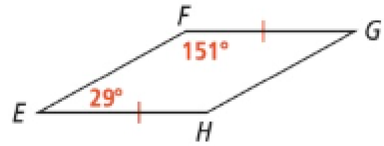
Although diagonal $\overline{WY}$ bisects $\overline{XZ}$, diagonal $\overline{XZ}$ does not necessarily bisect $\overline{WY}$. Quadrilateral *WXYZ* does not meet the conditions of Theorem 6-12, and so it is not necessarily a parallelogram.



Try It! **5. a.** Is *ABCD* a parallelogram? Explain.

b. Is *EFGH* a parallelogram? Explain.

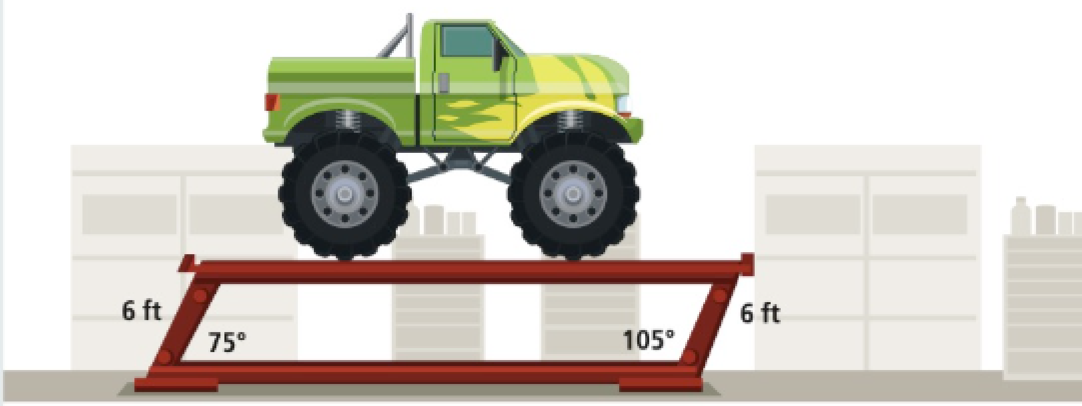




APPLICATION

EXAMPLE 6 Verify a Parallelogram

A mechanic raises a truck using a lift. For safety, the floor must be horizontal and the top of the lift must be parallel to the floor. Is the lift shown in a safe position? Explain.



Formulate The lift and the floor form a quadrilateral. If the quadrilateral is a parallelogram, then the side holding the truck will be parallel to the floor and the lift will be safe.

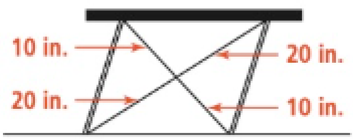
Compute Find the sum of the given angles.

$$105 + 75 = 180$$

Since a pair of same-side alternate interior angles are supplementary, the 6-ft sides of the lift are parallel. The lift is a parallelogram by Theorem 6-13.

Interpret Opposite sides of a parallelogram are parallel, so the side of the lift holding the truck is parallel to the floor. The lift is in a safe position.

Try It! **6.** A carpenter builds the table shown. If the floor is level, how likely is it that a ball placed on the table will roll off?

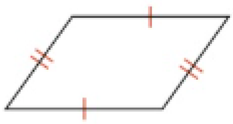


CONCEPT SUMMARY Identifying Quadrilaterals That Are Parallelograms

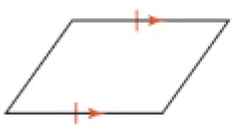
SIDES AND DIAGONALS

A quadrilateral is a parallelogram if

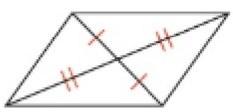
- two pairs of opposite sides are congruent



- one pair of opposite sides is congruent and parallel



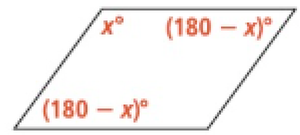
- the diagonals bisect each other



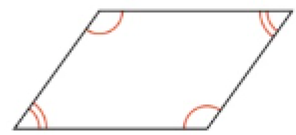
ANGLES

A quadrilateral is a parallelogram if

- one angle is supplementary to both consecutive angles



- two pairs of opposite angles are congruent

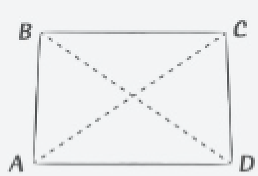


Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** Which properties determine whether a quadrilateral is a parallelogram?

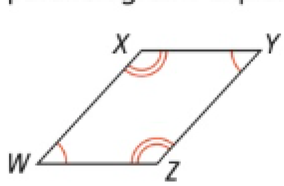
2. **Error Analysis** Explain why Rochelle is incorrect.

Given: $AC = BD$



ABCD is a parallelogram, because quadrilaterals with congruent diagonals are parallelograms.

3. **Analyze and Persevere** Is the information in the diagram enough to show *WXYZ* is a parallelogram? Explain.



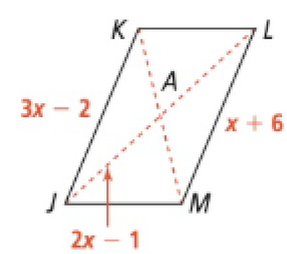
Do You KNOW HOW?

What must each angle measure be in order for quadrilateral *DEFG* to be a parallelogram?

4. $m\angle D$

5. $m\angle E$

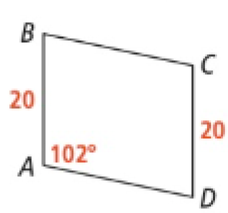
What must each length be in order for quadrilateral *JKLM* to be a parallelogram?



6. *JK*

7. *JL*

Use the diagram for Exercises 8 and 9.



8. If $AB \parallel DC$, is *ABCD* a parallelogram? Explain.

9. If *ABCD* is a parallelogram, how does *AC* compare to *BD*? Explain.

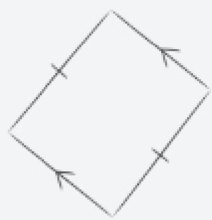
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. Represent and Connect If you are given a drawing of a quadrilateral, how can you determine whether or not it is a parallelogram? What tool or tools can you use?

11. Error Analysis Ahmed uses the following explanation to prove that a figure is a parallelogram. What is Ahmed's error?

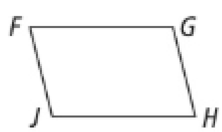
The quadrilateral has a pair of opposite sides congruent and a pair of opposite sides parallel. According to Theorem 6-13, the figure is a parallelogram.



12. Communicate and Justify Write a proof of Theorem 6-12.

Given: $m\angle F + m\angle G = 180$
$m\angle F + m\angle J = 180$

Prove: $FGHJ$ is a parallelogram.

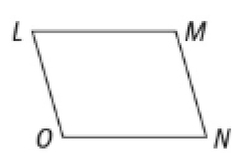


13. Mathematical Connections A rectangle is defined as a quadrilateral with four right angles. Which theorem or theorems from the lesson explain why a rectangle is a parallelogram? Explain how the theorem or theorems apply.

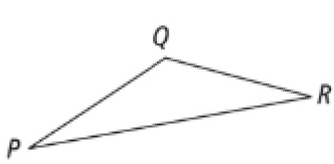
14. Communicate and Justify Write a proof of Theorem 6-11.

Given: $\angle L \cong \angle N$, $\angle M \cong \angle O$

Prove: $LMNO$ is a parallelogram.



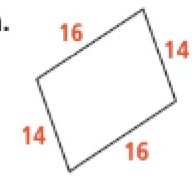
15. Higher Order Thinking Describe rigid motions you can apply to $\triangle PQR$ to construct three different parallelograms by combining the preimage and image. Explain why the resulting figures are parallelograms.



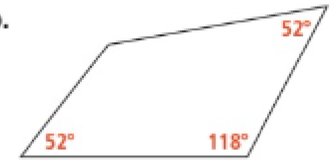
PRACTICE

16. Is each quadrilateral a parallelogram? Explain.
SEE EXAMPLES 1 AND 2

a.

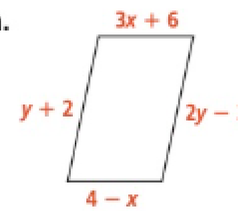


b.



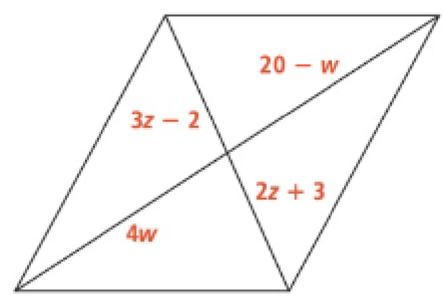
17. In each figure, for what values of x and y is the figure a parallelogram? SEE EXAMPLE 3

a.

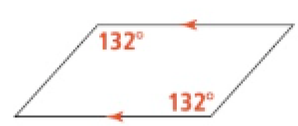


b.

18. Given the lengths shown, for what values of w and z is the figure a parallelogram?
SEE EXAMPLE 4



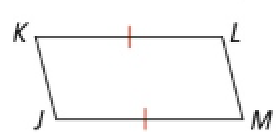
19. Is the figure below a parallelogram? Explain.
SEE EXAMPLES 5 AND 6



20. Write a proof of Theorem 6-13.

Given: $\overline{KL} \parallel \overline{JM}$, $\overline{KL} \cong \overline{JM}$

Prove: $JKLM$ is a parallelogram.

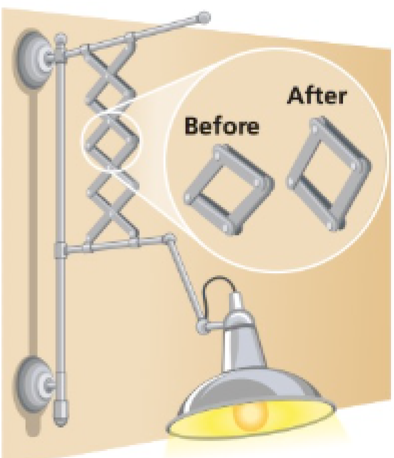


Hint: Construct diagonal $\overline{JL}$.

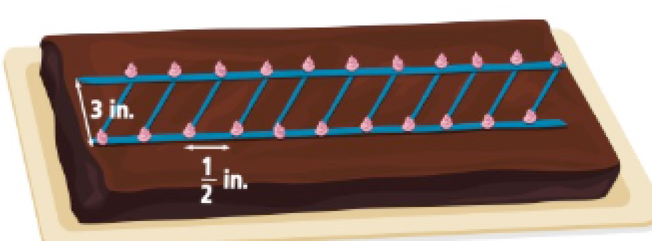
PRACTICE & PROBLEM SOLVING

APPLY

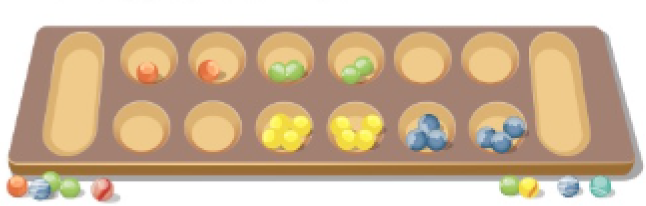
21. **Analyze and Persevere** A lamp on a wall is suspended from an extendable arm that allows the lamp to slide up and down. When it expands, does the shape shown remain a parallelogram? Explain.



22. **Represent and Connect** Simon wants to decorate a cake with a pattern of parallelograms. He first pipes two parallel lines that are 3 inches apart. He then makes a mark every $\frac{1}{2}$ inch along each line. He pipes a line from one mark to the next on the opposite side. Does this ensure that the lines will be parallel? Explain your answer.

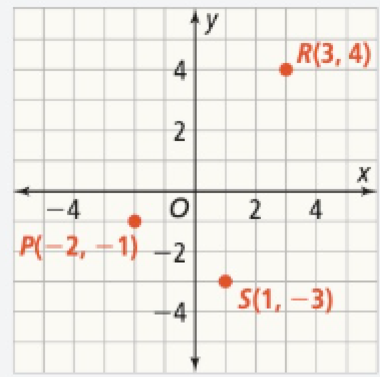


23. **Communicate and Justify** In the game shown, the arrangement of marbles on the board is called a *parallelogram formation*. Why is that name appropriate? Explain.



ASSESSMENT PRACTICE

24. Copy the graph and plot all possible coordinate pairs for point Q on the coordinate plane so that points P, Q, R, and S form the vertices of a parallelogram. GR.1.4



25. **SAT/ACT** In quadrilateral $ABCD$, $\angle A \cong \angle C$. Which additional statement can be used to show that $ABCD$ is a parallelogram?

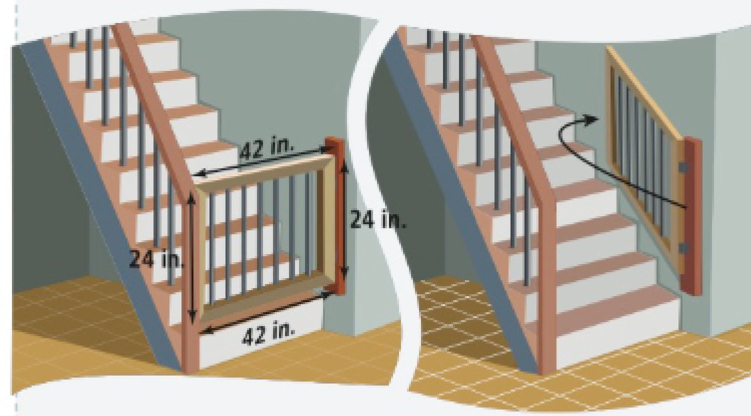
Ⓐ $m\angle A + m\angle C = 180$

Ⓑ $\overline{BD}$ bisects $\overline{AC}$

Ⓒ $m\angle B + m\angle D = 180$

Ⓓ $\angle B \cong \angle D$

26. **Performance Task** Margaret helps her sister build a baby gate that is built from dowels hinged at the top and bottom, so the gate can open up against the wall along the stairs. They call it the parallelogram gate.



Part A Are they correct to call it a parallelogram gate? Explain.

Part B What are the measurements of the sides of the gate when the gate is open? Explain.

Part C Margaret's father suggests that they add two diagonal slats at the front of the baby gate. What would that do to the gate? Explain.

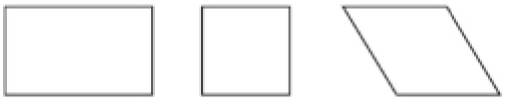
6-4 Properties of Special Parallelograms

I CAN… use the properties of rhombuses, rectangles, and squares to solve problems.

MA.912.GR.1.4–Prove relationships and theorems about parallelograms. Solve mathematical and real-world problems involving postulates, relationships and theorems of parallelograms.
MA.K12.MTR.5.1, MTR.1.1, MTR.6.1

EXPLORE & REASON

Consider these three figures.



A. What questions would you ask to determine whether each figure is a parallelogram?

B. **Communicate and Justify** What questions would you ask to determine whether Figure 1 is a rectangle? What additional questions would you ask to determine whether Figure 2 is a square?

C. If all three figures are parallelograms, what are all the possible names for Figure 3? How do you know?

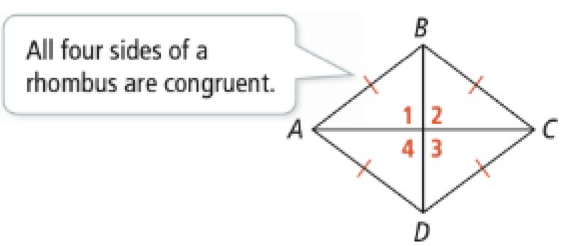
ESSENTIAL QUESTION

What properties of rhombuses, rectangles, and squares differentiate them from other parallelograms?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Find the Diagonals of a Rhombus

A. Parallelogram *ABCD* is a rhombus. What are the measures of ∠1, ∠2, ∠3, and ∠4?



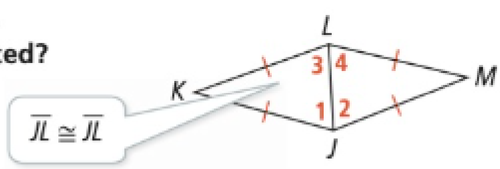
STUDY TIP
Recall that a rhombus is a parallelogram, so it has all the properties of parallelograms.

By the Converse of the Perpendicular Bisector Theorem, *B* and *D* are on the perpendicular bisector of $\overline{AC}$, so $\overline{AC} \perp \overline{BD}$.

All four angles formed by the intersection of the diagonals are right angles, so the measure of ∠1, ∠2, ∠3, and ∠4 is 90.

B. Parallelogram *JKLM* is a rhombus. How are ∠1, ∠2, ∠3, and ∠4 related?

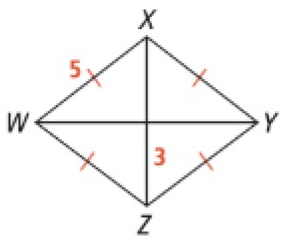
By SSS, △*JKL* ≅ △*JML*, so ∠1 ≅ ∠2 and ∠3 ≅ ∠4.



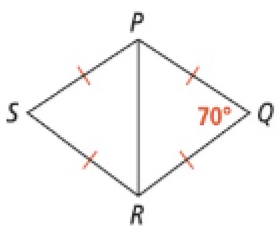
The diagonal $\overline{JL}$ bisects ∠*KLM* and ∠*KJM*.

Try It!

1. a. What is *WY*?



b. What is *m*∠*RPS*?

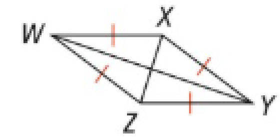


THEOREM 6-14

If a parallelogram is a rhombus, then its diagonals are perpendicular bisectors of each other.

PROOF: SEE EXERCISE 13.

If...



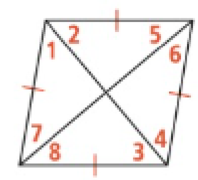
Then... $\overline{WY}$ and $\overline{XZ}$ are perpendicular bisectors of each other.

THEOREM 6-15

If a parallelogram is a rhombus, then each diagonal bisects a pair of opposite angles.

PROOF: SEE EXERCISE 17.

If...



Then... $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$, $\angle 5 \cong \angle 6$, and $\angle 7 \cong \angle 8$.

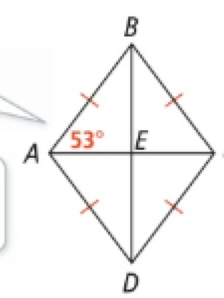
EXAMPLE 2 Find Lengths and Angle Measures in a Rhombus

A. Quadrilateral *ABCD* is a rhombus. What is $m\angle ADE$?

$\overline{AC}$ bisects $\angle BAD$, so $m\angle DAC = 53$.

$$m\angle DAE + m\angle AED + m\angle ADE = 180$$
$$53 + 90 + m\angle ADE = 180$$
$$m\angle ADE = 37$$

$\overline{AC} \perp \overline{BD}$, so $m\angle AED = 90$.



COMMON ERROR
You may incorrectly state that $m\angle ADE = m\angle DAE$. Remember that consecutive angles are not necessarily congruent.

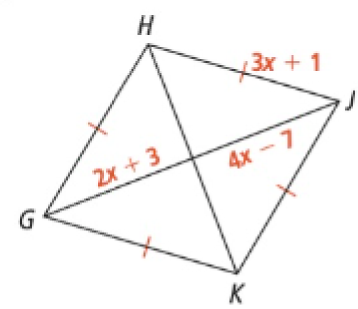
B. Quadrilateral *GHJK* is a rhombus. What is *GH*?

Step 1 Find *x*.

$$2x + 3 = 4x - 7$$
$$2x = 10$$
$$x = 5$$

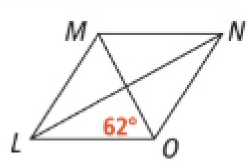
Step 2 Use the value of *x* to find *GH*.

$$GH = HJ = 3(5) + 1 = 16$$

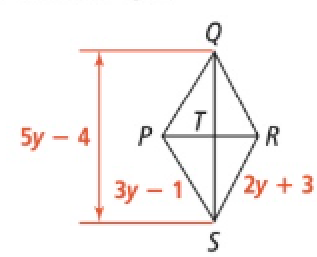


Try It! **2.** Each quadrilateral is a rhombus.

a. What is $m\angle MNO$?



b. What is *QT*?



THEOREM 6-16

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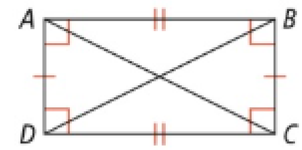
If a parallelogram is a rectangle, then its diagonals are congruent.

PROOF: SEE EXAMPLE 3.

If...



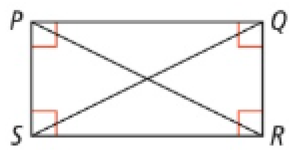
Then... $\overline{AC} \cong \overline{BD}$

PROOF

EXAMPLE 3 Prove Diagonals of a Rectangle Are Congruent

Write a proof for Theorem 6-18.

Given: *PQRS* is a rectangle.



Prove: $\overline{PR} \cong \overline{QS}$

Plan: To show that the diagonals are congruent, find a pair of congruent triangles that each diagonal is a part of. Both △*PSR* and △*QRS* appear to be congruent. Think about how to use properties of rectangles to show they are congruent. Draw each triangle separately and label the congruent sides.

Proof:

Statements	Reasons
1) *PQRS* is a rectangle.	1) Given
2) *PQRS* is a parallelogram.	2) Def. of rectangle
3) $\overline{PS} \cong \overline{QR}$	3) Opposite sides of a parallelogram are congruent.
4) ∠*PSR* and ∠*QRS* are right angles.	4) Def. of rectangle
5) ∠*PSR* ≅ ∠*QRS*	5) All right angles are congruent.
6) $\overline{SR} \cong \overline{RS}$	6) Reflexive Prop. of Congruence
7) △*PSR* ≅ △*QRS*	7) SAS Triangle Congruence Thm.
8) $\overline{PR} \cong \overline{QS}$	8) CPCTC

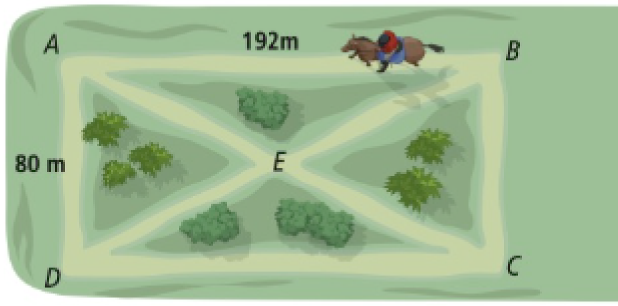
STUDY TIP
When you see triangles in a diagram for a proof, you can often use congruent triangles and CPCTC to complete the proof.

Try It! 3. A carpenter needs to check the gate his apprentice built to be sure it is rectangular. The diagonals measure 52 inches and 53 inches. Is the gate rectangular? Explain.

APPLICATION

EXAMPLE 4 Find Diagonal Lengths of a Rectangle

Paul is training his horse to run the course at a pace of 4 meters per second or faster. Paul rides his horse from *D* to *C* to *E* to *B* in 1 minute 30 seconds. The figure *ABCD* is a rectangle. Did he make his goal?



Formulate ◀ Use the Pythagorean Theorem to find *BD*. Then use properties of rectangles to find each segment length and the total distance. Finally, determine his speed.

Compute ◀

$(BD)^2 = 80^2 + 192^2$

$(BD)^2 = 43{,}264$

$BD = 208$

Apply the Pythagorean Theorem.

Use the properties of rectangles to find the total distance.

$CE = EB = 104$

$DC + CE + EB = 192 + 104 + 104 = 400$

Diagonals are congruent and bisect each other.

Determine the pace.

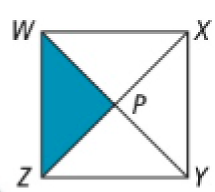
$400 \div 90 \approx 4.4$

Interpret ◀ Paul's horse ran at a pace of about 4.4 m/s, so he made his goal.

Try It! **4.** A rectangle with area 1,600 m^2 is 4 times as long as it is wide. What is the sum of the diagonals?

EXAMPLE 5 Diagonals and Angle Measures of a Square

Figure *WXYZ* is a square. If $WY + XZ = 92$, what is the area of $\triangle WPZ$?



Since the figure is also a rhombus, $\overline{WY} \perp \overline{XZ}$ and *WP* and *ZP* are the base and height of $\triangle WPZ$.

Step 1 Find the lengths of the diagonals.

$WY + XZ = 92$

$WY = XZ = 46$

WXYZ is a rectangle, so $\overline{WY} \cong \overline{XZ}$.

Step 2 Find *WP* and *ZP*.

$WP = \frac{1}{2}(WY) = 23$

$ZP = \frac{1}{2}(XZ) = 23$

WXYZ is a parallelogram, so $\overline{WY}$ and $\overline{XZ}$ bisect each other.

Step 3 Find the area of $\triangle WPZ$.

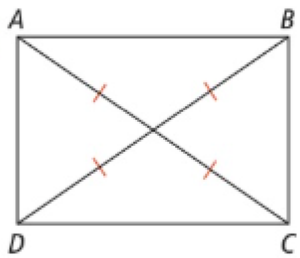
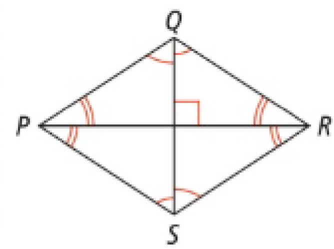
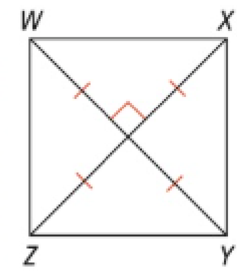
$\text{area}(\triangle WPZ) = \frac{1}{2}(23)(23) = 264.5$

The area of $\triangle WPZ$ is 264.5 square units.

USE PATTERNS AND STRUCTURE
Consider the four triangles formed by the diagonals of a square. What observations do you make about these triangles?

Try It! **5.** Square *ABCD* has diagonals $\overline{AC}$ and $\overline{BD}$. What is $m\angle ABD$? Explain.

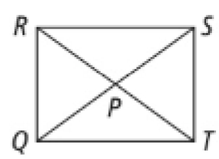
CONCEPT SUMMARY Properties of Special Parallelograms

	Rectangle	Rhombus	Square
WORDS	If a parallelogram is a rectangle, then the diagonals are congruent.	If a parallelogram is a rhombus, then the diagonals are perpendicular and bisect each pair of opposite angles.	If a parallelogram is a square, the properties of both a rectangle and a rhombus apply.
DIAGRAMS			
SYMBOLS	$\overline{AC} \cong \overline{BD}$	$\overline{PR} \perp \overline{QS}$	$\overline{WY} \cong \overline{XZ}$ $\overline{WY} \perp \overline{XZ}$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What properties of rhombuses, rectangles, and squares differentiate them from other parallelograms?

2. **Error Analysis** Figure *QRST* is a rectangle. Ramona wants to show that the four interior triangles are congruent. What is Ramona's error?



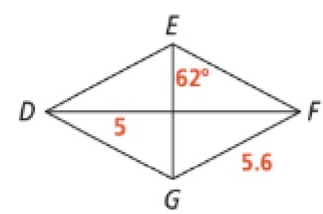
Diagonals of a rectangle are congruent and bisect each other, so $\overline{RP} \cong \overline{TP} \cong \overline{QP} \cong \overline{SP}$. Because the diagonals are perpendicular bisectors, $\angle RPS$, $\angle SPT$, $\angle TPQ$, and $\angle QPR$ are right angles. Therefore, by SAS,

$\triangle RPS \cong \triangle SPT \cong \triangle TPQ \cong \triangle PQR$. ✗

3. **Analyze and Persevere** Is any quadrilateral with four congruent sides a rhombus? Explain.

Do You KNOW HOW?

Find each length and angle measure for rhombus *DEFG*. Round to the nearest tenth.

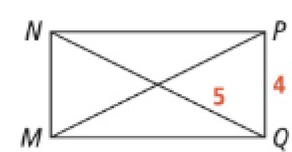


4. *DF*

5. $m\angle DFG$

6. *EG*

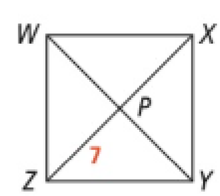
Find each length for rectangle *MNPQ*. Round to the nearest tenth.



7. *MP*

8. *MQ*

Find each length and angle measure for square *WXYZ*.

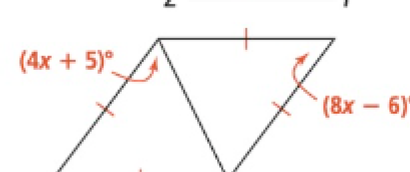


9. $m\angle YPZ$

10. $m\angle XWP$

11. *XZ*

12. What is the value of *x*?



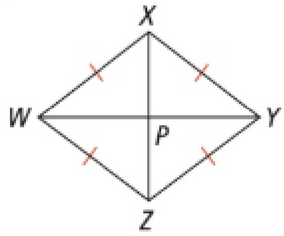
PRACTICE & PROBLEM SOLVING

UNDERSTAND

13. Communicate and Justify Write a proof of Theorem 6-14.

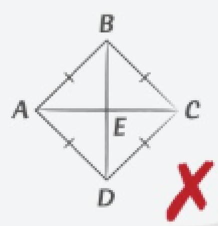
Given: *WXYZ* is a rhombus.

Prove: $\overline{WY}$ and $\overline{XZ}$ are perpendicular bisectors of each other.

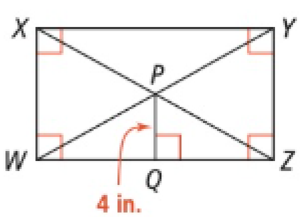


14. Error Analysis Figure *ABCD* is a rhombus. What is Malcolm's error?

Since *ABCD* is a rhombus, $\overline{AB} \cong \overline{CD}$. Since the diagonals of a rhombus bisect each other, $\overline{AE} \cong \overline{BE} \cong \overline{CE} \cong \overline{DE}$. So, by SSS, $\triangle ABE \cong \triangle CDE$.



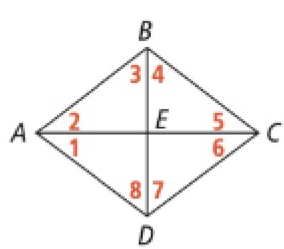
15. Mathematical Connections The area of rectangle *WXYZ* is 115.5 in.^{2}. What is the perimeter of $\triangle XYZ$? Explain your work.



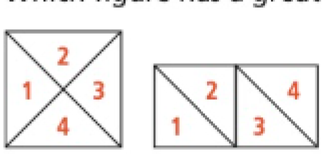
16. Communicate and Justify Write a proof of Theorem 6-15.

Given: *ABCD* is a rhombus.

Prove: $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$, $\angle 5 \cong 6$, $\angle 7 \cong \angle 8$



17. Higher Order Thinking A square is cut apart and reassembled into a rectangle as shown. Which figure has a greater perimeter? Explain.



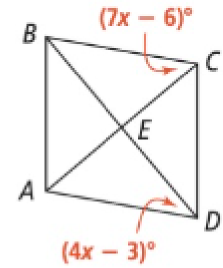
PRACTICE

For Exercises 18–20, find each angle measure for rhombus *ABCD*. SEE EXAMPLES 1 AND 2

18. $m\angle ACD$

19. $m\angle ABC$

20. $m\angle BEA$

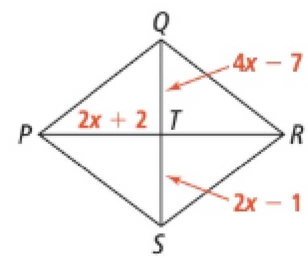


For Exercises 21–23, find each length for rhombus *PQRS*. Round to the nearest tenth.
SEE EXAMPLES 1 AND 2

21. *TR*

22. *QS*

23. *PS*



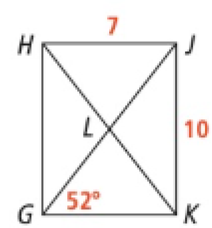
For Exercises 24–27, find each length and angle measure for rectangle *GHJK*. Round to the nearest tenth. SEE EXAMPLES 3 AND 4

24. $m\angle GHK$

25. $m\angle HLJ$

26. *GJ*

27. *HL*

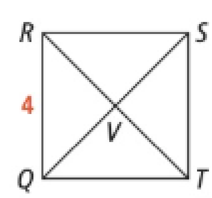


For Exercises 28–30, find each length and value for square *QRST*. Round to the nearest tenth.
SEE EXAMPLE 5

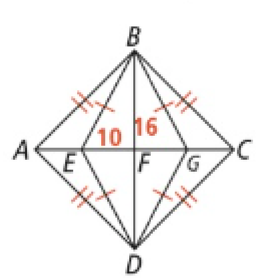
28. *SV*

29. *RT*

30. perimeter of $\triangle RVS$



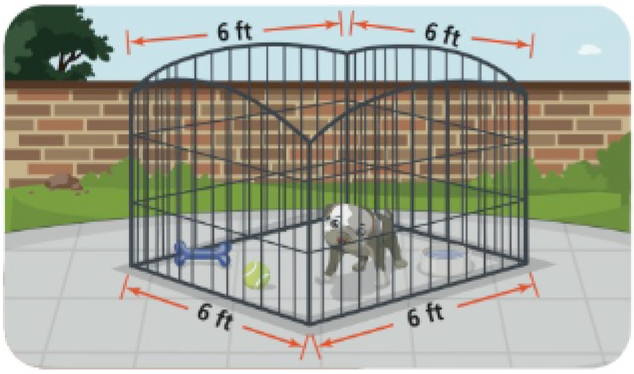
31. If *ABCD* is a square, what is *GC*?



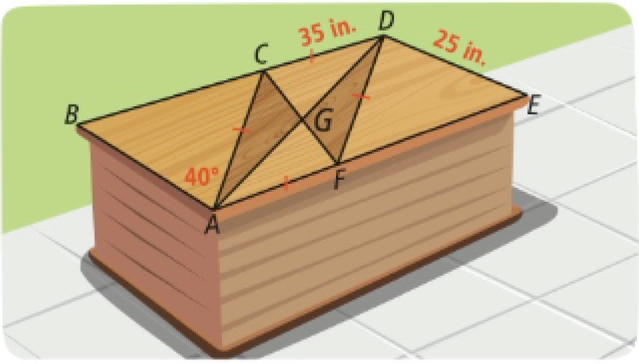
PRACTICE & PROBLEM SOLVING

APPLY

32. Use Patterns and Structure Jordan wants a collapsible puppy pen that gives his puppy at least 35 square feet of area and at least 10 feet of diagonal length. Should Jordan buy the pen shown? Explain.



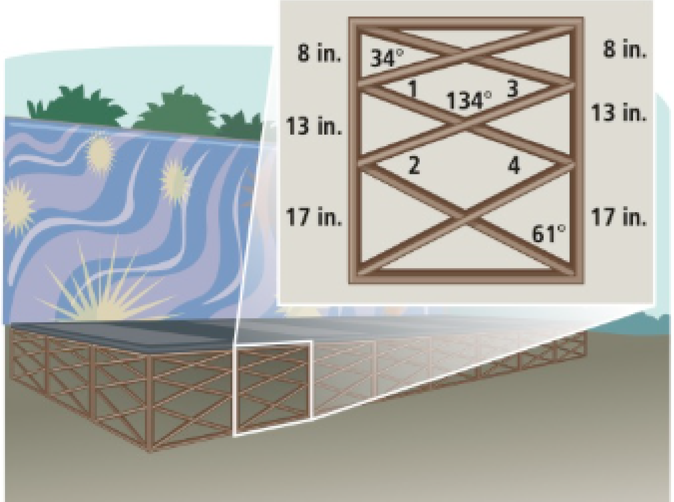
33. Analyze and Persevere Luis is using different types of wood to make a rectangular inlay top for a chest with the pattern shown.



a. What angle should he cut for $\angle CDG$? Explain.

b. If he makes the table top correctly, what will the length of the completed top be?

34. Choose Efficient Methods A carpenter is building a support for a stage. What should be the measures of $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$? Explain your answers.



ASSESSMENT PRACTICE

35. Select all the true statements about relationships within rectangles. GR.1.4

☐ **A.** Diagonals bisect each other.

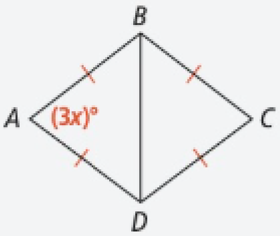
☐ **B.** Adjacent sides are perpendicular.

☐ **C.** Diagonals are perpendicular.

☐ **D.** Consecutive angles are supplementary.

☐ **E.** Consecutive angles are congruent.

36. SAT/ACT Which expression gives $m\angle DBC$?



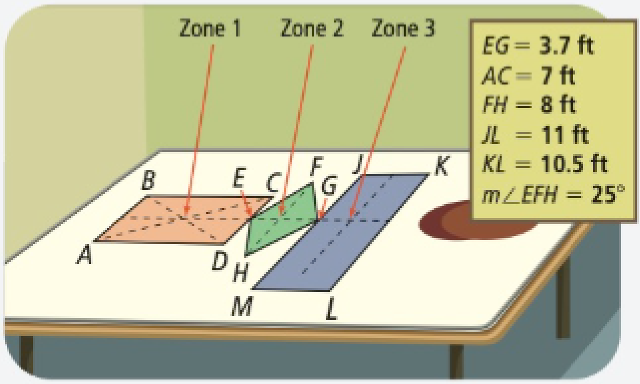
Ⓐ $\left(180 - \frac{3x}{2}\right)^\circ$

Ⓑ $(180 - 3x)^\circ$

Ⓒ $\left(\frac{180 - 3x}{2}\right)^\circ$

Ⓓ $\left(\frac{3x}{2} - 180\right)^\circ$

37. Performance Task At a carnival, the goal is to toss a disc into one of three zones to win a prize. Zone 1 is a square, zone 2 is a rhombus, and zone 3 is a rectangle. Some measurements have been provided.



Part A What are the lengths of the sides of each zone?

Part B What are the angle measures of each zone?

Part C What is the area of each zone?

6-5 Conditions of Special Parallelograms

I CAN… identify rhombuses, rectangles, and squares by the characteristics of their diagonals.

MA.912.GR.1.4–Prove relationships and theorems about parallelograms. Solve mathematical and real-world problems involving postulates, relationships and theorems of parallelograms.
MA.K12.MTR.4.1, MTR.2.1, MTR.7.1

MODEL & DISCUSS

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The sides of the lantern are identical quadrilaterals.

A. Communicate and Justify How could you check to see whether a side is a parallelogram? Justify your answer.

B. Does the side appear to be rectangular? How could you check?

C. Do you think that diagonals of a quadrilateral can be used to determine whether the quadrilateral is a rectangle? Explain.

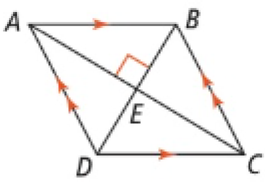
ESSENTIAL QUESTION

Which properties of the diagonals of a parallelogram help you to classify a parallelogram?

CONCEPTUAL UNDERSTANDING

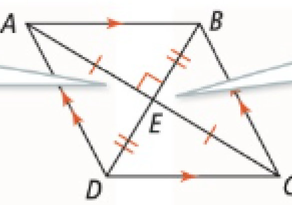
EXAMPLE 1 Use Diagonals to Identify Rhombuses

Information about diagonals can help to classify a parallelogram. In parallelogram *ABCD*, $\overline{AC}$ is perpendicular to $\overline{BD}$. What else can you conclude about the parallelogram?



STUDY TIP
Parallelograms have several properties, and some properties may not help you solve a particular problem. Here, the fact that diagonals bisect each other allows the use of SAS.

The diagonals of a parallelogram bisect each other, so $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$.

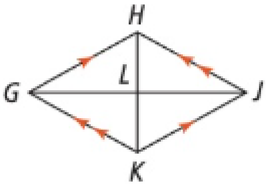


Any angle at *E* either forms a linear pair or is a vertical angle with ∠*AEB*, so all four angles are right angles.

The four triangles are congruent by SAS, so $\overline{AB} \cong \overline{CB} \cong \overline{CD} \cong \overline{AD}$.

Since *ABCD* is a parallelogram with four congruent sides, *ABCD* is a rhombus.

Try It! **1.** If ∠*JHK* and ∠*JGK* are complementary, what else can you conclude about *GHJK*? Explain.

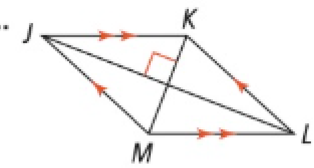


THEOREM 6-17 Converse of Theorem 6-14

If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

PROOF: SEE EXERCISE 9.

If...



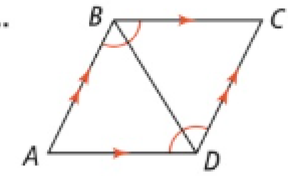
Then... $\overline{JK} \cong \overline{KL} \cong \overline{LM} \cong \overline{MJ}$

THEOREM 6-18 Converse of Theorem 6-15

If a diagonal of a parallelogram bisects two angles of the parallelogram, then the parallelogram is a rhombus.

PROOF: SEE EXAMPLE 2.

If...



Then... $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$

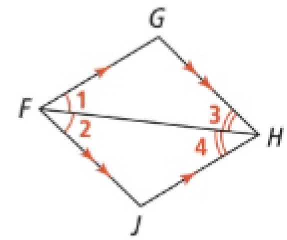
PROOF

EXAMPLE 2 Prove Theorem 6-18

Write a proof of Theorem 6-18.

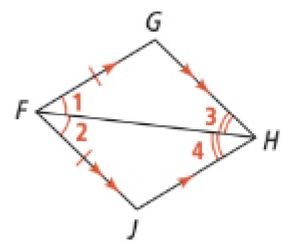
Given: Parallelogram *FGHJ* with $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$

Prove: *FGHJ* is a rhombus.

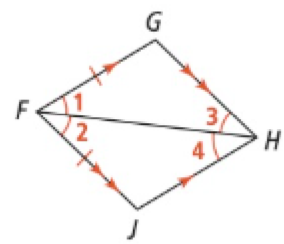


Proof:

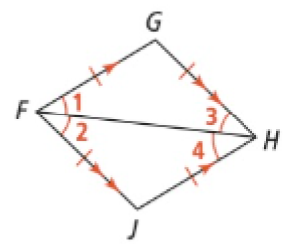
STUDY TIP
Drawing diagonals in parallelograms can help you see additional information that is useful in solving problems.



By ASA, $\triangle FHJ \cong \triangle FHG$. Thus, $\overline{FJ} \cong \overline{FG}$.



By the Alternate Interior Angles Theorem, $\angle 1 \cong \angle 4$, so $\angle 1 \cong \angle 2 \cong \angle 3 \cong \angle 4$.



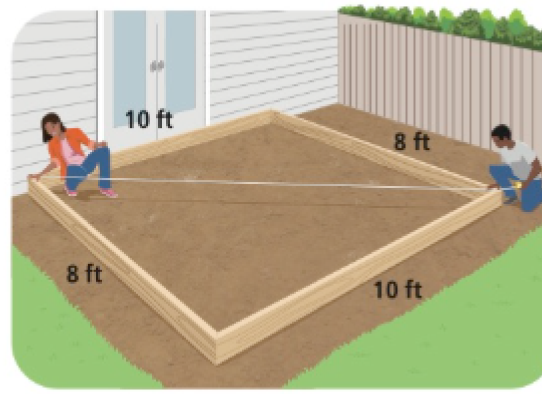
By the Converse of the Isosceles Triangle Theorem, $\overline{FG} \cong \overline{HG}$ and $\overline{FJ} \cong \overline{HJ}$.

Using the Transitive Property of Congruence, $\overline{FG} \cong \overline{HG} \cong \overline{FJ} \cong \overline{HJ}$. Since *FGHJ* is a parallelogram with congruent sides, it is a rhombus.

Try It! 2. Refer to the figure *FGHJ* in Example 2. Use properties of parallelograms to show that if $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$, then the four angles are congruent.

EXAMPLE 3 Use Diagonals to Identify Rectangles

Ashton measures the diagonals for his deck frame and finds that they are congruent. Will the deck be rectangular?



Since opposite sides are congruent, the supports form a parallelogram. To show that the structure is rectangular, show that the angles are right angles.

In a parallelogram, consecutive angles are supplementary. Angles that are congruent and supplementary are right angles. Similarly, $\angle DAB$ and $\angle CBA$ are also right angles.

The frame forms a parallelogram with four right angles, which is a rectangle.

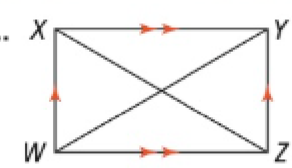
Try It! **3.** If the diagonals of any quadrilateral are congruent, is the quadrilateral a rectangle? Justify your answer.

THEOREM 6-19 Converse of Theorem 6-16

If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

PROOF: SEE EXERCISE 11.

If...



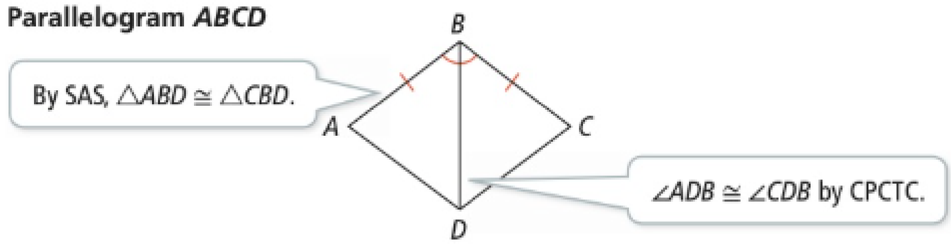
$\overline{XZ} \cong \overline{WY}$

Then... $\angle XWZ$, $\angle WZY$, $\angle XYZ$, and $\angle WXY$ are right angles

EXAMPLE 4 Identify Special Parallelograms

Can you conclude whether each parallelogram is a rhombus, a square, or a rectangle? Explain.

A. Parallelogram *ABCD*



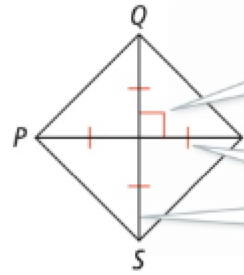
Diagonal $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$, so parallelogram *ABCD* is a rhombus.

CONTINUED ON THE NEXT PAGE

COMMUNICATE AND JUSTIFY
There are often multiple ways to prove something. How could you use properties of parallelograms to show the figure is a rhombus without the congruent angles shown?

EXAMPLE 4 CONTINUED

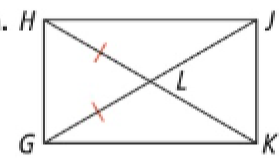
B. Parallelogram *PQRS*



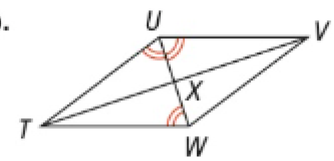
Since the parallelogram is a rhombus and a rectangle, it is a square.

Try It! **4.** Is each parallelogram a rhombus, a square, or a rectangle? Explain.

a.

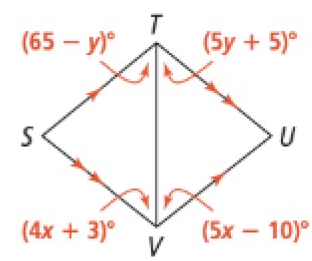


b.



EXAMPLE 5 Use Properties of Special Parallelograms

Quadrilateral *STUV* is a rhombus. What are the values of *x* and *y*?



ANALYZE AND PERSEVERE
Consider the information given in the diagram. How can you determine whether $\overline{TV}$ bisects the angles?

If a parallelogram is a rhombus then each diagonal bisects opposite angles. So, $\overline{TV}$ bisects $\angle SVU$ and $\angle STU$.

Solve for *x*.

$m\angle SVT = m\angle UVT$

$4x + 3 = 5x - 10$

$-x = -13$

$x = 13$

$\overline{TV}$ bisects $\angle SVU$ and $\angle STU$.

Solve for *y*.

$m\angle STV = m\angle UTV$

$65 - y = 5y + 5$

$-6y = -60$

$y = 10$

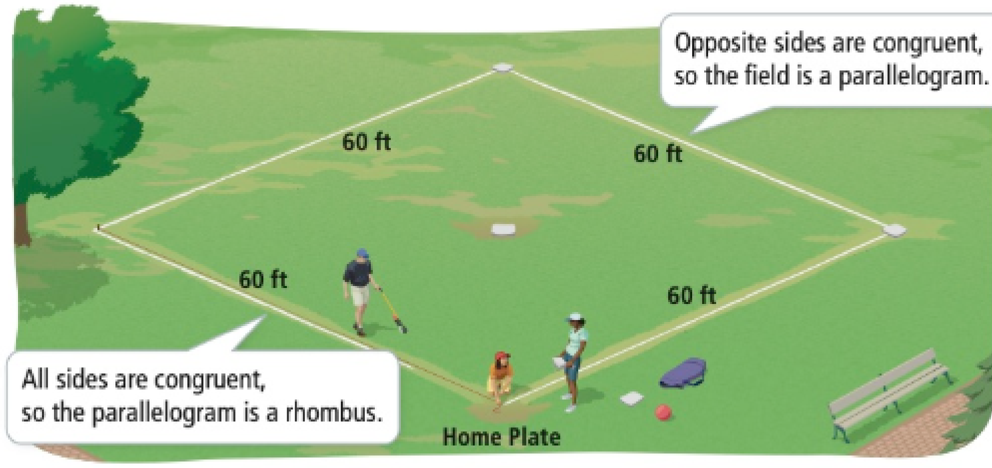
Try It! **5.** In parallelogram *ABCD*, $AC = 3w - 1$ and $BD = 2(w + 6)$. What must be true for *ABCD* to be a rectangle?

APPLICATION

EXAMPLE 6 Apply Properties of Special Parallelograms

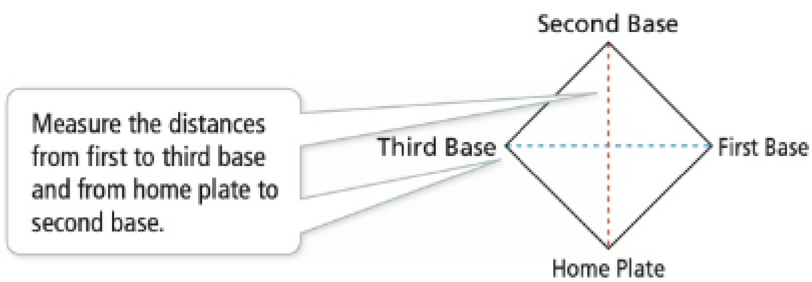
COMMON ERROR
The order in which the quadrilateral is identified matters. Be sure to first show that the quadrilateral is a parallelogram before applying the theorems to identify the quadrilateral as a rhombus or a rectangle.

A group of friends set up a kickball field with bases 60 ft apart. How can they verify that the field is a square?



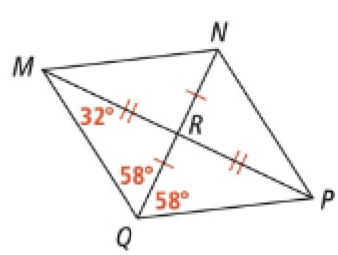
The field is a rhombus. To show that the rhombus is a square, show that it is also a rectangle.

A parallelogram is a rectangle if the diagonals are congruent.



The group of friends can verify the field is a square if they find that the distances from first base to third base and from second base to home plate are equal.

Try It! **6.** Is *MNPQ* a rhombus? Explain.

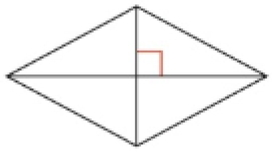


CONCEPT SUMMARY Conditions of Special Parallelograms

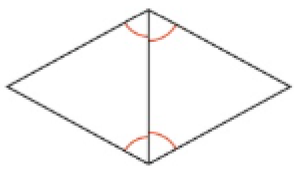
RHOMBUS

A parallelogram is a rhombus if

- diagonals are perpendicular



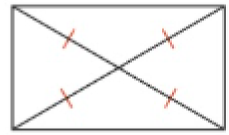
- a diagonal bisects angles



RECTANGLE

A parallelogram is a rectangle if

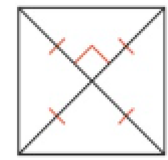
- diagonals are congruent



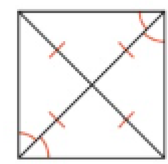
SQUARE

A parallelogram is a square if

- diagonals are perpendicular and congruent



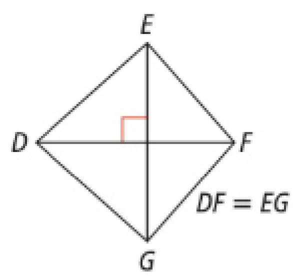
- a diagonal bisects angles and diagonals are congruent



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** Which properties of the diagonals of a parallelogram help you to classify a parallelogram?

2. **Error Analysis** Sage was asked to classify *DEFG*. What was Sage's error?



> Since $DF = EG$, *DEFG* is a rectangle.
> Since $\overline{EG} \perp \overline{DF}$, *DEFG* is also a rhombus.
> Therefore, *DEFG* is a square. ✗

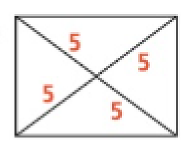
3. **Communicate and Justify** Write a biconditional statement about the diagonals of rectangles. What theorems justify your statement?

4. **Represent and Connect** Make a concept map showing the relationships among quadrilaterals, parallelograms, trapezoids, isosceles trapezoids, kites, rectangles, squares, and rhombuses.

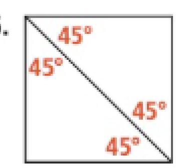
Do You KNOW HOW?

For Exercises 5–8, is the parallelogram a rhombus, a square, or a rectangle?

5.



6.



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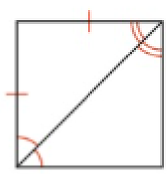
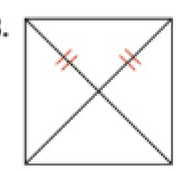
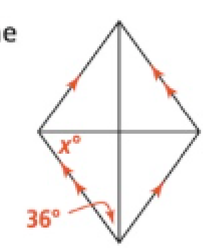


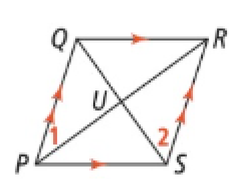
8.



9. What value of x will make the parallelogram a rhombus?



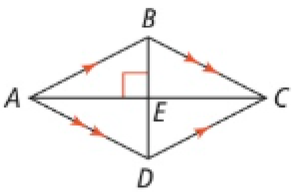
10. If $m\angle 1 = 36$ and $m\angle 2 = 54$, is *PQRS* a rhombus, a square, a rectangle, or none of these? Explain.



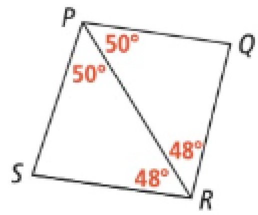
PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. Communicate and Justify Write a proof for Theorem 6-17 using the following diagram.

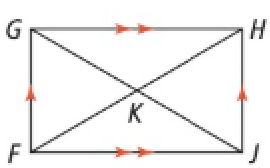


12. Error Analysis Becky is asked to classify *PQRS*. What is her error?

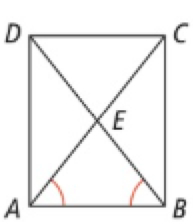


> $\overline{PR}$ bisects opposite angles $\angle SPQ$ and $\angle QRS$, so *PQRS* must be a rhombus. ✗

13. Communicate and Justify Write a proof for Theorem 6-19 using the following diagram.



14. Communicate and Justify Write a proof to show that if *ABCD* is a parallelogram and $\angle ABE \cong \angle BAE$, then *ABCD* is a rectangle.



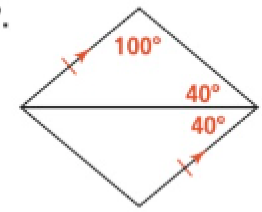
15. Mathematical Connections If *WXYZ* is a rhombus with $W(-1, 3)$ and $Y(9, 11)$, what must be an equation of $\overleftrightarrow{XZ}$ in order for *WXYZ* to be a rhombus? Explain how you found your answer.

16. Higher Order Thinking The longer diagonal of a rhombus is three times the length of the shorter diagonal. If the shorter diagonal is *x*, what expression gives the perimeter of the rhombus?

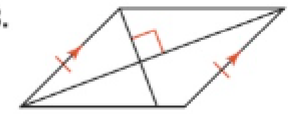
PRACTICE

For Exercises 17 and 18, determine whether each figure is a rhombus. Explain your answer.
SEE EXAMPLES 1 AND 2

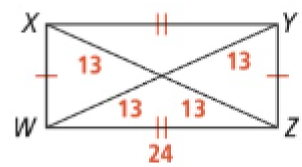
17.



18.

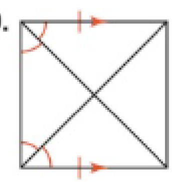


19. What is the perimeter of parallelogram *WXYZ*? SEE EXAMPLE 3

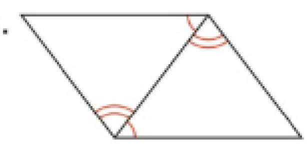


For Exercises 20 and 21, determine the name that best describes each figure: parallelogram, rectangle, square, or rhombus. SEE EXAMPLE 4

20.

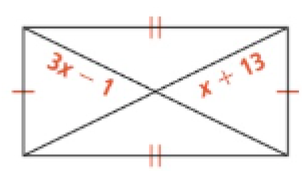


21.

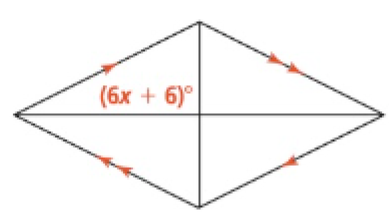


For Exercises 22–24, give the condition required for each figure to be the specified shape.
SEE EXAMPLES 5 AND 6

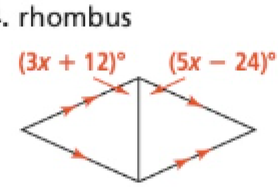
22. rectangle



23. rhombus



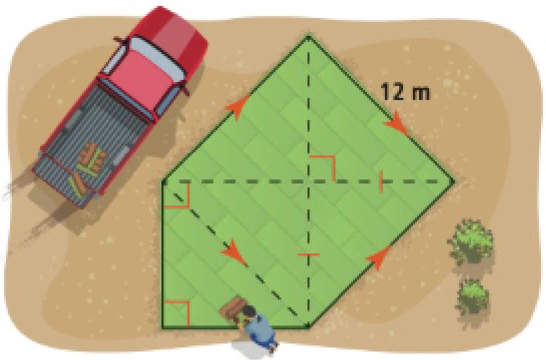
24. rhombus



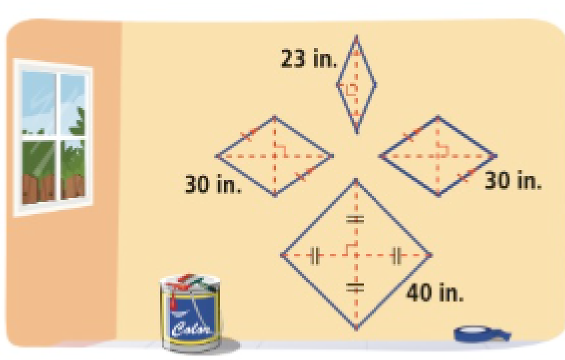
PRACTICE & PROBLEM SOLVING

APPLY

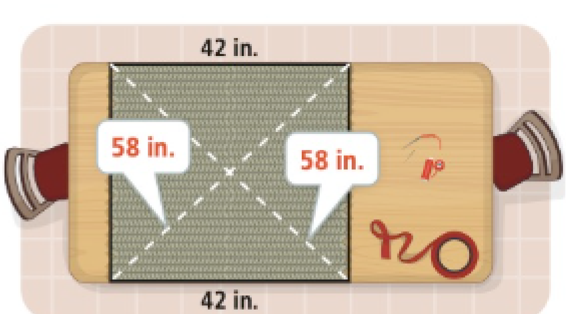
25. Check for Reasonableness Melissa charges $1.50 per square meter for laying sod. She says she can compute the amount to charge for the pentagonal lawn by evaluating $1.50(12^2 + 0.25(12^2))$. Do you agree? Explain.



26. Analyze and Persevere Jeffery is making a wall design with tape. How much tape does he need to put the design shown on his wall? Explain how you used the information in the diagram to find your answer.

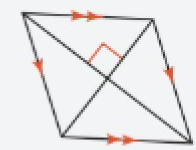


27. Apply Math Models After knitting a blanket, Monisha washes and stretches it out to the correct size and shape. Opposite sides line up with the edges of a rectangular table. She plans to sew a ribbon around the edge of the blanket. How much ribbon will she need?



ASSESSMENT PRACTICE

28. Select all the ways you can classify the figure shown. GR.1.4



- ☐ **A.** Square
- ☐ **B.** Rhombus
- ☐ **C.** Parallelogram
- ☐ **D.** Rectangle
- ☐ **E.** Trapezoid

29. SAT/ACT Parallelogram *ABCD* has diagonals with lengths $AC = 7x + 6$ and $BD = 9x - 2$. For which value of x is *ABCD* a rectangle?

Ⓐ 2   Ⓑ 4   Ⓒ 7   Ⓓ 9   Ⓔ 34

30. Performance Task Zachary is using the two segments shown as diagonals of quadrilaterals he is making for a decal design for the cover of his smart phone.

Part A Make a table showing at least four types of different quadrilaterals that Zachary can make using the segments as diagonals. For each type of quadrilateral, draw a diagram showing an example. Label angle measures where the diagonals intersect, and label segment lengths of the diagonals.

Part B Are some types of quadrilaterals not possible using these diagonals? Explain.

Part C Which has the greater area, a square or a rectangle? Explain.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.1.4–Prove relationships and theorems about parallelograms. Solve mathematical and real-world problems involving postulates, relationships and theorems of parallelograms.

Also GR.1.5

MA.K12.MTR.7.1

Picture This

There are many sizes and designs of picture frames to enhance a piece of art. While frames are often rectangular, there is no limit to the shape of a picture frame. From art galleries to personal homes, many people carefully arrange and display art, choosing shapes and styles that harmonize with and balance the surroundings.

How might you use the geometric properties of picture frames to organize a collection of pictures? Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, how is it different? What might explain the difference?

TOPIC
6

Topic Review

TOPIC ESSENTIAL QUESTION

1. How are properties of parallelograms used to solve problems and to classify quadrilaterals?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A parallelogram is a(n) __________ if its diagonals are perpendicular and congruent.
3. A(n) __________ is a quadrilateral with one pair of parallel sides and one pair of congruent non-parallel sides.
4. The length of the __________ is the average of its two bases.

- isosceles trapezoid
- midsegment of a trapezoid
- rectangle
- square

Concepts & Skills Review

LESSON 6-1 Kites and Trapezoids

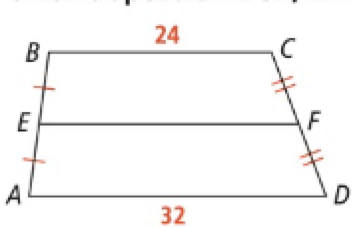
Quick Review

A kite is a quadrilateral with two pairs of adjacent sides congruent.

A trapezoid is a quadrilateral with at least one pair of parallel sides. The length of the midsegment of a trapezoid is the average of the two bases.

Example

Given trapezoid *ABCD*, what is *EF*?

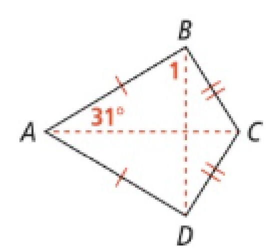


The midsegment of trapezoid *ABCD* is $\overline{EF}$, so $EF = \frac{1}{2}(24 + 32) = 28$.

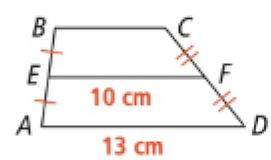
Practice & Problem Solving

Find each measure.

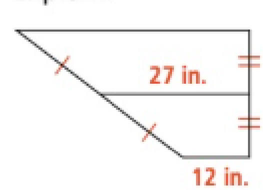
5. $m\angle 1$



6. *BC*



7. **Use Patterns and Structure** Shannon wants to hang curtains using a tension rod across the top of the trapezoid-shaped window that is shown. Is a 36-inch tension rod long enough to go across the top of the window? Explain.



LESSON 6-2 Properties of Parallelograms

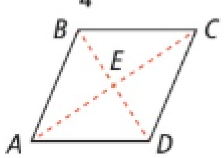
Quick Review

The sides, diagonals, and angles of parallelograms have special relationships.

- Opposite sides are congruent.
- Diagonals bisect each other.
- Consecutive angles are supplementary.
- Opposite angles are congruent.

Example

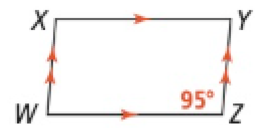
Given parallelogram *ABCD*, if $ED = 3$ and $BD = \frac{3}{4}(AC)$, what is *AC*?



Since $ED = BE$, $BD = 6$. Substitute into $BD = \frac{3}{4}(AC)$ to get $6 = \frac{3}{4}(AC)$, and then solve to get $AC = 8$.

Practice & Problem Solving

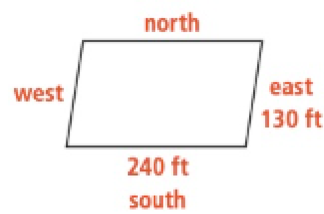
Use the diagram to find each angle measure.



8. $m\angle W$

9. $m\angle X$

10. **Communicate and Justify** The outline of a planned parking lot in the shape of a parallelogram is shown. Elijah says the north side of the lot is 130 ft and the west side of the lot is 240 ft. What is Elijah's mistake?



LESSON 6-3 Proving a Quadrilateral is a Parallelogram

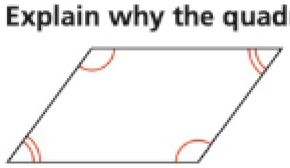
Quick Review

A quadrilateral is a parallelogram if any of the following conditions is true.

- Both pairs of opposite sides are congruent.
- One pair of opposite sides is congruent and parallel.
- An angle is supplementary to both of its consecutive angles.
- Both pairs of opposite angles are congruent.
- The diagonals bisect each other.

Example

Explain why the quadrilateral is a parallelogram.

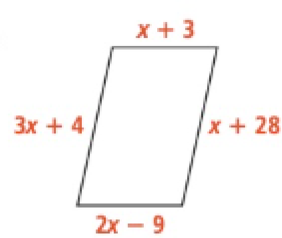


Two pairs of opposite angles are congruent, so the quadrilateral is a parallelogram.

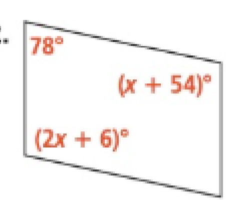
Practice & Problem Solving

For what value of *x* is each quadrilateral a parallelogram?

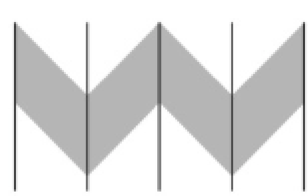
11.



12.



13. **Use Patterns and Structure** All the black lines in the pattern shown are vertical. What measurements can be used to show that each gray quadrilateral is a parallelogram? Explain.



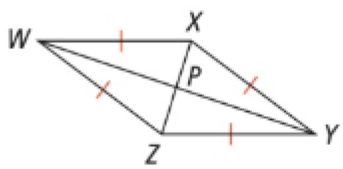
LESSON 6-4 Properties and Conditions of Special Parallelograms

Quick Review

The diagonals of a rhombus are perpendicular bisectors of each other. The diagonals of a rectangle are congruent. The diagonals of a square are perpendicular bisectors of each other and congruent.

Example

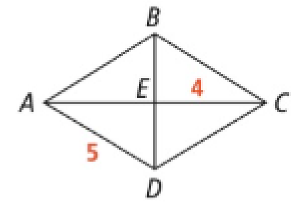
Given that *WXYZ* is a rhombus, show that $\overline{PW} \cong \overline{PY}$.



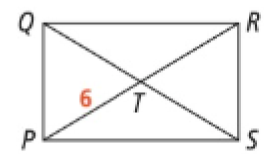
Quadrilateral *WXYZ* is a rhombus, so it is a parallelogram. Diagonals of parallelograms bisect each other, so $\overline{PW} \cong \overline{PY}$ by definition of bisect.

Practice & Problem Solving

14. Given that *ABCD* is a rhombus, what is *BD*?



15. Given that *PQRS* is a rectangle, what is *QS*?



16. Communicate and Justify Is every quadrilateral with four congruent sides a square? Explain.

LESSON 6-5 Conditions of Special Parallelograms

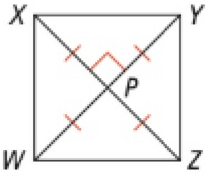
Quick Review

Suppose a figure is a parallelogram.

- If the diagonals are perpendicular, then the parallelogram is a rhombus.
- If the diagonals are congruent, then the parallelogram is a rectangle.
- If the diagonals are perpendicular and congruent, then the parallelogram is a square.

Example

Show that the parallelogram is a square.

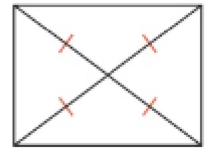


Since $XP = ZP = WP = YP$, $XZ = WY$, the diagonals are congruent. Since the diagonals are congruent, the parallelogram is a rectangle. The diagonals are also given to be perpendicular to each other, so the rectangle is a square.

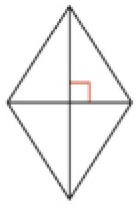
Practice & Problem Solving

Is the parallelogram a rhombus, a rectangle, or a square?

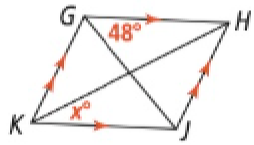
17.



18.



19. Analyze and Persevere For what value of *x* is *GHJK* a rhombus? Explain.



20. Communicate and Justify Can Nora construct a kite with diagonals that bisect each other? Explain.

TOPIC
7

Similarity

TOPIC ESSENTIAL QUESTION

How are properties of similar figures used to solve problems?

Topic Overview

enVision® STEM Project:
Design With a 3D Printer

7-1 Dilations
GR.2.1, GR.2.2, GR.2.5, GR.4.3, GR.4.4, MTR.1.1, MTR.5.1, MTR.3.1

7-2 Similarity Transformations
GR.2.8, GR.1.6, GR.2.3, GR.2.5, GR.6.5, MTR.4.1, MTR.1.1, MTR.7.1

7-3 Proving Triangles Similar
GR.1.6, GR.1.2, GR.2.8, GR.2.9 MTR.4.1, MTR.2.1, MTR.6.1

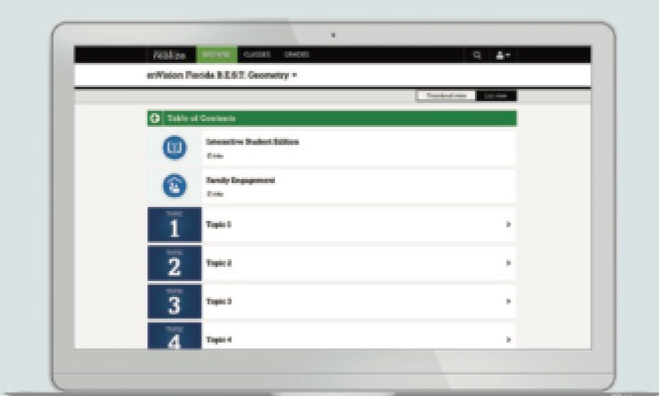
7-4 Similarity in Right Triangles
GR.1.3, GR.1.2, MTR.3.1, MTR.4.1, MTR.5.1

Mathematical Modeling in 3 Acts:
Make It Right
GR.1.6, MTR.7.1

7-5 Proportions in Triangles
GR.1.3, GR.1.2, MTR.5.1, MTR.1.1, MTR.4.1

Topic Vocabulary

- center of dilation
- geometric mean
- similarity transformation



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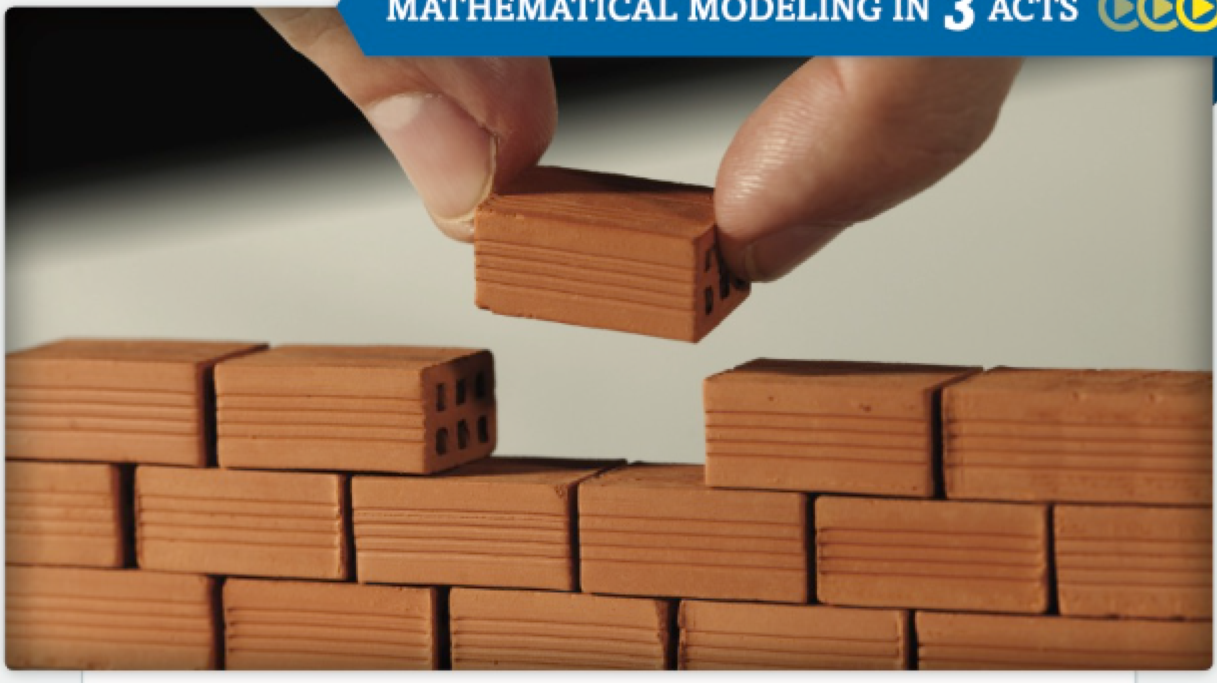
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Make It Right

Architects often make a scale physical model of a new building project. The scale model is usually a miniature version of the project it is representing.

When making a model, architects need to make sure that all of the parts of the model are the right size. Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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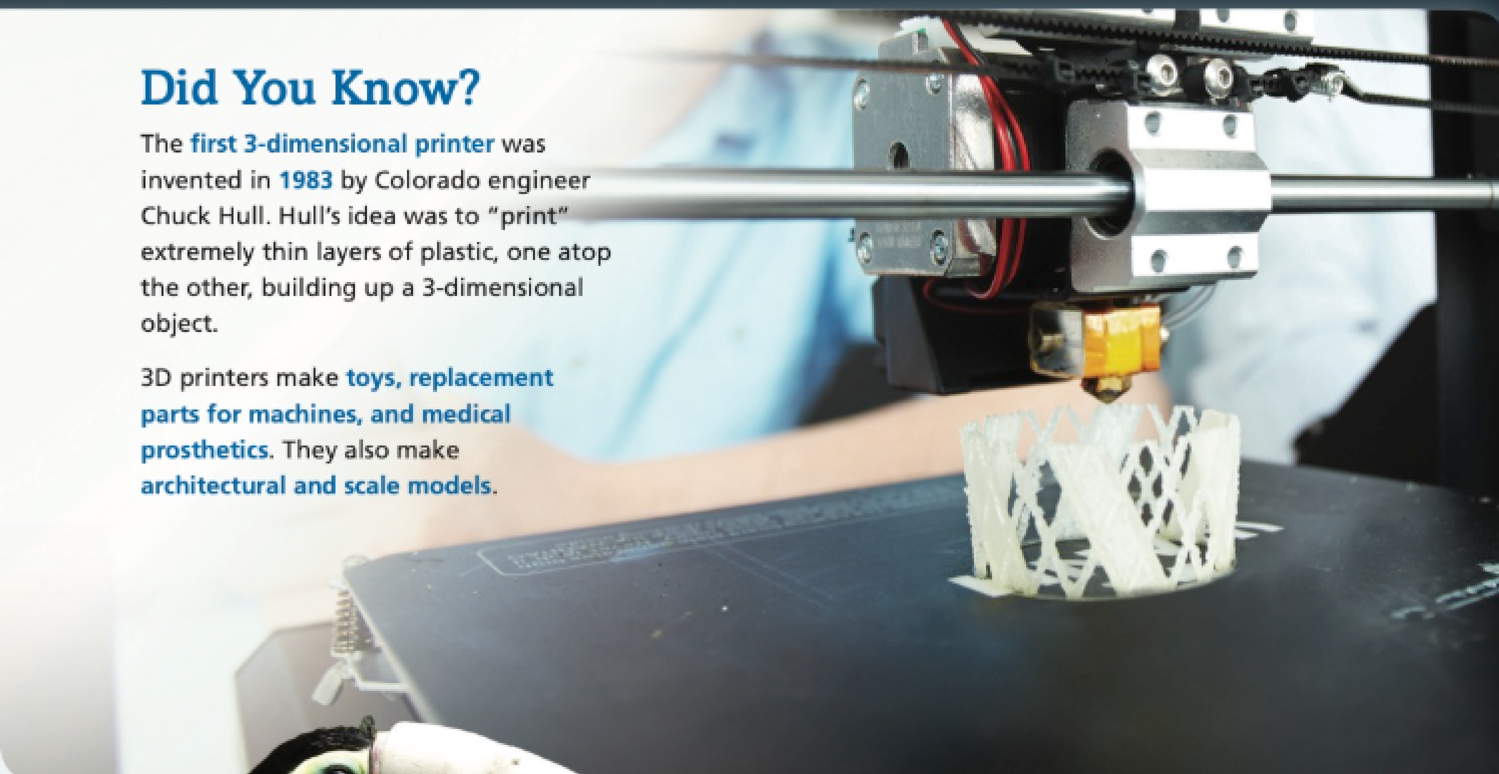
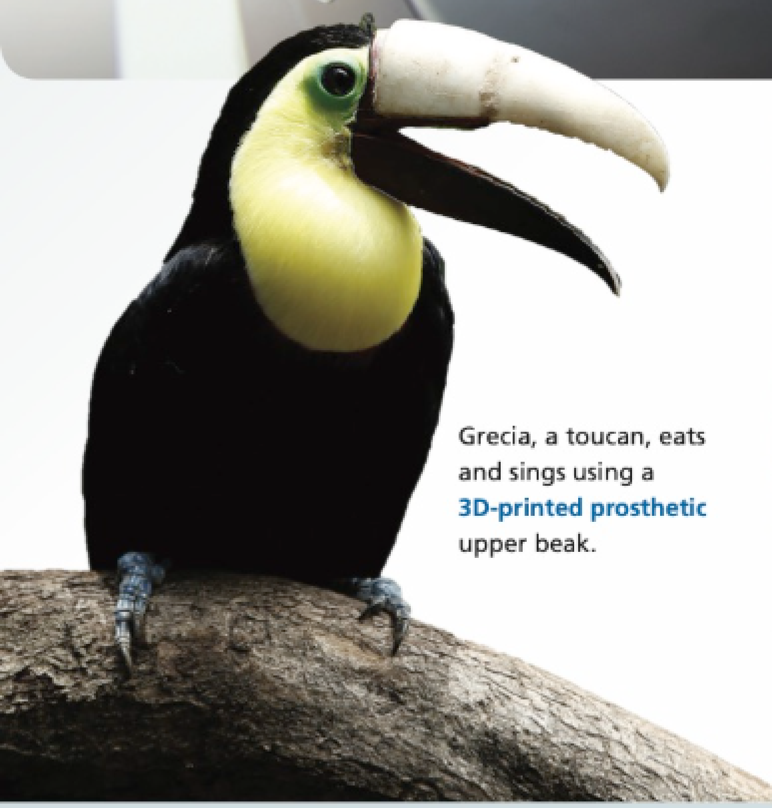


Did You Know?

The **first 3-dimensional printer** was invented in **1983** by Colorado engineer Chuck Hull. Hull's idea was to "print" extremely thin layers of plastic, one atop the other, building up a 3-dimensional object.

3D printers make **toys, replacement parts for machines, and medical prosthetics.** They also make **architectural and scale models**.



Grecia, a toucan, eats and sings using a **3D-printed prosthetic** upper beak.

The **first printing press** was invented by Johannes Gutenberg around the year **1450**. To print a page, Gutenberg made individual letters from metal and arranged the letters **on a block**. Then he inked the letters and **stamped them** on paper.

Your Task: Design With a 3D Printer

An engineer has built a scale model of a part for a rocket engine. Full-size, the part will be mass-produced using 3D printing. You and your classmates will use similarity to scale up the dimensions of the part. Then you'll describe and draw steps for the production of the part.

7-1 Dilations

I CAN… dilate figures and identify characteristics of dilations.

VOCABULARY

- center of dilation

MA.912.GR.2.1–Given a preimage and image, describe the transformation and represent the transformation algebraically using coordinates. Also GR.2.2, GR.2.5, GR.4.3, GR.4.4

MA.K12.MTR.1.1, MTR.5.1, MTR.3.1

EXPLORE & REASON

Roosevelt High School sells a sticker and a larger car decal with the school logo.

A. Use Patterns and Structure How are the sticker and the car decal alike? How are they different?

B. Suppose the sticker and decal are shown next to each other on a computer screen. If you zoom in to 125%, what would stay the same on the figures? What would be different?

ESSENTIAL QUESTION

How does a dilation affect the side lengths and angle measures of a figure?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Dilate a Figure

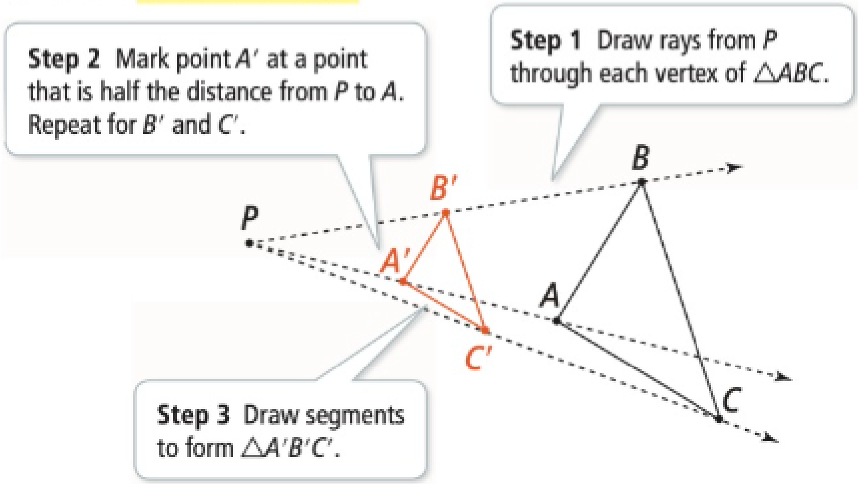
How can you draw a dilated image?

A dilation produces an image that is a different size than the preimage.

Method 1 The Ratio Method

Dilate $\triangle ABC$ by a scale factor of $\frac{1}{2}$ with fixed center P. This fixed center is called the **center of dilation**.

STUDY TIP
Recall that in a dilation, the preimage is enlarged or reduced in size by a given scale factor.



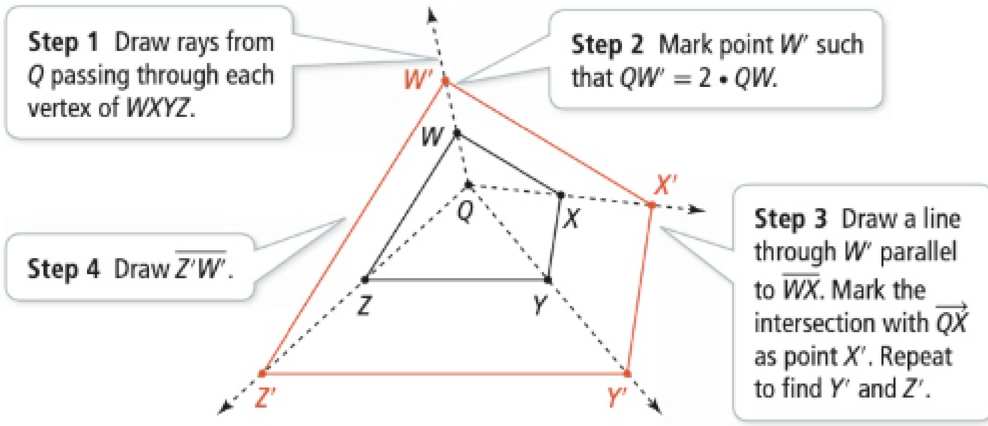
Triangle $A'B'C'$ is a copy of $\triangle ABC$ with side lengths that are $\frac{1}{2}$ the lengths of the corresponding sides of $\triangle ABC$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

Method 2 The Parallel Method

Dilate *WXYZ* by 2 with center of dilation *Q*.



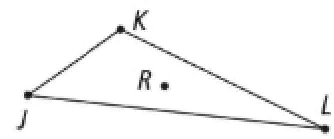
Segment $W'Z'$ is parallel to $\overline{WZ}$.

Quadrilateral $W'X'Y'Z'$ is a copy of *WXYZ* with side lengths that are twice the lengths of the corresponding sides of *WXYZ*.

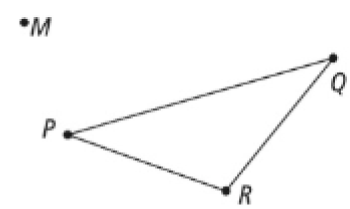
ANALYZE AND PERSEVERE Consider the two methods for dilating figures. How are they alike? How are they different?

Try It!

1. a. Trace $\triangle JKL$ and point *R*. Use Method 1 to dilate $\triangle JKL$ by a scale factor of 3 with center of dilation *R*.



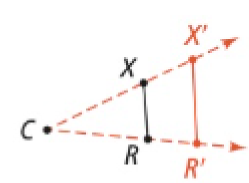
b. Trace $\triangle PQR$ and point *M*. Use Method 2 to dilate $\triangle PQR$ by a scale factor of $\frac{1}{3}$ with center of dilation *M*.



CONCEPT Dilations

A dilation $D_{(n, C)}$ is a transformation that has center of dilation *C* and scale factor *n*, where $n > 0$, with the following properties:

- Point *R* maps to R' in such a way that R' is on $\overrightarrow{CR}$ and $CR' = n \cdot CR$.
- Each length in the image is *n* times the corresponding length in the preimage (i.e., $X'R' = n \cdot XR$).

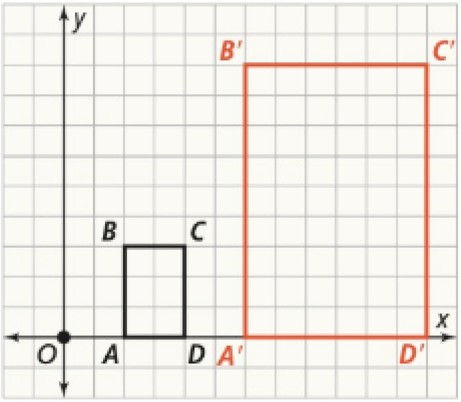


- The image of the center of dilation is the center itself (i.e., $C' = C$).
- If $n > 1$, the dilation is an *enlargement*.
- If $0 < n < 1$, the dilation is a *reduction*.
- Every angle is congruent to its image under the dilation.

On a coordinate plane, the notation D_n describes the dilation with the origin as center of dilation.

EXAMPLE 2 Analyze Dilations

Rectangle $A'B'C'D'$ is a dilation with center at the origin of $ABCD$. How are the side lengths and angle measures of $ABCD$ related to those of $A'B'C'D'$?



Step 1 Compare the angles.

All angles of rectangles are right angles, so each angle in image $A'B'C'D'$ is congruent to the corresponding angle in $ABCD$.

Step 2 Compare the side lengths.

Find the side lengths in the preimage and image.

$AB = 3$ $BC = 2$ $CD = 3$ $DA = 2$

$A'B' = 9$ $B'C' = 6$ $C'D' = 9$ $D'A' = 6$

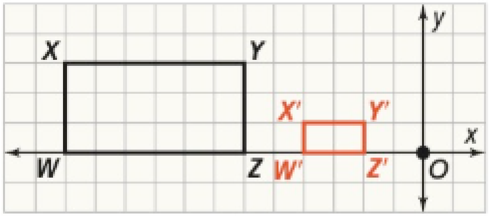
Find the ratios of the corresponding side lengths.

$\frac{A'B'}{AB} = \frac{9}{3} = 3$ $\frac{B'C'}{BC} = \frac{6}{2} = 3$ $\frac{C'D'}{CD} = \frac{9}{3} = 3$ $\frac{D'A'}{DA} = \frac{6}{2} = 3$

The ratios are equal, so the lengths of corresponding sides of the two figures are proportional.

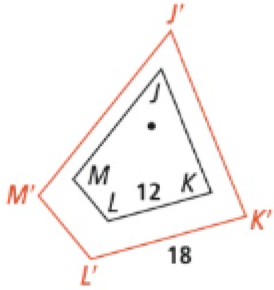
HAVE A GROWTH MINDSET
How can you use mistakes as opportunities to learn and grow?

Try It! 2. Rectangle $W'X'Y'Z'$ is a dilation with center O of $WXYZ$. How are the side lengths and angle measures of the two figures related?



EXAMPLE 3 Find a Scale Factor

Quadrilateral $J'K'L'M'$ is a dilation of $JKLM$. What is the scale factor?



The scale factor is the ratio of side lengths in the image to the corresponding side lengths in the preimage.

$\frac{K'L'}{KL} = \frac{18}{12} = \frac{3}{2}$

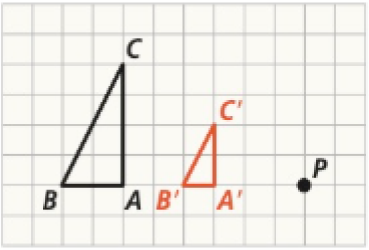
The scale factor is $\frac{3}{2}$.

COMMON ERROR
Be careful to find the scale factor and not its reciprocal. Think about whether the dilation is an enlargement or reduction to see whether the scale factor makes sense.

Try It! 3. Consider the dilation shown.

a. Is the dilation an enlargement or a reduction?

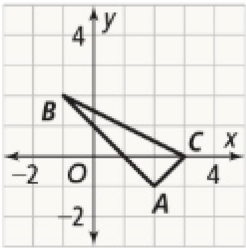
b. What is the scale factor?



EXAMPLE 4 Dilate a Figure With Center at the Origin

What are the vertices of $D_3(\triangle ABC)$?

The notation $D_3(\triangle ABC)$ means the image of $\triangle ABC$ after a dilation centered at the origin, with scale factor 3.



For a dilation with scale factor 3 centered at the origin, each image point is 3 times farther away from the origin than the corresponding preimage point.

ANALYZE AND PERSEVERE
Think about distances on a coordinate plane. What does the scale factor n indicate about the distance of the image from the origin?

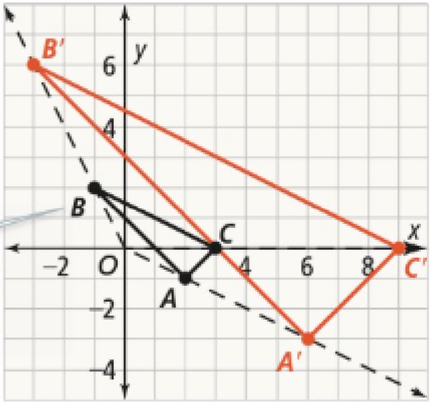
Multiply each coordinate of each preimage point by 3 to find the coordinates of the image points.

The dilation D_n maps $(x, y) \rightarrow (nx, ny)$

$A(2, -1) \rightarrow A'(3 \cdot 2, 3 \cdot -1) = A'(6, -3)$

$B(-1, 2) \rightarrow B'(3 \cdot -1, 3 \cdot 2) = B'(-3, 6)$

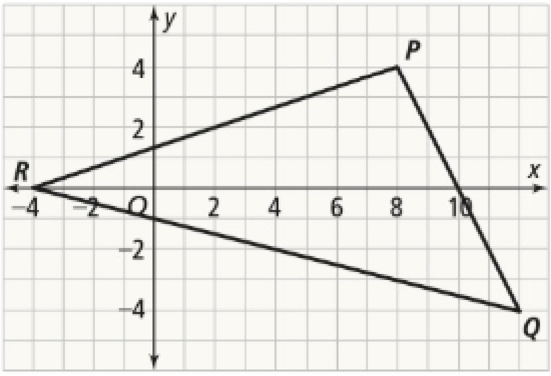
$C(3, 0) \rightarrow C'(3 \cdot 3, 3 \cdot 0) = C'(9, 0)$



D_3 maps each point $(x, y) \rightarrow (3x, 3y)$

The vertices of $D_3(\triangle ABC)$ are $A'(6, -3)$, $B'(-3, 6)$, and $C'(9, 0)$.

Try It! 4. Use $\triangle PQR$.



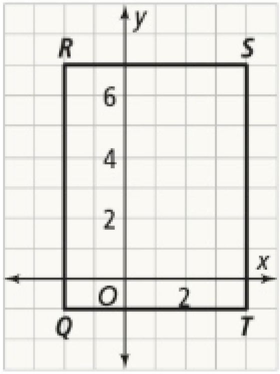
a. What are the vertices of $D_{\frac{1}{4}}(\triangle PQR)$?

b. How can you express this dilation algebraically?

c. How are the distances to the origin from each image point related to the distance to the origin from each corresponding preimage point?

EXAMPLE 5 Dilate a Figure With Center Not at the Origin

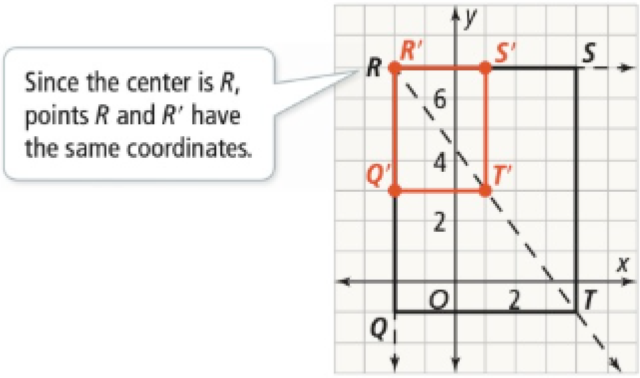
What are the vertices of $D_{(\frac{1}{2}, R)}(QRST)$?



The dilation is centered at $R(-2, 7)$ with a scale factor of $\frac{1}{2}$. So each image point is half the distance from R as the corresponding preimage point is. For each preimage point, multiply the horizontal and vertical changes from R by $\frac{1}{2}$. Then add the horizontal and vertical half-changes to the coordinates of the center of dilation.

Preimage Point	Distance From R(−2, 7)		Half-Distances From R(−2, 7)		Add to R(−2, 7)	Image Point
	horiz.	vert.	horiz.	vert.		
$Q(-2, -1)$	0	−8	0	−4	$(-2 + 0, 7 - 4)$	$Q'(-2, 3)$
$S(4, 7)$	6	0	3	0	$(-2 + 3, 7 + 0)$	$S'(1, 7)$
$T(4, -1)$	6	−8	3	−4	$(-2 + 3, 7 - 4)$	$T'(1, 3)$

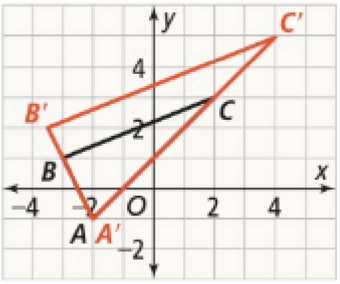
Graph the preimage and image on the same coordinate plane.



The vertices of $D_{(\frac{1}{2}, R)}(QRST)$ are $Q'(-2, 3)$, $R'(-2, 7)$, $S'(1, 7)$, and $T'(1, 3)$.

STUDY TIP
Remember to check your answers by measuring the distances between the center of dilation and the image and preimage vertices.

Try It! 5. A dilation of $\triangle ABC$ is shown.



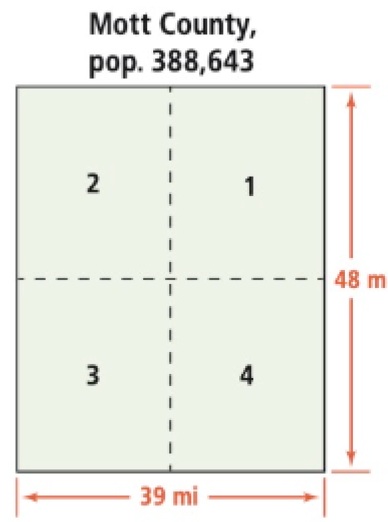
a. What is the center of dilation?

b. What is the scale factor?

APPLICATION **EXAMPLE 6** Solve Population Density Problems

County planners are considering dividing Mott County into four equally sized administrative zones. Each zone is a dilation of the county as a whole with scale factor $\frac{1}{2}$.

What would be the population of each zone if it has the same population density of the county as a whole?



Population density is a measure of the number of people in region divided by the total area.

Step 1 Find the area of Mott County.

$$\text{Area of Mott County} = (39\text{ mi})(48\text{ mi})$$
$$= 1{,}872\text{ sq. mi}$$

Step 2 Find population density of Mott County.

$$\text{Pop. density Mott County} = \frac{388{,}643}{1{,}872\text{ sq. mi}}$$
$$\approx 207.6/\text{sq. mi}$$

Step 3 Find the area of one of the administrative zones.

Multiply each dimension of the county by $\frac{1}{2}$ to find the dimensions of the administrative zones.

$$\text{Area of admin. zone} = \left(\frac{1}{2} \cdot 39\text{ mi}\right)\left(\frac{1}{2} \cdot 48\text{ mi}\right)$$
$$= 468\text{ sq. mi}$$

Step 4 Find the population of each administrative zone.

$$\text{Population density} = \frac{\text{population}}{\text{area}}$$

$$207.6/\text{sq. mi} = \frac{\text{population}}{468\text{ sq. mi}}$$

Substitute for population density and area. Then solve for population.

$$\text{population} = (207.6/\text{sq. mi})(468\text{ sq. mi})$$
$$= 97{,}156.8$$

STUDY TIP
Round to the nearest unit since you cannot have a fraction of a person in a population.

The population of each administrative zone is about 97,157, if each zone has the same population density of the county as a whole.

Try It! **6.** Suppose the population of Zone 1 is 102,981.

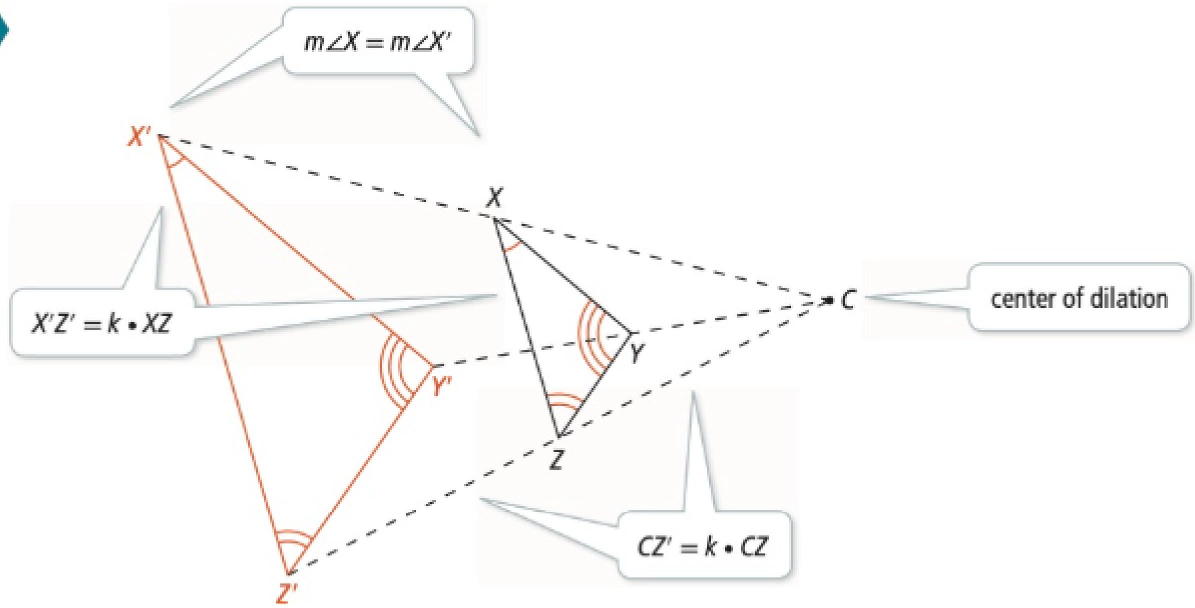
a. What is the population density of Zone 1?

b. What is the population density of Zones 2, 3, and 4 taken as a whole?

CONCEPT SUMMARY Dilations

WORDS A dilation is a transformation that maps point X to point X' such that X' lies on $\overrightarrow{CX}$ and $CX' = k \cdot CX$ for a center of dilation C and a scale factor k. Dilations preserve angle measures.

DIAGRAM



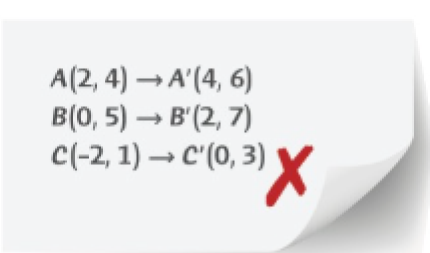
NOTATION Dilation centered at the origin: $D_k(X)$

Dilation centered at point C: $D_{(k,\ C)}(X)$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does a dilation affect the side lengths and angle measures of a figure?

2. **Error Analysis** Emilia was asked to find the coordinates of $D_2(\triangle ABC)$ for $A(2, 4)$, $B(0, 5)$, and $C(-2, 1)$. What is Emilia's error?

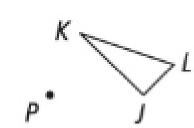


3. **Vocabulary** In the definition of a dilation D_n, why is n not equal to 0? What would a transformation like D_0 look like?

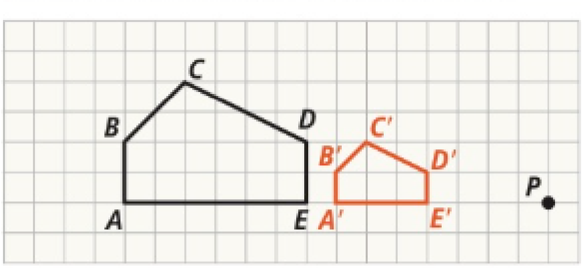
4. **Communicate and Justify** Compare the vertices of $D_1(\triangle ABC)$ for any points A, B, and C. Justify your answer.

Do You KNOW HOW?

5. Trace $\triangle JKL$ and point P. Draw the dilation of $\triangle JKL$ using scale factor 3 and P as the center of dilation.



6. What is the scale factor for the dilation?



Give the coordinates of the dilation.

7. $D_5(\triangle PQR)$ for $P(1, -3)$, $Q(-5, -4)$, $R(6, 2)$

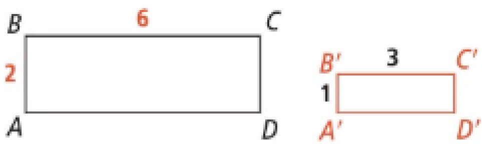
8. $D_{(3,\ B)}(\triangle ABC)$ for $A(0, 4)$, $B(0, 2)$, $C(-3, 2)$

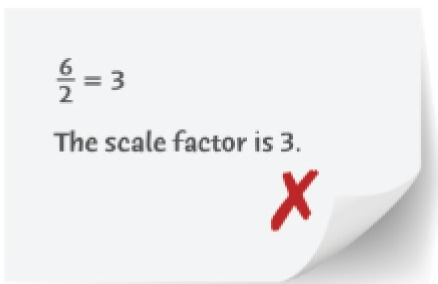
9. $D_{(4,\ F)}(FGHJ)$ for $F(0, -1)$, $G(4, -1)$, $H(4, -3)$, $J(0, -3)$

PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. Error Analysis Kendall was asked to find the scale factor for the dilation. What is Kendall's error?

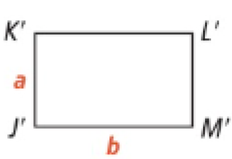




11. Higher Order Thinking Points $M(a, b)$ and $N(c, d)$ are dilated by scale factor k, with the origin as the center of dilation. Show algebraically that $\overleftrightarrow{MN} \parallel \overleftrightarrow{M'N'}$.

12. Choose Efficient Methods Suppose you want to dilate a figure on the coordinate plane with a center of dilation at point (a, b) that is not the origin and with a scale factor k. Describe how you can use a composition of translations and a dilation centered at the origin to dilate the figure. Then write the transformation rule.

13. Analyze and Persevere Rectangle $J'K'L'M'$ is a dilation of $JKLM$ with scale factor k. What are the perimeter and area of $JKLM$?

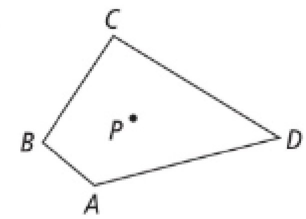


14. Mathematical Connections Carolina says that when a figure is dilated using a scale factor of 2, the angle measures in the image are twice the angle measures in the preimage. How could you use the Triangle Angle Sum Theorem to explain why this cannot be true?

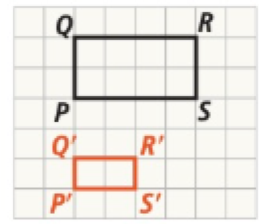
15. Generalize Is it always true that $(D_m \circ D_n)(X) = D_{mn}(X)$? Explain.

PRACTICE

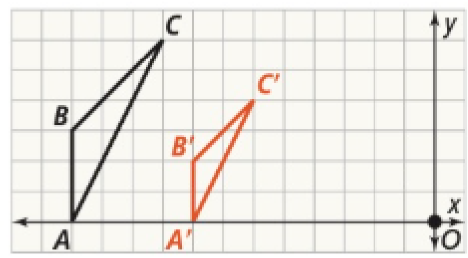
16. Trace $ABCD$ and point P. Draw the dilation of $ABCD$ using P as the center of dilation and sides that are two times as long. SEE EXAMPLE 1



17. How are the side lengths of the preimage and dilated image related? SEE EXAMPLE 2



18. What is the scale factor of the dilation shown? SEE EXAMPLE 3

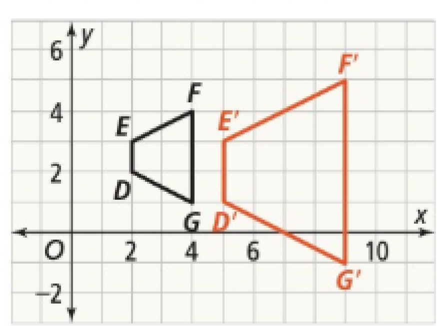


19. What are the coordinates of $D_{1.5}(ABCD)$ for $A(2, 0)$, $B(8, -4)$, $C(4, -6)$, and $D(-5, -10)$? SEE EXAMPLE 4

20. What are the coordinates of $D_{(2, X)}(\triangle XYZ)$ for $X(1, 1)$, $Y(2, 2)$, and $Z(3, 0)$? SEE EXAMPLE 5

21. The population density of the city of Wellington is 12,000 people per square mile. The neighboring city Morrison is a dilation of Wellington with scale factor 1.2. If Morrison has $\frac{3}{4}$ of the population of Wellington, what is the population density of Morrison? SEE EXAMPLE 6

22. What are the coordinates of the center of dilation for the dilation shown?



PRACTICE & PROBLEM SOLVING

APPLY

23. Check for Reasonableness The images on Henry's digital camera have a width-to-length ratio of 2 : 3. He wants to make an 8 in.-by-10 in. print of one of his photographs.

a. Is this possible? Explain.

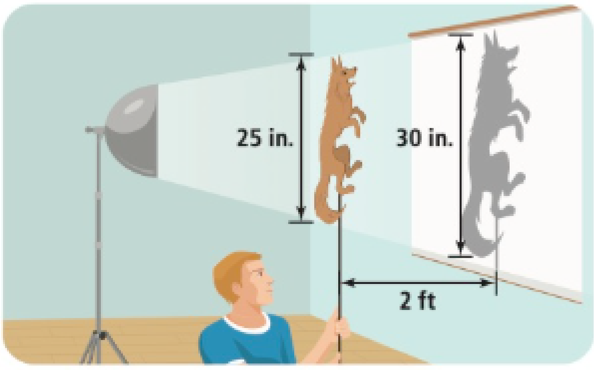
b. How can Henry crop an image so that an 8 in.-by-10 in. print can be made?

24. Represent and Connect Alex draws the scale model shown as a plan for a large wall mosaic.

a. The wall is 10 m wide and 7 m high. What are the dimensions of the largest mosaic he can make on that wall? Explain.

b. He will use 2-cm square tiles to make his mosaic. How many tiles will he need? Explain how you found your answer.

25. Use Patterns and Structure How far from the screen should the light be placed in order for the shadow of the puppet to be 30 in. tall? Explain how you found your answer.



ASSESSMENT PRACTICE

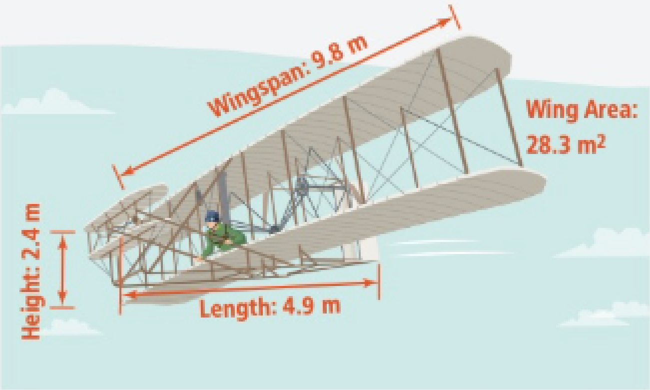
26. Copy and complete the table to show information about dilations centered at the origin. GR.2.1

Preimage Coordinates	Scale Factor	Image Coordinates
(5, –2)	■	(20, –8)
(9, 3)	■	(3, 1)
(–4, 0)	■	(–6, 0)
(–1, 2)	■	(–5, 10)

27. SAT/ACT A dilation maps $\triangle ABC$ to $\triangle A'B'C'$. The area of $\triangle ABC$ is 13 square units, and the area of $\triangle A'B'C'$ is 52 square units. What is the scale factor?

Ⓐ 2
Ⓑ 13
Ⓒ 4
Ⓓ 26

28. Performance Task Alberto wants to make a scale model of the Wright brothers' glider.



Part A The wingspan of the scale model must be between 15 cm and 18 cm. What scale factor should he use? Explain.

Part B Use your scale factor from Part A. What will be the length, wingspan, and height of the model glider?

Part C What will be the wing area of the model glider? If both wing sections are the same size, what will be the dimensions of each wing section?

7-2 Similarity Transformations

I CAN… determine whether figures are similar.

VOCABULARY

- similarity transformation

MA.912.GR.2.8–Apply an appropriate transformation to map one figure onto another to justify that the two figures are similar. Also GR.1.6, GR.2.3, GR.2.5, GR.6.5

MA.K12.MTR.4.1, MTR.1.1, MTR.7.1

CRITIQUE & EXPLAIN

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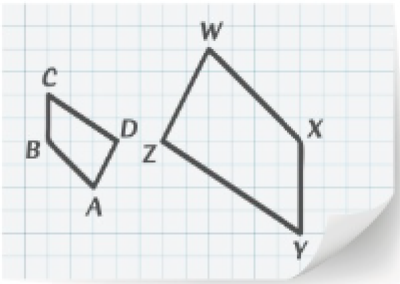
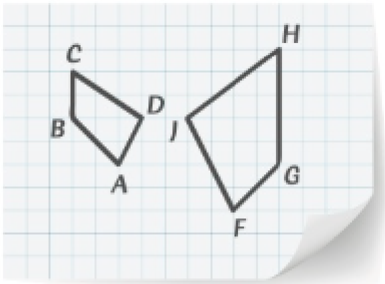
Helena and Edwin were asked to apply a composition of transformations to *ABCD*.





A. Represent and Connect Is there a composition of transformations that maps *ABCD* to the second figure in each student's work? If so, what is it?

B. For each student whose work shows a composition of transformations, describe the relationship between the figures.

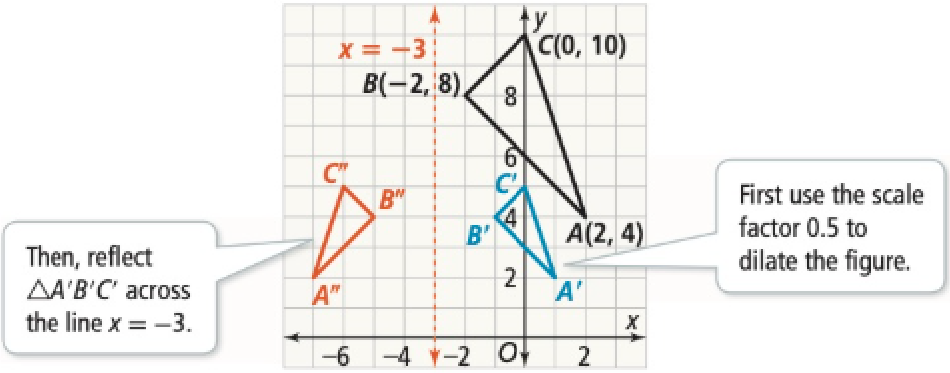
ESSENTIAL QUESTION

What makes a transformation a similarity transformation? What is the relationship between a preimage and the image resulting from a similarity transformation?

EXAMPLE 1 Graph a Composition of a Rigid Motion and a Dilation

If line *m* is represented by the equation $x = -3$, what is a graph of the image $(r_m \circ D_{0.5})(\triangle ABC)$?

COMMON ERROR
Be careful to use the correct center of dilation. When the notation does not specify the center of dilation, the center of dilation is at the origin.



The graph of the image $(r_m \circ D_{0.5})(\triangle ABC)$ is $\triangle A''B''C''$.

Try It!

1. The vertices of $\triangle XYZ$ are $X(3, 5)$, $Y(-1, 4)$, and $Z(1, 7)$.

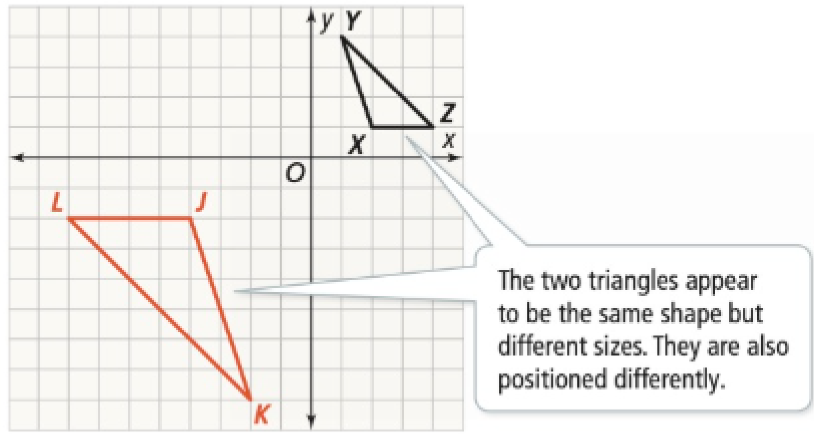
   a. What is the graph of the image $(D_2 \circ T_{\langle 1, -2\rangle})(\triangle XYZ)$?

   b. What is the graph of the image $(D_3 \circ R_{(90°, O)})(\triangle XYZ)$?

EXAMPLE 2 Describe a Composition of a Rigid Motion and a Dilation

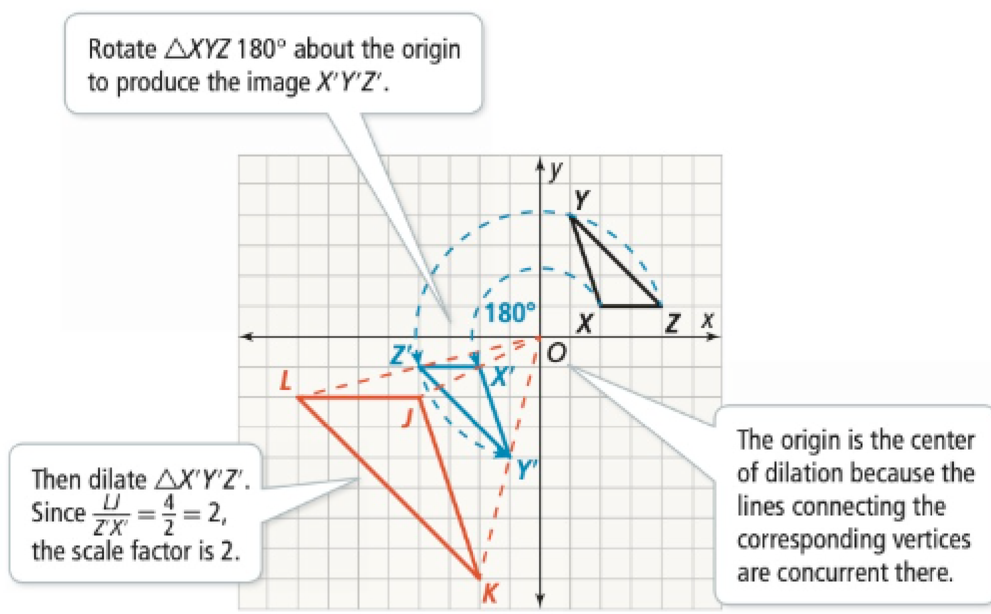
Powered By desmos

Is there a composition of transformations that maps $\triangle XYZ$ to $\triangle JKL$? Explain.



ANALYZE AND PERSEVERE
Think about how you could use reflection, translation, or rotation to create an image of $\triangle XYZ$ with the orientation opposite of $\triangle JKL$. Which rigid motion would you use?

Notice that Y in $\triangle XYZ$ is in the upper left of the first quadrant, but its corresponding vertex K in $\triangle JKL$ is in the lower right of the third quadrant, so it appears that $\triangle XYZ$ is rotated. Since $\triangle JKL$ is larger than $\triangle XYZ$, it is also dilated.



So, the composition of transformations $D_2 \circ R_{(180°,\ O)}$ maps $\triangle XYZ$ to $\triangle JKL$.

Try It! **2.** If the transformations in Example 2 are performed in the reverse order, are the results the same? Do you think your answer holds for all compositions of transformations? Explain.

CONCEPTUAL UNDERSTANDING

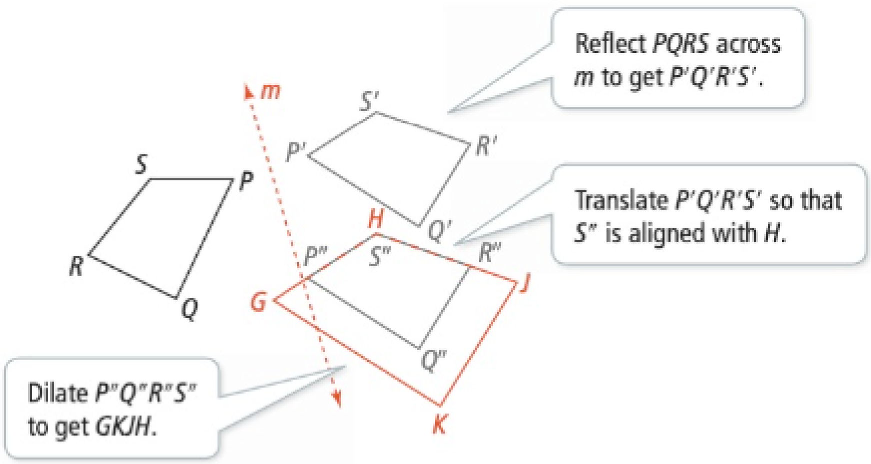
STUDY TIP
A similarity transformation is more precisely a rigid motion, a dilation, or a composition of a rigid motion and a dilation. Furthermore, the phrase "one or more rigid motions" is not necessary due to Theorem 3-3.

EXAMPLE 3 Find Similarity Transformations

Why is *PQRS* similar to *GKJH*?

A **similarity transformation** is a composition of one or more rigid motions and a dilation. A similarity transformation results in an image that is similar to the preimage.

Measure the angles of the figures to determine that ∠*S* corresponds to ∠*H* and ∠*R* corresponds to ∠*J*. The orientation is reversed in *GHJK*, so the rigid motion includes a reflection.



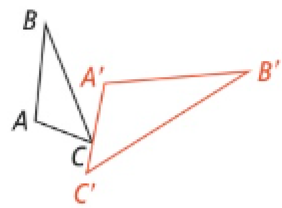
A composition of a reflection, a translation, and a dilation maps *PQRS* to *GKJH*, so *PQRS* and *GKJH* are similar, or *PQRS* ~ *GKJH*.

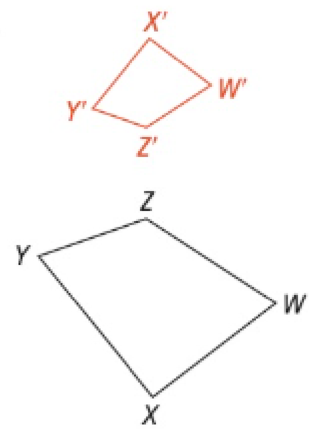
The symbol ~ is used to indicate similarity.

For any figures *A*, *B*, and *C*, the following properties hold.

Reflexive Property of Similarity: *A* ~ *A*
Symmetric Property of Similarity: If *A* ~ *B*, then *B* ~ *A*.
Transitive Property of Similarity: If *A* ~ *B* and *B* ~ *C*, then *A* ~ *C*.

Try It! 3. Describe a possible similarity transformation for each pair of similar figures shown, and then write a similarity statement.

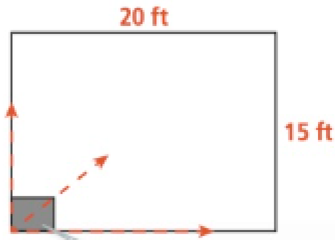
a. 

b. 

APPLICATION

EXAMPLE 4 Determine Similarity

Can the artist copy her sketch to cover an entire wall measuring 15 ft high by 20 ft wide so her wall mural is similar to her sketch? Explain.



If the artist can map her sketch onto the wall, then she can place her sketch at the bottom left corner and dilate it.

Determine whether the sketch can be dilated to fit the wall

Step 1 Compare the corresponding angles.
Both figures are rectangles, so the corresponding angles are congruent.

ANALYZE AND PERSEVERE
Why can you compare angle measures and the ratios of the corresponding side lengths to determine whether two figures are similar?

Step 2 Compare the ratios of the corresponding side lengths.

vertical sides: $\frac{11}{15}$ horizontal sides: $\frac{7}{10}$

Because $\frac{11}{15} \neq \frac{7}{10}$ corresponding side lengths are not equal.

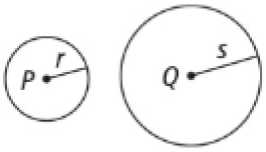
The sketch and the wall are not similar, so the sketch cannot be copied to cover the entire wall.

Try It! **4.** Suppose the artist cuts 2 inches from the width of her sketch in Example 4. How much should she cut from the height so she can copy a similar image to cover the wall?

PROOF

EXAMPLE 5 Identify Similar Circles

Write a proof that any two circles are similar.



Given: $\odot P$ with radius r, $\odot Q$ with radius s

Prove: $\odot P \sim \odot Q$

Proof: Translate P to Q, so P' coincides with Q. Then find a scale factor that dilates $\odot P'$ to the circle with radius s.

Let $k = \frac{s}{r}$. Then the translation followed by a dilation centered at Q with scale factor k maps $\odot P$ onto $\odot Q$. Since a similarity transformation exists, $\odot P \sim \odot Q$.

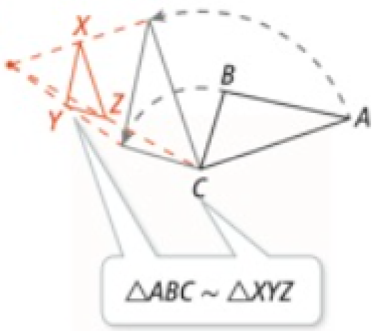
Try It! **5.** Write a proof that any two squares are similar.

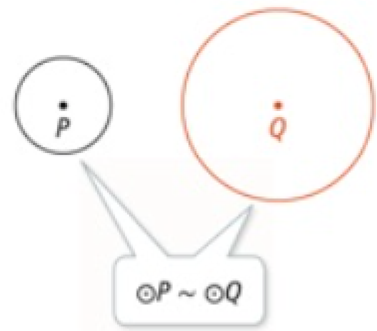
CONCEPT SUMMARY Similarity

WORDS

- A similarity transformation is a composition of one or more rigid motions and a dilation.
- Two figures are similar if there is a similarity transformation that maps one to the other.
- All circles are similar to each other.

DIAGRAMS

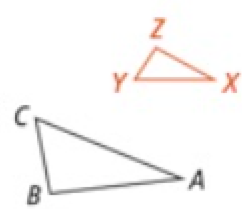




Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** What makes a transformation a similarity transformation? What is the relationship between a preimage and the image resulting from a similarity transformation?

2. **Error Analysis** Reese described the similarity transformation that maps $\triangle ABC$ to $\triangle XZY$. What is Reese's error?



$\triangle ABC$ is dilated and then rotated to produce the image $\triangle XYZ$.

3. **Vocabulary** How are similarity transformations and congruence transformations alike? How are they different?

4. **Communicate and Justify** A similarity transformation consisting of a reflection and a dilation is performed on a figure, and one point maps to itself. Describe one way this could happen.

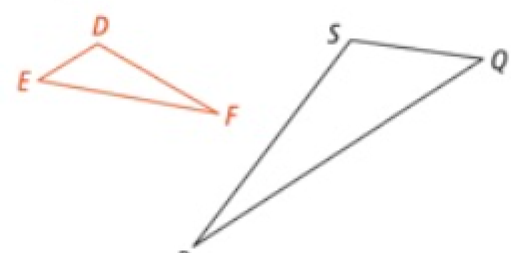
Do You KNOW HOW?

For Exercises 5 and 6, what are the vertices of each image?

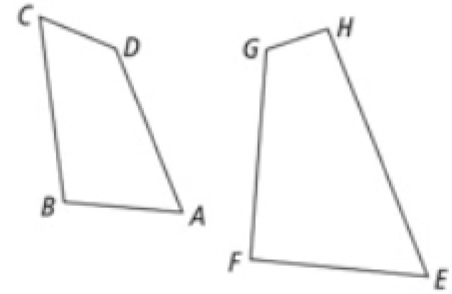
5. $R_{(90°, O)} \circ D_{0.5}(ABCD)$ for $A(5, 1)$, $B(-3, 4)$, $C(0, 2)$, $D(4, 6)$

6. $(D_3 \circ r_{x\text{-axis}})(\triangle GHJ)$ for $G(3, 5)$, $H(1, -2)$, $J(-1, 6)$

7. Describe a similarity transformation that maps $\triangle SQR$ to $\triangle DEF$.



8. Do the two figures appear to be similar? Use transformations to explain.

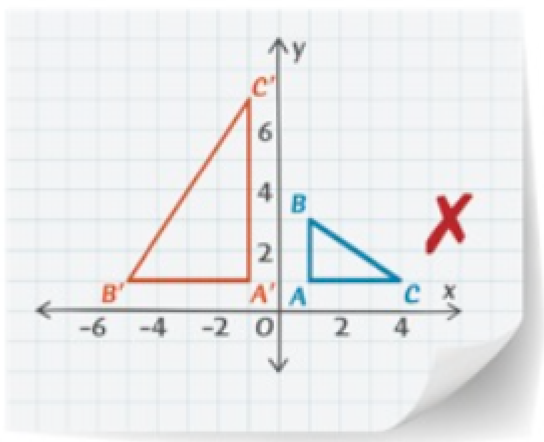


PRACTICE & PROBLEM SOLVING

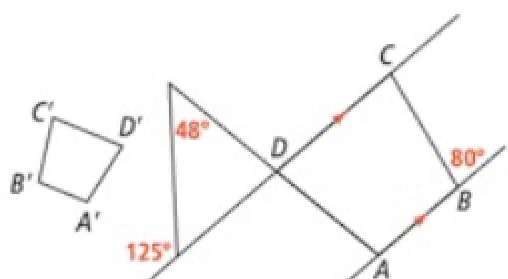
UNDERSTAND

9. **Communicate and Justify** Is it possible to use only translations and dilations to map one circle to another? Explain.

10. **Error Analysis** Keegan was asked to graph $(R_{90^\circ} \circ D_2)(\triangle ABC)$. Explain Keegan's error.



11. **Mathematical Connections** In the diagram, $ABCD \sim A'B'C'D'$. What are the angle measures of $A'B'C'D'$?



12. **Communicate and Justify** Are all equilateral triangles similar? Use transformations to explain.

13. **Generalize** Show whether a composition of a dilation and a translation can be performed in either order and result in the same image. (Hint: Test whether the equation $(D_k \circ T_{\langle a,\, b \rangle})(x, y) = (T_{\langle a,\, b \rangle} \circ D_k)(x, y)$ is true.)

14. **Higher Order Thinking** Given isosceles triangle ABC with $\overline{AB} \cong \overline{BC}$. Point D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{BC}$, and F is the midpoint of $\overline{CA}$. Use a similarity transformation and triangle congruence to show that $\triangle ABC \sim \triangle FEC$.

PRACTICE

What are the vertices of each image? SEE EXAMPLE 1

15. $(T_{\langle 5, -4 \rangle} \circ D_{1.5})(\triangle XYZ)$ for $X(6, -2)$, $Y(4, 1)$, $Z(-2, 3)$

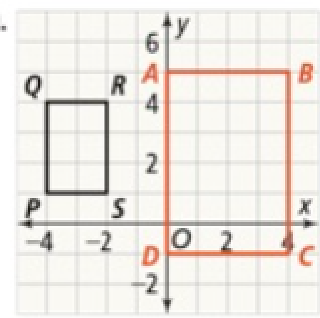
16. $(r_{x\text{-axis}} \circ D_{0.5})(LMNP)$ for $L(2, 4)$, $M(4, 4)$, $N(4, -4)$, $P(2, -4)$

17. $(r_{y\text{-axis}} \circ D_2 \circ r_{270^\circ})(\triangle PQR)$ for $P(1, 3)$, $Q(-4, 2)$, $R(0, 5)$

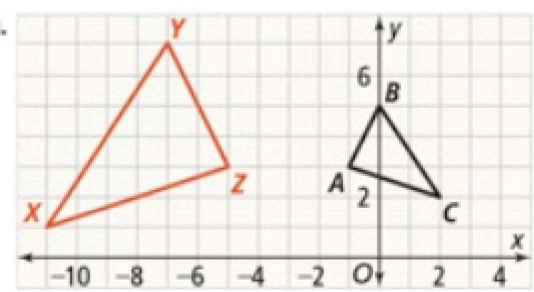
18. $(D_{0.25} \circ r_{x\text{-axis}})(ABCD)$ for $A(2, 6)$, $B(0, 0)$, $C(-5, 8)$, $D(-2, 10)$

For each black pre-image and red image describe a similarity transformation, and write a similarity statement. SEE EXAMPLES 2 AND 3

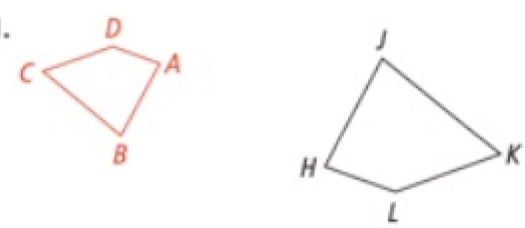
19.



20.

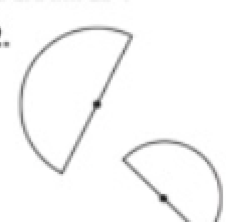


21.

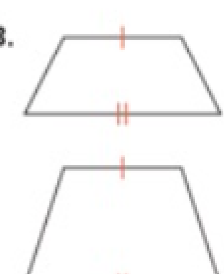


Do the figures appear to be similar? Explain.
SEE EXAMPLE 4

22.



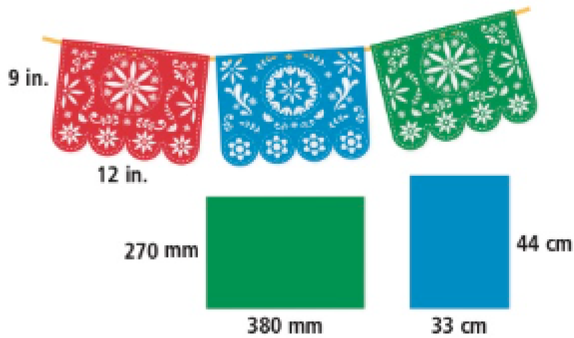
23.



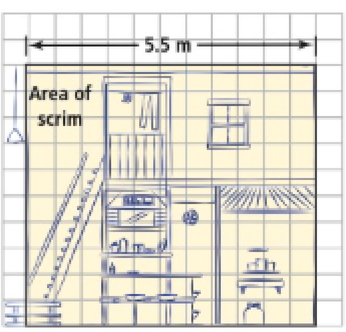
PRACTICE & PROBLEM SOLVING

APPLY

24. Apply Math Models Can Lucas use the other sheets of paper shown to make paper cutouts, or papel picado, similar to those in the banner? If not, how can he trim the sheets of paper so he can use them? Justify your answer.

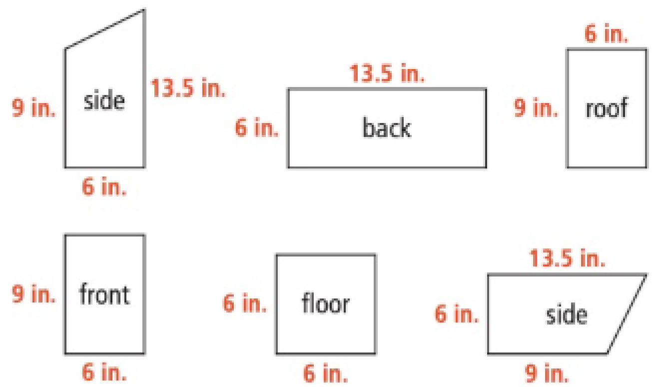


25. Analyze and Persevere Rachel makes a sketch for a stage set design on a grid. She plans to have a gauze fabric called a scrim drop down from a beam that is 5.5 m wide. Assuming that her sketch is similar to the actual set, how much scrim is needed? Explain.



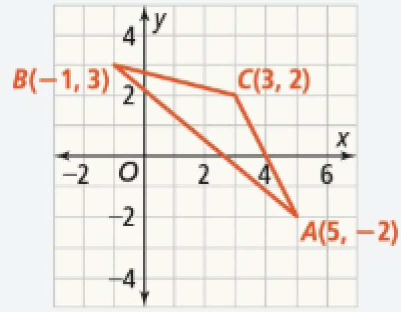
26. Use Patterns and Structure Juanita wants to make a dollhouse following the pattern shown but with a reduced size so that the floor has an area of 25 in.^{2}. Make a sketch showing the dimensions of the pieces for the smaller dollhouse.

Dollhouse Pattern



ASSESSMENT PRACTICE

27. Are $\triangle ABC$ and $(T_{\langle 1, -3\rangle} \circ D_2)(\triangle ABC)$ similar triangles? Explain. GR.2.8



28. SAT/ACT What are the coordinates of $(D_4 \circ r_{x\text{-axis}})(8, 2)$?

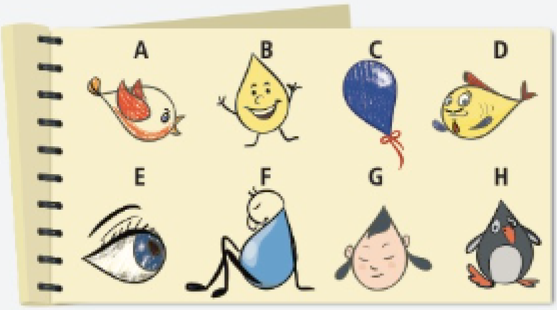
Ⓐ (−32, 8)

Ⓑ (32, 8)

Ⓒ (−32, −8)

Ⓓ (32, −8)

29. Performance Task Lourdes makes sketches for her graphic novel using a repeating similar shape as a motif.



Part A On a separate sheet of paper, draw a small simple figure. Label it A.

Part B Use transformations, including similarity transformations, to create at least five images that are similar to figure A. Label the images B, C, D, E, and F.

Part C Is it possible to select any two of your figures and find a similarity transformation that maps one to the other? Explain.

7-3 Proving Triangles Similar

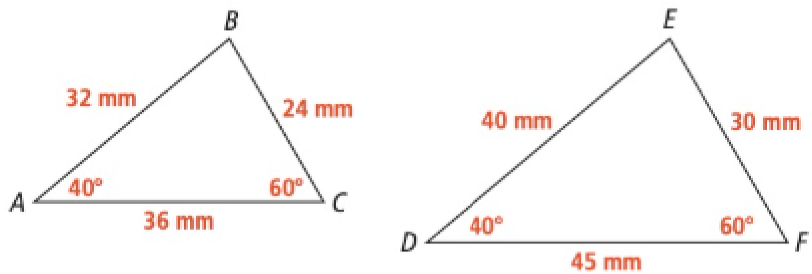
I CAN… use dilation and rigid motion to establish triangle similarity theorems.

MA.912.GR.GR.1.6–Solve mathematical and real-world problems involving congruence or similarity in two-dimensional figures.
Also GR.1.2, GR.2.8, GR.2.9
MA.K12.MTR.4.1, MTR.2.1, MTR.6.1

EXPLORE & REASON

Powered By desmos

The measurements of two triangles are shown.



A. Are the triangles similar? Explain.

B. **Communicate and Justify** Would any triangle with 40°- and 60°-angles be similar to △*ABC*? Explain.

ESSENTIAL QUESTION

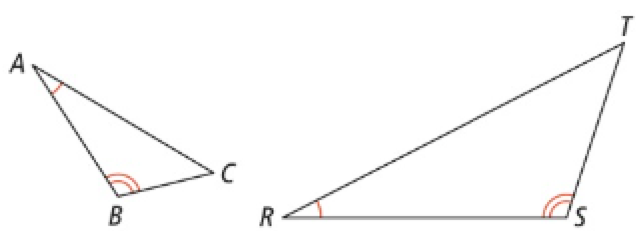
How can you use the angles and sides of two triangles to determine whether they are similar?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Establish the Angle-Angle Similarity (AA~) Theorem

If ∠*A* ≅ ∠*R* and ∠*B* ≅ ∠*S*, is △*ABC* ~ △*RST*? Explain.

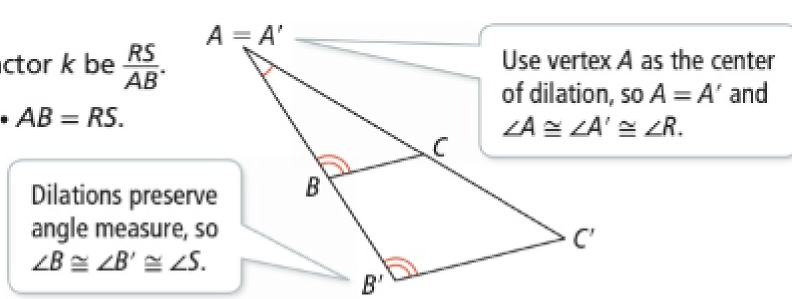
To show that the triangles are similar, determine whether there is a similarity transformation that maps △*ABC* to △*RST*.



Determine the center of dilation and the scale factor that map △*ABC* to image △*A′B′C′* such that *A′B′* = *RS*.

Let the scale factor k be $\frac{RS}{AB}$.

Then, $A'B' = k \cdot AB = RS$.



The dilation $D_{(k,\ A)}$ maps △*ABC* to △*A′B′C′*, and △*A′B′C′* ≅ △*RST* by ASA, so there is a rigid motion that maps △*A′B′C′* to △*RST*. Thus, the composition is a similarity transformation that maps △*ABC* to △*RST*. So, △*ABC* ~ △*RST*.

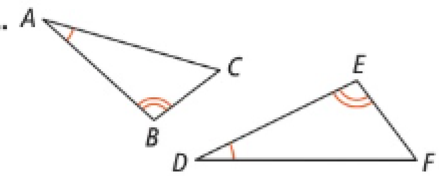
ANALYZE AND PERSEVERE
Think about what it means for figures to be similar. Are congruent triangles similar to the same triangle?

Try It! 1. If ∠*A* is congruent to ∠*R*, and ∠*C* is congruent to ∠*T*, how would you prove the triangles are similar?

THEOREM 7-1 Angle-Angle Similarity (AA~) Theorem

If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

If...



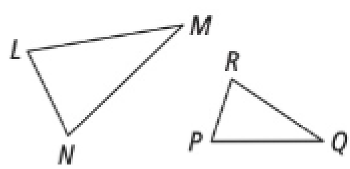
$\angle A \cong \angle D$ and $\angle B \cong \angle E$

Then... $\triangle ABC \sim \triangle DEF$

PROOF: SEE EXERCISE 10.

EXAMPLE 2 Establish the Side-Side-Side Similarity (SSS~) Theorem

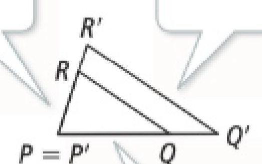
If $\frac{LM}{PQ} = \frac{MN}{QR} = \frac{LN}{PR}$, is there a similarity transformation that maps $\triangle PQR$ to $\triangle LMN$? Explain.



Dilate $\triangle PQR$ by scale factor $k = \frac{LM}{PQ} = \frac{MN}{QR} = \frac{LN}{PR}$ to map $\triangle PQR$ to $\triangle P'Q'R'$.

$P'R' = k \cdot PR = \frac{LN}{PR} \cdot PR = LN$

$Q'R' = k \cdot QR = \frac{MN}{QR} \cdot QR = MN$



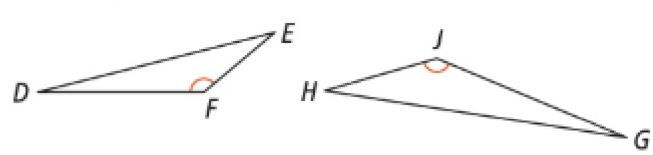
$P'Q' = k \cdot PQ = \frac{LM}{PQ} \cdot PQ = LM$

Because $P'R' = LN$, $P'Q' = LM$, and $Q'R' = MN$, $\triangle P'Q'R' \cong \triangle LMN$ by SSS. By the definition of congruence, there is a rigid motion that maps $\triangle P'Q'R'$ to $\triangle LMN$.

So, $\triangle PQR$ was mapped to $\triangle LMN$ by a similarity transformation and $\triangle PQR$ is similar to $\triangle LMN$.

STUDY TIP
Remember, two figures are similar if there is a similarity transformation between them.

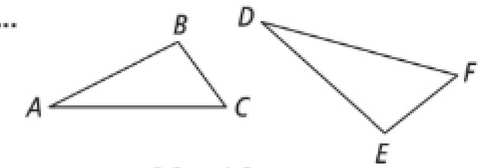
Try It! **2.** If $\frac{DF}{GJ} = \frac{EF}{HJ}$ and $\angle F \cong \angle J$, is there a similarity transformation that maps $\triangle DEF$ to $\triangle GHJ$? Explain.



THEOREM 7-2 Side-Side-Side Similarity (SSS~) Theorem

If the corresponding sides of two triangles are proportional, then the triangles are similar.

If...



$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

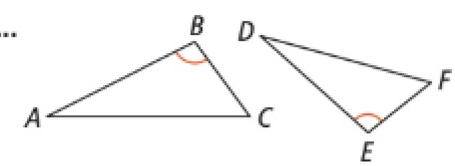
Then... $\triangle ABC \sim \triangle DEF$

PROOF: SEE EXERCISE 21.

THEOREM 7-3 Side-Angle-Side Similarity (SAS~) Theorem

If an angle of one triangle is congruent to an angle of a second triangle, and the sides that include the two angles are proportional, then the triangles are similar.

If...



$\angle B \cong \angle E$ and $\frac{AB}{DE} = \frac{BC}{EF}$

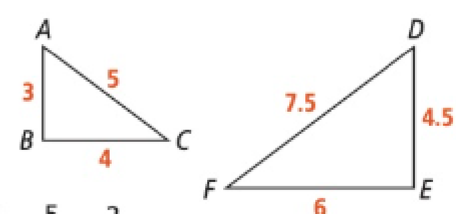
Then... $\triangle ABC \sim \triangle DEF$

PROOF: SEE EXERCISE 14.

EXAMPLE 3 Verify Triangle Similarity

A. Are $\triangle ABC$ and $\triangle DEF$ similar?

Determine whether the ratios of the corresponding side lengths are equal.



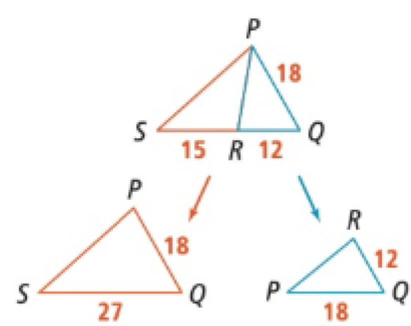
$\frac{AB}{DE} = \frac{3}{4.5} = \frac{2}{3}$ $\quad \frac{BC}{EF} = \frac{4}{6} = \frac{2}{3}$ $\quad \frac{AC}{DF} = \frac{5}{7.5} = \frac{2}{3}$

The ratios are equal, so the corresponding side lengths are proportional. $\triangle ABC \sim \triangle DEF$ by SSS~.

COMMON ERROR
When setting up ratios to check if triangles are similar, be sure you place sides from the same triangle in the same position in each ratio.

B. Are $\triangle PQS$ and $\triangle RQP$ similar?

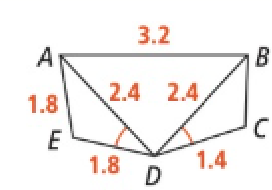
The two triangles share an included angle, $\angle Q$. Separate the triangles and see whether the lengths of the corresponding sides are in proportion.



Since $\frac{SQ}{PQ} = \frac{27}{18} = \frac{3}{2}$ and $\frac{PQ}{RQ} = \frac{18}{12} = \frac{3}{2}$, the lengths of the sides that include $\angle Q$ are proportional. By SAS~, $\triangle PQS \sim \triangle RQP$.

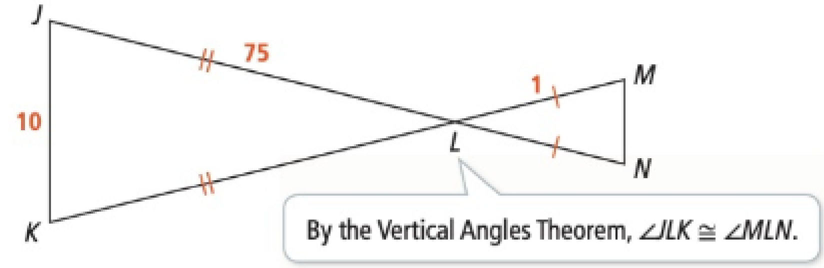
Try It! 3. a. Is $\triangle ADE \sim \triangle ABD$? Explain.

b. Is $\triangle ADE \sim \triangle BDC$? Explain.



EXAMPLE 4 Find Lengths in Similar Triangles

What is *MN*?



STUDY TIP
The measures of the two pairs of corresponding legs of any two isosceles triangles will always be proportional, even if the triangles are not similar.

The sides that include $\angle JLK$ and $\angle MLN$ are proportional.

By SAS~, $\triangle JLK \sim \triangle MLN$.

Write a proportion using corresponding sides, and solve for *MN*.

$$\frac{MN}{10} = \frac{1}{75}$$

$$MN = \frac{2}{15}$$

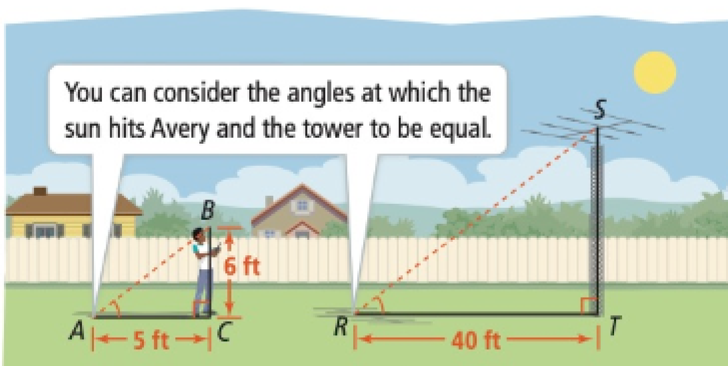
Try It! **4. a.** In Example 4, if the measure of $\overline{JL}$ were 150 instead of 75, how would the value of *MN* be different?

b. In Example 4, if the measure of $\overline{JK}$ were 20 instead of 10, how would the value of *MN* be different?

APPLICATION

EXAMPLE 5 Solve Problems Involving Similar Triangles

Avery puts up a radio antenna tower in his yard. Ella tells him that their city has a law limiting towers to 50 ft in height. How can Avery use the lengths of his shadow and the shadow of the tower to show that his tower is within the limit without directly measuring it?



Art not drawn to scale.

REPRESENT AND CONNECT
Think about how the situation is modeled with triangles. What information do you need to be able to apply a similarity theorem?

Since $\angle BAC \cong \angle SRT$ and $\angle ACB \cong \angle RTS$, you can apply the AA~ Theorem.

$$\frac{ST}{BC} = \frac{RT}{AC}$$

$$\frac{ST}{6} = \frac{40}{5}$$

$$ST = 48$$

Corresponding sides of $\triangle ABC$ and $\triangle RST$ are proportional.

The antenna tower is 48 ft high. Avery's tower is within the 50-ft limit.

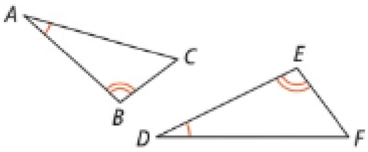
Try It! **5.** If the tower were 50 ft tall, how long would the shadow of the tower be?

CONCEPT SUMMARY Triangle Similarity Theorems

THEOREM 7-1

Angle-Angle Similarity

If...



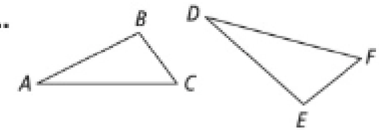
$\angle A \cong \angle D$ and $\angle B \cong \angle E$

Then... $\triangle ABC \sim \triangle DEF$

THEOREM 7-2

Side-Side-Side Similarity

If...



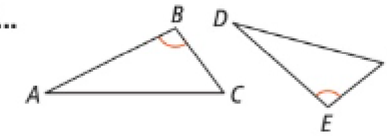
$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

Then... $\triangle ABC \sim \triangle DEF$

THEOREM 7-3

Side-Angle-Side Similarity

If...



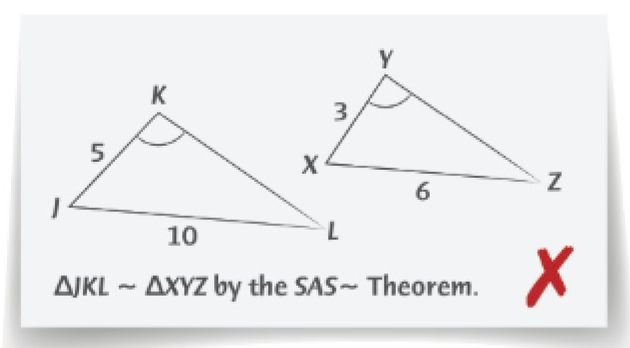
$\angle B \cong \angle E$ and $\frac{AB}{DE} = \frac{BC}{EF}$

Then... $\triangle ABC \sim \triangle DEF$

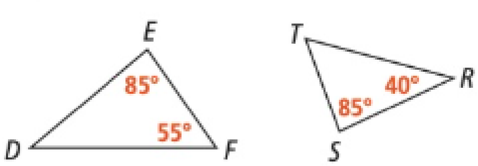
Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you use the angles and sides of two triangles to determine whether they are similar?

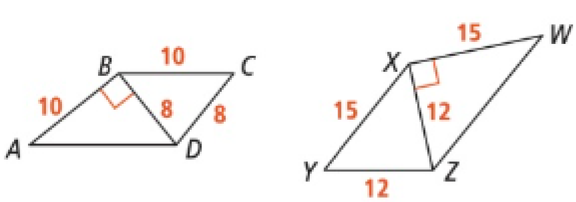
2. **Error Analysis** Allie says $\triangle JKL \sim \triangle XYZ$. What is Allie's error?



3. **Analyze and Persevere** Is any additional information needed to show $\triangle DEF \sim \triangle RST$? Explain.



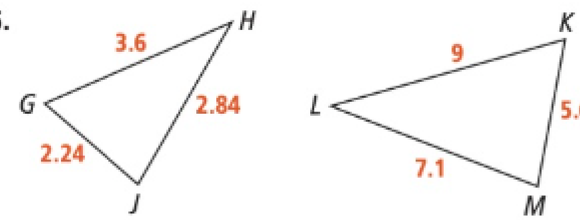
4. **Communicate and Justify** Explain how you can use triangle similarity to show that $ABCD \sim WXYZ$.



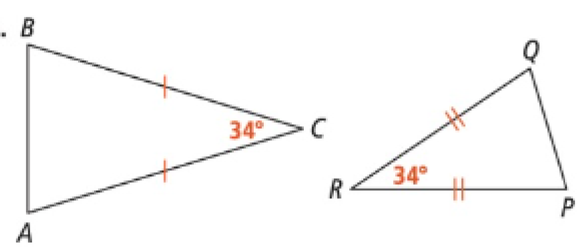
Do You KNOW HOW?

For Exercises 5 and 6, explain whether the two triangles are similar.

5.



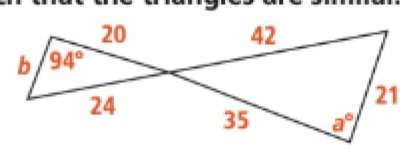
6.



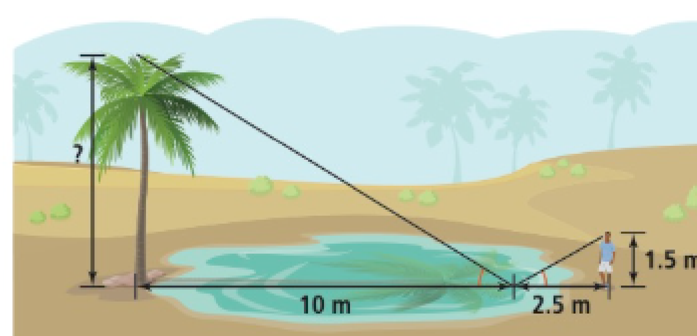
For Exercises 7 and 8, find the value of each variable such that the triangles are similar.

7. a

8. b



9. When Esteban looks at the puddle, he sees a reflection of the top of the palm tree. How tall is the palm tree?



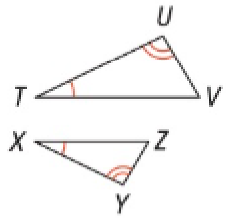
PRACTICE & PROBLEM SOLVING

UNDERSTAND

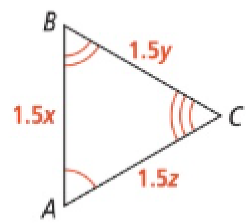
10. Communicate and Justify Write a proof of the Angle-Angle Similarity Theorem.

Given: $\angle T \cong \angle X$
$\angle U \cong \angle Y$

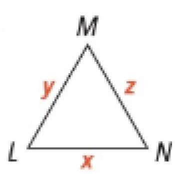
Prove: $\triangle TUV \sim \triangle XYZ$



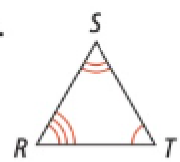
11. Use Patterns and Structure For each triangle, name the triangle similar to $\triangle ABC$ and explain why it is similar.



a.

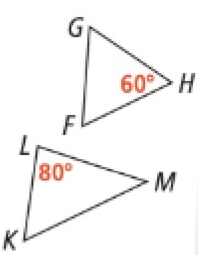


b.



12. Communicate and Justify If two triangles are congruent by ASA, are the triangles similar? Explain.

13. Error Analysis What is Russel's error?

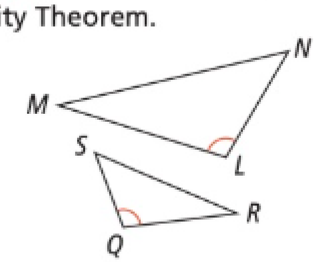


> $180 - 80 - 60 = 40$,
> so the unlabeled angle in each triangle is 40°. So, $m\angle M = 60$, and thus $\triangle FGH \sim \triangle KLM$ by AA ~.

14. Communicate and Justify Write a proof of the Side-Angle-Side Similarity Theorem.

Given: $\frac{LM}{QR} = \frac{LN}{QS}$
$\angle L \cong \angle Q$

Prove: $\triangle LMN \sim \triangle QRS$

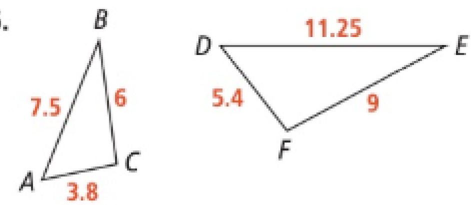


15. Higher Order Thinking Explain why there is no Side-Side-Angle Similarity Theorem.

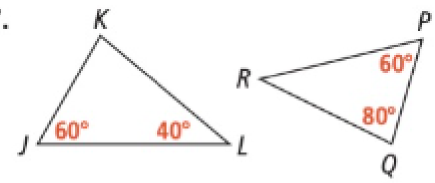
PRACTICE

For Exercise 16–18, explain whether each pair of triangles is similar. SEE EXAMPLES 1–3

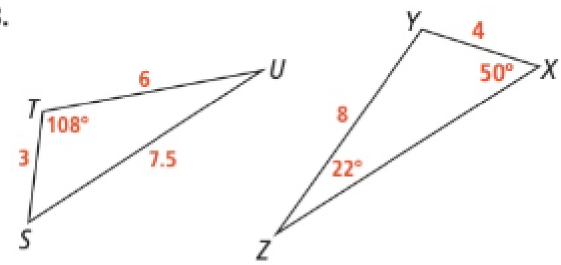
16.



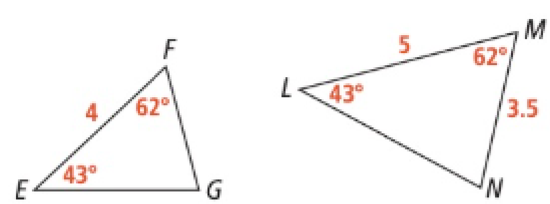
17.



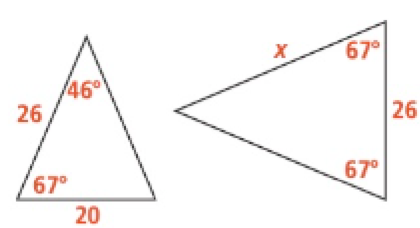
18.



19. What is FG? SEE EXAMPLES 4 AND 5



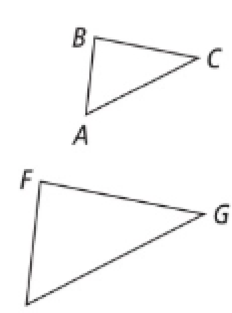
20. What is the value of x? SEE EXAMPLES 4 AND 5



21. Write a proof of the Side-Side-Side Similarity Theorem.

Given: $\frac{AB}{EF} = \frac{BC}{FG} = \frac{AC}{AG}$

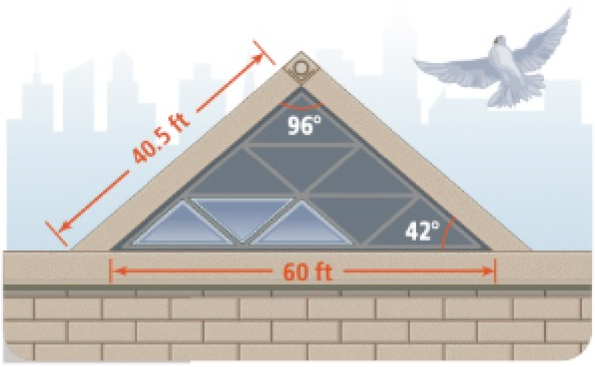
Prove: $\triangle ABC \sim \triangle EFG$



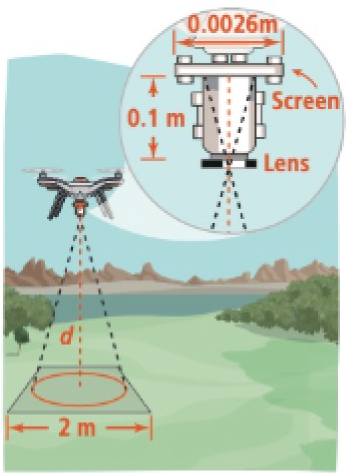
PRACTICE & PROBLEM SOLVING

APPLY

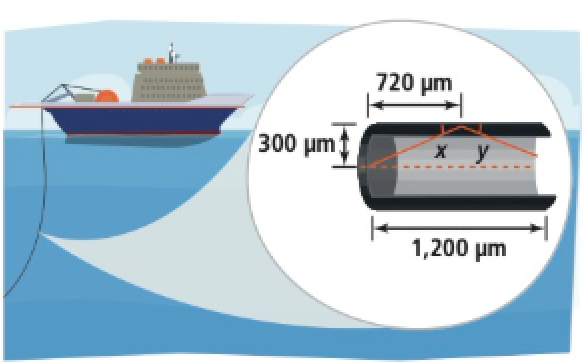
22. Check for Reasonableness A building manager needs to order 9 replacement panes that are all the same size, each similar to the window itself. At what angles should each pane be cut in order to fit in the window? What are the dimensions of each pane? Explain.



23. Use Patterns and Structure The screen of a surveying device is 0.0026 m wide and is 0.1 m away from the lens. If the surveyor wants the image of the 2-m target to fit on the screen, what distance d should the lens be from the target? Explain.

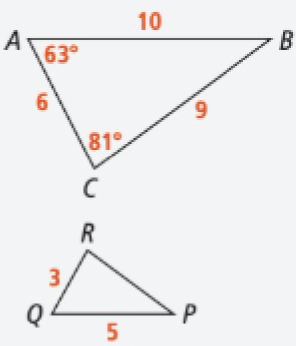


24. Mathematical Connections If a light beam strikes the inside of a fiber optic cable, it bounces off at the same angle. In a piece of cable 1,200 micrometers (µm) long, if the beam strikes the wall after 720 µm what distance $x + y$ does the beam travel? Explain.



ASSESSMENT PRACTICE

25. Which condition is sufficient to show that $\triangle ABC \sim \triangle QPR$? Select all that apply. GR.2.9



☐ **A.** $RP = 4.5$

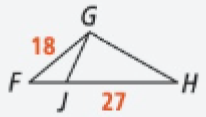
☐ **B.** $m\angle Q = 63$

☐ **C.** $m\angle P = 81$

☐ **D.** $m\angle R = 63$

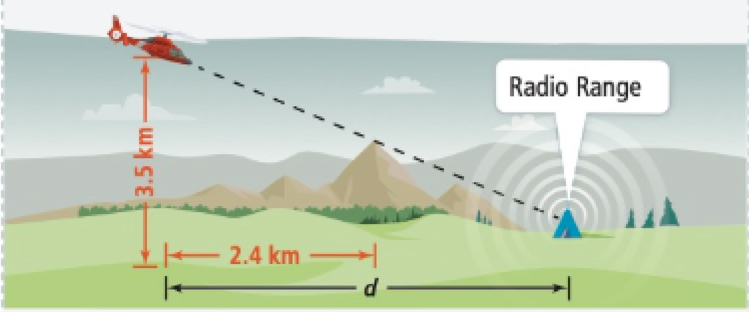
☐ **E.** $RP = 4$

26. SAT/ACT For which value of FJ must $\triangle FGJ$ be similar to $\triangle FHG$?



Ⓐ 6 Ⓑ 8 Ⓒ 9 Ⓓ 12

27. Performance Task A rescue helicopter hovering at an altitude of 3.5 km sights a campsite just over the peak of a mountain.



Part A The horizontal distance of the helicopter from the mountain is 2.4 km. If the height of the mountain is 2.8 km, what is the horizontal distance d of the helicopter from the campsite? Explain.

Part B The groundspeed (horizontal speed) of the helicopter is 1.6 km/min. When will the helicopter reach the campsite? Explain.

Part C The radio at the campsite can only transmit to a distance of 5 km. If the helicopter begins immediately to descend toward the campsite (along the diagonal line), how far will the pilot be, horizontally, when he contacts the campsite?

7-4 Similarity in Right Triangles

I CAN… use similarity and the geometric mean to solve problems involving right triangles.

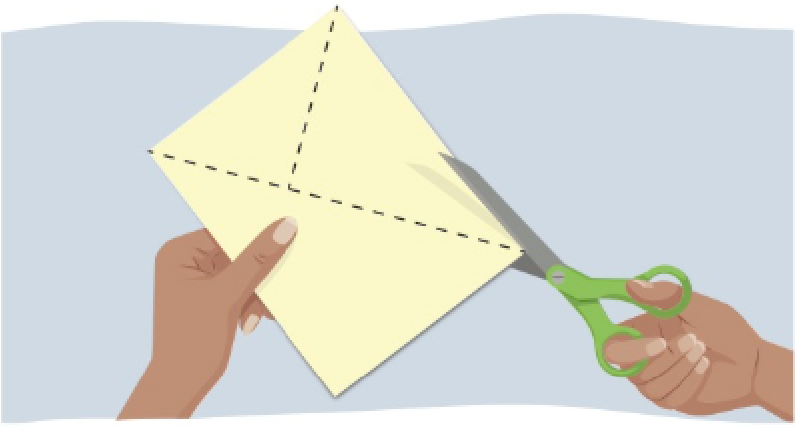
VOCABULARY

- geometric mean

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles. Also GR.1.2
MA.K12.MTR.3.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

Suppose you cut a rectangular sheet of paper to create three right triangles.



A. Choose Effective Methods How can you compare leg lengths and angle measures among the triangles?

B. Are any of the triangles similar to each other? Explain.

ESSENTIAL QUESTION In a right triangle, what is the relationship between the altitude to the hypotenuse, triangle similarity, and the geometric mean?

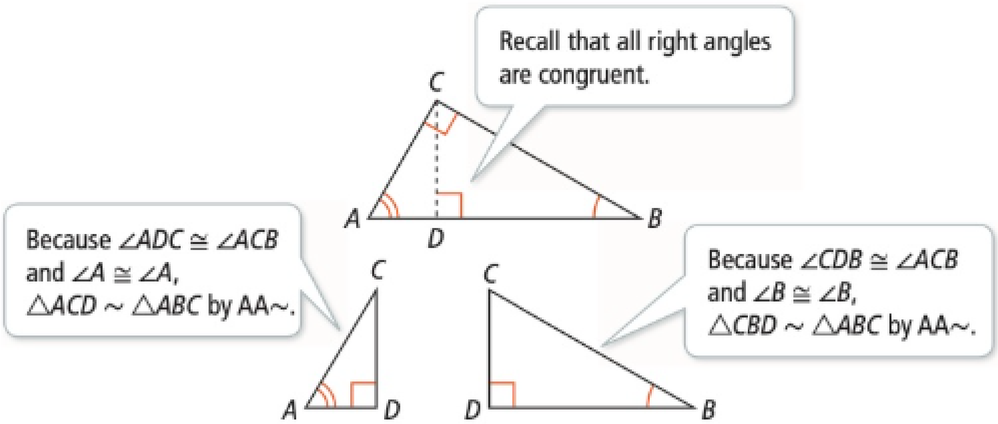
CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Identify Similar Triangles Formed by an Altitude

When you draw an altitude to the hypotenuse of a right triangle, you create three right triangles. How are the triangles related?

The altitude $\overline{CD}$ divides $\triangle ABC$ into two right triangles, $\triangle ACD$ and $\triangle CBD$. Compare each triangle to $\triangle ABC$.

USE PATTERNS AND STRUCTURE
Think about an altitude of a triangle. What type of angle does it form with the base?



$\triangle ACD$ and $\triangle CBD$ are each similar to $\triangle ABC$.

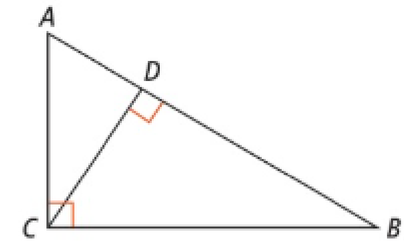
Try It! **1.** In Example 1, how is $\triangle ACD$ related to $\triangle CBD$? Explain.

THEOREM 7-4

The altitude to the hypotenuse of a right triangle divides the triangle into two triangles that are similar to the original triangle and to each other.

PROOF: SEE EXAMPLE 1.

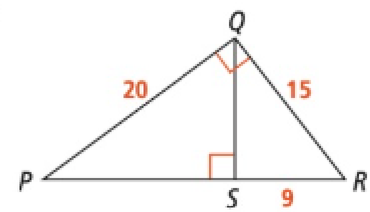
If...



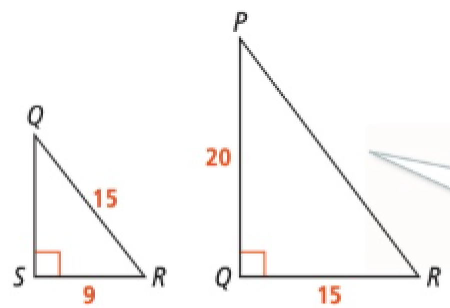
Then... $\triangle CAB \sim \triangle DAC \sim \triangle DCB$

EXAMPLE 2 Find Missing Lengths Within Right Triangles

Given that $\triangle PQR \sim \triangle QSR$, what is QS?



Draw $\triangle PQR$ and $\triangle QSR$.



Because $\triangle QSR \sim \triangle PQR$, $\overline{QS}$ corresponds to $\overline{PQ}$, $\overline{SR}$ corresponds to $\overline{QR}$, and $\overline{QR}$ corresponds to $\overline{PR}$.

To find QS, write a proportion using corresponding legs of $\triangle QSR$ and $\triangle PQR$.

$$\frac{QS}{PQ} = \frac{SR}{QR}$$

$$\frac{QS}{20} = \frac{9}{15}$$

$$QS = \frac{9}{15} \cdot 20$$

$$QS = 12$$

The length of altitude $\overline{QS}$ is 12.

STUDY TIP

With right triangles, you can apply the Pythagorean Theorem to verify your results.

Try It! **2.** Refer to $\triangle PQR$ in Example 2.

a. Write a proportion that you can use to solve for PS.

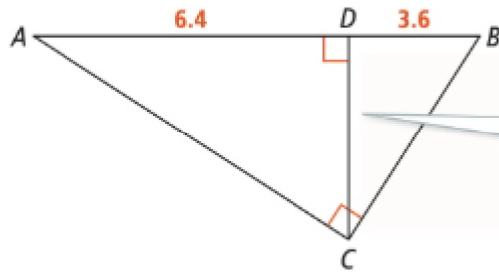
b. What is PS?

DEFINITION

The **geometric mean** of a and b is the number x such that $\frac{a}{x} = \frac{x}{b}$, where a, b, and x are positive numbers.

EXAMPLE 3 Relate Altitude and Geometric Mean

Given $\triangle ACB$, what is CD?



Notice that CD is the length of the longer leg of $\triangle CDB$ and is also the length of the shorter leg of $\triangle ADC$.

By Theorem 7-4, $\triangle ADC \sim \triangle CDB$. Use the properties of similar triangles to write a proportion.

$$\frac{AD}{CD} = \frac{CD}{BD}$$

$$\frac{6.4}{CD} = \frac{CD}{3.6}$$

$$(CD)^2 = 23.04$$

$$CD = 4.8$$

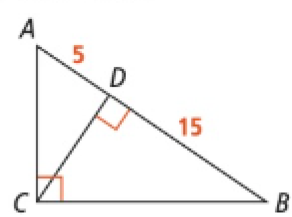
CD is the geometric mean of AD and BD.

The length of altitude $\overline{CD}$ is 4.8.

STUDY TIP

You can think about the geometric mean in another way. If $x^2 = ab$, then x is the geometric mean of a and b.

Try It! **3.** Use $\triangle ABC$.



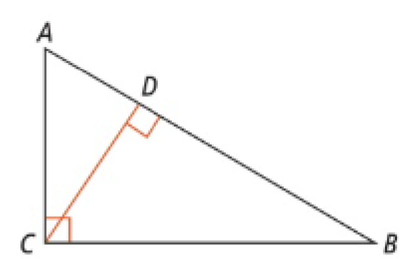
a. What is CD?

b. Describe how you can use the value you found for CD to find AC and CB.

COROLLARY 1 TO THEOREM 7-4

The length of the altitude to the hypotenuse of a right triangle is the geometric mean of the lengths of the segments of the hypotenuse.

If...

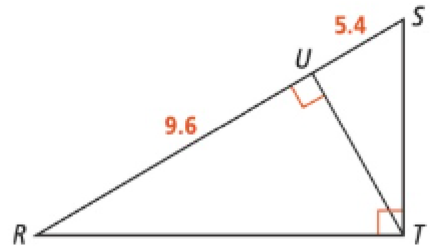


Then... $\frac{AD}{CD} = \frac{CD}{DB}$

PROOF: SEE EXERCISE 14.

EXAMPLE 4 Relate Side Lengths and Geometric Mean

Given $\triangle RST$, what is RT?



STUDY TIP
If you have difficulty identifying the similar triangles, remember that you can draw them separately as was done in Example 1.

By Theorem 7-4, $\triangle RST \sim \triangle RTU$. Use the properties of similar triangles to write a proportion.

$$\frac{RS}{RT} = \frac{RT}{RU}$$

$$\frac{15}{RT} = \frac{RT}{9.6}$$

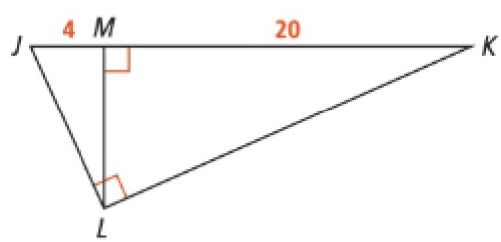
RT is the geometric mean of RS and RU.

$$(RT)^2 = 144$$

$$RT = 12$$

The length of $\overline{RT}$ is 12.

Try It! **4.** Use $\triangle JKL$.



a. What is JL?

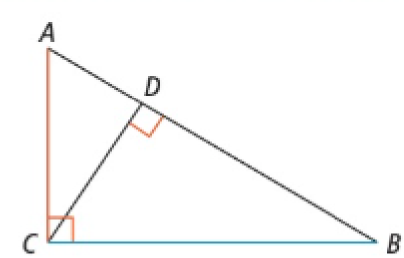
b. What is KL?

COROLLARY 2 TO THEOREM 7-4

The altitude to the hypotenuse of a right triangle divides the hypotenuse so that the length of a given leg is the geometric mean of the length of the hypotenuse and the length of the segment of the hypotenuse that is adjacent to the leg.

PROOF: SEE EXERCISE 14.

If...

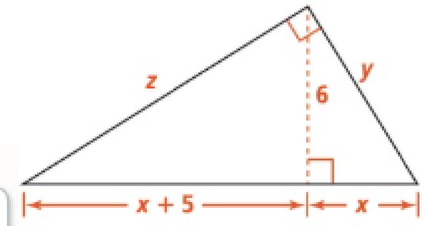


Then... $\frac{AB}{AC} = \frac{AC}{AD}$ and $\frac{AB}{CB} = \frac{CB}{DB}$

EXAMPLE 5 Use the Geometric Mean to Solve Problems

STUDY TIP
The length of the shared leg of the two smaller triangles (the altitude of the larger triangle) is the geometric mean of the lengths of the two triangles' non-shared legs.

What is the value of x?



Use the formula for the geometric mean.

$$\frac{x+5}{6} = \frac{6}{x}$$

$$x(x+5) = 36$$

$$x^2 + 5x - 36 = 0$$

$$(x+9)(x-4) = 0$$

$$x = -9 \text{ or } x = 4$$

The geometric mean of x and $x + 5$ is 6.

The value of x is 4.

Length is always positive.

Try It! **5.** Use the geometric mean and Example 5 to find each unknown.

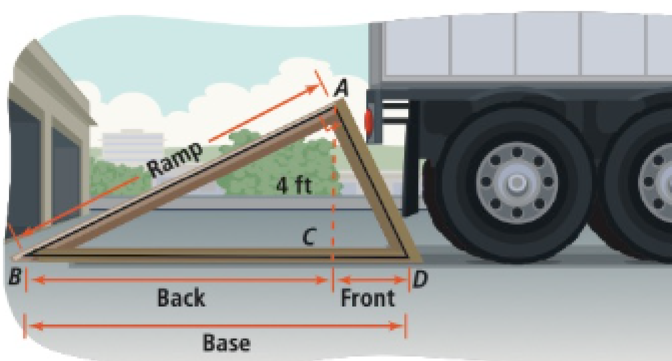
a. Find the value of y.

b. Find the value of z.

APPLICATION

EXAMPLE 6 Apply Geometric Mean to Find a Distance

Zhang is constructing a 4-ft high loading ramp. The length of the back of the base must be 12.8 ft. How long must the entire base be?



Formulate The ramp and the base form the long leg and the hypotenuse of a right triangle. The height of the ramp is the altitude of the triangle. The base of the ramp is composed of the front base x and the back base 12.8.

Compute **Step 1** Find the length of the front base x. Use the fact that the height is the geometric mean of the lengths of the front base and the back base.

$$\frac{x}{4} = \frac{4}{12.8}$$

$$12.8x = 16$$

$$x = 1.25$$

The geometric mean of x and 12.8 is 4.

Step 2 Find the length of the base.

$$12.8 + 1.25 = 14.05$$

Interpret The base of the ramp should be 14.05 feet long.

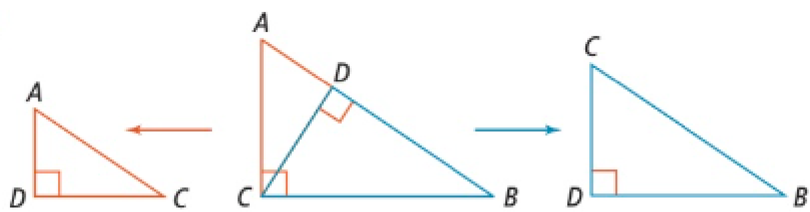
Try It! **6.** In Example 6, how long should Zhang make the ramp?

CONCEPT SUMMARY Similarity in Right Triangles

WORDS

- The altitude to the hypotenuse of a right triangle divides the triangle into two triangles that are similar to the original triangle and to each other.
- The length of the altitude is the geometric mean of the lengths of the segments of the hypotenuse.
- The length of each leg is the geometric mean of the length of the hypotenuse and the length of the segment adjacent to the leg.

DIAGRAMS



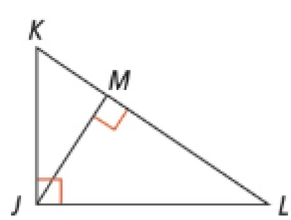
SYMBOLS $\triangle ACD \sim \triangle ABC \sim \triangle CBD$

$$\frac{AB}{AC} = \frac{AC}{AD}, \frac{AD}{CD} = \frac{CD}{DB}, \text{ and } \frac{AB}{CB} = \frac{CB}{DB}$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** In a right triangle, what is the relationship between the altitude to the hypotenuse, triangle similarity, and the geometric mean?

2. **Error Analysis** Chris is asked to find a geometric mean in $\triangle JKL$. What is his error?



$\Delta JKL \sim \Delta MKJ$

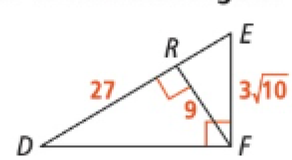
$$\frac{KL}{JK} = \frac{JK}{JM}$$ ✗

3. **Vocabulary** Do the altitudes to the legs of a right triangle also create similar triangles? Explain.

Do You KNOW HOW?

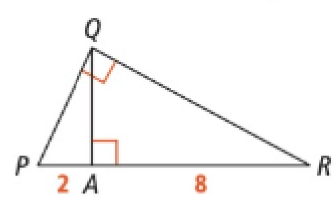
For Exercises 4–6, use $\triangle DEF$ to find the lengths.

4. ER
5. DF
6. DE

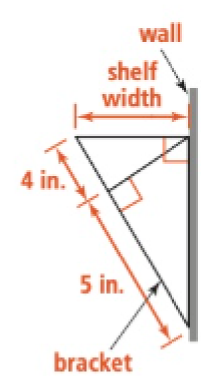


For Exercises 7–9, use $\triangle PQR$ to find the lengths.

7. QA
8. PQ
9. QR



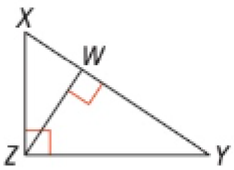
10. Deshawn installs a shelf bracket. What is the widest shelf that will fit without overhang? Explain.



PRACTICE & PROBLEM SOLVING

UNDERSTAND

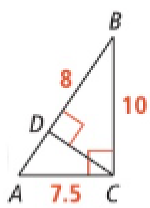
11. **Mathematical Connections** Consider $\triangle XYZ$ with altitude to the hypotenuse $\overline{ZW}$.



a. Describe a sequence of transformations that maps $\triangle XYZ$ to $\triangle XZW$.

b. Describe a sequence of transformations that maps $\triangle XYZ$ to $\triangle ZYW$.

12. **Error Analysis** Amaya was asked to find DC. What is Amaya's error?



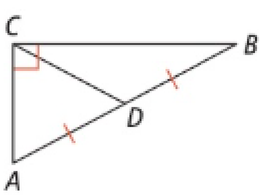
$\Delta ABC \sim \Delta ACD$ by Theorem 7-4.

$\frac{AC}{BC} = \frac{AC}{DC} \rightarrow \frac{7.5}{10} = \frac{7.5}{DC}$

$7.5 \times DC = 7.5 \times 10$,

so $DC = 10$. ✗

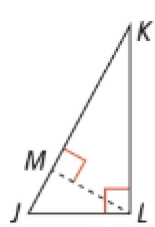
13. **Analyze and Persevere** Is CD the geometric mean of AD and BD? Explain.



14. **Communicate and Justify** Write proofs of Theorem 7-4 and its corollaries.

a. Given: $m\angle JLK = 90$ and $\overline{LM} \perp \overline{JK}$

Prove: $\triangle JKL \sim \triangle JLM \sim \triangle LKM$



b. Given: $\triangle JLM \sim \triangle LKM$

Prove: $\frac{JM}{LM} = \frac{LM}{KM}$

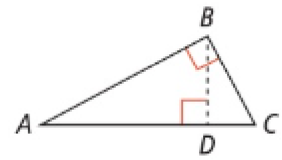
c. Given: $\triangle JKL \sim \triangle JLM \sim \triangle LKM$

Prove: $\frac{JK}{JL} = \frac{JL}{JM}$ and $\frac{JK}{LK} = \frac{LK}{MK}$

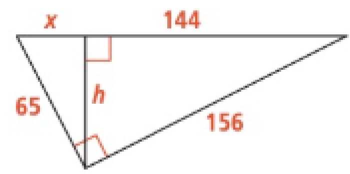
15. **Higher Order Thinking** Suppose the altitude to the hypotenuse of a right triangle also bisects the hypotenuse. What type of right triangle is it? Use the similarity of right triangles to explain your answer.

PRACTICE

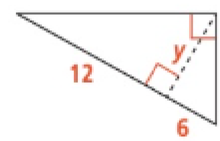
16. In the figure, what two smaller triangles are similar to $\triangle ABC$? Explain. SEE EXAMPLE 1



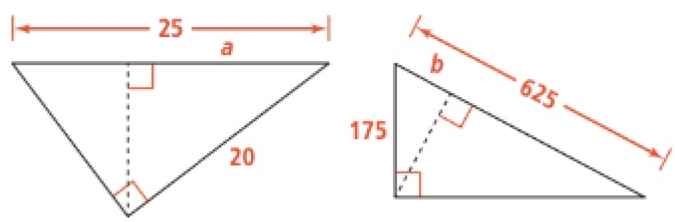
17. What are the values of h and x in the right triangle? Explain. SEE EXAMPLE 2



18. What is the value of y in the right triangle? Explain. SEE EXAMPLE 3

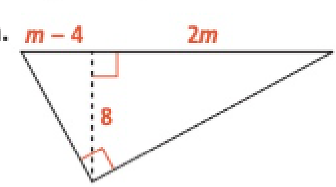


19. What are the values of a and b in each right triangle? Explain. SEE EXAMPLES 4 AND 6

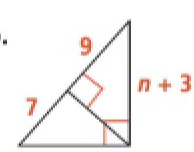


20. What are the values of m and n in each right triangle? Explain. SEE EXAMPLE 5

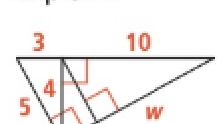
a.



b.



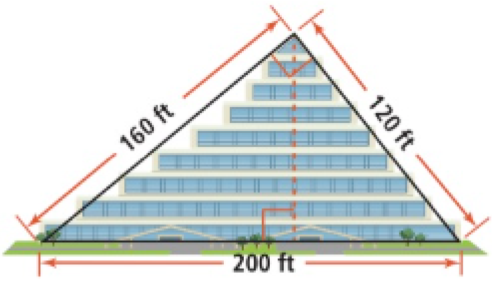
21. What is the value of w in the right triangle? Explain.



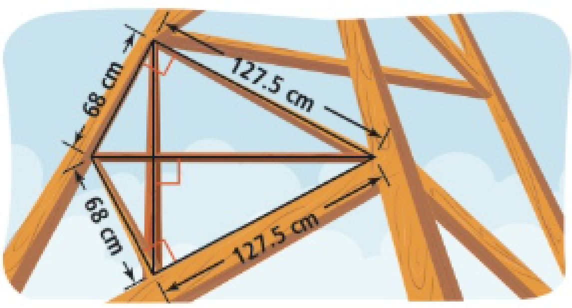
PRACTICE & PROBLEM SOLVING

APPLY

22. **Choose Efficient Methods** Jake wants the profile of a hotel he is planning to be a right triangle with the dimensions shown. The city prohibits structures over 100 ft at the location where he would like to build. Can the hotel be located there? Explain.



23. **Use Patterns and Structure** Kiyo is repairing a wooden climbing tower.



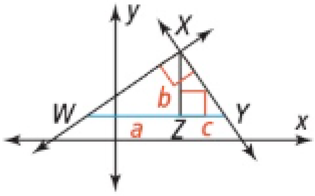
a. He needs to cut two crossbars. What should the lengths of the two crossbars be? Explain.

b. Kiyo will make a notch in each crossbar in order to fit them together. Where should he make the notch on each crossbar? Explain.

24. **Higher Order Thinking** Write a proof for Theorem 2-14.

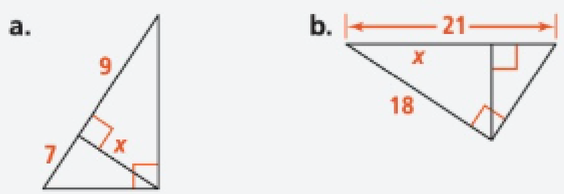
Given: Right $\triangle WXY$ with altitude $\overline{XZ}$ to hypotenuse $\overline{WY}$

Prove: The product of the slopes of perpendicular lines is -1.

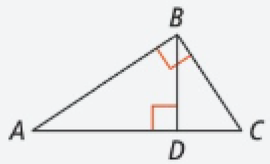


ASSESSMENT PRACTICE

25. For each figure, write an equation that you could use to find the value of x. GR.1.3

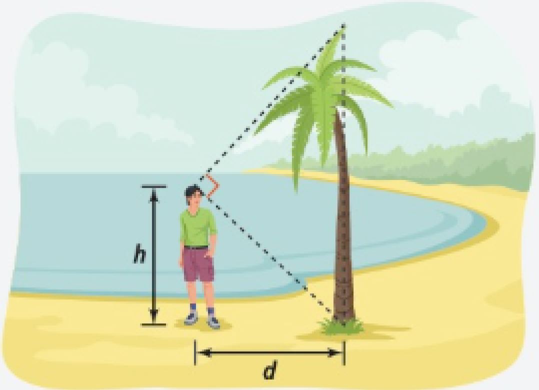


26. **SAT/ACT** Which triangle is similar to $\triangle ABC$?



Ⓐ $\triangle CBA$
Ⓑ $\triangle ABD$
Ⓒ $\triangle CDB$
Ⓓ $\triangle BDC$

27. **Performance Task** To estimate the height of a tree, Tia and Felix walk away from the tree until the angle of sight with the top and bottom of the tree is a right angle. Let h represent the height of a person's eyes and d represent the distance away from the tree.



Part A If the height of Tia's eyes is 1.6 m and her distance away from the tree is 2.5 m, what is the height of the tree? Round to the nearest hundredth of a meter.

Part B If the height of Felix's eyes is 1.7 m, about how far from the tree is Felix if his angle of sight is a right angle? Round to the nearest hundredth of a meter.

Part C Suppose Tia and Felix stand equally distant from another tree and their angles of sight are right angles, what is the height of the tree? Explain.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.1.6–Solve mathematical and real-world problems involving congruence or similarity in two-dimensional figures.
MA.K12.MTR.7.1

Make It Right

Architects often make a scale physical model of a new building project. The scale model is usually a miniature version of the project it is representing.

When making a model, architects need to make sure that all of the parts of the model are the right size. Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

7-5 Proportions in Triangles

I CAN... find the lengths of segments using proportional relationships in triangles resulting from parallel lines.

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles. Also GR.1.2
MA.K12.MTR.5.1, MTR.1.1, MTR.4.1

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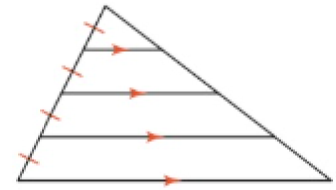
EXPLORE & REASON

Draw a triangle, like the one shown, by dividing one side into four congruent segments and drawing lines parallel to one of the other sides.



A. How many similar triangles are in the figure? Explain.

B. Use Patterns and Structure How are the lengths of the parallel segments related to each other?

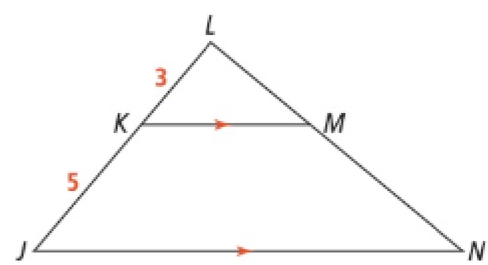
ESSENTIAL QUESTION

When parallel lines intersect two transversals, what are the relationships among the lengths of the segments formed?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Explore Proportions from Parallel Lines

In △*JLN*, if *LN* = 9.6, what are *LM* and *MN*? Are the sides divided proportionally? Explain.



Step 1 Determine how △*JLN* and △*KLM* are related.

The triangles share ∠*L*, and ∠*LJN* ≅ ∠*LKM* by the Corresponding Angles Theorem. Therefore, △*JLN* ~ △*KLM* by AA~.

Step 2 Write a proportion to relate the corresponding sides.

$$\frac{LK}{LJ} = \frac{LM}{LN}$$

Step 3 Use the proportion to find *LM* and *MN*.

$$\frac{3}{8} = \frac{LM}{9.6}$$

$$LM = 3.6$$

$LK = 3$, $LJ = LK + KJ = 3 + 5 = 8$, and $LN = 9.6$

Then $MN = LN - LM = 9.6 - 3.6 = 6$.

Step 4 Find the ratios between the segments of each side divided by $\overline{KM}$.

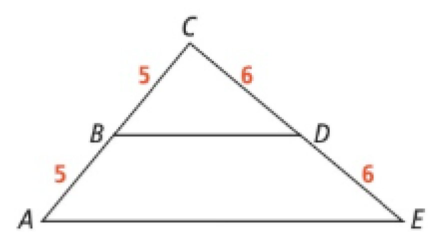
$$\frac{LK}{KJ} = \frac{3}{5} = 0.6 \text{ and } \frac{LM}{MN} = \frac{3.6}{6} = 0.6$$

Since $\frac{LK}{KJ} = \frac{LM}{MN}$, $\overline{KM}$ divides $\overline{LJ}$ and $\overline{LN}$ proportionally.

USE PATTERNS AND STRUCTURE
The length of the shorter segments, $\overline{LK}$ and $\overline{LM}$, are each the same fraction of the lengths of the corresponding longer segments, $\overline{KJ}$ and $\overline{MN}$.

Try It!

1. $\overline{BD}$ bisects sides $\overline{AC}$ and $\overline{CE}$. Is $\overline{AE} \parallel \overline{BD}$? Explain.

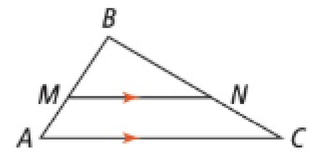


THEOREM 7-5 Side-Splitter Theorem

If a line is parallel to one side of a triangle and intersects the other two sides, then it divides those sides proportionally.

PROOF: SEE EXERCISE 14.

If... $\overline{MN} \parallel \overline{AC}$



Then... $\frac{AM}{MB} = \frac{CN}{NB}$

THEOREM 7-6 Triangle Midsegment Theorem

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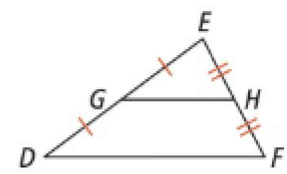
If a segment joins the midpoints of two sides of a triangle, then the segment is parallel to the third side and is half as long.

PROOF: SEE EXERCISE 24.

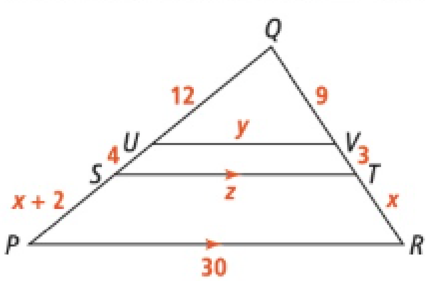
If... $\overline{DG} \cong \overline{GE}$ and $\overline{FH} \cong \overline{HE}$



Then... $\overline{GH} \parallel \overline{DF}$ and $GH = \frac{1}{2}DF$

EXAMPLE 2 Use the Side-Splitter Theorem

What is the value of *x* in $\triangle PQR$?



LEARN TOGETHER
How do you value other perspectives and points of view respectfully?

Since $\overline{ST} \parallel \overline{PR}$, $\frac{PS}{SQ} = \frac{RT}{TQ}$ by the Side-Splitter Theorem. Write a proportion in terms of *x* and solve.

$$\frac{PS}{SQ} = \frac{RT}{TQ}$$

$SQ = 4 + 12 = 16$ and $TQ = 3 + 9 = 12$

$$\frac{x+2}{16} = \frac{x}{12}$$

$$48\left(\frac{x+2}{16}\right) = 48\left(\frac{x}{12}\right)$$

$$3x + 6 = 4x$$

$$x = 6$$

Try It! **2.** Refer to $\triangle PQR$ in Example 2.

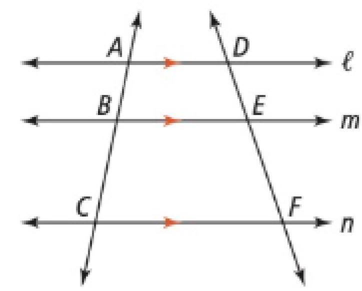
a. What is the value of *y*? Explain.

b. What is the value of *z*? Explain.

COROLLARY TO THE SIDE-SPLITTER THEOREM

If three parallel lines intersect two transversals, then the segments intercepted on the transversals are proportional.

If... $\ell \parallel m \parallel n$



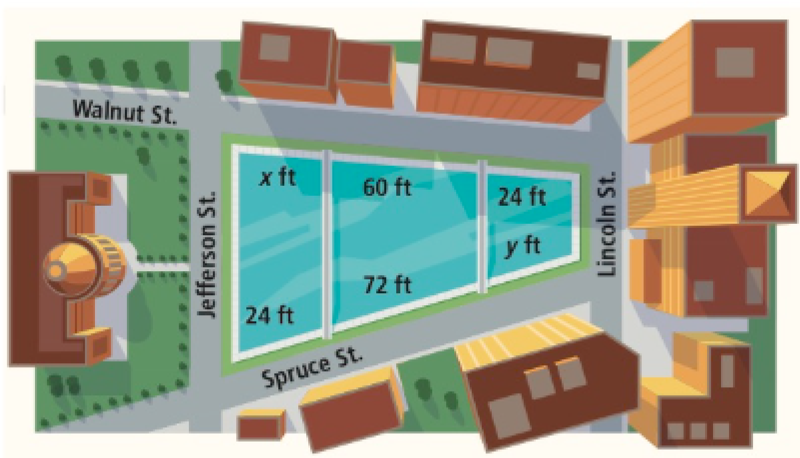
Then... $\frac{AB}{BC} = \frac{DE}{EF}$

PROOF: SEE EXERCISE 25.

APPLICATION

EXAMPLE 3 Find a Length

A reflecting pool is separated by walkways parallel to Lincoln St. and Jefferson St., which are parallel to each other. The city wants to add additional tiling around the pool. How much tiling does *x* ft represent?



Formulate Walnut St. and Spruce St. are transversals of Jefferson St., Lincoln St., and the walkways that separate the pool.

Compute Write an equation with *x*.

$$\frac{x}{60} = \frac{24}{72}$$

Apply the Corollary to the Side-Splitter Theorem.

$$x = 60\left(\frac{24}{72}\right)$$

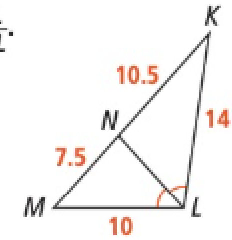
$$x = 20$$

Interpret The amount of tiling represented by *x* ft is 20 ft.

Try It! **3.** In Example 3, how much tiling does *y* ft represent?

EXAMPLE 4 Investigate Proportionality with an Angle Bisector

In $\triangle KLM$, $\overline{NL}$ bisects $\angle KLM$. Compare the ratios $\frac{KN}{MN}$ and $\frac{KL}{ML}$. Is $\triangle LKN \sim \triangle LMN$? Explain.



Compute the ratios of the corresponding sides for the given measures.

$$\frac{KN}{MN} = \frac{10.5}{7.5} = 1.4 \qquad \frac{KL}{ML} = \frac{14}{10} = 1.4$$

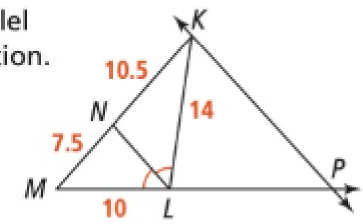
The ratios $\frac{KN}{MN}$ and $\frac{KL}{ML}$ are equal. However, the third pair is $\overline{NL}$ and $\overline{NL}$, and that ratio is 1. So, the two triangles are not similar.

Try It!

4. Draw $\overrightarrow{ML}$ and a line through K parallel to $\overline{NL}$. Let P be the point of intersection.

   a. Is $\triangle MNL \sim \triangle MKP$? Explain.

   b. Is $\angle LKP \cong \angle LPK$? Explain.

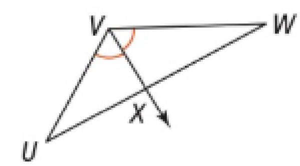


THEOREM 7-7 Triangle-Angle-Bisector Theorem

If a ray bisects an angle of a triangle, then it divides the opposite side into two segments such that the ratio between the segments is the same as the ratio between the sides adjacent to each segment.

PROOF: SEE EXERCISE 16.

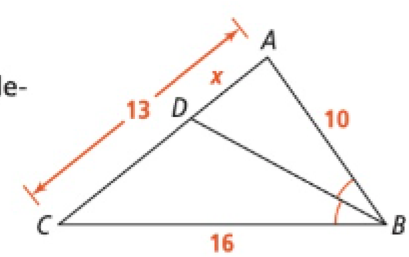
If... $\angle UVX \cong \angle WVX$



Then... $\frac{UX}{WX} = \frac{UV}{WV}$

EXAMPLE 5 Use the Triangle-Angle-Bisector Theorem

What are the values of AD and DC?



Since $\overline{BD}$ bisects $\angle ABC$, use the Triangle-Angle-Bisector Theorem to write a proportion.

$$\frac{AD}{CD} = \frac{AB}{CB}$$

$$\frac{x}{13 - x} = \frac{10}{16}$$

$CD = AC - AD$ or $CD = 13 - x$

$$16x = 130 - 10x$$

$$26x = 130$$

$$x = 5$$

In the figure, $AD = 5$ and $CD = 13 - 5 = 8$.

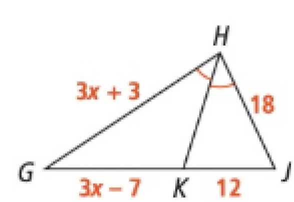
COMMON ERROR

You may incorrectly use 13 as the length of one of the shorter segments. Remember to correctly identify the lengths of all of the segments.

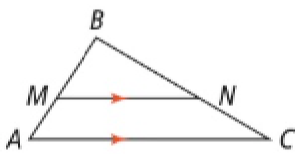
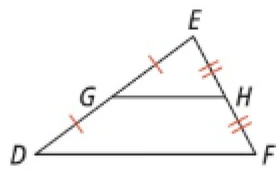
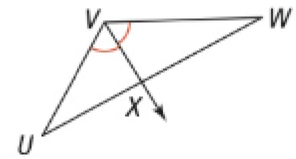
Try It!

5. a. What is the value of x?

   b. What are the values of GH and GK?



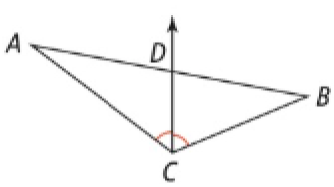
CONCEPT SUMMARY Proportions in Triangles

THEOREM 7-5	THEOREM 7-6	THEOREM 7-7
Side-Splitter Theorem	Triangle Midsegment Theorem	Triangle-Angle-Bisector Theorem
If... $\overline{MN} \parallel \overline{AC}$	If... $\overline{DG} \cong \overline{GE}$ and $\overline{FH} \cong \overline{HE}$	If... $\angle UVX \cong \angle WVX$
		
Then... $\frac{AM}{MB} = \frac{CN}{NB}$	Then... $\overline{GH} \parallel \overline{DF}$ and $GH = \frac{1}{2}DF$	Then... $\frac{UX}{WX} = \frac{UV}{WV}$

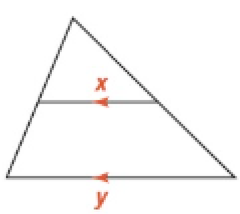
Do You UNDERSTAND?

1. ESSENTIAL QUESTION When parallel lines intersect two transversals, what are the relationships among the lengths of the segments formed?

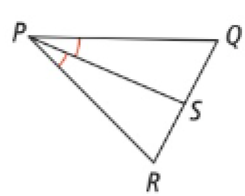
2. **Error Analysis** Carmen thinks that $AD = BD$. What is Carmen's error?



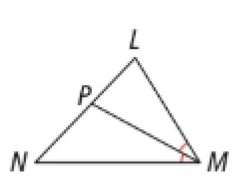
3. **Analyze and Persevere** What information is needed to determine if x is half of y?



4. **Communicate and Justify** If $\overline{RS} \cong \overline{QS}$, what type of triangle is $\triangle PQR$? Use the Triangle-Angle-Bisector Theorem to explain your reasoning.



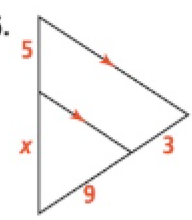
5. **Communicate and Justify** Explain why LP must be less than LM.



Do You KNOW HOW?

For Exercises 6–11, find each value of x.

6.



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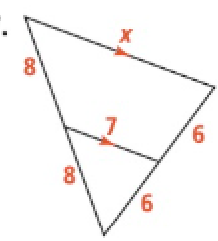

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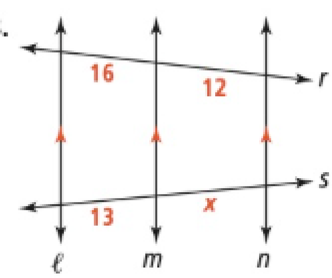

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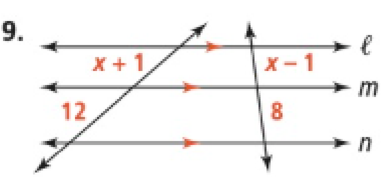

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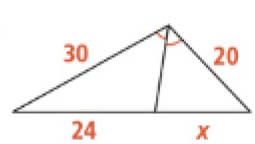
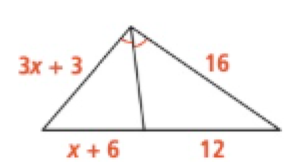


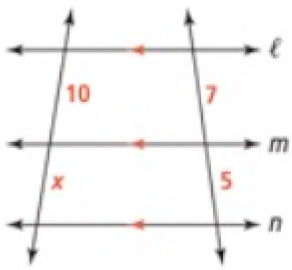
11.



PRACTICE & PROBLEM SOLVING

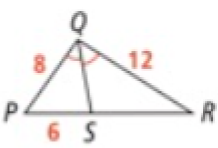
UNDERSTAND

12. **Error Analysis** What is Benson's error?



$$\frac{5}{10} = \frac{x}{7}$$
$$10x = 35$$
$$x = 3.5$$

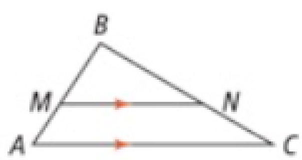
13. **Mathematical Connections** What percent of the area of $\triangle PQR$ is the area of $\triangle QRS$? Explain.



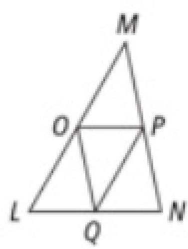
14. **Communicate and Justify** Write a proof of the Side-Splitter Theorem.

Given: $\overline{MN} \parallel \overline{AC}$

Prove: $\frac{AM}{MB} = \frac{CN}{NB}$



15. **Higher Order Thinking** Suppose *O*, *P*, and *Q* are midpoints of the sides of $\triangle LMN$. Show that $\triangle LOQ$, $\triangle OMP$, $\triangle QPN$, and $\triangle PQO$ are congruent to each other.

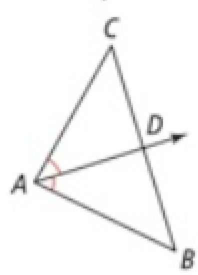


16. **Communicate and Justify** Write a proof for the Triangle-Angle-Bisector Theorem.

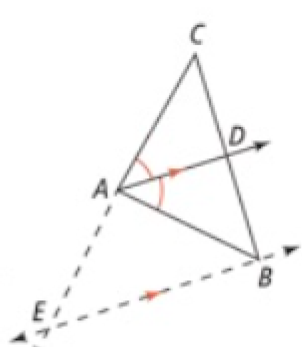
Given: $\overrightarrow{AD}$ bisects $\angle A$.

Prove: $\frac{CA}{AB} = \frac{CD}{DB}$

Use the following outline.



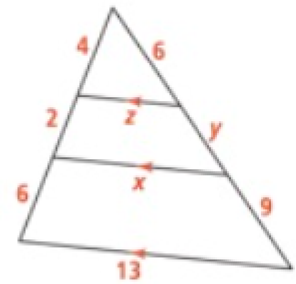
- Extend $\overrightarrow{CA}$ and draw a line through point *B* parallel to $\overrightarrow{AD}$ that intersects $\overrightarrow{CA}$ at point *E*.
- Show that $\frac{CA}{AE} = \frac{CD}{DB}$.
- Then show that $\triangle AEB$ is isosceles.



PRACTICE

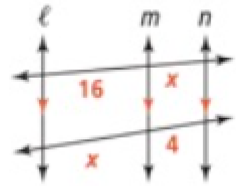
For Exercises 17–19, find each value.

SEE EXAMPLES 1 AND 2



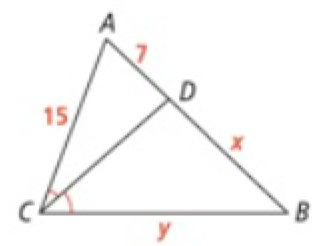
17. x 18. y 19. z

20. What is the value of x? SEE EXAMPLE 3



For Exercises 21–23, find each value of x for the given value of y. Round to the nearest tenth.

SEE EXAMPLES 4 AND 5

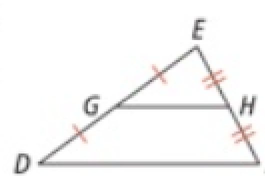


21. $y = 16$ 22. $y = 20$ 23. $y = 18$

24. Write a proof of the Triangle Midsegment Theorem.

Given: $\overline{DG} \cong \overline{GE}$, $\overline{FH} \cong \overline{HE}$

Prove: $\overline{GH} \parallel \overline{DF}$, $GH = \frac{1}{2}DF$

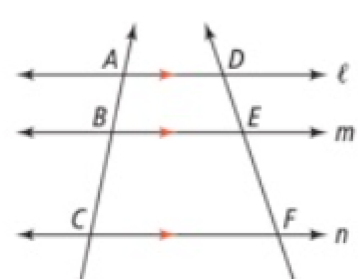


25. Write a proof of the Corollary to the Side-Splitter Theorem.

Given: $\ell \parallel m \parallel n$

Prove: $\frac{AB}{BC} = \frac{DE}{EF}$

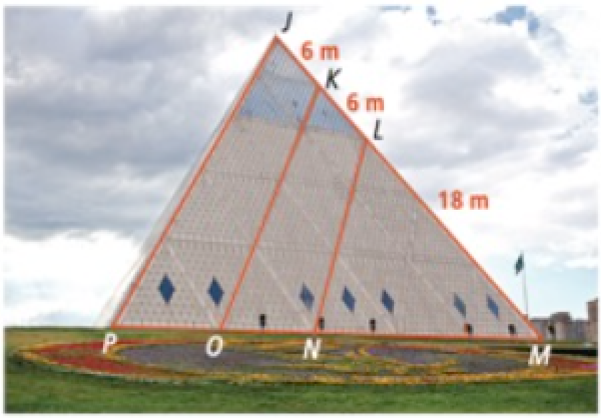
Hint: Draw $\overline{AF}$. Label the intersection of $\overline{AF}$ and $\overleftrightarrow{BE}$ point *G*.



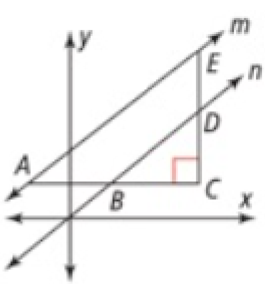
PRACTICE & PROBLEM SOLVING

APPLY

26. Use Patterns and Structure A building in the shape of a pyramid needs to have supports repaired, and two parallel sections need to be reinforced. The face of the building is an equilateral triangle. What are the lengths of $\overline{KO}$ and $\overline{LN}$?



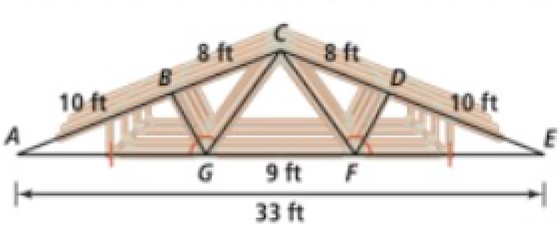
27. Higher Order Thinking Use the figure to prove Theorem 2-13: Two non-vertical lines are parallel if and only if they have the same slope.



a. Assume the slopes of lines m and n are equal. Use proportions in $\triangle ACE$ and $\triangle BCD$ to show that $m \parallel n$.

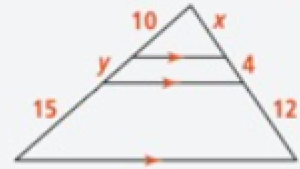
b. Now assume that $m \parallel n$. Show that the slopes of m and n are equal.

28. Use Patterns and Structure Aisha is building a roof and needs to determine the lengths of $\overline{CG}$ and $\overline{CF}$ from the design shown. How can she determine CG and CF? What are CG and CF?

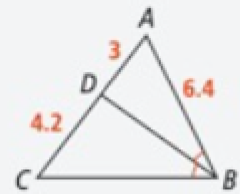


ASSESSMENT PRACTICE

29. What is the value of x? GR.1.3

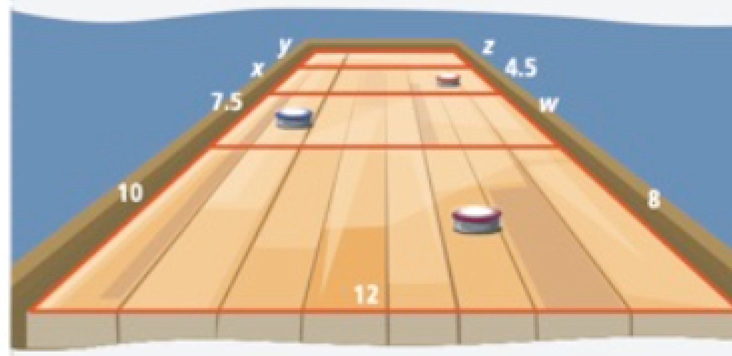


30. SAT/ACT What is the measure of side CB?



Ⓐ 4.57 Ⓑ 6.4 Ⓒ 8.96 Ⓓ 9.4

31. Performance Task Emma is determining measurements needed to simulate the distances in a shuffleboard computer game that she is programming.



Part A The horizontal lines must be parallel and in proportion so that each zone of the shuffleboard appears to be the same length. What are the lengths w, x, and y?

Part B What is the length of each horizontal segment?

Part C Which horizontal segment is closest to the midsegment of the triangle that extends off of the screen? How do you know?

TOPIC 7

Topic Review

? TOPIC ESSENTIAL QUESTION

1. How are properties of similar figures used to solve problems?

Vocabulary Review

Choose the correct term to complete each sentence.

2. Two triangles that are ________ have two pairs of corresponding congruent angles.
3. A ________ is a composition of a dilation and one or more rigid motions.
4. A point that is its own image in a dilation is the ________.
5. As a result of a dilation, if $A'B' = n \cdot AB$, then n is the ________.

- center of dilation
- dilation
- geometric mean
- scale factor
- similar
- similarity transformation

Concepts & Skills Review

LESSONS 7-1 & 7-2 Dilations and Similarity Transformations

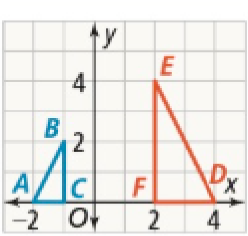
Quick Review

A **dilation** is a transformation that maps a point X to X' such that X' lies on $\overrightarrow{CX}$ and $CX' = k \cdot CX$, with **center of dilation** C and **scale factor** k.

Two figures are **similar** if there is a **similarity transformation** that maps one figure onto the other.

Example

Are $\triangle ABC$ and $\triangle DEF$ similar? Explain.

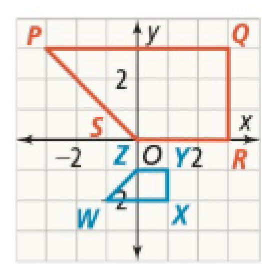


The reflection $r_{y\text{-axis}}$ maps $\triangle ABC$ to a triangle with vertices $A'(2, 0)$, $B'(1, 2)$ and $C'(1, 0)$. The dilation D_2 maps the image to $\triangle DEF$. Since the composition $D_2 \circ r_{y\text{-axis}}$ maps $\triangle ABC$ to $\triangle DEF$, the triangles are similar.

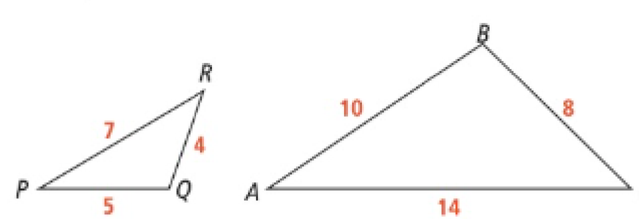
Practice & Problem Solving

Give the coordinates of each image.

6. $D_{\frac{1}{2}}(\triangle FGH)$ for $F(5, -2)$, $G(-2, -4)$, $H(0, 6)$
7. $D_{(3,\ K)}(\triangle KLM)$ for $K(0, 4)$, $L(3, 0)$, $M(-2, 4)$
8. What is a similarity transformation from $PQRS$ to $WXYZ$?



9. **Communicate and Justify** Isabel says that the scale factor in the similarity transformation that maps $\triangle ABC$ to $\triangle PQR$ is 2. Is she correct? Explain.



LESSON 7-3 Proving Triangles Similar

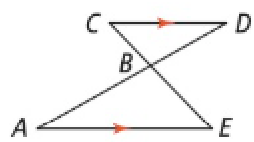
Quick Review

A pair of triangles can be shown to be similar by using the following criteria.

- Two pairs of corresponding angles are congruent.
- All corresponding sides are proportional.
- Two pairs of corresponding sides are proportional and the included angles are congruent.

Example

Explain whether $\triangle ABE$ and $\triangle DBC$ are similar.

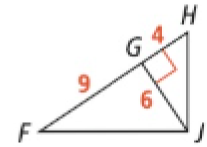


By the Alternate Interior Angles Theorem, $\angle A \cong \angle D$ and $\angle E \cong \angle C$. Since two pairs of corresponding angles are congruent, $\triangle ABE \sim \triangle DBC$.

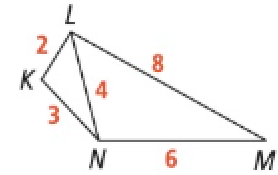
Practice & Problem Solving

For Exercises 10 and 11, explain whether each triangle similarity is true.

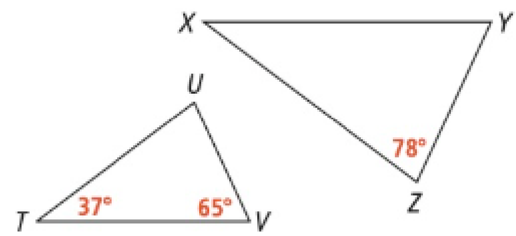
10. $\triangle FGJ \sim \triangle JGH$



11. $\triangle KLN \sim \triangle NLM$



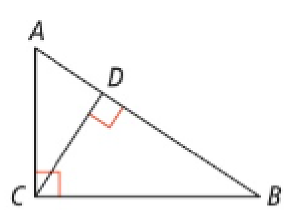
12. Analyze and Persevere Explain what additional information is needed to use AA~ to show that $\triangle TUV \sim \triangle XZY$.



LESSONS 7-4 & 7-5 Similarity in Right Triangles and Proportions in Triangles

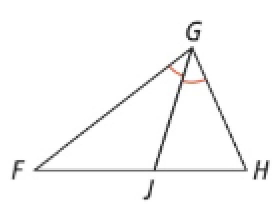
Quick Review

For right triangle $\triangle ABC$, $\triangle ABC \sim \triangle ACD \sim \triangle CBD$.



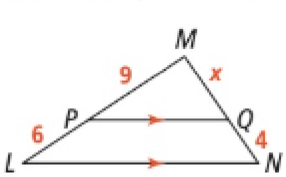
Also, CD is the **geometric mean** of AD and BD, AC is the geometric mean of AB and AD, and CB is the geometric mean of AB and DB.

For $\triangle FGH$, $\frac{FJ}{HJ} = \frac{FG}{HG}$.



Example

For $\triangle LMN$, what is x?



By the Side-Splitter Theorem, $\frac{LP}{PM} = \frac{NQ}{QM}$, so $\frac{6}{9} = \frac{4}{x}$. Solve for x to get $x = \frac{9}{6}(4) = 6$.

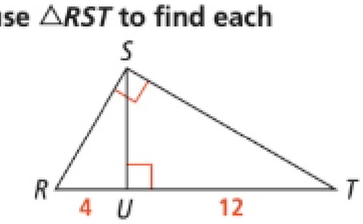
Practice & Problem Solving

For Exercises 13–15, use $\triangle RST$ to find each length.

13. RS

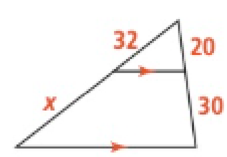
14. ST

15. SU

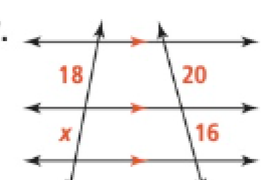


For Exercises 16–19, find the value of x.

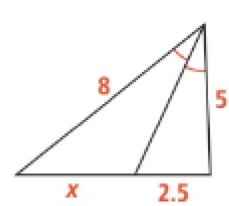
16.



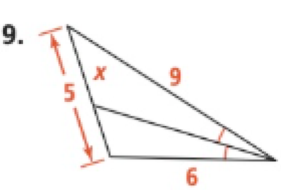
17.



18.



19.



20. Use Patterns and Structure Given right triangle $\triangle GHJ$ with $\overline{JK}$ the altitude to hypotenuse $\overline{GH}$, what is GJ the geometric mean of? Explain.

TOPIC

8

Right Triangles and Trigonometry

TOPIC ESSENTIAL QUESTION

How are the Pythagorean Theorem and trigonometry useful?

Topic Overview

enVision® STEM Project:
Measure a Distance

8-1 Right Triangles and the Pythagorean Theorem
GR.1.3, T.1.2, MTR.5.1, MTR.4.1, MTR.7.1

8-2 Trigonometric Ratios
T.1.1, T.1.2, MTR.1.1, MTR.2.1, MTR.5.1

8-3 Problem Solving With Trigonometry
T.1.2, T.1.4, MTR.2.1, MTR.6.1, MTR.1.1

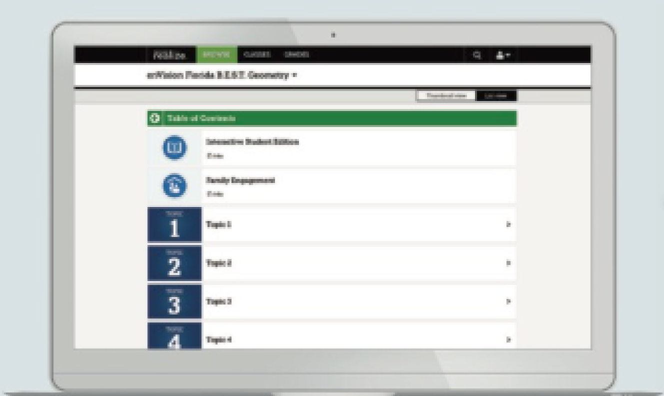
8-4 The Law of Sines
T.1.3, MTR.1.1, MTR.7.1, MTR.4.1

8-5 The Law of Cosines
T.1.3, MTR.5.1, MTR.3.1, MTR.7.1

Mathematical Modeling in 3 Acts:
The Impossible Measurement
T.1.2, MTR.7.1

Topic Vocabulary

- angle of depression
- angle of elevation
- cosine
- Law of Cosines
- Law of Sines
- Pythagorean triple
- sine
- tangent
- trigonometric ratios



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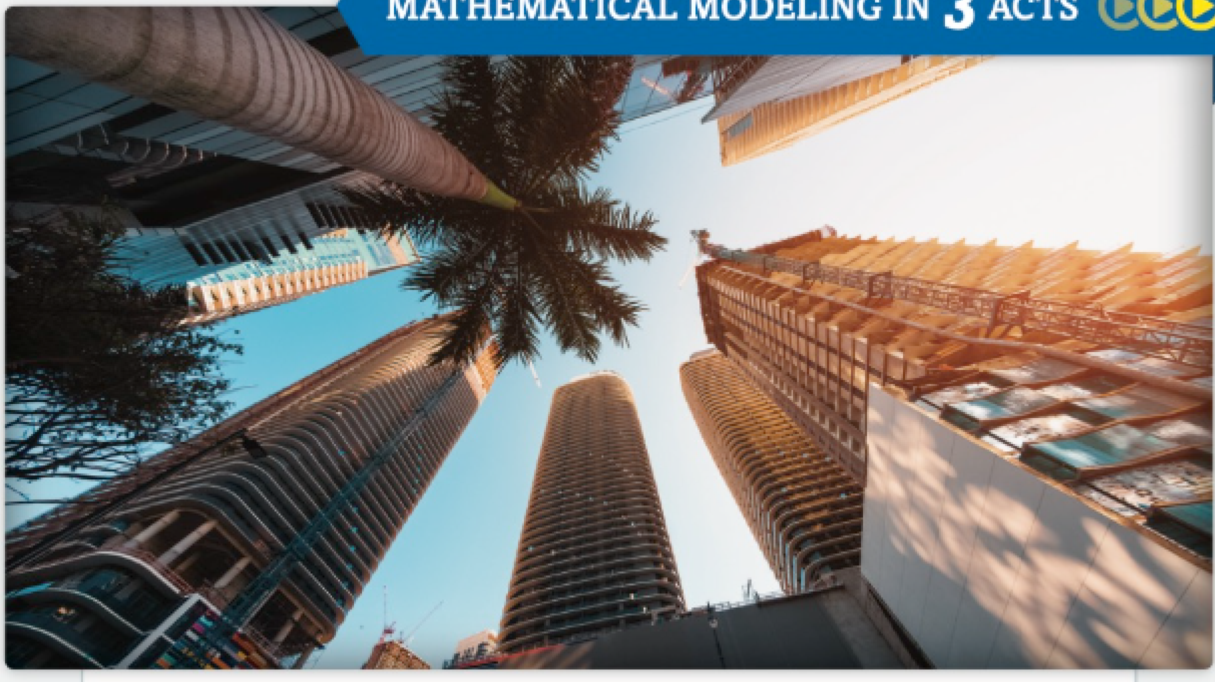
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



The Impossible Measurement

Tall buildings are often some of the most recognizable structures of cities. The Empire State Building in New York City, the Transamerica Pyramid in San Francisco, and the JPMorgan Chase Tower in Houston are all famous landmarks in those cities.

Cities around the world compete for the tallest building bragging rights. Which city currently has the tallest building? This Mathematical Modeling in 3 Acts lesson will get you thinking about the height of structures, including tall buildings such as these.

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ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

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CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

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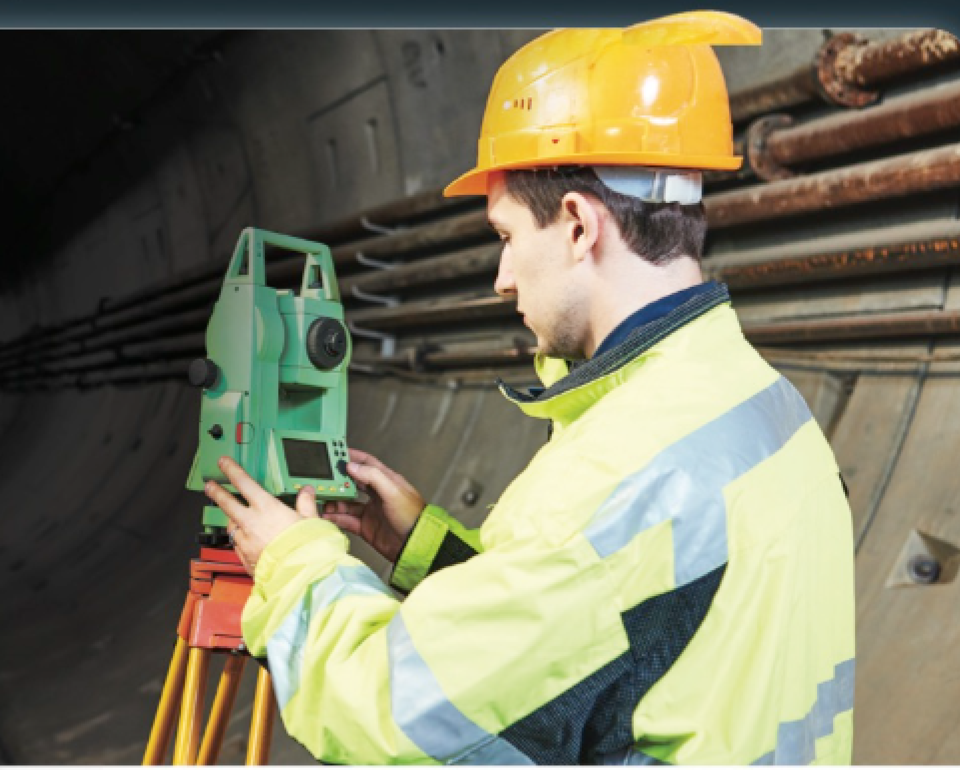
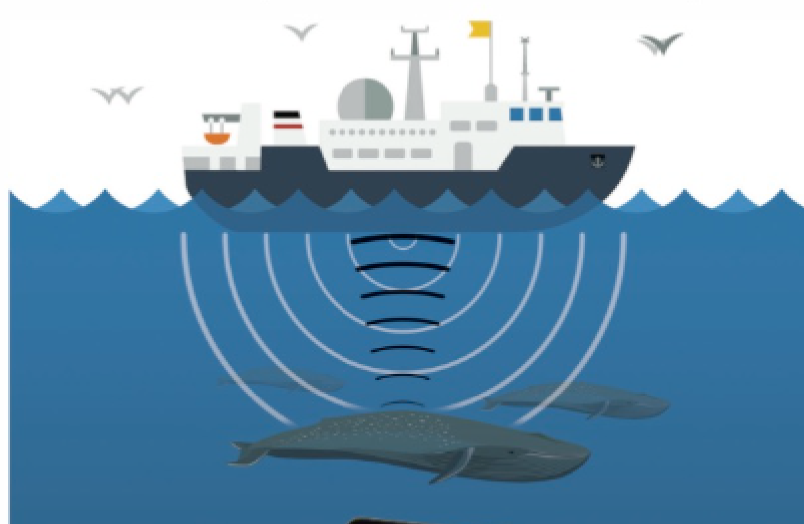
Did You Know?

A **laser rangefinder** measures the distance to an object indirectly. The instrument measures the time it takes a laser pulse to travel to the object and return, at a speed of approximately **186,000 miles per second.**



SONAR is used to measure distances underwater. The speed of sound in the ocean depends on the **temperature**, **pressure**, and **salt content** of the water. Once you know the speed of sound, you can bounce SONAR waves off an underwater object, measure the time it takes to return, and calculate the distance to the object.



In the 3rd Century B.C., the Greek mathematician **Eratosthenes** calculated the circumference of the Earth indirectly. He couldn't actually measure the distance foot by foot. So he used **geometry and the shadow of the sun** to make a determination of the circumference.

Your Task: Measure a Distance

Trigonometry is a powerful tool for measuring lengths and distances indirectly. You and your classmates will use trigonometry and indirect measurement to find the height of an object that is too tall to measure directly.

8-1 Right Triangles and the Pythagorean Theorem

I CAN… prove the Pythagorean Theorem using similarity and establish the relationships in special right triangles.

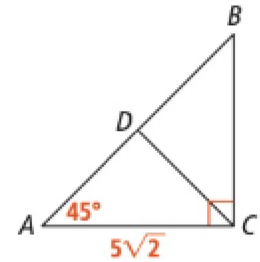
VOCABULARY

- Pythagorean triple

MA.912.GR.1.3–Prove relationships and theorems about triangles. Solve mathematical and real-world problems involving postulates, relationships and theorems of triangles. Also T.1.2
MA.K12.MTR.5.1, MTR.4.1, MTR.7.1

EXPLORE & REASON

Consider $\triangle ABC$ with altitude $\overline{CD}$ as shown.



A. What is the area of $\triangle ABC$? Of $\triangle ACD$? Explain your answers.

B. Find the lengths of $\overline{AD}$ and $\overline{AB}$.

C. **Use Patterns and Structure** Divide the length of the hypotenuse of $\triangle ABC$ by the length of one of its sides. Divide the length of the hypotenuse of $\triangle ACD$ by the length of one of its sides. Make a conjecture that explains the results.

ESSENTIAL QUESTION

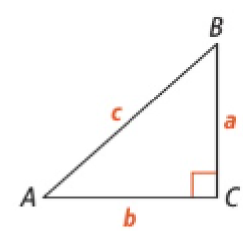
How are similarity in right triangles and the Pythagorean Theorem related?

Remember that the Pythagorean Theorem and its converse describe how the side lengths of right triangles are related.

THEOREM 8-1 Pythagorean Theorem

If a triangle is a right triangle, then the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

If... $\triangle ABC$ is a right triangle.



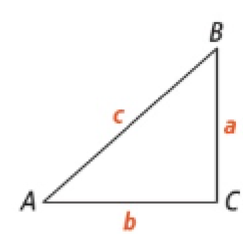
Then... $a^2 + b^2 = c^2$

PROOF: SEE EXAMPLE 1.

THEOREM 8-2 Converse of the Pythagorean Theorem

If the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third side, then the triangle is a right triangle.

If... $a^2 + b^2 = c^2$



Then... $\triangle ABC$ is a right triangle.

PROOF: SEE EXERCISE 17.

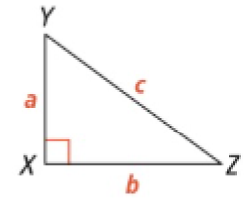
PROOF

EXAMPLE 1 Use Similarity to Prove the Pythagorean Theorem

Use right triangle similarity to write a proof of the Pythagorean Theorem.

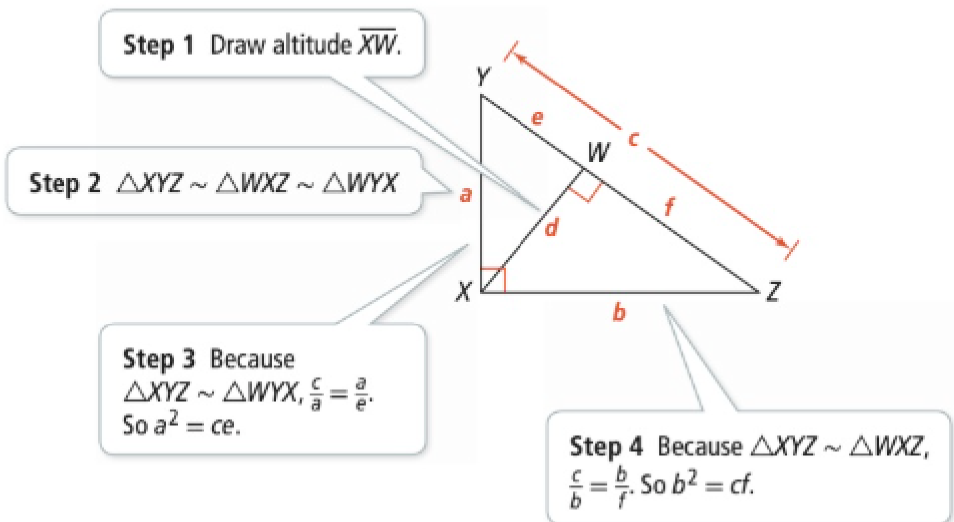
Given: $\triangle XYZ$ is a right triangle.

Prove: $a^2 + b^2 = c^2$



Plan: To prove the Pythagorean Theorem, draw the altitude to the hypotenuse. Then use the relationships in the resulting similar right triangles.

Proof:



Step 5 Write an equation that relates a^2 and b^2 to ce and cf.

$$a^2 + b^2 = ce + cf$$

$$a^2 + b^2 = c(e + f)$$

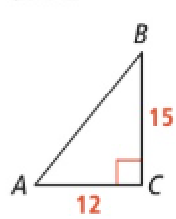
$$a^2 + b^2 = c(c)$$

$$a^2 + b^2 = c^2$$

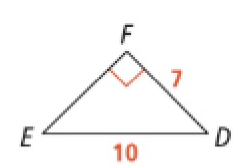
GENERALIZE
Think about how you can apply properties of similar triangles. What is the relationship between corresponding sides of similar triangles?

Try It! **1.** Find the unknown side length of each right triangle.

a. AB



b. EF



APPLICATION

EXAMPLE 2 Use the Pythagorean Theorem and Its Converse

A. To satisfy safety regulations, the distance from the wall to the base of a ladder should be at least one-fourth the length of the ladder. Did Drew set up the ladder correctly?

The floor, the wall, and the ladder form a right triangle.

Step 1 Find the length of the ladder.

$$a^2 + b^2 = c^2$$
$$2.5^2 + 9^2 = c^2$$
$$87.25 = c^2$$
$$9.34 \approx c$$

Use the Pythagorean Theorem with $a = 2.5$ and $b = 9$.

Step 2 Find $\frac{1}{4}$ the length of the ladder.

$$\frac{1}{4}c \approx \frac{1}{4}(9.34)$$
$$\approx 2.335$$

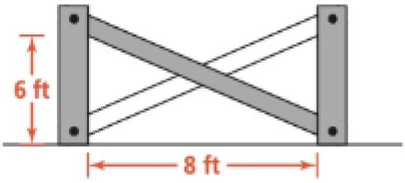
The length of the ladder is 9.34 ft.

Since $2.5 > 2.335$, Drew set up the ladder correctly.

B. The length of each crosspiece of the fence is 10 ft. Why would a rancher build this fence with the measurements shown?

STUDY TIP
Learn and recognize common Pythagorean triples such as 3, 4, and 5; and 5, 12, and 13 to speed calculations.

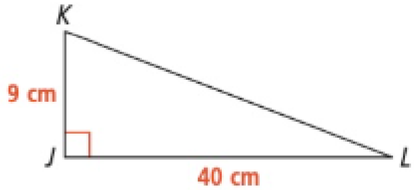
The numbers 6, 8, and 10 form a *Pythagorean triple*. A **Pythagorean triple** is a set of three nonzero whole numbers that satisfy the equation $a^2 + b^2 = c^2$.



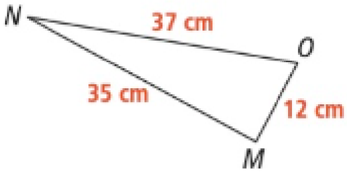
Since $6^2 + 8^2 = 10^2$, the posts, the ground, and the crosspieces form right triangles.

By using those measurements, the rancher knows that the fence posts are perpendicular to the ground, which stabilizes the fence.

Try It! **2. a.** What is *KL*?



b. Is $\triangle MNO$ a right triangle? Explain.

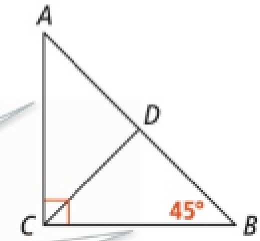


CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Investigate Side Lengths in 45°-45°-90° Triangles

Is there a relationship between the lengths of $\overline{AB}$ and $\overline{AC}$ in $\triangle ABC$? Explain.

Draw altitude $\overline{CD}$ to form similar right triangles $\triangle ABC$, $\triangle ACD$, and $\triangle CBD$.



Notice that $\triangle ABC$ is a 45°-45°-90° triangle, and that $AC = BC$.

USE PATTERNS AND STRUCTURE
Think about the properties of a triangle with two congruent angles. How do the properties of the triangle help you relate the side lengths?

Use right-triangle similarity to write an equation.

$$\frac{AB}{AC} = \frac{AC}{AD}$$

Since $\triangle ABC \sim \triangle ACD$, AC is the geometric mean of AB and AD.

$$\frac{AB}{AC} = \frac{AC}{\frac{1}{2}AB}$$

Because $\triangle ABC$ is isosceles, $\overline{CD}$ bisects $\overline{AB}$.

$$\frac{1}{2}AB^2 = AC^2$$

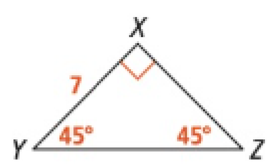
$$AB^2 = 2AC^2$$

$$AB = \sqrt{2} \cdot AC$$

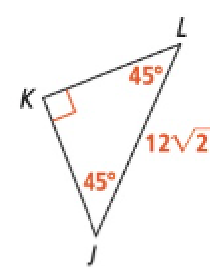
The length of $\overline{AB}$ is $\sqrt{2}$ times the length of $\overline{AC}$.

Try It! 3. Find the side lengths of each 45°-45°-90° triangle.

a. What are XZ and YZ?



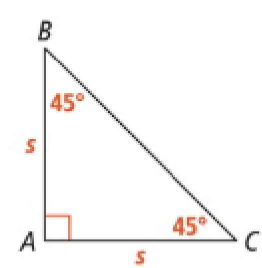
b. What are JK and LK?



THEOREM 8-3 45°-45°-90° Triangle Theorem

In a 45°-45°-90° triangle, the legs are congruent and the length of the hypotenuse is $\sqrt{2}$ times the length of a leg.

If...

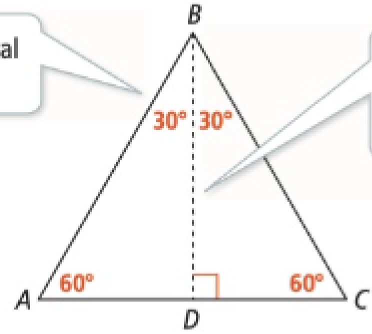


Then... $BC = s\sqrt{2}$

PROOF: SEE EXERCISE 18.

EXAMPLE 4 Explore the Side Lengths of a 30°-60°-90° Triangle

Using an equilateral triangle, show how the lengths of the short leg, the long leg, and the hypotenuse of a 30°-60°-90° triangle are related.



STUDY TIP
Recall that an altitude of a triangle is perpendicular to a side. Think about what properties of the triangle result in the altitude also being a segment bisector.

Look at △ADB. Let the length of the short leg $\overline{AD}$ be *s*.

Find the relationship between *AD* and *AB*.

$AD = CD = s$ — $\overline{BD}$ bisects $\overline{AC}$.

$AC = AD + CD$

$AC = 2s$

$AB = 2s$ — △ABC is equilateral, so $AB = AC = 2s$.

Find the relationship between *AD* and *BD*.

$AD^2 + BD^2 = AB^2$ — Use the Pythagorean Theorem.

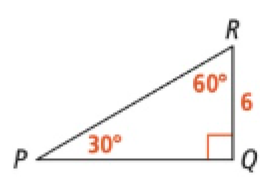
$s^2 + BD^2 = (2s)^2$

$BD^2 = 3s^2$

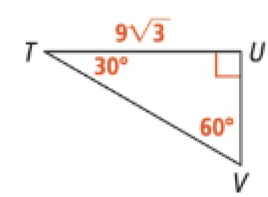
$BD = s\sqrt{3}$

In △ADB, the length of hypotenuse $\overline{AB}$ is twice the length of the short leg $\overline{AD}$. The length of the long leg $\overline{BD}$ is $\sqrt{3}$ times the length of the short leg.

Try It! **4. a.** What are *PQ* and *PR*?



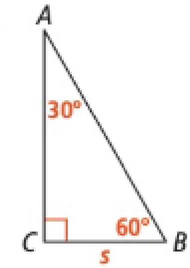
b. What are *UV* and *TV*?



THEOREM 8-4 30°-60°-90° Triangle Theorem

In a 30°-60°-90° triangle, the length of the hypotenuse is twice the length of the short leg. The length of the long leg is $\sqrt{3}$ times the length of the short leg.

If...



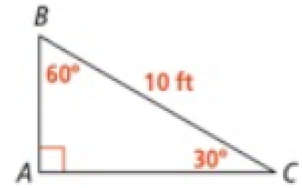
Then... $AC = s\sqrt{3}$, $AB = 2s$

PROOF: SEE EXERCISE 19.

EXAMPLE 5 Apply Special Right Triangle Relationships

COMMON ERROR
Be careful not to mix up the relationship of the shorter and longer legs. Remember that the longer leg is $\sqrt{3}$ times as long as the shorter leg, so the longer leg is between $1\frac{1}{2}$ and 2 times as long as the short leg.

A. Alejandro needs to make both the horizontal and vertical supports, $\overline{AC}$ and $\overline{AB}$, for the ramp. Is one 12-foot board long enough for both supports? Explain.



The ramp and supports form a 30°-60°-90° triangle.

$BC = 2AB$ $\quad\quad$ $AC = AB\sqrt{3}$

$10 = 2AB$ $\quad\quad$ $AC = 5\sqrt{3}$ ft

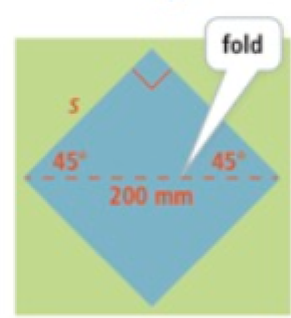
$AB = 5$ ft

Find the total length of the supports.

$$AB + AC = 5 + 5\sqrt{3}$$
$$\approx 13.66 \text{ ft}$$

Since $13.66 > 12$, the 12-foot board will not be long enough for Alejandro to make both supports.

B. Olivia starts an origami paper crane by making the 200-mm diagonal fold. What are the side length and area of the paper square?



Step 1 Find the length of one side of the paper.

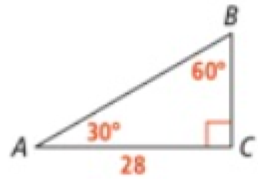
$$s\sqrt{2} = 200$$
$$s = \frac{200}{\sqrt{2}}$$
$$s \approx 141.4 \text{ mm}$$

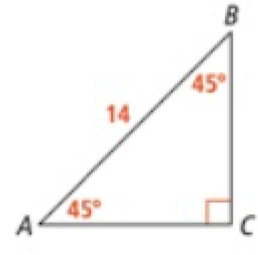
Step 2 Find the area of the paper square.

$$A = s^2$$
$$A = (100\sqrt{2})^2$$
$$A = 20{,}000 \text{ mm}^2$$

The paper square has side length 141.4 mm and area 20,000 mm^2.

Try It! 5. a. What are AB and BC? $\quad$ b. What are AC and BC?

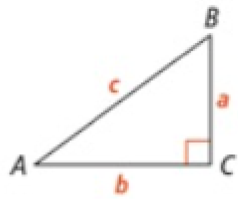




CONCEPT SUMMARY The Pythagorean Theorem and Special Right Triangles

THEOREM 8-1 Pythagorean Theorem

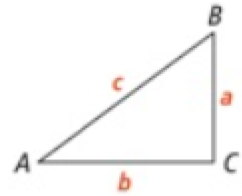
If... $\triangle ABC$ is a right triangle



Then... $a^2 + b^2 = c^2$

THEOREM 8-2 Converse of the Pythagorean Theorem

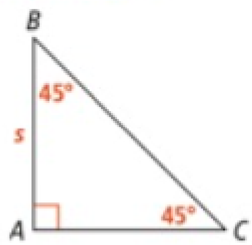
If... $a^2 + b^2 = c^2$



Then... $\triangle ABC$ is a right triangle.

THEOREM 8-3 45°-45°-90° Triangle Theorem

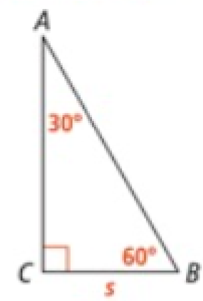
If...



Then... $BC = s\sqrt{2}$

THEOREM 8-4 30°-60°-90° Triangle Theorem

If...

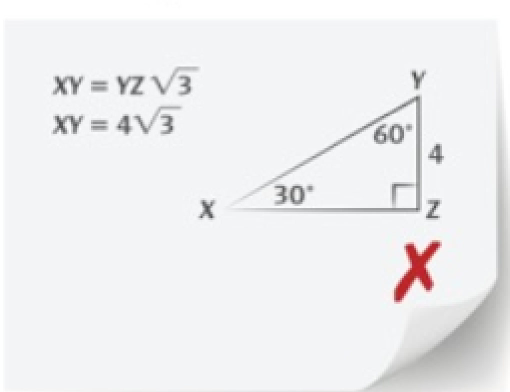


Then... $AC = s\sqrt{3}$, $AB = 2s$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are similarity in right triangles and the Pythagorean Theorem related?

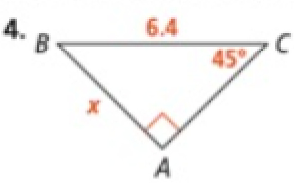
2. **Error Analysis** Casey was asked to find XY. What is Casey's error?

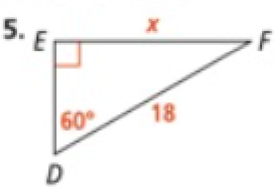


3. **Use Patterns and Structure** A right triangle has leg lengths 4.5 and $4.5\sqrt{3}$. What are the measures of the angles? Explain.

Do You KNOW HOW?

For Exercises 4 and 5, find the value of x.

4. 

5. 

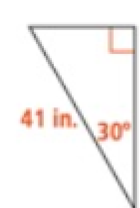
For Exercises 6–8, is $\triangle RST$ a right triangle? Explain.

6. $RS = 20$, $ST = 21$, $RT = 29$

7. $RS = 35$, $ST = 36$, $RT = 71$

8. $RS = 40$, $ST = 41$, $RT = 11$

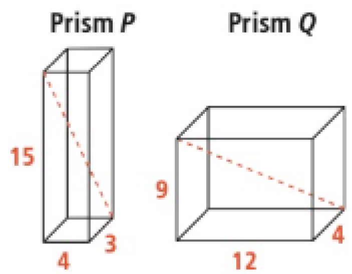
9. Charles wants to hang the pennant shown vertically between two windows that are 19 inches apart. Will the pennant fit? Explain.



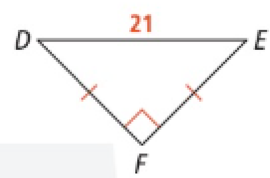
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. Mathematical Connections Which rectangular prism has the longer diagonal? Explain.

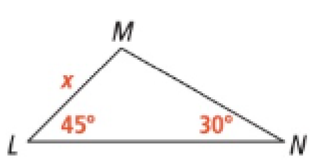


11. Error Analysis Dakota is asked to find *EF*. What is her error?

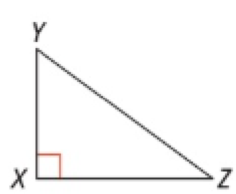


> There is not enough information to find *EF* because you need to know either the length of $\overline{DF}$ or one of the other angle measures.

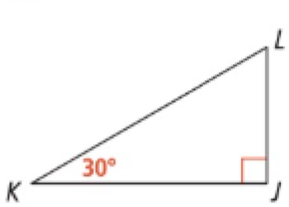
12. Analyze and Persevere What are expressions for *MN* and *LN*? *Hint*: Construct the altitude from *M* to $\overline{LN}$.



13. Higher Order Thinking Triangle *XYZ* is a right triangle. For what kind of triangle would $XZ^2 + XY^2 > YZ^2$? For what kind of triangle would $XZ^2 + XY^2 < YZ^2$? Explain.



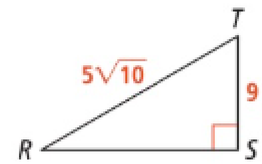
14. Represent and Connect Write an equation that represents the relationship between *JK* and *KL*.



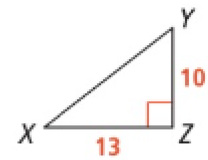
PRACTICE

For Exercises 15 and 16, find the unknown side length of each triangle. SEE EXAMPLE 1

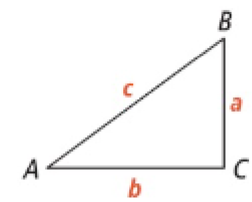
15. *RS*



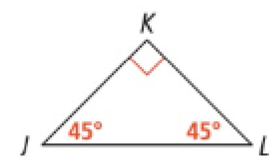
16. *XY*



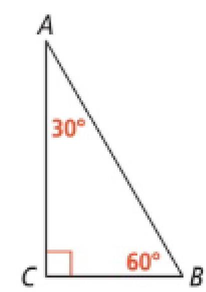
17. Given $\triangle ABC$ with $a^2 + b^2 = c^2$, write a paragraph proof of the Converse of the Pythagorean Theorem. SEE EXAMPLE 2



18. Write a two-column proof of the 45°-45°-90° Triangle Theorem. SEE EXAMPLE 3

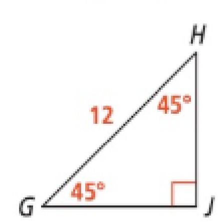


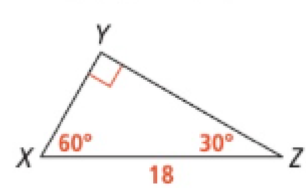
19. Write a paragraph proof of the 30°-60°-90° Triangle Theorem. SEE EXAMPLE 4



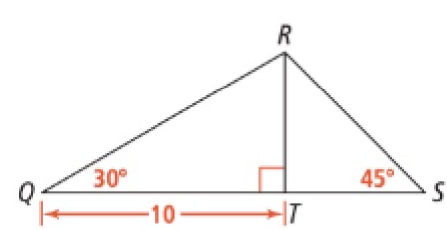
For Exercise 20 and 21, find the side lengths of each triangle. SEE EXAMPLES 3 AND 4

20. What are *GJ* and *HJ*? **21.** What are *XY* and *YZ*?





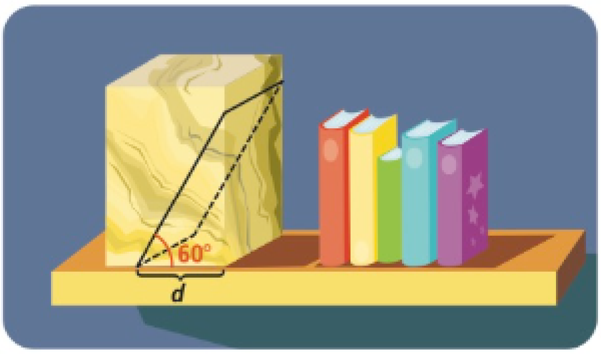
22. What is *QS*? SEE EXAMPLE 5



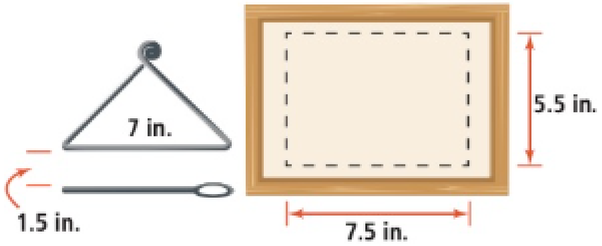
PRACTICE & PROBLEM SOLVING

APPLY

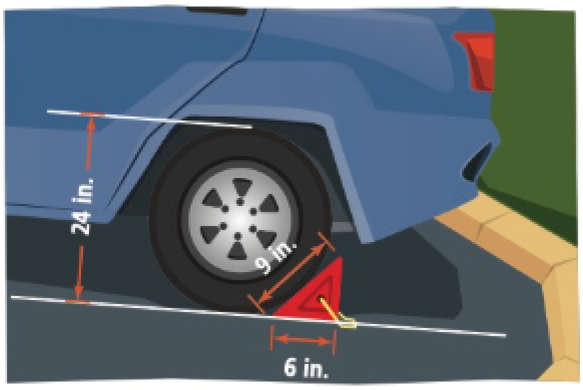
23. **Apply Math Models** Esteban wants marble bookends cut at a 60° angle, as shown. If Esteban wants his bookends to be between 7.5 in. and 8 in. tall, what length *d* should the marble cutter make the base of the bookends? Explain.



24. **Communicate and Justify** Sarah finds an antique dinner bell that appears to be in the shape of an isosceles right triangle, but the only measurement given is the longest side. Sarah wants to display the bell and wand in a 5.5-in. by 7.5-in. picture frame. Assuming that the bell is an isosceles right triangle, can Sarah display the bell and wand within the frame? Explain.



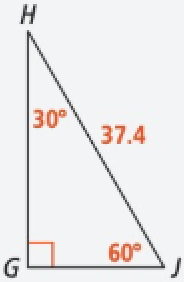
25. **Communicate and Justify** When Carmen parks on a hill, she places chocks behind the wheels of her car. The height of the chocks must be at least one-fourth of the height of the wheels to hold the car securely in place. The chock shown has the shape of a right triangle. Is it safe for Carmen to use? Explain.



ASSESSMENT PRACTICE

26. A kite is flying at the end of a string that is 200 ft long. The wind pulls the kite and the string taut so that the string forms a straight line. The line makes a 60-degree angle with the ground. About how high above the ground is the kite flying? GR.1.3

27. **SAT/ACT** What is *GJ*?



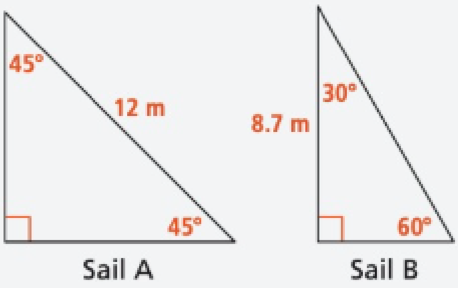
Ⓐ 18.7

Ⓑ $18.7\sqrt{2}$

Ⓒ $18.7\sqrt{3}$

Ⓓ 74.8

28. **Performance Task** Emma designed two triangular sails for a boat.



Part A What is the area of Sail A?

Part B What is the area of Sail B?

Part C Is it possible for Emma to cut both sails from one square of sailcloth with sides that are 9 meters in length? Draw a diagram to explain.

8-2 Trigonometric Ratios

I CAN...
use trigonometric ratios to find lengths and angle measures of right triangles.

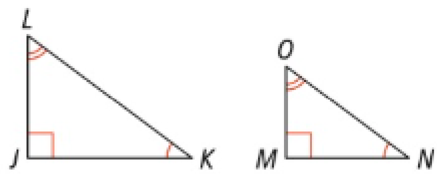
VOCABULARY

- cosine
- sine
- tangent
- trigonometric ratios

MA.912.T.1.1–Define trigonometric ratios for acute angles in right triangles. Also T.1.2
MA.K12.MTR.1.1, MTR.2.1, MTR.5.1

CRITIQUE & EXPLAIN

A teacher asked students to write a proportion using the lengths of the legs of the two right triangles.



Two students' responses are shown.

Diego

$$\frac{JK}{MN} = \frac{JL}{MO}$$

Rebecca

$$\frac{JK}{JL} = \frac{MN}{MO}$$

A. Do you think that the proportion that Diego wrote is correct? Explain.

B. Do you think that the proportion that Rebecca wrote is correct? Explain.

C. **Use Patterns and Structure** If $\frac{a}{b} = \frac{c}{d}$, how can you get an equivalent equation such that the left side of the equation is $\frac{a}{c}$?

ESSENTIAL QUESTION

How do trigonometric ratios relate angle measures to side lengths of right triangles?

CONCEPT Trigonometric Ratios

The **trigonometric ratios**, or functions, relate the side lengths of a right triangle to its acute angles.

sine of $\angle A$

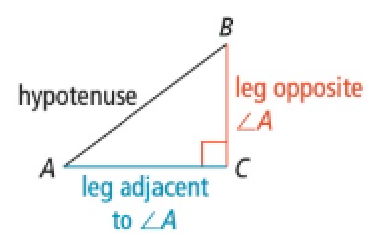
$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB}$$

cosine of $\angle A$

$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB}$$

tangent of $\angle A$

$$\tan A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A} = \frac{BC}{AC}$$

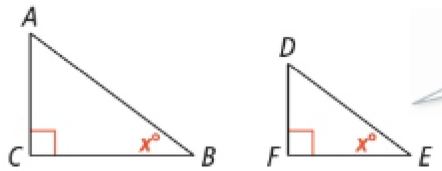


CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Understand Trigonometric Ratios Using Similarity

How are the sines of two different angles with the same measure related?

Let $\triangle ABC$ and $\triangle DEF$ be right triangles with $m\angle B = m\angle E$.



Since $\angle C \cong \angle F$ and $\angle B \cong \angle E$, $\triangle ABC \sim \triangle DEF$ by AA.

You can use properties of similarity to determine the relationship between sin B and sin E.

ANALYZE AND PERSEVERE
There are many proportional relationships in similar triangles. Look for one that uses all the side lengths in the expressions for sin B and sin E.

By the definition of the sine ratio, $\sin B = \frac{AC}{AB}$ and $\sin E = \frac{DF}{DE}$.

Because $\triangle ABC \sim \triangle DEF$, you know that corresponding side lengths are proportional. In particular, $\frac{AC}{DF} = \frac{AB}{DE}$. Rewrite this equation to compare the sides in $\triangle ABC$ to the sides in $\triangle DEF$.

$$\frac{AC}{DF} \cdot \frac{DF}{AB} = \frac{AB}{DE} \cdot \frac{DF}{AB}$$

$$\frac{AC}{AB} = \frac{DF}{DE}$$

$$\sin B = \sin E$$

Substitute sin B for $\frac{AC}{AB}$ and sin E for $\frac{DF}{DE}$.

Any two acute angles with the same measure have the same sine.

Try It! 1. Show that any two acute angles with the same measure have the same cosine.

EXAMPLE 2 Write Trigonometric Ratios

What are the sine, cosine, and tangent ratios for $\angle H$?

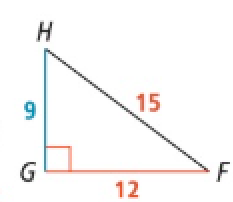
COMMON ERROR
You may incorrectly assume the horizontal leg and vertical leg to be the adjacent and opposite legs respectively. Remember that adjacent and opposite are relative to the angle.

Use the definitions of the trigonometric ratios.

$$\sin H = \frac{\text{length of leg opposite } \angle H}{\text{length of hypotenuse}} = \frac{12}{15}$$

$$\cos H = \frac{\text{length of leg adjacent to } \angle H}{\text{length of hypotenuse}} = \frac{9}{15}$$

$$\tan H = \frac{\text{length of leg opposite } \angle H}{\text{length of leg adjacent to } \angle H} = \frac{12}{9}$$

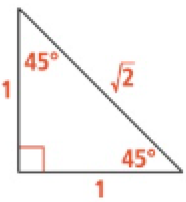


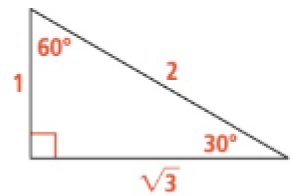
Try It! 2. In Example 2, what are the sine, cosine, and tangent ratios of $\angle F$?

EXAMPLE 3 Trigonometric Ratios of Special Angles

A. What are the sine, cosine, and tangent ratios for 30°, 45°, and 60° angles?

You can use what you know about 45°-45°-90° and 30°-60°-90° right triangles to find the ratios.





$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ $\sin 30° = \frac{1}{2}$ $\quad$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$

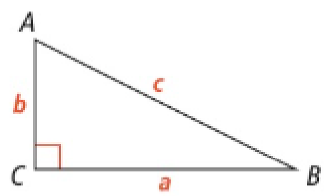
$\tan 45° = \frac{1}{1} = 1$ $\quad$ $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\quad$ $\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

VOCABULARY

Recall that two angles are *complementary* if the sum of their measures is 90. If an angle has measure x, its complement has measure $90 - x$.

B. How are the sine and cosine of complementary angles related?

By the Triangle Angle-Sum Theorem, the two acute angles in any right triangle are complementary.



$$m\angle A + m\angle B + m\angle C = 180$$

$$m\angle A + m\angle B + 90 = 180$$

$$m\angle A + m\angle B = 90$$

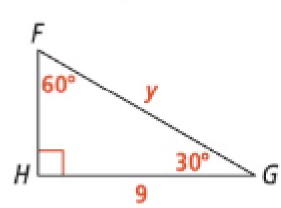
Find the sine and cosine of the complementary angles $\angle A$ and $\angle B$.

$\sin A = \frac{a}{c}$ $\quad$ $\sin B = \frac{b}{c}$

$\cos A = \frac{b}{c}$ $\quad$ $\cos B = \frac{a}{c}$

So $\sin A = \cos B$ and $\sin B = \cos A$. The sine of an angle is equal to the cosine of its complement, and vice versa.

Try It! **3. a.** In $\triangle FGH$, what is the value of y?

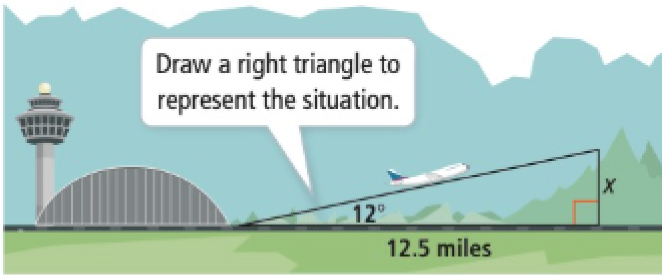


b. How can you write an equivalent expression for cos 70° using sine? An equivalent expression for sin 34° using cosine?

APPLICATION

EXAMPLE 4 Use Trigonometric Ratios to Find Distances

A plane takes off and climbs at a 12° angle. Is that angle sufficient enough to fly over an 11,088-foot mountain that is 12.5 miles from the runway or does the plane need to increase its angle of ascent?



STUDY TIP
To choose the right trigonometric ratio, consider which side of the triangle you know and which side you need to find.

Step 1 You know the length of the side adjacent to the 12° angle, so use the tangent ratio to find the altitude of the plane as it passes the mountain.

$$\tan 12° = \frac{x}{12.5}$$

$$x = 12.5 \cdot \tan 12°$$

$$\approx 2.66$$

Use a calculator.
12.5 TAN 12 ENTER

The altitude of the plane is about 2.66 miles.

Step 2 Compare the altitude of the plane to the height of the mountain.

$$11{,}088 \text{ ft} \cdot \frac{1 \text{ mi}}{5{,}280 \text{ ft}} = 2.1 \text{ mi}$$

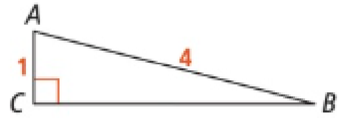
Write the height of the mountain in miles.

A 12° angle is sufficient because 2.1 mi < 2.66 mi.

Try It! **4.** If a plane climbs at 5° and flies 20 miles through the air as it climbs, what is the altitude of the airplane, to the nearest foot?

EXAMPLE 5 Use Trigonometric Inverses to Find Angle Measures

What are $m\angle A$ and $m\angle B$?



If you know the sine, cosine, or tangent of an angle, you can use a *trigonometric inverse* ($\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$) to find the angle measure.

REPRESENT AND CONNECT
On many graphing calculators, you can use the 2nd key and the sin, cos, and tan keys to use the trigonometric inverses.

Since you know that $\cos A = \frac{1}{4}$, use $\cos^{-1}$ to find $m\angle A$.

$$m\angle A = \cos^{-1}\left(\frac{1}{4}\right)$$

$$\approx 75.5$$

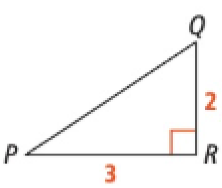
Since you know that $\sin B = \frac{1}{4}$, use $\sin^{-1}$ to find $m\angle B$.

$$m\angle B = \sin^{-1}\left(\frac{1}{4}\right)$$

$$\approx 14.5$$

Try It! **5. a.** What is $m\angle P$?

b. What is $m\angle Q$?

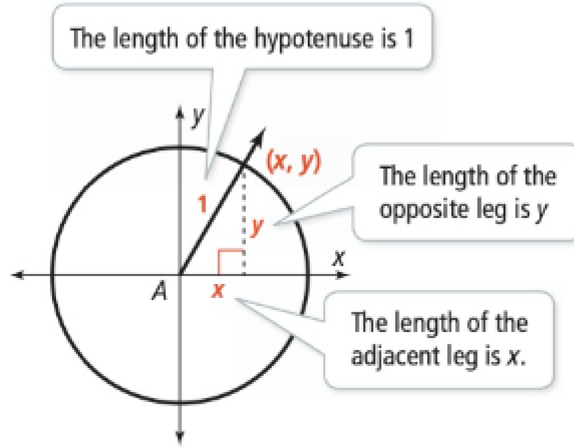


EXAMPLE 6 Use the Unit Circle to Find Trigonometric Ratios

VOCABULARY
A **unit circle** is a circle of radius 1 unit.

A. How are cosine and the sine of an angle related to the coordinates on a unit circle?

An acute angle measured from the x-axis creates a right triangle. Use the triangle to write an expression for the cosine and sine of $\angle A$.



$\cos A = \frac{x}{1}$
$\cos A = x$

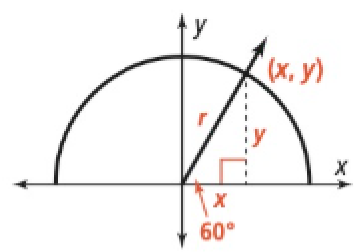
Cos A is the x-coordinate where $\angle A$ intersects the circle.

$\sin A = \frac{y}{1}$
$\sin A = y$

Sin A is the y-coordinate where $\angle A$ intersects the circle.

B. What are the coordinates at the point of intersection on the unit circle for a 60° angle?

Use trigonometric ratios of special triangles to find x and y.



$x = \cos 60°$ $\quad y = \sin 60°$

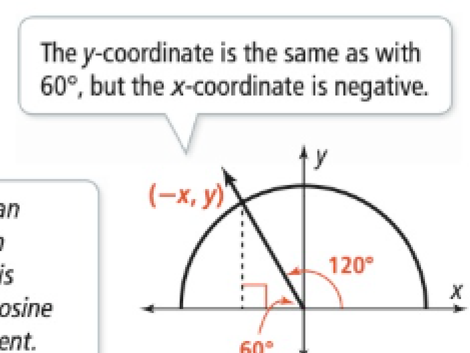
$x = \frac{1}{2}$ $\quad y = \frac{\sqrt{3}}{2}$

The point of intersection is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

LEARN TOGETHER
How do you listen actively as others share ideas?

C. What is cos 120° and sin 120°?

A 120° angle forms a 30°-60°-90° right triangle in Quadrant 2.



Find cos 120°.

$\cos 120° = -\cos 60°$

$\cos 120° = -\frac{1}{2}$

The cosine of an angle between 90° and 180° is opposite the cosine of its supplement.

Find sin 120°

$\sin 120° = \sin 60°$

$\sin 120° = \frac{\sqrt{3}}{2}$

The sine of an angle between 90° and 180° is equal to the sine of its supplement.

Try It! **6. a.** What are the x- and y-coordinates on the unit circle at 45°?

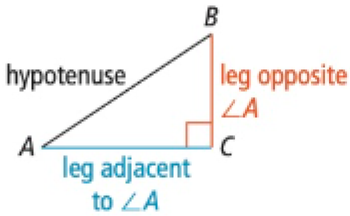
b. Find sin 135°.

c. Find cos 135°.

CONCEPT SUMMARY Trigonometric Ratios

WORDS For a right triangle, **the trigonometric ratios sine, cosine,** and **tangent** relate the measure of an acute angle of the triangle to the lengths of the sides.

DIAGRAM Triangle *ABC* is a right triangle.



SYMBOLS

$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}}$$

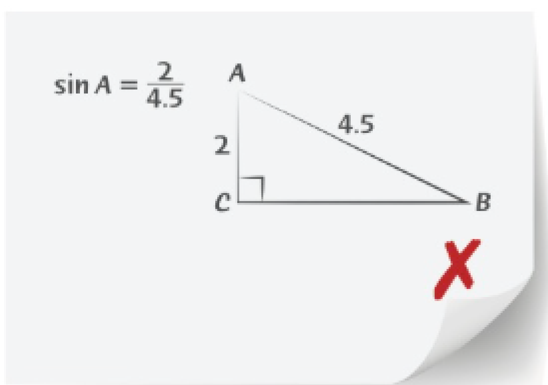
$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}}$$

$$\tan A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How do trigonometric ratios relate angle measures to side lengths of right triangles?

2. **Error Analysis** What is the error in this equation for a trigonometric ratio?

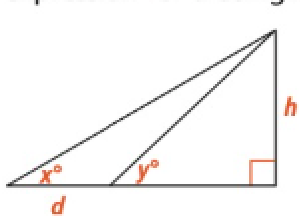


3. **Vocabulary** How are finding the inverses of trigonometric ratios similar to using inverse operations?

4. **Communicate and Justify** How is the sine ratio similar to the cosine ratio? How is it different?

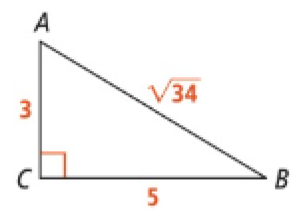
5. **Use Patterns and Structure** If $\angle A$ is acute and $\sin A = \frac{a}{c}$, how could you use a and c to find $\cos A$?

6. **Analyze and Persevere** What is an expression for d using $x°$, $y°$, and h?



Do You KNOW HOW?

For Exercises 7–12, use △*ABC* to find each trigonometric ratio or angle measure.



7. tan *B*

8. cos *B*

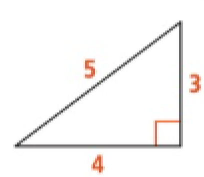
9. sin *A*

10. tan *A*

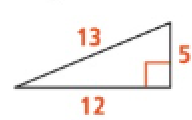
11. $m\angle B$

12. $m\angle A$

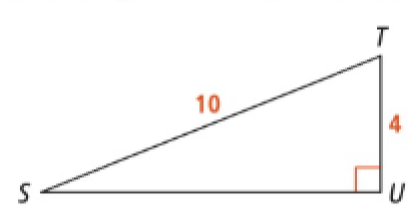
13. What are the sine and cosine of the smallest angle in the right triangle shown?



14. What is the measure of the largest acute angle in the right triangle shown?



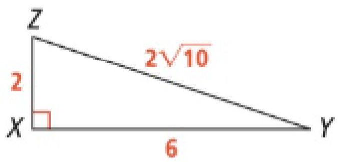
15. In the figure shown, what are $m\angle S$ and $m\angle T$?



PRACTICE & PROBLEM SOLVING

UNDERSTAND

16. Error Analysis Jacinta's teacher asks her to find the tangent of $\angle Y$. What is her error?

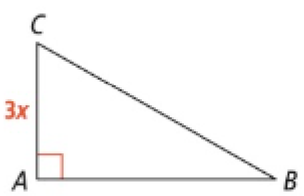


$\tan Y = \frac{XY}{XZ}$

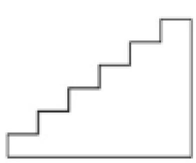
$\tan Y = \frac{6}{2}$

$\tan Y = 3$

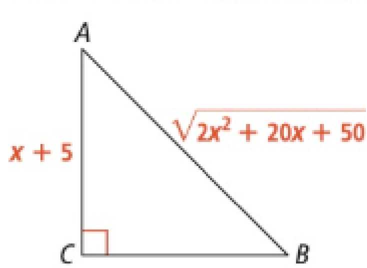
17. Analyze and Persevere If $\sin B = 0.5$ in the triangle shown, what is an expression for AB?



18. Apply Math Models Every tread of a staircase is 8 in. deep, and every riser is 6 in. high. How would you find the angle the staircase makes with the floor? Explain.

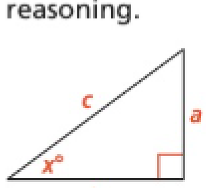


19. Mathematical Connections Find the values.



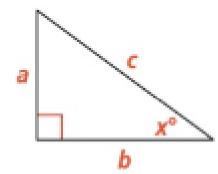
a. $\sin B$ b. $m\angle B$

20. Higher Order Thinking Why are the sine and cosine ratios of $x°$ never greater than one? Use the triangle below to explain your reasoning.



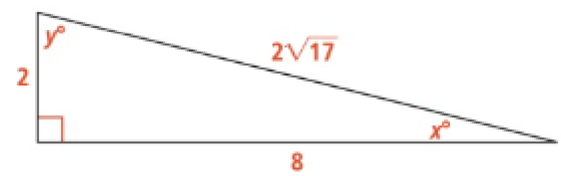
PRACTICE

For Exercises 21–23, write each ratio. SEE EXAMPLE 1



21. $\sin x°$ **22.** $\cos x°$ **23.** $\tan x°$

For Exercises 24–29, find each value. SEE EXAMPLE 2



24. $\sin x°$ **25.** $\cos x°$ **26.** $\tan x°$

27. $\sin y°$ **28.** $\cos y°$ **29.** $\tan y°$

For Exercises 30–35, find each value. SEE EXAMPLE 3

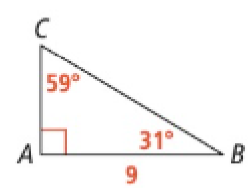
30. $\sin 30°$ **31.** $\cos 60°$ **32.** $\sin 45°$

33. $\tan 45°$ **34.** $\cos 30°$ **35.** $\tan 60°$

36. Write an expression for $\cos 68°$ using sine.

37. Write an expression for $\sin 44°$ using cosine.

For Exercises 38 and 39, find each length.
SEE EXAMPLE 4

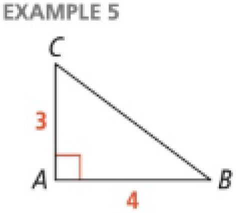


38. AC **39.** BC

For Exercises 40–41, find the angle measures in the triangle. SEE EXAMPLE 5

40. $m\angle B$

41. $m\angle C$



For Exercises 42–43, find each value. SEE EXAMPLE 6

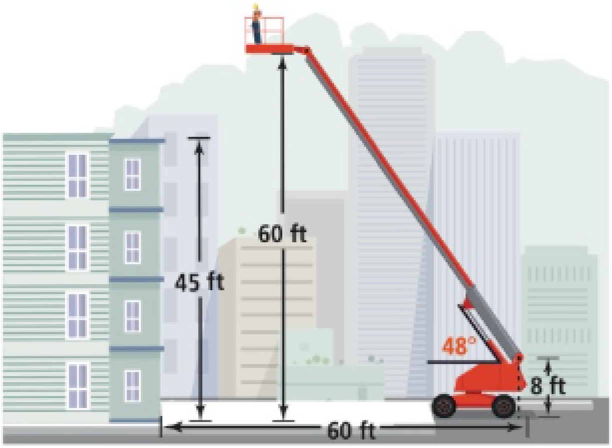
42. $\sin 135°$
$\cos 135°$

43. $\sin 150°$
$\cos 150°$

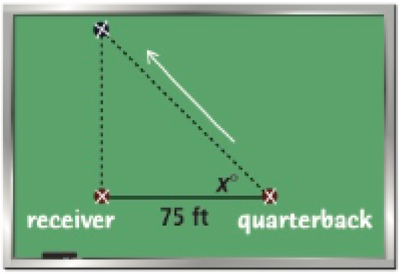
PRACTICE & PROBLEM SOLVING

APPLY

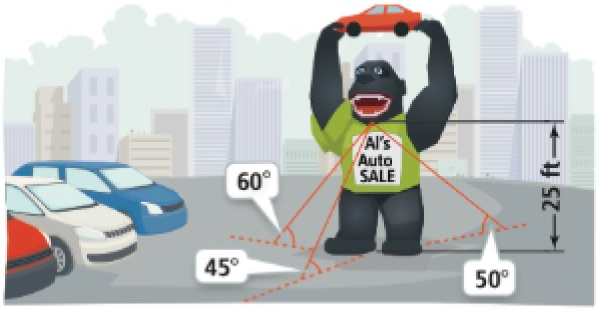
44. Analyze and Persevere Workers need to make repairs on a building. A boom lift has maximum height of 60 ft at an angle of 48°. If the bottom of the boom is 60 ft from the building, can the boom reach the top of the building? Explain.



45. Represent and Connect A coach draws up a play so a quarterback throws the football at the same time a receiver runs straight down the field. Suppose the quarterback throws the football at a speed of 20 ft/s and the receiver runs at a speed of 12 ft/s. At what angle $x°$ to the horizontal line must the quarterback throw the football in order for the receiver to catch it? Explain.

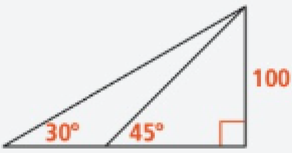


46. Use Patterns and Structure Kelsey puts up an inflatable gorilla to advertise a sale. She realizes that she needs to secure the figure with rope. She estimates she needs to attach three pieces at the angles shown. How much rope does Kelsey need? Round to the nearest foot.

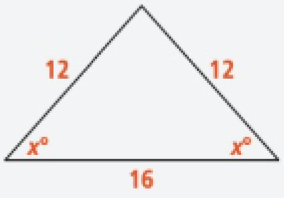


ASSESSMENT PRACTICE

47. Melody and her friends are building 2 paintball arenas; a small arena for kids, and a larger arena for adults. Each arena will need to be enclosed by fencing. What is the total length of fencing in meters that will be needed? Round your answer to the nearest meter. T.1.2



48. SAT/ACT What is the value of $\cos x°$?



Ⓐ $\frac{\sqrt{5}}{2}$ Ⓑ $\frac{3}{2}$ Ⓒ $\frac{\sqrt{5}}{3}$ Ⓓ $\frac{2}{3}$

49. Performance Task Jacy anchors a retractable leash to a tree and attaches the leash to her dog's collar. When the dog fully extends the leash, the angle between the leash and the tree is 84°.

Part A A gap in the fence to a neighbor's yard is 18 feet away from the tree. Can Jacy's dog get into her neighbor's yard? If so, how far into the yard can the dog go? Round to nearest tenth of a foot.

Part B If Jacy wants to make sure her dog cannot get within 1 foot of her neighbor's yard, how high up the tree must she anchor the leash? Round to the nearest tenth of a foot.

8-3 Problem Solving With Trigonometry

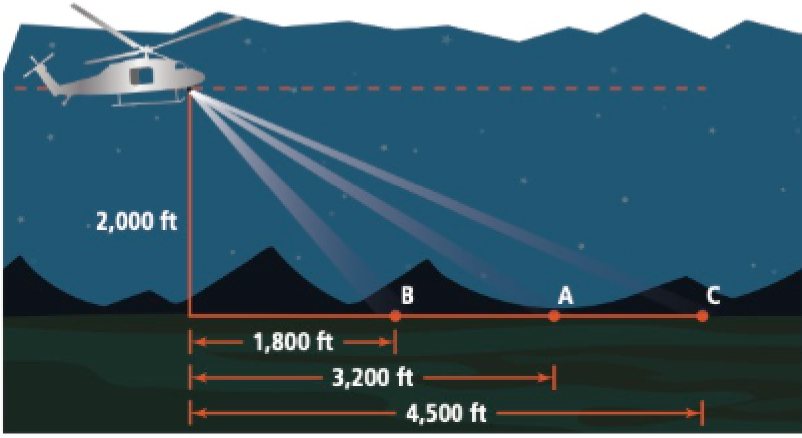
I CAN… use trigonometry to solve problems.

VOCABULARY
- angle of depression
- angle of elevation

MA.912.T.1.2–Solve mathematical and real-world problems involving right triangles using trigonometric ratios and the Pythagorean Theorem. Also T.1.4
MA.K12.MTR.2.1, MTR.6.1, MTR.1.1

MODEL & DISCUSS

A search-and-rescue team is having a nighttime practice drill. Two members of the team are in a helicopter that is hovering at 2,000 feet above ground level.



A. The team first tries to locate object A. At what angle from the horizontal line even with the helicopter should they position the spotlight so that it shines on object A?

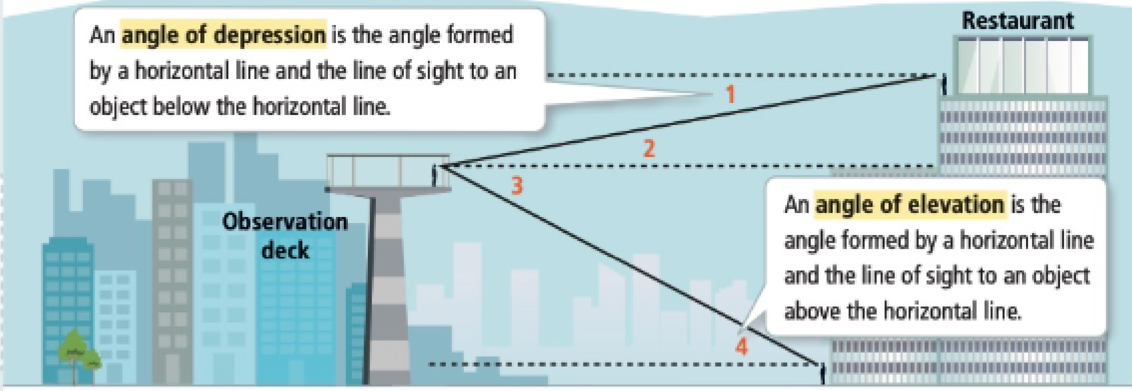
B. Next, they shine the spotlight on object B. How does the angle of the spotlight from the horizontal line change?

C. **Generalize** In general, how does the angle of the spotlight from the horizontal change as the light moves from object A to object B? From object A to object C?

ESSENTIAL QUESTION How can trigonometry be used to solve real-world and mathematical problems?

EXAMPLE 1 Identify Angles of Elevation and Depression

Identify ∠2 as an angle of elevation or an angle of depression. Do the same for ∠3. Explain your reasoning.



STUDY TIP
When solving problems involving angle of elevation or angle of depression, use a diagram and look for right triangles.

To see the person above, the person on the observation deck is looking up from the horizontal, so ∠2 is an angle of elevation.

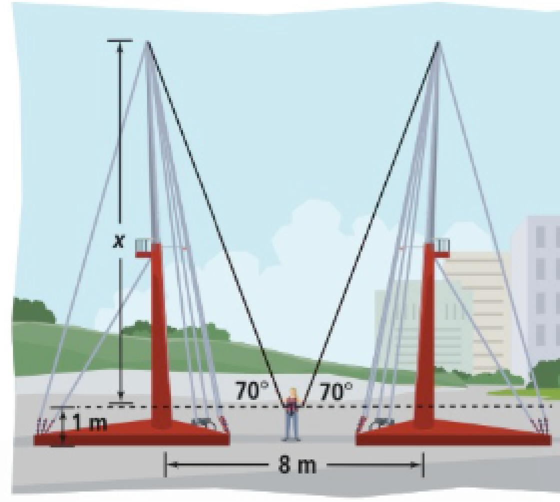
To see the person below, the person on the observation deck is looking down from the horizontal, so ∠3 is an angle of depression.

Try It! 1. In Example 1, how does the angle of depression, ∠1, compare with the angle of elevation, ∠2? Explain your reasoning.

APPLICATION

EXAMPLE 2 Use Angles of Elevation and Depression

For a reverse bungee ride, Reagan stands halfway between two vertical posts. Two bungee cords extend from the top of the posts to Reagan's waist at a height 1 m above the ground. How tall are the vertical posts?



Write an equation to determine *x* m, the vertical distance from the top of a post to a point 1 meter above the ground.

$$\tan 70° = \frac{x}{4}$$

$$x = 4 \tan 70°$$

$$x \approx 10.9899$$

The unknown length and the 4-m length are opposite and adjacent to a 70° angle. So use the tangent function.

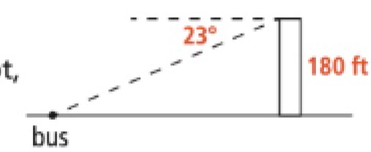
Find the height of the vertical posts.

$$11 + 1 = 12$$

The vertical posts are about 12 meters tall.

COMMON ERROR
Be careful not to forget the distance between Reagan's waist and the ground.

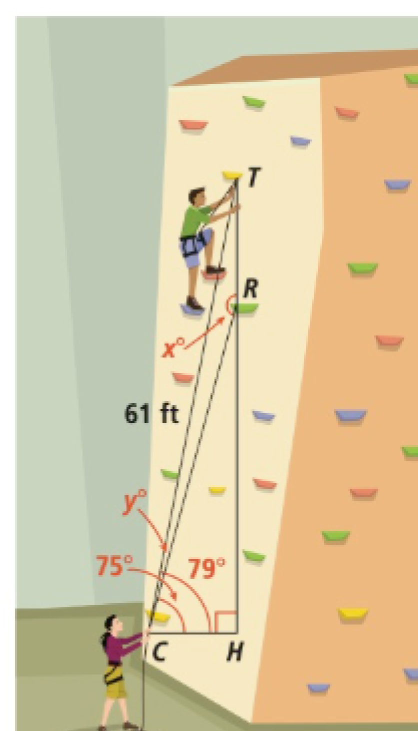
Try It! 2. Nadeem sees the tour bus from the top of the tower. To the nearest foot, how far is the bus from the base of the tower?



APPLICATION

EXAMPLE 3 Use Trigonometry to Solve Problems

An instructor holds a safety rope at point *C* for a student to rappel from the anchor point *T*. The rope between them currently measures 61 ft. How much more rope should the instructor let out so the student can make it to a resting point at point *R*?



The the distance the student will rappel to get to point *R* is *TR*.

Step 1 Find *TH*.

$$\sin 79° = \frac{TH}{61}$$

$$61 \sin 79° = TH$$

$$59.9 \text{ ft} \approx TH$$

Step 2 Find *RH*.

$$\cos 79° = \frac{CH}{61} \qquad \tan 75° = \frac{RH}{11.6}$$

$$61 \cos 79° = CH \qquad 11.6 \tan 75° = RH$$

$$11.6 \text{ ft} \approx CH \qquad 43.3 \text{ ft} \approx RH$$

Step 3 Use *TH* and *RH* to find *TR*.

$$TR = TH - RH$$

$$TR = 59.9 - 43.3$$

$$TR = 16.6$$

The instructor should let out about 17 ft of rope.

CHECK FOR REASONABLENESS
Think about how you could check the reasonableness of your answer. What theorems or definitions could you use?

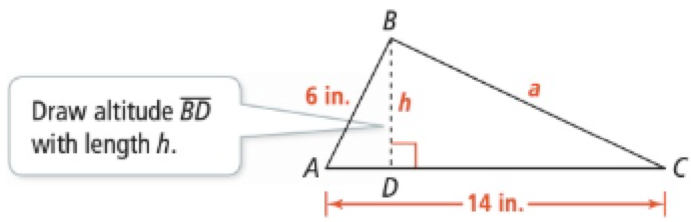
CONTINUED ON THE NEXT PAGE

Try It! 3. In Example 3, how far is the student from the instructor at the resting point?

CONCEPTUAL UNDERSTANDING

EXAMPLE 4 Use Trigonometry to Find Triangle Area

A. How can you use trigonometry to find the area of △*ABC* if you know the measure of ∠*A* is 65°?



To write a formula with side lengths *b* and *c* and included ∠*A*, apply the area formula for a triangle.

$$\text{area} = \frac{1}{2}bh$$
$$= \frac{1}{2}b(c \sin A)$$
$$= \frac{1}{2}14(6 \sin 65°)$$
$$\approx 38.1 \text{ in.}^2$$

In △*ABD*, $\sin A = \frac{h}{c}$, so $h = c \sin A$.

STUDY TIP
In order to apply the formula area $= \frac{1}{2}bc \sin A$, you must know two side lengths and the measure of the included angle.

You can apply the same reasoning to ∠*B* and ∠*C* to write the following area formulas.

$\text{area} = \frac{1}{2}ac \sin B$ $\qquad$ $\text{area} = \frac{1}{2}ab \sin C$

B. What is the area of △*FEG*?

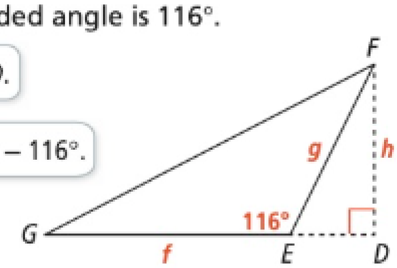
In the triangle, the lengths of sides *g* and *f* are 3 cm and 4 cm, respectively, and the measure of the included angle is 116°.

$$\text{area} = \frac{1}{2}f(g \sin \angle FED)$$
$$= \frac{1}{2}(4)(3) \sin 64°$$
$$\approx 5.4$$

$h = g \sin \angle FED$.

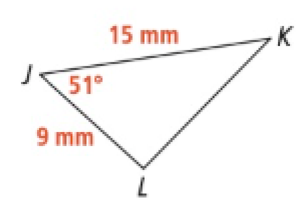
$m\angle FED = 180° - 116°$.

The area of the triangle is about 5.4 cm^2.

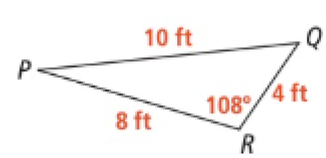


GENERALIZE
How can you use the fact that sin 64° = sin 116° to show area $= \frac{1}{2} fg \sin E$?

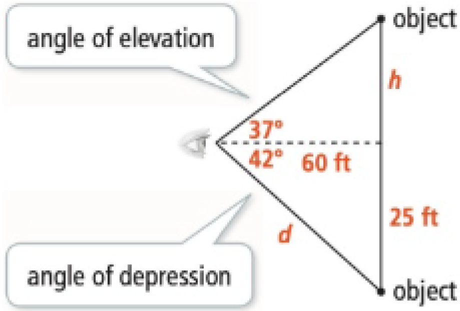
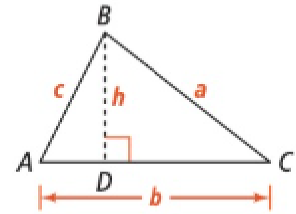
Try It! 4. a. What is the area of △*JKL*?



b. What is the area of △*PQR*?



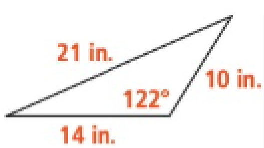
CONCEPT SUMMARY Using Trigonometry to Solve Problems

	Angles of Elevation or Depression	Area Formulas
DIAGRAMS		
SYMBOLS	$\tan 37° = \frac{h}{60}$ $\sin 42° = \frac{25}{d}$	$\text{area} = \frac{1}{2}bc \sin A$ $\text{area} = \frac{1}{2}ac \sin B$ $\text{area} = \frac{1}{2}ab \sin C$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can trigonometry be used to solve real-world and mathematical problems?

2. **Error Analysis** What error does Jamie make in finding the area?



$A = \frac{1}{2} bc \sin A$

$A = \frac{1}{2} \cdot 210 \cdot 0.8480$

Area is about 89 in.^{2}

✗

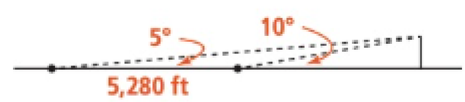
3. **Vocabulary** A person on a balcony and a person on a street look at each other. Draw a diagram to represent the situation and label the angles of elevation and depression.

4. **Analyze and Persevere** How do you find the area of an obtuse triangle when given the obtuse angle measure and all side lengths?

Do You KNOW HOW?

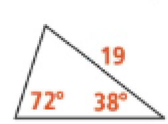
5. A person rides a glass elevator in a hotel lobby. As the elevator goes up, how does the angle of depression to a fixed point on the lobby floor change?

6. A person observes the top of a radio antenna at an angle of elevation of 5°. After getting 1 mile closer to the antenna, the angle of elevation is 10°. How tall is the antenna to the nearest tenth of a foot?



7. Triangle ABC has $AB = 13$, $AC = 15$, and $m\angle A = 59$. What is the area of the triangle to the nearest tenth?

8. A temporary pen for cattle is built using 10-foot sections of fence arranged in a triangle. Each side of the pen has a different number of 10-foot sections. What is the area enclosed by the pen?

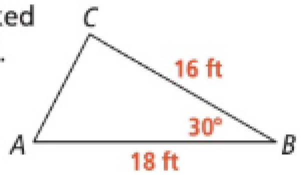


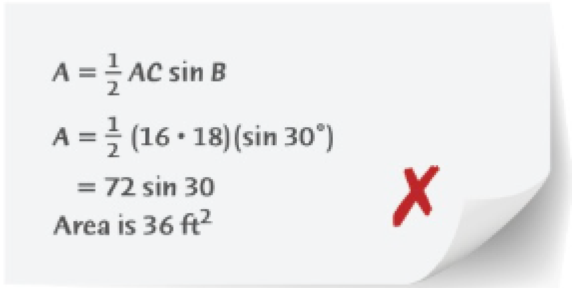
PRACTICE & PROBLEM SOLVING

UNDERSTAND

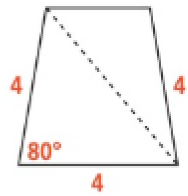
9. **Communicate and Justify** How is the area of a triangle determined if the lengths of two sides and the measure of the included angle are given?

10. **Error Analysis** Leah is asked to find the area of $\triangle ABC$. What is her error?



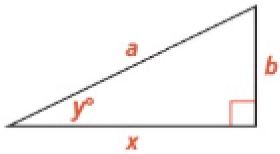


11. **Mathematical Connections** Find the length of the diagonal of the isosceles trapezoid. Then find the length of the fourth side.

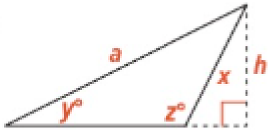


12. **Represent and Connect** For each triangle, write an equation for x using a trigonometric function.

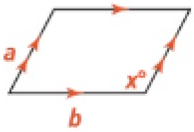
a.



b.



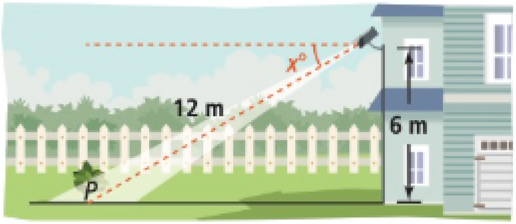
13. **Higher Order Thinking** What is a formula for the area of the parallelogram in the figure? Explain.



PRACTICE

14. What is the angle of elevation to a building 1,000 m away that is 300 m high? SEE EXAMPLE 1

15. To what angle of depression should the security camera be adjusted in order to have the lens aimed at point P on the ground? SEE EXAMPLE 2

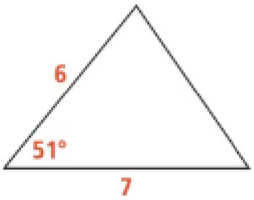


16. The angle of elevation to the sun is 21.5°. What is the length of the shadow cast by a person 5 ft 6 in. tall? SEE EXAMPLE 2

17. Libby's eyes are 5 ft above the ground, and the angle of elevation of her line of sight to the top of the monument is 74°. How far is she from the monument? SEE EXAMPLE 3

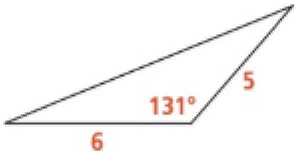
18. Triangle GHJ has $GH = 13$, $GJ = 15$, and $m\angle G = 74$. What is the area of the triangle? SEE EXAMPLE 4

19. What is the area of the triangle? SEE EXAMPLE 4



20. Triangle KLM has $KL = 22$, $KM = 27$, and $m\angle K = 108$. What is the area of the triangle? SEE EXAMPLE 4

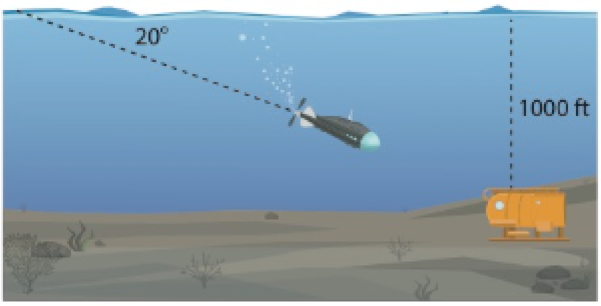
21. What is the area of the triangle? SEE EXAMPLE 4



PRACTICE & PROBLEM SOLVING

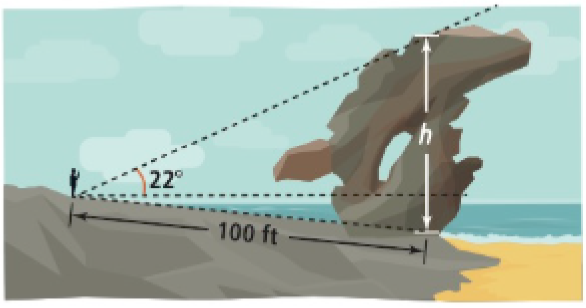
APPLY

22. **Represent and Connect** A research submarine dives at a speed of 100 ft/min directly toward the research lab. How long will it take the submarine to reach the lab from the surface of the ocean to the nearest tenth of a minute?



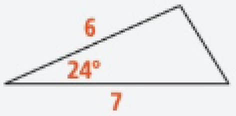
23. **Analyze and Persevere** Benito aims for the center of the target from a distance of 70 meters. Benito releases the arrow at the same height as the center of the target, at a 0.055° angle of depression below the center. Will he hit the yellow section? Explain.

24. **Apply Math Models** Ramona is climbing a hill with a 10° incline and wants to know the height of the rock formation. She walks 100 ft up the hill and uses a clinometer to measure the angle of elevation to the top of the formation. She then walks another 229.4 ft to the top of the hill. What is the height h of the rock formation?

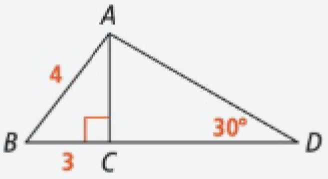


ASSESSMENT PRACTICE

25. Mandy is designing a new window for a space shuttle using high-temperature quartz glass. What is the area of the piece of glass she needs? Round to the nearest one hundredth of a square foot. T.1.4



26 **SAT/ACT** Which of the following equations is true?



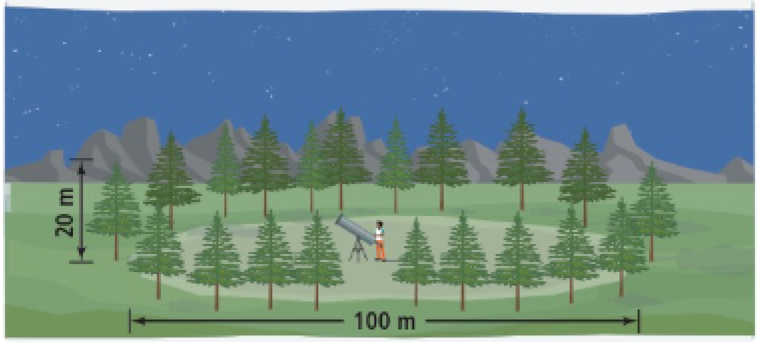
I. $\tan B = \frac{4}{3}$

II. $AD = 2\sqrt{7}$

III. $\frac{7}{CD} = \tan 30°$

Ⓐ I only
Ⓑ II only
Ⓒ III only
Ⓓ II and III only

27. **Performance Task** An amateur astronomer sets up his telescope in the center of a circular field. The field is surrounded by trees 20 m tall. The tripod holding the telescope pivots 1 m above the ground.



Part A What is the lowest angle of elevation at which the astronomer can observe a star?

Part B If the astronomer wants to observe a star 15° above the horizon to the east, how far west must the astronomer move the telescope to see the star?

Part C If the astronomer sets up the telescope in the center of the field on the bed of a truck 1.5 meters above the ground, what is the lowest angle at which he can observe?

8-4 The Law of Sines

I CAN… usethe Law of Sines to solve problems.

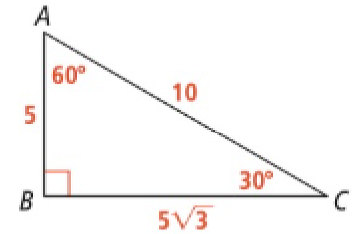
VOCABULARY

- Law of Sines

MA.912.T.1.3–Apply the Law of Sines and the Law of Cosines to solve mathematical and real-world problems involving triangles.
MA.K12.MTR.1.1, MTR.7.1, MTR.4.1

EXPLORE & REASON

Consider the 30°-60°-90° triangle shown.



A. Calculate the values of the ratios $\frac{\sin A}{BC}$ and $\frac{\sin C}{AB}$. How are the values of the ratios related?

B. **Analyze and Persevere** Do you think the ratios would have the same relationship in any 30°-60°-90° right triangle? Explain your answer.

ESSENTIAL QUESTION

How can the Law of Sines be used to determine side lengths and angle measures in acute and obtuse triangles?

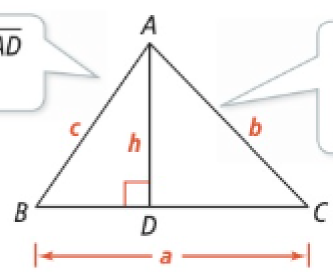
CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Explore the Sine Ratio

How can you use the sine ratio to relate the lengths and angle measures in $\triangle ABC$?

STUDY TIP
Recall that the sine ratio relates the length of the opposite side to the length of the hypotenuse.

Construct the altitude $\overline{AD}$ from vertex *A*.



The altitude intersects $\overline{BC}$ at point *D*, forming right triangles $\triangle ADB$ and $\triangle ADC$.

Write equations for sin *B* and sin *C* using the right triangles.

$$\sin B = \frac{h}{c} \qquad \sin C = \frac{h}{b}$$

$$c \sin B = h \qquad b \sin C = h$$

Solve each equation for *h*.

Set the two expressions for *h* equal to each other.

$$c \sin B = b \sin C$$

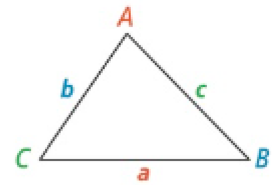
$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

The definition of the sine ratio only includes acute angles. For triangles with right and obtuse angles, you can extend the definition of sine and cosine to include these angles.

Try It! **1.** For Example 1, show that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

CONCEPT Law of Sines

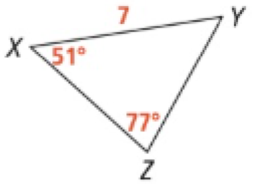
For any $\triangle ABC$ with side lengths a, b, and c opposite angles A, B, and C, respectively, the **Law of Sines** relates the sine of each angle to the length of the opposite side.



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

EXAMPLE 2 Apply the Law of Sines to Find a Side Length

For $\triangle XYZ$, what is YZ to the nearest tenth?



Identify pairs of known sides and angles.

- The measure of $\angle Z$ and the length of its opposite side $\overline{XY}$ are known.
- The measure of $\angle X$ is known. The length of $\overline{YZ}$ is the unknown quantity.

Apply the Law of Sines to write and solve an equation for YZ.

$$\frac{\sin 77^\circ}{7} = \frac{\sin 51^\circ}{YZ}$$

$\frac{\sin Z}{XY} = \frac{\sin X}{YZ}$

$$YZ = \frac{7 \sin 51^\circ}{\sin 77^\circ}$$

$$YZ \approx 5.6$$

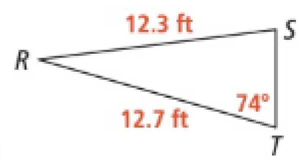
Use a calculator.
7 × SIN 51 ÷ SIN 77

COMMON ERROR
Be careful to correctly write the ratio of the sine of an angle and its opposite side length. It may be helpful to draw an arrow from each angle that points to its opposite side.

Try It! **2.** In Example 2, what is XZ to the nearest tenth?

EXAMPLE 3 Apply Law of Sines to Find the Measure of an Angle

What are $m\angle R$ and $m\angle S$ in $\triangle RST$?



HAVE A GROWTH MINDSET
In what ways do you give your best effort and persist?

Step 1 Apply the Law of Sines to write and solve an equation to find $\sin S$.

$$\frac{\sin S}{RT} = \frac{\sin T}{RS}$$

$$\frac{\sin S}{12.7} = \frac{\sin 74^\circ}{12.3}$$

$$\sin S = \frac{12.7 \sin 74^\circ}{12.3}$$

$$\sin S \approx 0.9925$$

You know $m\angle T$ and the length of the side opposite $\angle T$. You also know the length of the side opposite $\angle S$.

Step 2 Use the inverse function $\sin^{-1}$ to find $m\angle S$.

$$\sin^{-1}(\sin S) = \sin^{-1}(0.9925)$$

$$m\angle S \approx 83$$

Step 3 Find $m\angle R$.

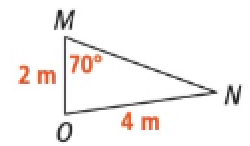
$$m\angle R \approx 180 - 74 - 83$$

$$\approx 23$$

CONTINUED ON THE NEXT PAGE

Try It! **3. a.** What is $m\angle N$?

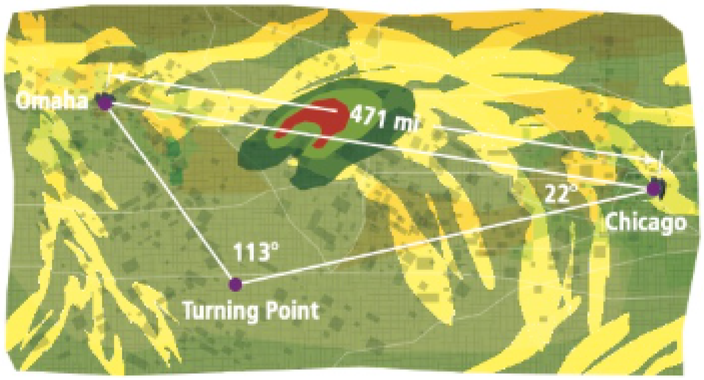
b. What is $m\angle O$?



APPLICATION

EXAMPLE 4 Apply the Law of Sines

The map shows the path a pilot flew between Omaha and Chicago in order to avoid a thunderstorm. How much longer is this route than the direct route to Chicago?



ANALYZE AND PERSEVERE
Think about what the problem is asking for. Which distances are you supposed to find?

Step 1 Let x represent the distance from the turning point to Omaha. Find x.

Use the Law of Sines.

$$\frac{\sin 113^\circ}{471} = \frac{\sin 22^\circ}{x}$$

$$x = \frac{471 \sin 22^\circ}{\sin 113^\circ}$$

$$x \approx 191.7$$

Step 2 Let y represent the distance from Chicago to the turning point. Find y.

First determine the measure of the angle opposite y.

$$180^\circ - 113^\circ - 22^\circ = 45^\circ$$

Then use the Law of Sines.

$$\frac{\sin 113^\circ}{471} = \frac{\sin 45^\circ}{y}$$

$$y = \frac{471 \sin 45^\circ}{\sin 113^\circ}$$

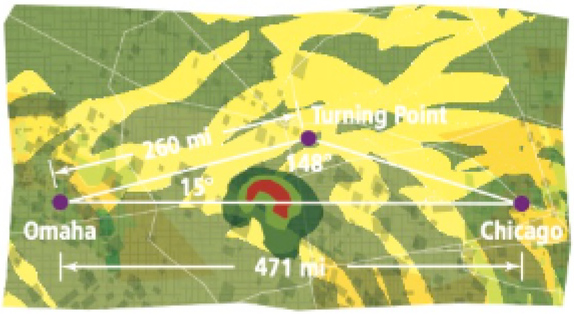
$$y \approx 361.8$$

Step 3 Find the total distance the pilot flew.

$$191.7 \text{ mi} + 361.8 \text{ mi} = 553.5 \text{ mi}$$

The pilot flew 553.5 mi − 471 mi = 82.5 mi farther.

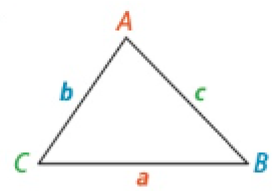
Try It! **4.** Suppose the pilot chose to fly north of the storm. How much farther is that route than the direct route?



CONCEPT SUMMARY Law of Sines

WORDS For any $\triangle ABC$ with side lengths a, b, and c opposite angles A, B, and C, respectively, the Law of Sines relates the sine of each angle to the length of the opposite side.

DIAGRAM



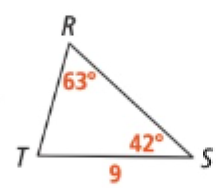
SYMBOLS

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can the Law of Sines be used to determine side lengths and angle measures in acute and obtuse triangles?

2. **Error Analysis** Amelia is asked to find a missing side length in $\triangle RST$. What is her error?



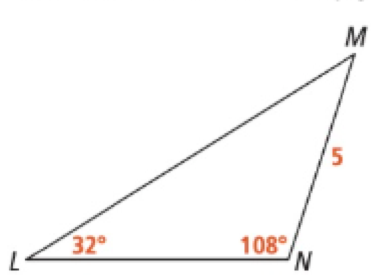
$$\frac{\sin R}{RT} = \frac{\sin S}{ST}$$

$$\frac{\sin 63^\circ}{RT} = \frac{\sin 42^\circ}{9}$$

$$RT = \frac{9 \sin 63^\circ}{\sin 42^\circ} \approx 12$$

✗

3. **Vocabulary** What are the pairs of opposite angles and side lengths in $\triangle LMN$? What does the Law of Sines help you find?

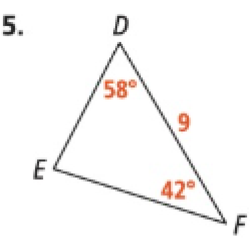


4. **Analyze and Persevere** Can you find all the missing parts of a triangle using the Law of Sines if you know the lengths of all three sides? Explain.

Do You KNOW HOW?

For Exercises 5 and 6, list the parts of each triangle you can determine using only the Law of Sines.

5.



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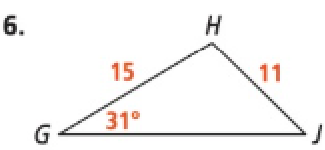
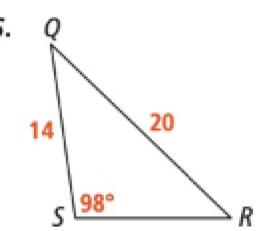


For Exercises 7 and 8, use $\triangle QRS$.

7. What are $m\angle Q$ and $m\angle R$?

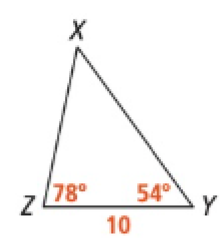
8. What is the perimeter of $\triangle QRS$?



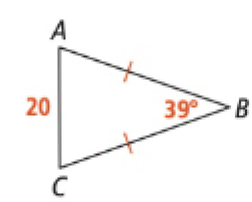
For Exercises 9 and 10, use $\triangle XYZ$.

9. What is XY?

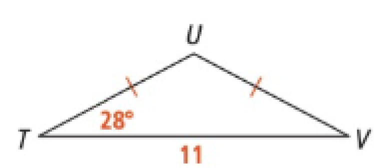
10. What is XZ?



11. What are AB and BC?



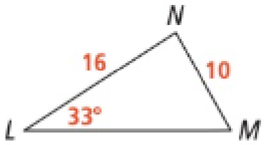
12. What is the perimeter of $\triangle TUV$?



PRACTICE & PROBLEM SOLVING

UNDERSTAND

13. Error Analysis Kimberly is asked to find $m\angle M$. What is her error?

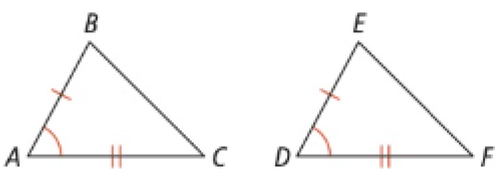


$$\frac{\sin M}{16} = \frac{\sin L}{10}$$

$$\sin M = \frac{16 \cdot \sin 33^\circ}{10}$$

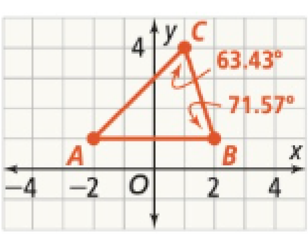
$$m\angle M = 0.8714$$

14. Communicate and Justify Suppose you only know the lengths and angle measures of two triangles used to show they are congruent by SAS. Can you find the missing angle measures and side lengths of the triangles using the Law of Sines? Explain.

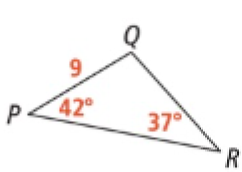


15. Choose Efficient Methods The measures of two angles are given along with the measure of the side opposite the third angle of a triangle. How can the Law of Sines be used to find missing angle measures and side lengths of the triangle? Explain.

16. Mathematical Connections What is AC? Use both the Distance Formula and the Law of Sines. How do the values compare? Explain.



17. Use Patterns and Structure Explain how to use the Law of Sines to find the perimeter of $\triangle PQR$. Then write an expression for the perimeter.

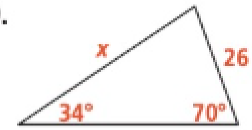


PRACTICE

For Exercises 18–23, find each length *x*. Round to the nearest tenth. SEE EXAMPLES 1 AND 2

18.

19.



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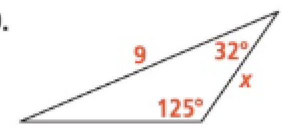

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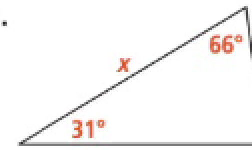

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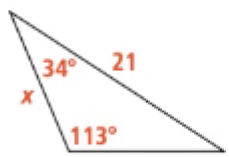

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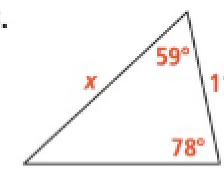
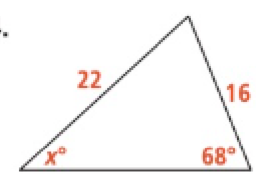


For Exercises 24–29, find each angle measure *x*°. Round to the nearest tenth. SEE EXAMPLE 3

24.



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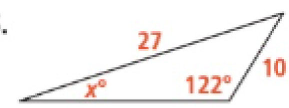

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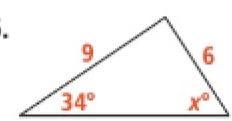

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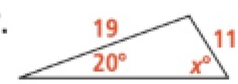

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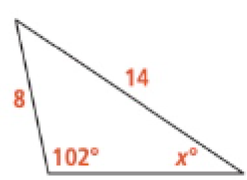

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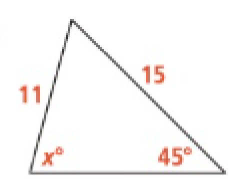

For Exercises 30–33, find the perimeter of each triangle. Round to the nearest tenth. SEE EXAMPLE 4

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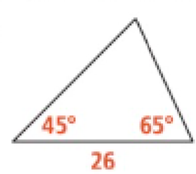

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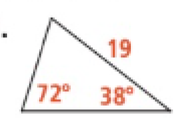

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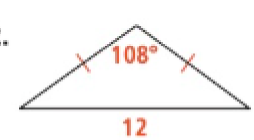
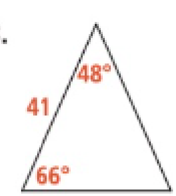


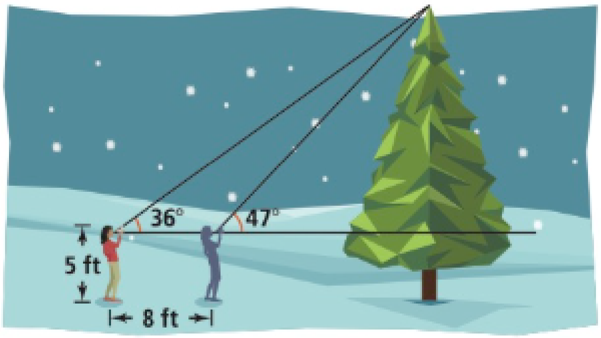
33.



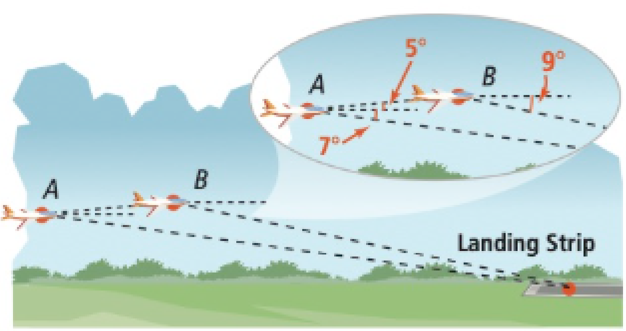
PRACTICE & PROBLEM SOLVING

APPLY

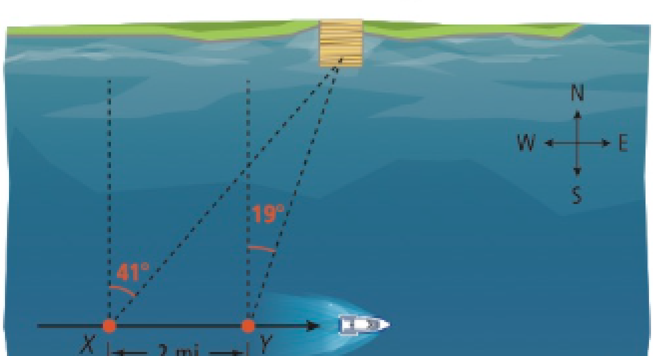
34. **Analyze and Persevere** To find the height of a tree, a forester uses a clinometer to measure the angle to the top of the tree. She then measures again at a distance 8 feet farther away. What is the height of the tree? Round to the nearest foot.



35. **Apply Math Models** At point A, the pilot of a plane looks down at an angle of 7° at the landing strip. After flying up to point B, the pilot looks down at an angle of 9° to see the landing strip. What is the distance from point B to the landing strip if the distance from point A to point B is 1,100 ft? Round to the nearest foot.

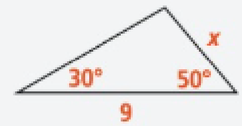


36. **Higher Order Thinking** A boat is cruising due east. At point X, the captain measures the angle east of north to a pier. At point Y, he again measures the angle east of north to the pier. A restricted area with radius 2.5 miles is centered at the end of the pier. Does the boat enter the restricted area? Explain.

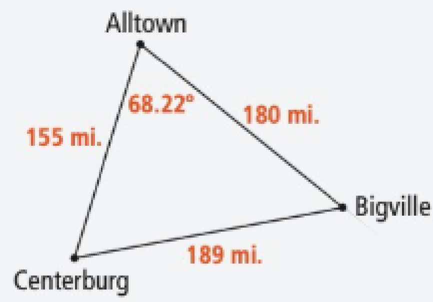


ASSESSMENT PRACTICE

37. What is the value of x? Round to the nearest tenth. T.1.3

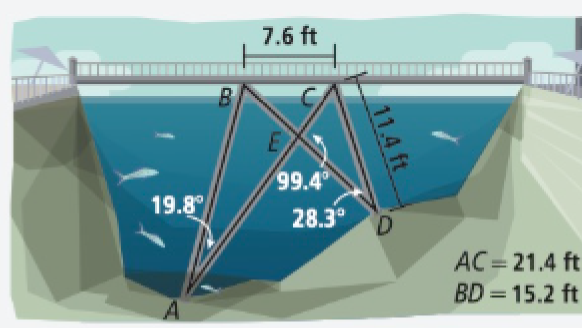


38. **SAT/ACT** What is the measure of the angle made by the roads at Bigville?



Ⓐ 55.10°
Ⓑ 72.28°
Ⓒ 49.60°
Ⓓ 40.40°

39. **Performance Task** An engineer is designing a walkway for an aquarium, but she only receives the partial information shown for the supports of the walkway. She must determine the lengths of the remaining supports to complete the design.



Part A What is AB to the nearest tenth of a foot?

Part B The engineer decides she wants beams from point A to the left end of the walkway and from point D to the right end of the walkway. Is there enough information given to find the lengths? If so, find the lengths. If not, explain what other information she needs.

8-5 The Law of Cosines

I CAN… use the Law of Cosines to solve problems.

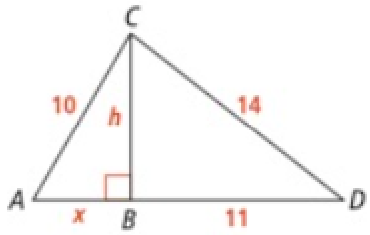
VOCABULARY

- Law of Cosines

MA.912.T.1.3–Apply the Law of Sines and the Law of Cosines to solve mathematical and real-world problems involving triangles.
MA.K12.MTR.5.1, MTR.3.1, MTR.7.1

EXPLORE & REASON

Use $\triangle ABC$ to answer the questions.



A. Write equations for the side lengths of $\triangle ABC$ and $\triangle CBD$ using the Pythagorean Theorem.

B. Use a system of equations to solve for x.

C. **Use Patterns and Structure** How can you use the information you found to determine $m\angle A$?

ESSENTIAL QUESTION

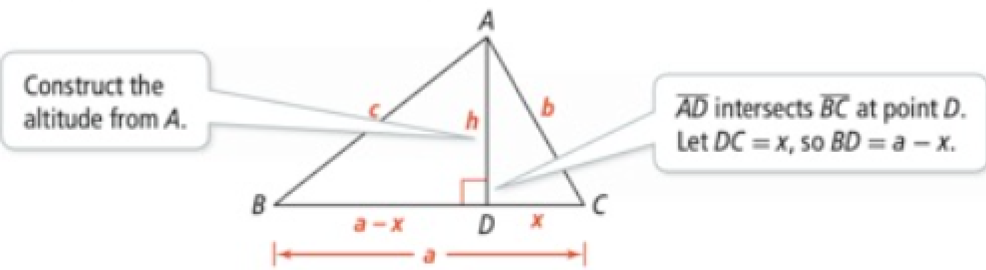
How can the Law of Cosines be used to determine side lengths and angle measures of acute and obtuse triangles?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Develop the Law of Cosines with Trigonometry

Triangle ABC is not a right triangle. How can you use a cosine ratio to write an equation relating the side lengths a, b, and c?

USE PATTERNS AND STRUCTURE
When constructing auxiliary lines, look for ones that result in right angles, right triangles, parallel lines, or other convenient geometric relationships.



Use the Pythagorean Theorem with $\triangle ABD$. Then distribute and combine like terms.

$c^2 = (a - x)^2 + h^2$

$c^2 = a^2 - 2ax + x^2 + h^2$

$c^2 = a^2 - 2ax + b^2$

$\triangle ACD$ is a right triangle, so $x^2 + h^2 = b^2$. Substitute b^2 for $x^2 + h^2$.

$c^2 = a^2 - 2ab(\cos C) + b^2$

$c^2 = a^2 + b^2 - 2ab \cos C$

Since $\cos C = \frac{x}{b}$, $x = b(\cos C)$. Substitute $b(\cos C)$ for x.

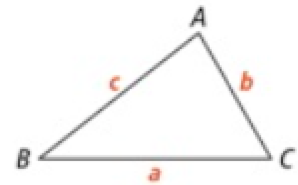
Thus, given side lengths a and b of a triangle and the included angle measure, you can find the length c of the third side by taking the square root of the expression on the right side of the equation.

Try It! 1. Use the same method as in Example 1 to write equations for a^2 using $\cos A$ and b^2 using $\cos B$.

CONCEPT Law of Cosines

For any $\triangle ABC$, the **Law of Cosines** relates the cosine of each angle to the side lengths of the triangle.

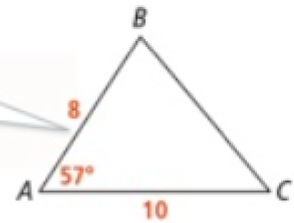
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$



EXAMPLE 2 Apply the Law of Cosines to Find a Side Length

What is BC to the nearest tenth?

The side length opposite $\angle A$ is unknown. You know two side lengths and the measure of their included angle, $\angle A$.

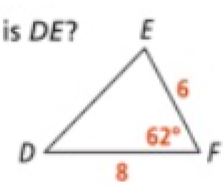


USE PATTERNS AND STRUCTURE
What information does the Law of Cosines provide that the Law of Sines does not?

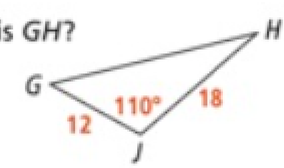
Use the Law of Cosines to write and solve an equation for BC.

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC) \cos A$$
$$BC^2 \approx 8^2 + 10^2 - 2(8)(10)(0.5446)$$
$$BC^2 \approx 76.864$$
$$BC \approx 8.8$$

Try It! 2. a. What is DE?

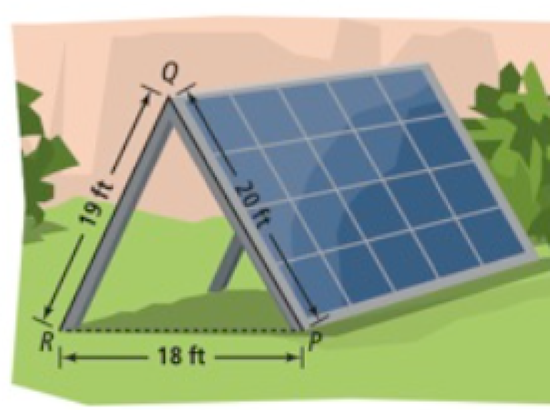


b. What is GH?



EXAMPLE 3 Apply the Law of Cosines to Find an Angle Measure

The optimal tilt for Keenan's solar panel is between 58° and 60° to the horizontal. Has Keenan placed his solar panel at an optimal angle?



Write an equation using the Law of Cosines. Then use the inverse cosine to find $m\angle P$.

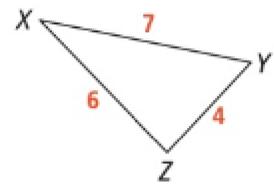
COMMON ERROR
In applying the Law of Cosines, be sure to correctly place the included angle measure and the length of the side opposite the include angle on opposite sides of the equation.

$$QR^2 = PR^2 + PQ^2 - 2(PQ)(QR)\cos P$$
$$19^2 = 18^2 + 20^2 - 2(18)(20)\cos P$$
$$\cos P = \frac{-363}{-720}$$
$$\cos P \approx 0.5042$$
$$m\angle P \approx 59.72°$$

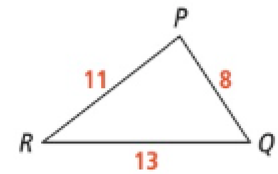
Keenan has placed his solar panel at an optimal angle.

CONTINUED ON THE NEXT PAGE

Try It! **3. a.** What is $m\angle X$?



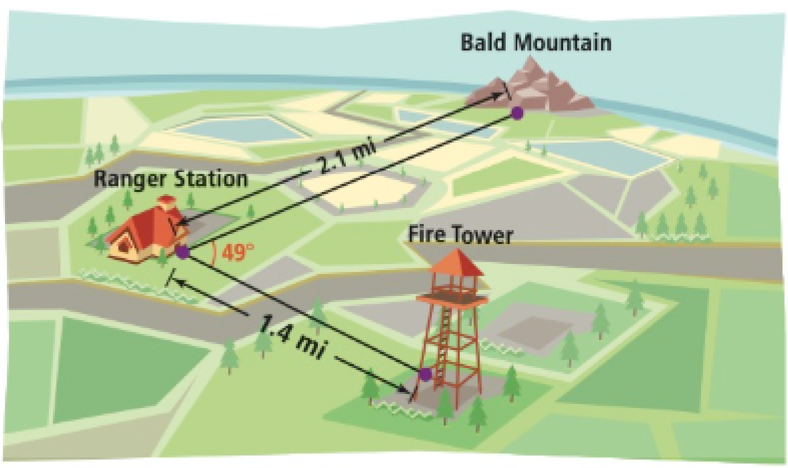
b. What is $m\angle P$?



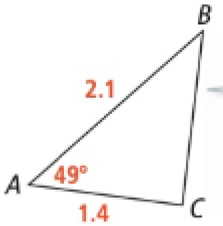
APPLICATION

EXAMPLE 4 Use the Law of Cosines to Solve a Problem

The district ranger wants to build a new ranger station at the location of the fire tower because it would be closer to Bald Mountain than the old station is. Is the district ranger correct? Explain.



Formulate The ranger station, Bald Mountain, and the fire tower form the vertices of a triangle. Label the ranger station vertex A, Bald Mountain vertex B, and the fire tower vertex C. Draw the triangle.



The triangle shows two sides and an included angle. The distance from the fire tower to Bald Mountain is unknown.

Compute Use the Law of Cosines to write an equation for the unknown.

$$BC^2 = AC^2 + AB^2 - 2(AC)(AB)\cos A$$

$$BC^2 = 1.4^2 + 2.1^2 - 2(1.4)(2.1)(\cos 49°)$$

Substitute the known values and solve.

$$BC^2 \approx 2.512$$

$$BC \approx 1.6$$

Interpret The distance from the fire tower to Bald Mountain is about 1.6 miles, so the district ranger is correct that a new station at the fire tower would be closer.

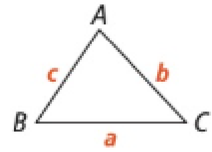
Try It! **4.** In Example 4, what is the angle that the new path forms with the old path at Bald Mountain?

CONCEPT SUMMARY Law of Cosines

WORDS For any $\triangle ABC$, the Law of Cosines relates the cosine of each angle to the side lengths of the triangle.

SYMBOLS

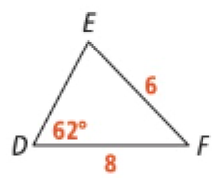
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can the Law of Cosines be used to determine side lengths and angle measures of acute and obtuse triangles?

2. **Error Analysis** Cameron is asked to find DE. What is his error?

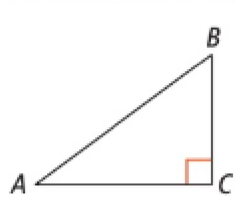


$DE^2 = DF^2 + EF^2 - 2(DF)(EF)\cos F$
$DE^2 = 8^2 + 6^2 - 2(8)(6)\cos 62^\circ$
$DE^2 = 54.930\ldots$
$DE = 7.411\ldots$

3. **Vocabulary** How would you describe the Law of Cosines in words?

4. **Communicate and Justify** With the Law of Sines and the Law of Cosines, can you find the missing side lengths and angle measures of any triangle for which you know any three parts? Explain.

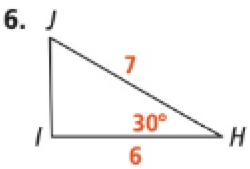
5. **Choose Efficient Methods** Use the Law of Cosines and the Pythagorean Theorem with $\triangle ABC$ to show that $\cos 90^\circ = 0$.



Do You KNOW HOW?

For Exercises 6–9, list the parts of each triangle you can determine using only the Law of Cosines.

6.



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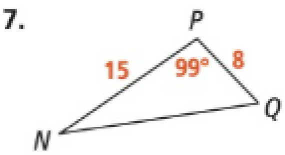

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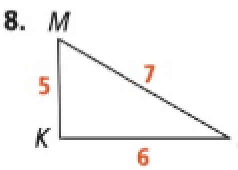

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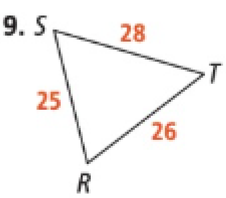
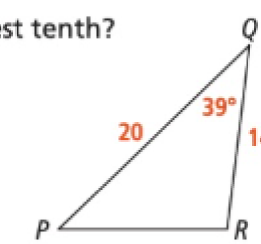
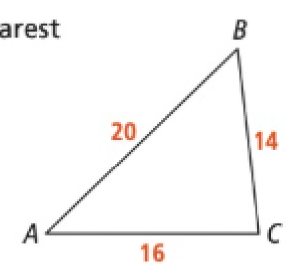


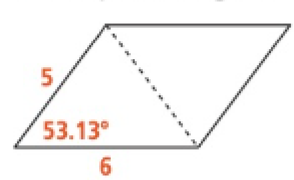
10. What is PR to the nearest tenth?



11. What is $m\angle B$ to the nearest tenth of a degree?



12. Use the Law of Cosines to find the diagonal of the parallelogram.

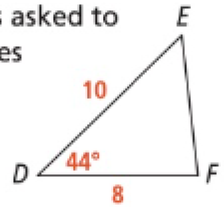


PRACTICE & PROBLEM SOLVING

UNDERSTAND

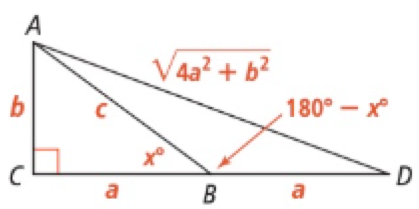
13. Communicate and Justify How is the Law of Cosines used to find missing angle measures if the side lengths of a triangle are given? Explain.

14. Error Analysis Tavon is asked to find *EF*. What error does Tavon make?



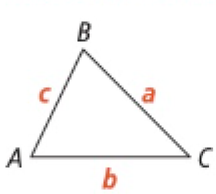
$EF^2 = DF^2 + DE^2 - 2(EF)(DE)\cos D$
$EF^2 = 8^2 + 10^2 - 2(8)(10)\cos 44^\circ$
$EF^2 \approx 64 + 100 - 160 \cdot 0.7193$
$EF \approx 48.91$ ✗

15. Higher Order Thinking Use the diagram to show that for acute angle x°, $\cos(180 - x)^\circ = -\frac{a}{c}$. *Hint:* Apply the Law of Cosines to $\triangle ABD$ and write an equation involving $\cos(180 - x)^\circ$. Then, for $\triangle ABC$, use the relationship $a^2 + b^2 = c^2$.



16. Generalize Suppose you know two of the side lengths of a triangle and the measure of one of the angles. How do you choose whether to find the third side length using the Law of Sines or the Law of Cosines? Explain.

17. Use Patterns and Structure Consider $\triangle ABC$.

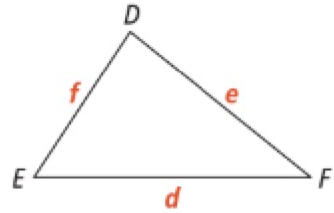


a. How would you find $m\angle A$ if you were given *a*, *b*, and *c*? Include an equation in your explanation.

b. How would you find *a* if you were given $m\angle A$, *b*, and *c*? Include an equation in your explanation.

PRACTICE

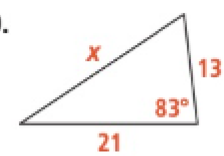
For Exercises 18 and 19, use $\triangle DEF$. Find an equation for each length. SEE EXAMPLE 1



18. length *e*

19. length *f*

For Exercises 20–23, find *x* to the nearest tenth.
SEE EXAMPLE 2

20.


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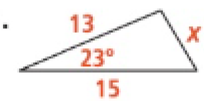

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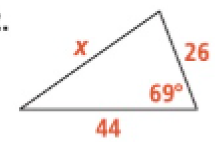

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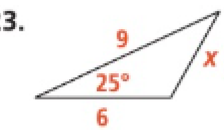
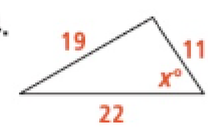


For Exercises 24–27, find *x* to the nearest tenth.
SEE EXAMPLE 3

24.


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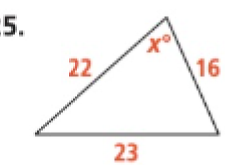

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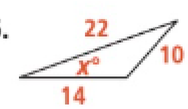

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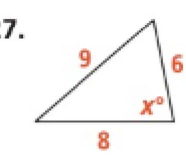
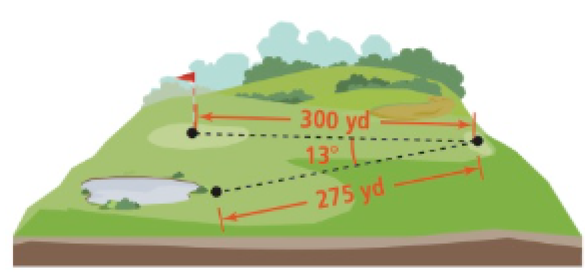


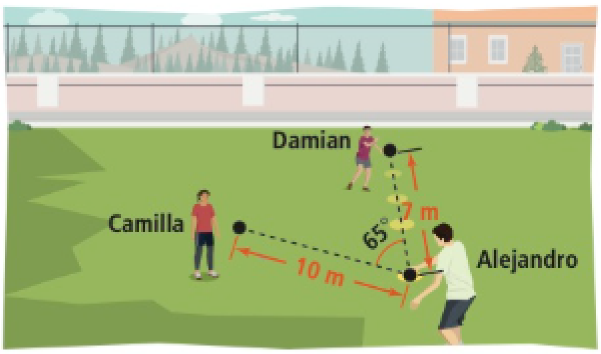
28. A golfer hits from the tee for a 300-yard hole. Her drive carries 275 yards but is 13° off line from the hole. How much farther must the golfer now hit the ball to reach the hole?
SEE EXAMPLE 4



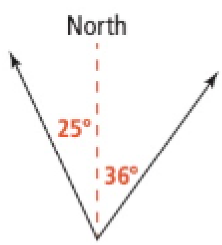
PRACTICE & PROBLEM SOLVING

APPLY

29. Apply Math Models Alejandro, Camilla, and Damian are practicing for a game of ultimate, which is played on a field with a flying disc. Alejandro has the disc and Camilla is in the end zone, but Alejandro can only throw accurately for distances of 7.5 m or less. He throws the disc to Damian, because Damian can throw with accuracy up to 9.5 m. Is Camilla within Damian's range of accuracy? Explain.

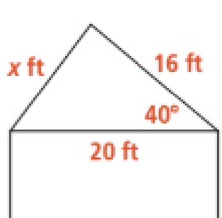


30. Mathematical Connections Two rescue workers leave at the same time to find an injured hiker. The first walks 25° west of north at 3.5 mi/h, and the second walks 36° east of north at 2.5 mi/h.



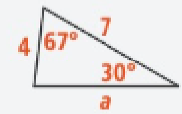
After 2 hours, the second worker finds the hiker and radios the first worker for help. If the first worker jogs directly to the second worker at 6 mi/h, how long will it take for her to arrive?

31. Analyze and Persevere An architect proposes the plan shown for a new roof with a 40° incline on one side. The owner of the house thinks that 40° is not steep enough and wants the incline to be 50°. If the length of the adjacent side does not change, by how much does the length, x ft, of the opposite side increase? What is the new angle of incline for the opposite side?



ASSESSMENT PRACTICE

32. Which equation is true for the triangle shown? Select all that apply. T.1.3



☐ **A.** $a^2 = 16 + 49 - 28 \cos 30°$

☐ **B.** $16 = a^2 + 49 - (14a)\cos 30°$

☐ **C.** $49 = a^2 + 16 - (4a)\cos 67°$

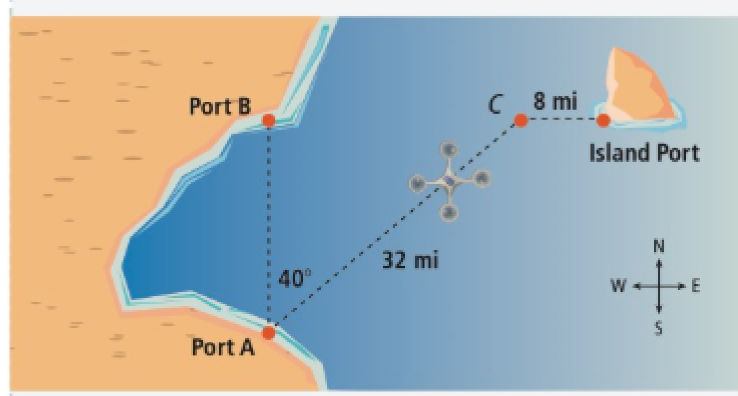
☐ **D.** $a^2 = 49 + 16 - 56 \cos 67°$

☐ **E.** $49 = a^2 + 16 - (8a)\cos 83°$

33. SAT/ACT A triangle has sides with lengths 12 cm and 15 cm. The measure of the included angle is 46°. What is the length of the third side, to the nearest tenth of a centimeter?

Ⓐ 24.9 cm
Ⓑ 10.5 cm
Ⓒ 13.0 cm
Ⓓ 10.9 cm

34. Performance Task A medical supply drone leaves Port A traveling 40 degrees east of north. After flying 32 mi on that course to point C, the drone turns 50° to the right to fly due east. It then flies another 8 mi, where it makes a drop at Island Port.



Part A The drone can fly a total distance of 75 miles before it needs to recharge. Can it fly directly back to Port A from Island Port without recharging? Explain.

Part B Port B is 24 miles due north of Port A. What is the distance from Island Port to Port B? Can the drone fly to Port B?

Part C Port B appears to be directly west of Island Port. Is it? If so, explain. If not, what direction is it from Island Port?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.T.1.2–Solve mathematical and real-world problems involving right triangles using trigonometric ratios and the Pythagorean Theorem.

MA.K12.MTR.7.1

The Impossible Measurement

Tall buildings are often some of the most recognizable structures of cities. The Empire State Building in New York City, the Transamerica Pyramid in San Francisco, and the JPMorgan Chase Tower in Houston are all famous landmarks in those cities.

Cities around the world compete for the tallest building bragging rights. Which city currently has the tallest building? This Mathematical Modeling in 3 Acts lesson will get you thinking about the height of structures, including tall buildings such as these.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in the topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

TOPIC

8

Topic Review

? TOPIC ESSENTIAL QUESTION

1. How are the Pythagorean Theorem and trigonometry useful?

Vocabulary Review

Choose the correct term to complete each sentence.

2. The ratio of the length of the leg adjacent to an acute angle in a right triangle to the length of the hypotenuse is the ________ of the angle.
3. The ________ gives a relationship between the sine of each angle in a triangle and the length of the side opposite the angle.
4. The angle formed by a horizontal line and a line of sight to an object above the line is a(n) ________.
5. A(n) ________ is a set of three nonzero whole numbers that satisfies the equation $a^2 + b^2 = c^2$.

- angle of depression
- angle of elevation
- cosine
- Law of Cosines
- Law of Sines
- Pythagorean triple
- sine
- tangent
- trigonometric ratio

Concepts & Skills Review

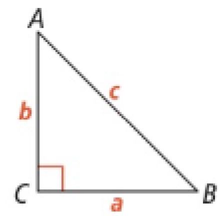
LESSONS 8-1 & 8-2 Right Triangles and The Pythagorean Theorem, and Trigonometric Ratios

Quick Review

Given $\triangle ABC$, the Pythagorean Theorem states $a^2 + b^2 = c^2$.

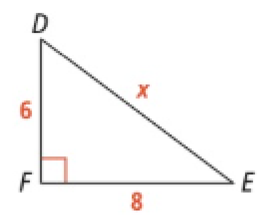
The **trigonometric ratios** are

$\sin A = \frac{a}{c}$ $\cos A = \frac{b}{c}$ $\tan A = \frac{a}{b}$.



Example

Given $\triangle DEF$, what is $\sin D$?



Step 1 Use the Pythagorean Theorem to find x.

$8^2 + 6^2 = x^2$

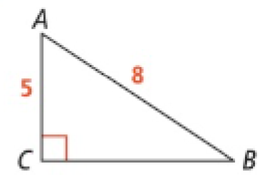
$10 = x$

Step 2 Use a trigonometric ratio to find $\sin D$.

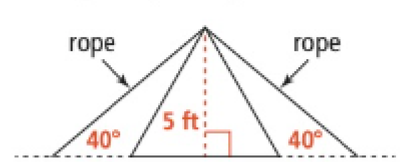
$\sin D = \frac{8}{10} = \frac{4}{5}$

Practice & Problem Solving

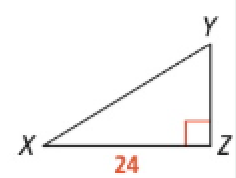
For Exercises 6–8, use $\triangle ABC$ to find each value.



6. BC
7. $\cos B$
8. $m\angle A$ to the nearest tenth
9. Ines has 16 feet of rope to stake out the front part of her tent. Does she have enough rope? Explain.



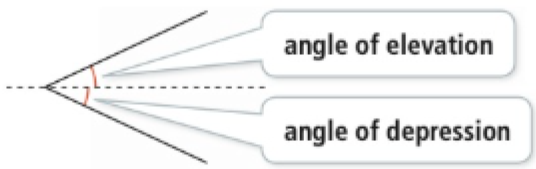
10. **Analyze and Persevere** Given $\triangle XYZ$, what additional information do you need to find YZ? Explain.



LESSON 8-3 Problem Solving With Trigonometry

Quick Review

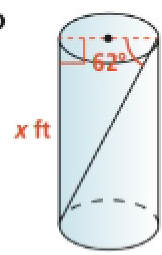
Many problems, such as those with angles of elevation or depression, can be modeled with triangles.



You can use trigonometric ratios to solve those problems.

Example

The angle of depression from the top to the bottom of a well is 62°. If the well is 4.5 feet in diameter, how deep is it?



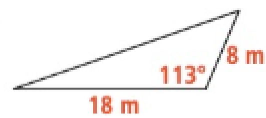
$$\tan 62^\circ = \frac{x}{4.5 \text{ ft}}$$

$$x \approx 8.5 \text{ ft}$$

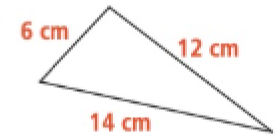
Practice & Problem Solving

For Exercises 11 and 12, find the area of each triangle.

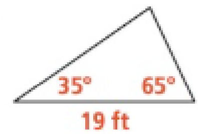
11.



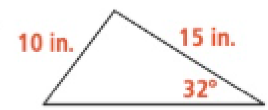
12.



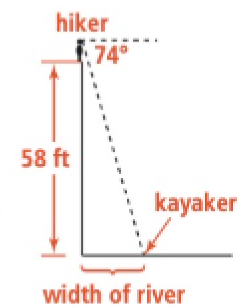
13.



14.



15. **Apply Math Models** A hiker whose eyes are $5\frac{1}{2}$ feet above ground looks down at a kayaker on the far side of the river below. How could you find the approximate width of the river?

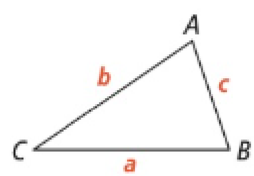


LESSON 8-4 The Law of Sines

Quick Review

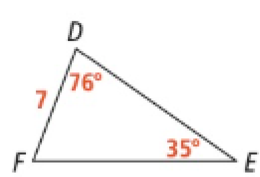
Given $\triangle ABC$, the **Law of Sines** states

$$\frac{(\sin A)}{a} = \frac{(\sin B)}{b} = \frac{(\sin C)}{c}.$$



Example

Given $\triangle DEF$, what is EF to the nearest tenth?



By the Law of Sines,

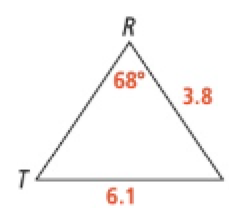
$$\frac{(\sin 35^\circ)}{7} = \frac{(\sin 76^\circ)}{EF}$$

$$EF = \frac{7(\sin 76^\circ)}{(\sin 35^\circ)}$$

$$EF \approx 11.8$$

Practice & Problem Solving

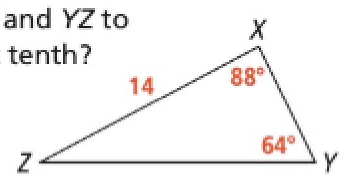
For Exercises 16 and 17, use $\triangle RST$ and the Law of Sines to find each measure to the nearest tenth.



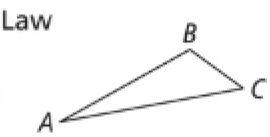
16. $m\angle T$

17. RT

18. What is XY and YZ to the nearest tenth?



19. **Generalize** How can the Law of Sines be applied to show that the expression is equivalent to the perimeter of $\triangle ABC$?



$$\frac{AB \sin A}{\sin C} + \frac{BC \sin B}{\sin A} + \frac{AC \sin C}{\sin B}$$

LESSON 8-5 The Law of Cosines

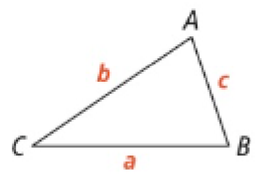
Quick Review

Given $\triangle ABC$, the **Law of Cosines** states

$a^2 = b^2 + c^2 - 2bc(\cos A)$

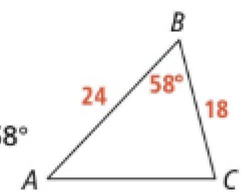
$b^2 = a^2 + c^2 - 2ac(\cos B)$

$c^2 = a^2 + b^2 - 2ab(\cos C)$.



Example

In $\triangle ABC$, what is PR to the nearest tenth?



By the Law of Cosines

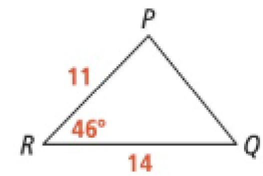
$b^2 = 18^2 + 24^2 - 2(18)(24)\cos 58°$

$b^2 \approx 442.15$

$b \approx 21$

Practice & Problem Solving

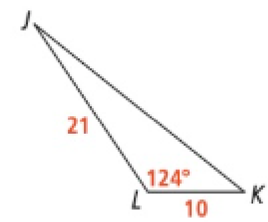
For Exercises 20 and 21, use $\triangle PQR$ to find each measure to the nearest tenth.



20. PQ

21. $m\angle P$

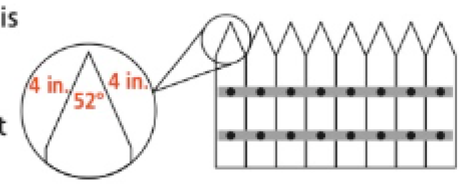
For Exercises 22 and 23, use $\triangle JKL$ and the Law of Cosines to find each measure to the nearest tenth.



22. JK

23. $m\angle J$

24. Use Patterns and Structure How wide is the gate shown, to the nearest inch?



TOPIC

9

Coordinate Geometry

TOPIC ESSENTIAL QUESTION

How can geometric relationships be proven by applying algebraic properties to geometric figures represented in the coordinate plane?

Topic Overview

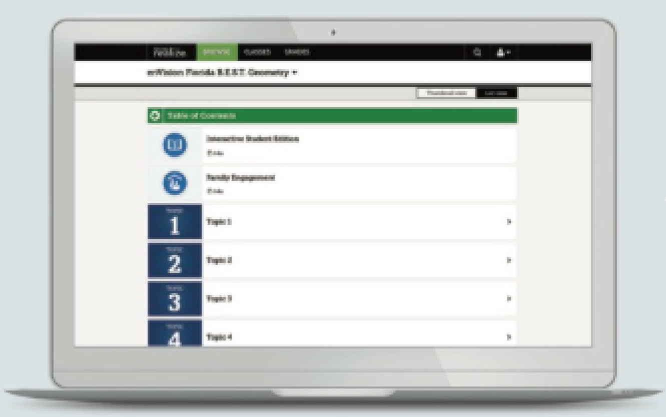
enVision® STEM Project:
Design a Solar Collector

9-1 Polygons in the Coordinate Plane
GR.3.2, GR.3.3, GR.3.4, GR.4.4, MTR.1.1, MTR.4.1, MTR.5.1

Mathematical Modeling in 3 Acts:
You Be the Judge
GR.3.3, MTR.7.1

9-2 Proofs Using Coordinate Geometry
GR.1.3, GR.3.2, GR.3.3, MTR1.1, MTR.3.1, MTR.5.1

9-3 Circles in the Coordinate Plane
GR.7.2, GR.7.3, GR.3.2, MTR.2.1, MTR.4.1, MTR.6.1



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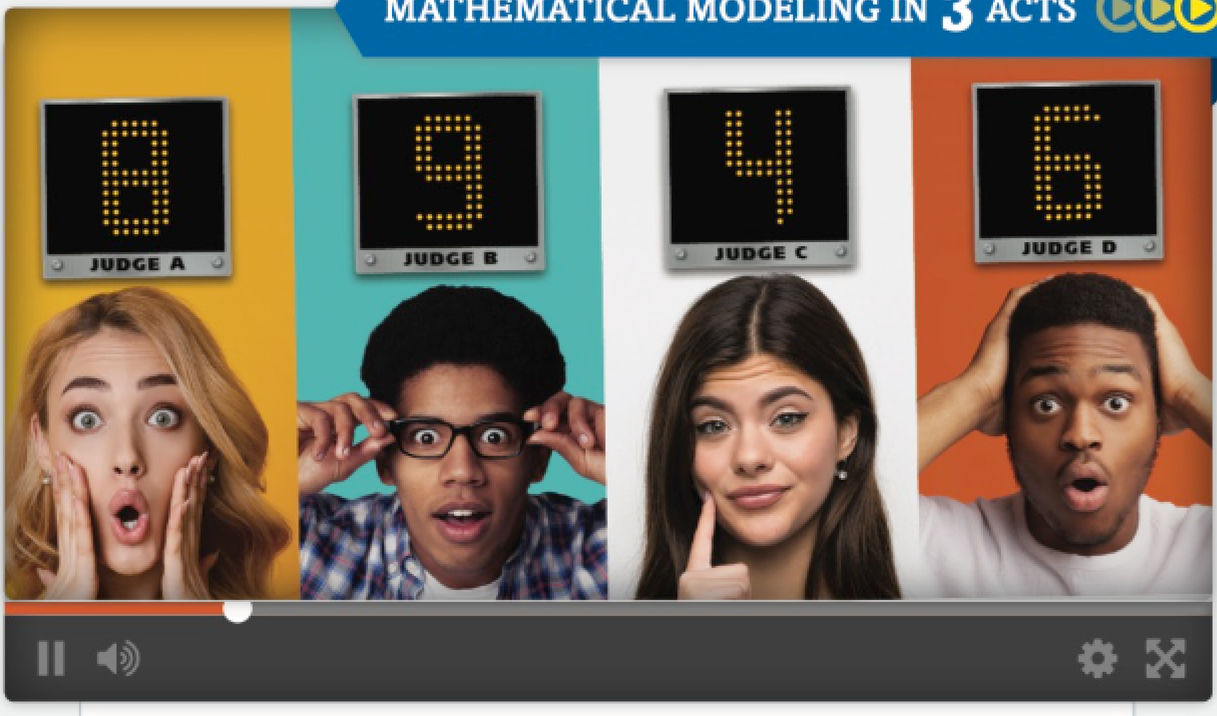
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



You Be the Judge

Have you ever been a judge in a contest or competition? What criteria did you use to decide the winner? If you were one of many judges, did you all agree on who should win?

Often there is a set of criteria that judges use to help them score the performances of the contestants. Having criteria helps all of the judges be consistent regardless of the person they are rating. Think of this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you.*

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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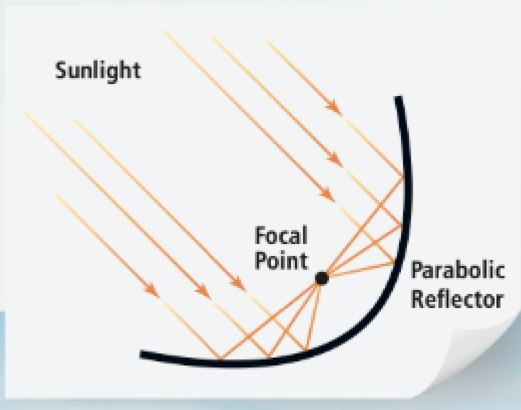
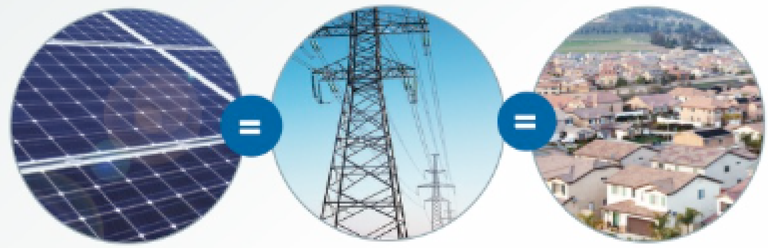
Did You Know?

Solar reflectors are made of mirrors or pieces of glass in many shapes and sizes. **Parabolic reflectors** collect the sun's rays from a wide area and **focus them on a small area**, concentrating the energy.





258,048 **mirrors** = 100 **megawatts** = 20,000 **homes**

The world's largest power station, the SHAMS 1 in the United Arab Emirates, uses **258,048 mirrors**. That's enough to generate **100 megawatts** of electricity per day and power **20,000 homes**.

In 2016, the United States produced more than 40 billion kilowatt-hours of **solar energy**, 40 times more than it did a decade earlier.

Your Task: Design a Solar Collector

Giant power plants are not the only place to see solar collectors—you might find a water purifier made from a single 6 ft-x-4 ft mirror in a neighbor's back yard! You and your classmates will design a solar collector for use in your school or community.

9-1 Polygons in the Coordinate Plane

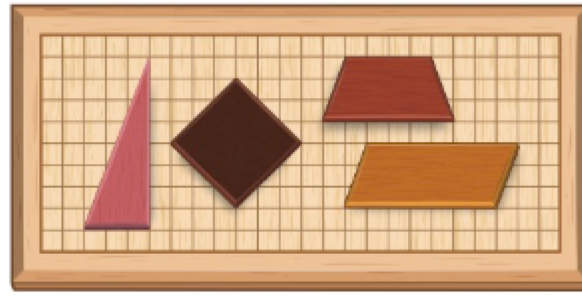
I CAN… use the coordinate plane to analyze geometric figures.

MA.912.GR.3.2–Given a mathematical context, use coordinate geometry to classify or justify definitions, properties and theorems involving circles, triangles or quadrilaterals. **Also GR.3.3, GR.3.4, GR.4.4**

MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

EXPLORE & REASON

Players place game pieces on the board shown and earn points from the attributes of the piece placed on the board.



- 1 point for a right angle
- 2 points for a pair of parallel sides
- 3 points for the shortest perimeter

A. Which game piece is worth the greatest total points? Explain.

B. Analyze and Persevere Describe a way to determine the perimeters that is different from the way you chose. Which method do you consider better? Explain.

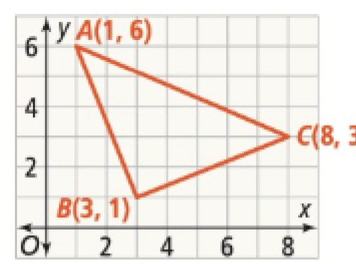
ESSENTIAL QUESTION

How are properties of geometric figures represented in the coordinate plane?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Connect Algebra and Geometry Through Coordinates

What formulas can you use to identify properties of figures on the coordinate plane?



A. Which formula can you use to find AB?

Use the Distance Formula to find segment length.

$$AB = \sqrt{(3-1)^2 + (1-6)^2} = \sqrt{29}$$

B. What point bisects $\overline{AB}$?

Use the Midpoint Formula to find a segment bisector.

$$\text{midpoint of } \overline{AB} = \left(\frac{1+3}{2}, \frac{6+1}{2}\right) = \left(2, \frac{7}{2}\right)$$

C. Why do slopes of $\overline{AB}$ and $\overline{BC}$ show that $m\angle ABC = 90°$?

Use the slopes of the two segments to show that they are perpendicular.

$$\text{slope of } \overline{AB} = \frac{1-6}{3-1} = -\frac{5}{2}$$

$$\text{slope of } \overline{BC} = \frac{3-1}{8-3} = \frac{2}{5}$$

The product of the slopes is -1. So $\overline{AB} \perp \overline{BC}$, and $m\angle ABC = 90°$.

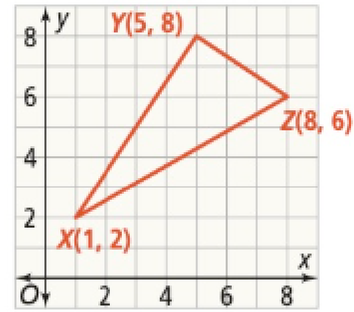
COMMON ERROR

Recall that the slope of a line is the ratio of the difference in the y-coordinates to the difference in the x-coordinates. Be careful not to reverse the ratio.

Try It! **1.** Given $\triangle ABC$ in Example 1, what is the length of the line segment connecting the midpoints of $\overline{AC}$ and $\overline{BC}$?

EXAMPLE 2 Classify a Triangle on the Coordinate Plane

A. Is △*XYZ* equilateral, isosceles, or scalene?



Find the length of each side.

$XY = \sqrt{(5-1)^2 + (8-2)^2} = \sqrt{52}$

$YZ = \sqrt{(8-5)^2 + (6-8)^2} = \sqrt{13}$

$XZ = \sqrt{(8-1)^2 + (6-2)^2} = \sqrt{65}$

No two sides are congruent. The triangle is scalene.

LEARN TOGETHER
Do you seek help when needed? Do you offer help and support others?

B. Is △*XYZ* a right triangle?

If △*XYZ* is a right triangle, then $\overline{XZ}$ is the hypotenuse because it is the longest side, and *XY*, *YZ*, and *XZ* satisfy the Pythagorean Theorem.

$$(\sqrt{52})^2 + (\sqrt{13})^2 \stackrel{?}{=} (\sqrt{65})^2$$

$$65 = 65 \checkmark$$

Triangle *XYZ* is a right triangle.

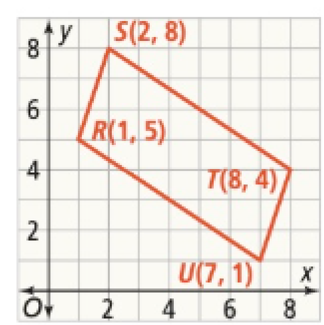
Try It! **2.** The vertices of △*PQR* are *P*(4, 1), *Q*(2, 7), and *R*(8, 5).

a. Is △*PQR* equilateral, isosceles, or scalene? Explain.

b. Is △*PQR* a right triangle? Explain.

EXAMPLE 3 Classify a Parallelogram on the Coordinate Plane

What type of parallelogram is *RSTU*?



Determine whether *RSTU* is a rhombus, a rectangle, or a square. First calculate *ST* and *SR*:

$ST = \sqrt{(2-8)^2 + (8-4)^2} = \sqrt{52}$

$RS = \sqrt{(2-1)^2 + (8-5)^2} = \sqrt{10}$

Since not all side lengths are equal, *RSTU* is not a rhombus or a square.

Check for right angles by finding the slopes.

slope of $\overline{ST} = \frac{4-8}{8-2} = -\frac{2}{3}$

slope of $\overline{RS} = \frac{8-5}{2-1} = 3$

The product of the slopes is not −1, so $\overline{ST}$ and $\overline{RS}$ are not perpendicular. At least one angle is not a right angle, so *RSTU* is not a rectangle. Therefore, quadrilateral *RSTU* is a parallelogram that is neither a square, nor a rhombus, nor a rectangle.

ANALYZE AND PERSEVERE
Consider other formulas you use on the coordinate plane. What are some ways to show that a quadrilateral is not a rectangle or a rhombus?

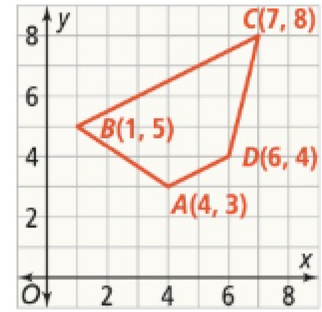
Try It! **3.** The vertices of a parallelogram are *A*(−2, 2), *B*(4, 6), *C*(6, 3), and *D*(0, −1).

a. Is *ABCD* a rhombus? Explain.

b. Is *ABCD* a rectangle? Explain.

EXAMPLE 4 Classify Quadrilaterals as Trapezoids and Kites on the Coordinate Plane

A. Is *ABCD* a trapezoid?



A trapezoid has at least one pair of parallel sides. Use the slope formula to determine if at least one pair of opposite sides is parallel.

slope of $\overline{AB} = \frac{5-3}{1-4} = -\frac{2}{3}$

slope of $\overline{BC} = \frac{8-5}{7-1} = \frac{1}{2}$

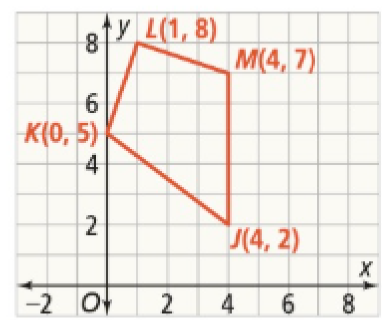
slope of $\overline{CD} = \frac{4-8}{6-7} = \frac{4}{1}$

slope of $\overline{AD} = \frac{4-3}{6-4} = \frac{1}{2}$

The slopes of $\overline{BC}$ and $\overline{AD}$ are equal, so $\overline{BC} \parallel \overline{AD}$. Since at least one pair of opposite sides is parallel, quadrilateral *ABCD* is a trapezoid.

COMMUNICATE AND JUSTIFY
Think about the properties of a trapezoid. Why do you need to find the slopes of opposite sides?

B. Is *JKLM* a kite?



A kite has two pairs of consecutive congruent sides. Use the Distance Formula to find the lengths of the sides.

$JK = \sqrt{(0-4)^2 + (5-2)^2} = 5$

$KL = \sqrt{(1-0)^2 + (8-5)^2} = \sqrt{10}$

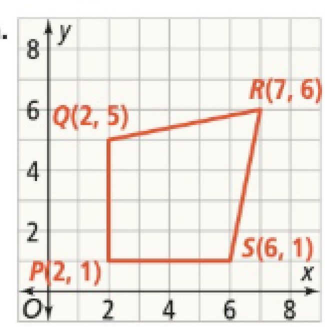
$LM = \sqrt{(4-1)^2 + (7-8)^2} = \sqrt{10}$

$MJ = \sqrt{(4-4)^2 + (2-7)^2} = 5$

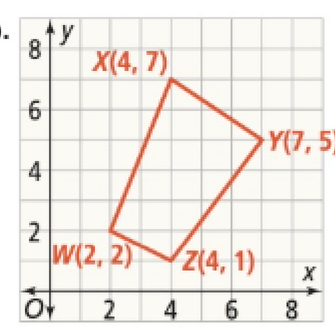
Consecutive pair $\overline{KL}$ and $\overline{LM}$ and consecutive pair $\overline{JK}$ and $\overline{MJ}$ are congruent. Quadrilateral *JKLM* is a kite.

Try It! 4. Is each quadrilateral a kite, trapezoid, or neither?

a.



b.

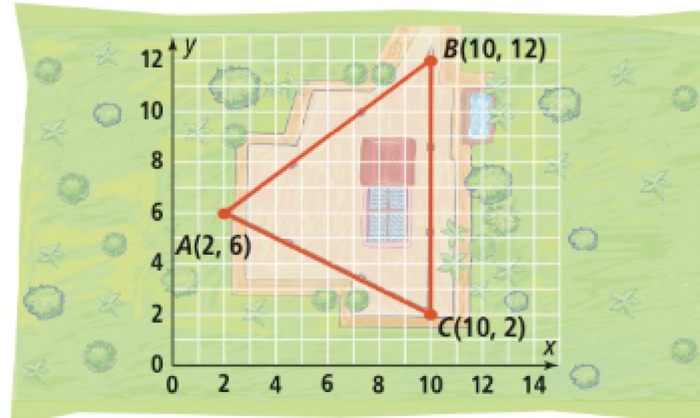


APPLICATION

EXAMPLE 5 Find Perimeter and Area

Dylan draws up a plan to fence in a yard for his chickens. The distance between grid lines is 1 foot.

A. Is 30 feet of fencing enough to enclose the yard?



Find the lengths of the sides.

$AB = \sqrt{(2 - 10)^2 + (6 - 12)^2} = 10$ ft

$BC = \sqrt{(10 - 10)^2 + (12 - 2)^2} = 10$ ft

$AC = \sqrt{(2 - 10)^2 + (6 - 2)^2} = \sqrt{80}$ ft

Find the perimeter of the yard.

$P = 10 + 10 + \sqrt{80}$

≈ 28.9 ft

The perimeter is about 28.9 feet, which is less than 30 feet. Dylan has enough fencing material.

B. For a healthy flock, each chicken needs at least 8 square feet of space. What is the maximum number of chickens Dylan can put in the yard?

COMMUNICATE AND JUSTIFY
Consider the properties of an isosceles triangle. What property of an isosceles triangle justifies that BX is a height of the triangle?

The yard is an isosceles triangle. To find the area, you need the height of the triangle. The height of $\triangle ABC$ is BX, where X is the midpoint of $\overline{AC}$.

Find the midpoint of $\overline{AC}$.

$X = \left(\frac{2 + 10}{2}, \frac{6 + 2}{2}\right) = (6, 4)$

Find the height of $\triangle ABC$.

$BX = \sqrt{(10 - 6)^2 + (12 - 4)^2} = \sqrt{80}$ ft

Then find the area of the yard.

area of $\triangle ABC = \frac{1}{2}(\sqrt{80})(\sqrt{80}) = 40$ ft^2

Divide 40 by 8 to find the number of chickens.

$40 \div 8 = 5$

Dylan can keep as many as 5 chickens in the yard.

Try It! **5.** The vertices of $WXYZ$ are $W(5, 4)$, $X(2, 9)$, $Y(9, 9)$, and $Z(8, 4)$.

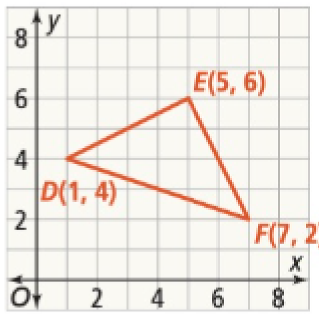
a. What is the perimeter of $WXYZ$?

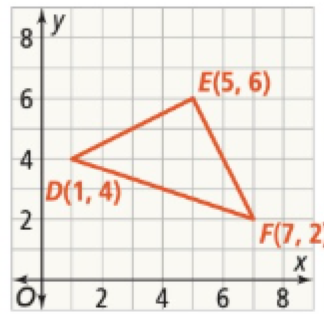
b. What is the area of $WXYZ$?

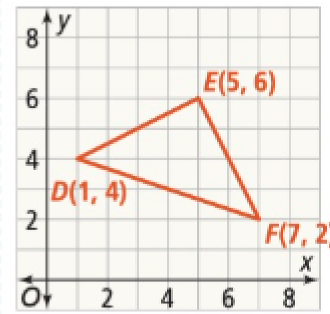
CONCEPT SUMMARY Connecting Algebra and Geometry

	Distance Formula	Slope Formula	Midpoint Formula
WORDS	Use the Distance Formula to find the lengths of segments to classify figures.	Use the Slope Formula to determine whether two lines or segments are parallel or perpendicular.	Use the Midpoint Formula to determine if a point bisects a segment.

GRAPH







NUMBERS

Distance Formula: $DE = \sqrt{(5-1)^2 + (6-4)^2}$ $= \sqrt{20}$

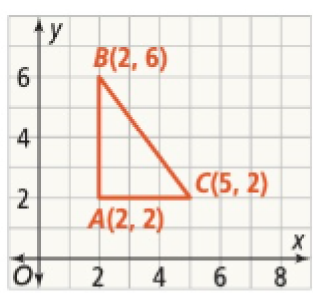
Slope Formula: slope of $\overline{DE} = \frac{6-4}{5-1}$ $= \frac{1}{2}$

Midpoint Formula: midpoint of $\overline{DF}$ $= \left(\frac{1+7}{2}, \frac{4+2}{2}\right) = (4, 3)$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are properties of geometric figures represented in the coordinate plane?

2. **Error Analysis** Chen is asked to describe two methods to find *BC*. Why is Chen incorrect?

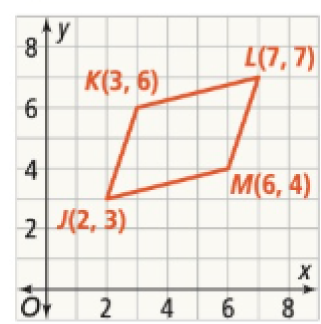


> The only possible method is to use the Distance Formula because you only know the endpoints of $\overline{BC}$.
>
> ✗

3. **Use Patterns and Structure** Describe three ways you can determine whether a quadrilateral is a parallelogram given the coordinates of the vertices.

Do You KNOW HOW?

Use *JKLM* for Exercises 4–6.

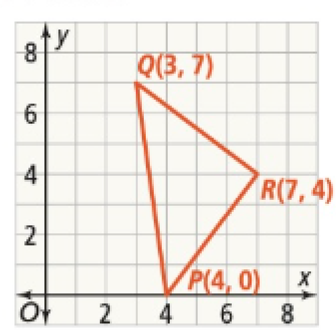


4. What is the perimeter of *JKLM*?

5. What is the relationship between $\overline{JL}$ and $\overline{KM}$? Explain.

6. What type of quadrilateral is *JKLM*? Explain.

Use △*PQR* for Exercises 7 and 8.



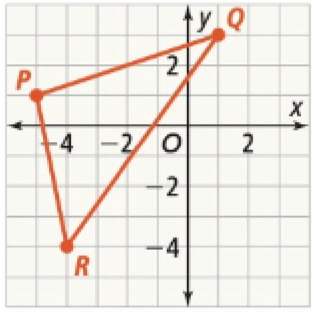
7. What kind of triangle is *PQR*? Explain.

8. What is the area of *PQR*?

PRACTICE & PROBLEM SOLVING

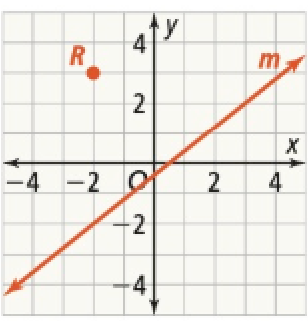
UNDERSTAND

9. Error Analysis What error did Kelley make in finding the area of $\triangle PQR$?

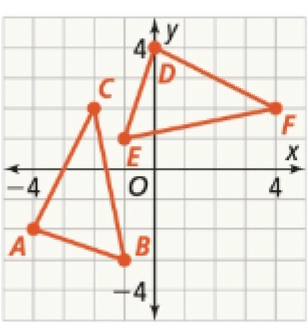


Area $= \frac{1}{2}bh = \frac{1}{2}(PR)(PQ) = \frac{1}{2}\sqrt{26}\sqrt{40}$
The area of ΔPQR is about
16.12 square units. ✗

10. Mathematical Connections Find the equation of the line that passes through point R and is perpendicular to line m.



11. Use Patterns and Structure Prove $\triangle ABC \cong \triangle DEF$.

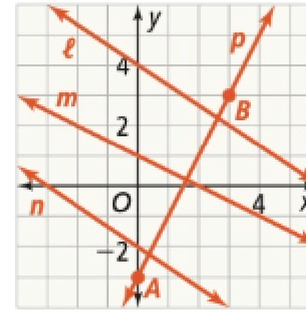


12. Communicate and Justify Given the coordinates of the vertices, how can you show that a quadrilateral is a kite without using the Distance Formula?

13. Higher Order Thinking Let line p be the perpendicular bisector of $\overline{AB}$ that has endpoints and $A(x_1, y_1)$ and $B(x_2, y_2)$. Describe the process for writing a general equation in slope-intercept form for line p.

PRACTICE

Use the figure shown for Exercises 14–17.
SEE EXAMPLE 1



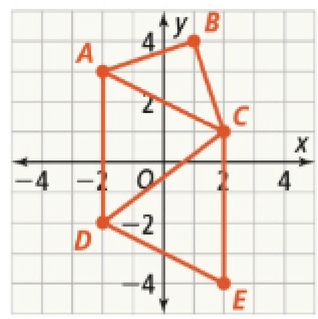
14. Which lines are parallel?

15. Which lines are perpendicular?

16. What is the length of $\overline{AB}$?

17. What is the midpoint of $\overline{AB}$?

Use the figure shown for Exercises 18–23.



18. Is $\triangle ABC$ a scalene, isosceles, or equilateral triangle? Is it a right triangle? Explain.
SEE EXAMPLE 2

19. Is $\triangle ADC$ a scalene, isosceles, or equilateral triangle? Is it a right triangle? Explain.
SEE EXAMPLE 2

20. What type of parallelogram is $ACED$? Explain. SEE EXAMPLE 3

21. What type of quadrilateral is $ABCD$? How do you know? SEE EXAMPLE 4

22. Find the area and perimeter of $\triangle ABC$.
SEE EXAMPLE 5

23. Find the area and perimeter of $ABCD$.
SEE EXAMPLE 6

Three vertices of a quadrilateral are $P(-2, 3)$, $Q(2, 4)$, and $R(1, 0)$. SEE EXAMPLE 3

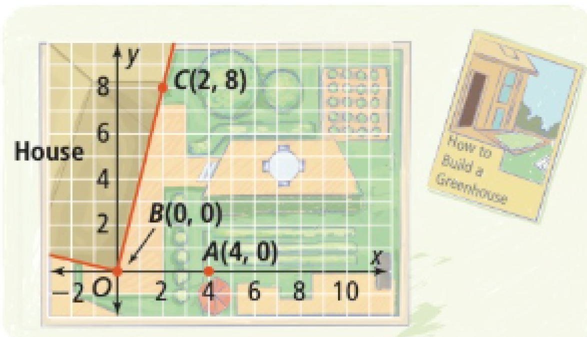
24. Suppose $PQRS$ is a parallelogram. What are the coordinates of vertex S? What type of parallelogram is $PQRS$?

25. Suppose $PQSR$ is a parallelogram. What are the coordinates of vertex S? What type of parallelogram is $PQSR$?

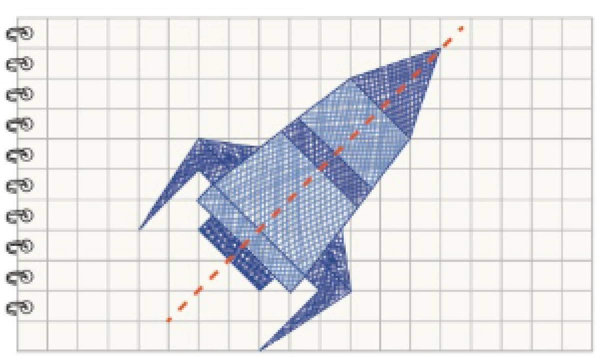
PRACTICE & PROBLEM SOLVING

APPLY

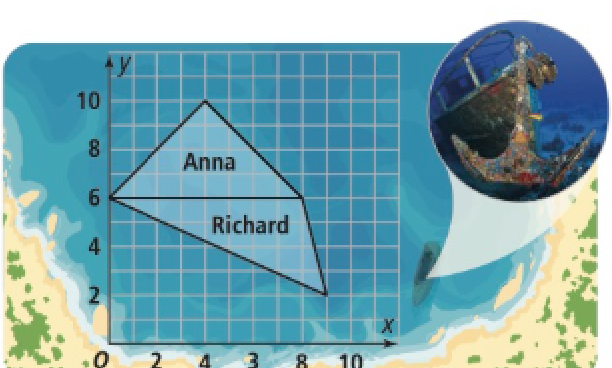
26. Represent and Connect An architect overlays a coordinate grid on her plans for attaching a greenhouse to the side of a house. She wants to locate point *D* so that *ABCD* is a trapezoid and $\overline{CD}$ is perpendicular to the house. What are the coordinates for point *D*?



27. Apply Math Models Yuson thinks the design she made is symmetric across the dashed line she drew. How can she use coordinates to show that her design is symmetric?



28. Communicate and Justify The map shows the regions that Anna and Richard have explored. Each claims to have explored the greater area. Who is correct? Explain.



ASSESSMENT PRACTICE

29. A quadrilateral has vertices at (–1, 4), (1, –2), (4, –1), and (2, 5). What kind of quadrilateral is it? Explain. GR.3.2

30. SAT/ACT Quadrilateral *JKLM* has vertices *J*(1, –2), *K*(7, 1), *L*(8, –1), and *M*(2, –4). Which is the most precise classification of *JKLM*?

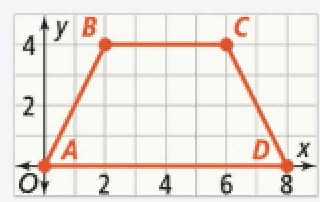
Ⓐ rectangle

Ⓑ rhombus

Ⓒ trapezoid

Ⓓ kite

31. Performance Task Dana draws the side view of a TV stand that has slanted legs. Each unit in his plan equals half of a foot.



Part A Dana thinks his TV stand is in the shape of isosceles trapezoid. Is he correct? Explain.

Part B Dana adds an additional support by connecting the midpoints of the legs. How long is the support?

Part C Dana decides he wants to make the TV stand a half foot higher by placing *B* at (2, 5) and *C* at (6, 5). How much longer will the legs and support connecting the midpoints be?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.3.3–Use coordinate geometry to solve mathematical and real-world geometric problems involving lines, circles, triangles and quadrilaterals.
MA.K12.MTR.7.1

You Be the Judge

Have you ever been a judge in a contest or competition? What criteria did you use to decide the winner? If you were one of many judges, did you all agree on who should win?

Often there is a set of criteria that judges use to help them score the performances of the contestants. Having criteria helps all of the judges be consistent regardless of the person they are rating. Think of this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

9-2 Proofs Using Coordinate Geometry

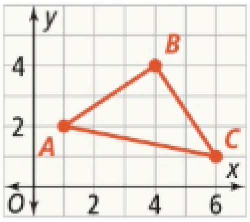
I CAN… prove geometric theorems using algebra and the coordinate plane.

MA.912.GR.3.2–Given a mathematical or real-world context, use coordinate geometry to classify or justify definitions, properties and theorems involving circles, triangles or quadrilaterals. Also GR.1.3, GR.3.3

MA.K12.MTR.1.1, MTR.3.1, MTR.5.1

CRITIQUE & EXPLAIN

Dakota and Jung are trying to show that $\triangle ABC$ is a right triangle. Each student uses a different method.



Dakota

slope of $\overline{AB} = \frac{2}{3}$, slope of $\overline{BC} = -\frac{3}{2}$

slope of $\overline{AB}$ • slope of $\overline{BC} = -1$

Triangle ABC is a right triangle.

Jung

$AB = BC = \sqrt{13}$, $AC = \sqrt{26}$

$(\sqrt{13})^2 + (\sqrt{13})^2 = (\sqrt{26})^2$

Triangle ABC is a right triangle.

A. Did Dakota and Jung both show $\triangle ABC$ is a right triangle? Explain.

B. Represent and Connect If the coordinates of $\triangle ABC$ were changed to (2, 3), (5, 5), and (7, 2), how would each student's method change? Explain.

ESSENTIAL QUESTION

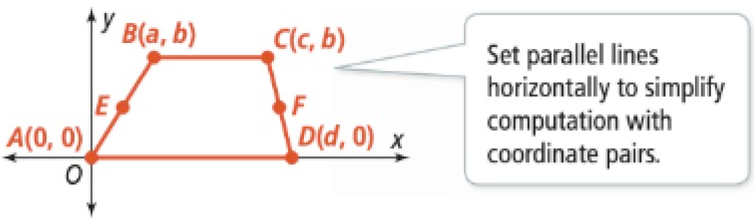
How can geometric relationships be proven algebraically in the coordinate plane?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Plan a Coordinate Proof

How can you use coordinates to prove geometric relationships algebraically? Plan a proof for the Trapezoid Midsegment Theorem.

Draw and label a diagram that names all points to be used in the proof.



USE PATTERNS AND STRUCTURE
Variables are used according to the properties of trapezoids. Why is the proof valid for all trapezoids?

Restate the Trapezoid Midsegment Theorem so the statement can be proved using algebra on the coordinate plane.

Theorem: The midsegment of a trapezoid is parallel to each base and its length is one half the sum of the lengths of the bases.

Restatement: If $\overline{EF}$ is the midsegment of trapezoid $ABCD$, then slope $\overline{EF}$ = slope $\overline{AD}$ = slope $\overline{BC}$ and $EF = \frac{AD + BC}{2}$.

Plan: To show that the midsegment is parallel to the bases, show that their slopes are equal. Then show that the mean of the base lengths is the length of the midsegment.

Try It! 1. Plan a proof to show that the diagonals of a square are congruent and perpendicular.

PROOF

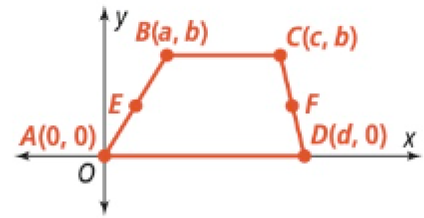
EXAMPLE 2 Write a Coordinate Proof

Write a coordinate proof of the Trapezoid Midsegment Theorem. Use the conditional statement from Example 1 to decide what is given and what is to be proved.

STUDY TIP
Placing one vertex at the origin makes calculations of slopes and distances easier. If a figure has a line of symmetry, align the line of symmetry along one of the axes to simplify calculations.

Given: Trapezoid *ABCD*, with midpoints *E* and *F*

Prove: $\overline{EF} \parallel \overline{AD} \parallel \overline{BC}$ and $EF = \frac{AD + BC}{2}$



Plan: Apply the plan and diagram from Example 1. Use the coordinates in the diagram to show that slope $\overline{EF}$ = slope $\overline{AD}$ = slope $\overline{BC}$, and $EF = \frac{AD + BC}{2}$.

Proof:

Step 1 Use the Midpoint Formula to find the coordinates of points *E* and *F*.

$$E = \left(\frac{0 + a}{2}, \frac{0 + b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right) \qquad F = \left(\frac{c + d}{2}, \frac{b + 0}{2}\right) = \left(\frac{c + d}{2}, \frac{b}{2}\right)$$

Step 2 The slopes of $\overline{AD}$, $\overline{BC}$, and $\overline{EF}$ all equal zero, because the segments are horizontal.

Therefore, $\overline{EF} \parallel \overline{AD} \parallel \overline{BC}$.

Step 3 Determine *AD*, *BC*, and *EF*.

Since the segments are horizontal lines, the lengths are differences of the *x*-coordinates.

$$AD = d - 0 = d \qquad BC = c - a \qquad EF = \frac{c + d}{2} - \frac{a}{2} = \frac{c + d - a}{2}$$

Step 4 Use algebra to show that $\frac{AD + BC}{2} = EF$.

$$\frac{AD + BC}{2} = \frac{d + (c - a)}{2} = \frac{c + d - a}{2} = EF$$

The bases and the midsegment are parallel, and the length of the midsegment is equal to the mean of the base lengths.

Therefore, $\overline{EF} \parallel \overline{AD} \parallel \overline{BC}$ and $EF = \frac{AD + BC}{2}$.

Try It! **2.** Use coordinate geometry to prove that the diagonals of a rectangle are congruent.

PROOF

EXAMPLE 3 Plan and Write a Coordinate Proof

Write a coordinate proof of the Concurrency of Medians Theorem.

Given: $\triangle ABC$ with medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$

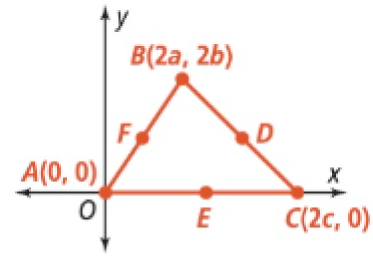
Prove: The medians are concurrent at point P such that

$$AP = \frac{2}{3}AD,\ BP = \frac{2}{3}BE, \text{ and } CP = \frac{2}{3}CF.$$

Plan: Draw and label a triangle in the coordinate plane. Then use the Midpoint Formula to locate the midpoints. Draw two medians and locate the point of intersection P. Use algebra to determine that the medians are concurrent at P. Finally, find the distance from P to each vertex.

COMMON ERROR
Do not confuse the Midpoint and Distance Formulas. The x-coordinate of a midpoint is the average of the x-coordinates of the endpoints and the y-coordinate is the average of the y-coordinates.

Proof: Draw the triangle with the coordinates shown.



Find the coordinates of D, E, and F using the Midpoint Formula.

$D = (a + c, b) \qquad E = (c, 0) \qquad F = (a, b)$

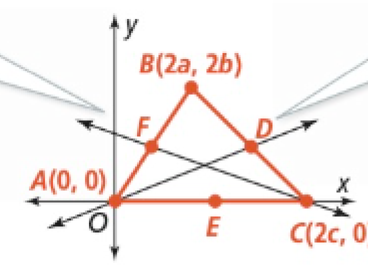
Then find the slopes of the lines containing the medians $\overline{AD}$ and $\overline{CF}$.

$$\text{slope of } \overline{AD} = \frac{b-0}{a+c-0} = \frac{b}{a+c} \qquad \text{slope of } \overline{CF} = \frac{b-0}{a-2c} = \frac{b}{a-2c}$$

Write equations for $\overleftrightarrow{AD}$ and $\overleftrightarrow{CF}$ using point-slope form.

equation for $\overleftrightarrow{CF}$:

$$y - 0 = \frac{b}{a-2c}(x - 2c)$$
$$y = \frac{bx}{a-2c} - \frac{2bc}{a-2c}$$



equation for $\overleftrightarrow{AD}$:

$$y - 0 = \frac{b}{a+c}(x - 0)$$
$$y = \frac{bx}{a+c}$$

Set the expressions for y equal to each other. Solve for x to get $x = \frac{2(a+c)}{3}$.
Then substitute the expression for x into $y = \frac{bx}{a+c}$ to get $y = \frac{2b}{3}$.
Let point P be $\left(\frac{2(a+c)}{3}, \frac{2b}{3}\right)$.

To show that point P is on $\overline{BE}$, find an equation for the line containing $\overline{BE}$. Start by finding the slope of $\overline{BE}$.

$$\text{slope of } \overline{BE} = \frac{0-2b}{c-2a} = -\frac{2b}{c-2a}$$

Then, using point-slope form, an equation for $\overleftrightarrow{BE}$ is $y = -\frac{2bx}{c-2a} + \frac{2bc}{c-2a}$.

Substituting $\frac{2(a+c)}{3}$ for x into the equation results in $y = \frac{2b}{3}$, so point P is on $\overline{BE}$. The three medians are concurrent at P.

Use the Distance Formula to complete the proof in the Try It.

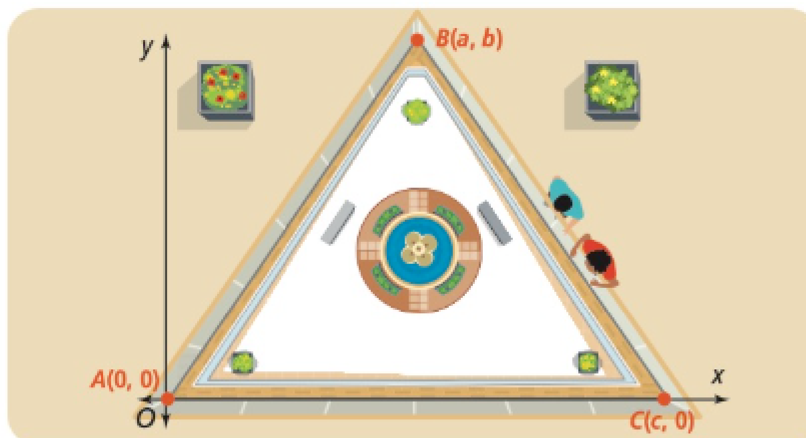
CONTINUED ON THE NEXT PAGE

Try It! **3.** Complete the proof in Example 3. Use the Distance Formula to show that $AP = \frac{2}{3}AD$, $BP = \frac{2}{3}BE$, and $CP = \frac{2}{3}CF$.

APPLICATION

EXAMPLE 4 Use Coordinate Proofs to Solve Problems

An interior designer wants the center of a circular fountain to be equidistant from the corners of a triangular lobby. Where should he place the center of the fountain?



Formulate ◀ The center of the fountain must be at the circumcenter of the triangle. Find the point of intersection of the perpendicular bisectors of two sides of the triangle.

Compute ◀ Determine the intersection of the perpendicular bisectors of $\overline{AC}$ and $\overline{AB}$.

An equation of the perpendicular bisector of $\overline{AC}$ is $x = \frac{c}{2}$.

The perpendicular bisector of $\overline{AB}$ contains the point $\left(\frac{a}{2}, \frac{b}{2}\right)$ and has slope $-\frac{a}{b}$. Its point-slope equation is $y - \frac{b}{2} = -\frac{a}{b}\left(x - \frac{a}{2}\right)$, which simplifies to $y = -\frac{a}{b}x + \frac{a^2 + b^2}{2b}$.

Calculate the intersection of the two lines.

$$y = -\frac{a}{b}x + \frac{a^2 + b^2}{2b}$$

$$y = -\frac{a}{b} \cdot \frac{c}{2} + \frac{a^2 + b^2}{2b}$$

Substitute $\frac{c}{2}$ for x, and then solve for y.

$$y = \frac{a^2 - ac + b^2}{2b}$$

Interpret ◀ The center of the fountain should be at the point $\left(\frac{c}{2}, \frac{a^2 - ac + b^2}{2b}\right)$.

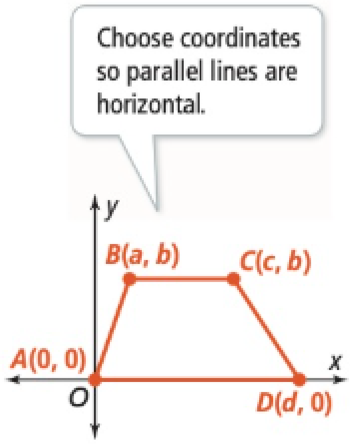
Try It! **4.** A table has a top that is a right triangle and a single support leg. Where should the center of the leg be placed so it corresponds with the center of gravity of the table top? Plan a coordinate geometry proof to find its location.

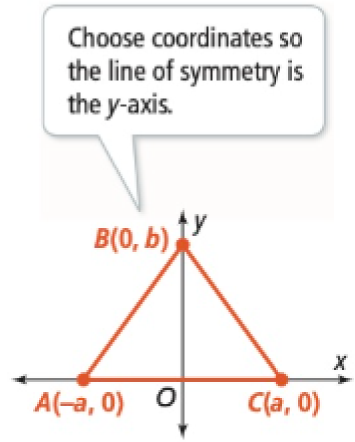
CONCEPT SUMMARY Writing a Coordinate Proof

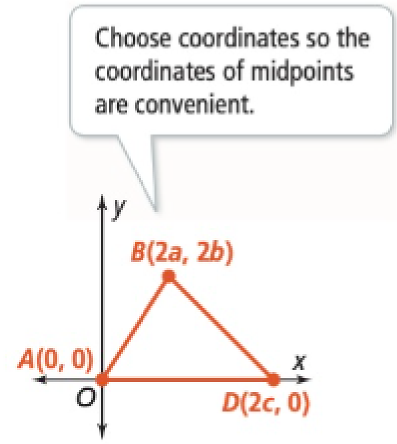
WORDS

- Determine which numerical relationships you must calculate to show the statement is true.
- Draw and label a figure on a coordinate plane. Choose coordinates that simplify computations.
- Calculate the numerical values needed to prove a statement or solve a problem.

DIAGRAMS



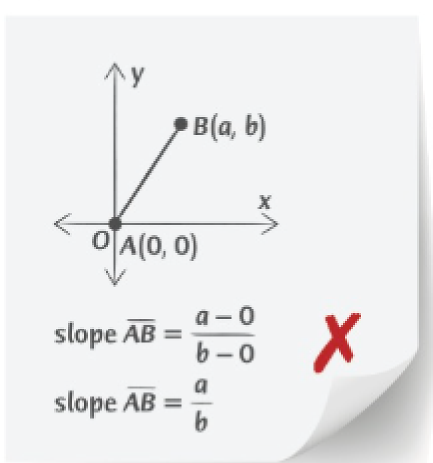




Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can geometric relationships be proven algebraically in the coordinate plane?

2. **Error Analysis** Venetta tried to find the slope of $\overline{AB}$. What is her error?



3. **Communicate and Justify** What is a coordinate geometry proof?

4. **Represent and Connect** Describe why it is important to plan a coordinate proof.

5. **Use Patterns and Structure** What coordinates would you use to describe an isosceles triangle on a coordinate plane? Explain.

Do You KNOW HOW?

For Exercises 6–8, write a plan for a coordinate proof.

6. The diagonals of a rhombus are perpendicular.

7. The area of a triangle with vertices $A(0, 0)$, $B(0, a)$ and $C(b, c)$ is $\frac{ab}{2}$.

8. The lines that contain the altitudes of a triangle are concurrent.

For Exercises 9–12, plan and write a coordinate proof.

9. A point on the perpendicular bisector of a segment is equidistant from the endpoints.

10. The diagonals of a kite are perpendicular.

11. All squares are similar.

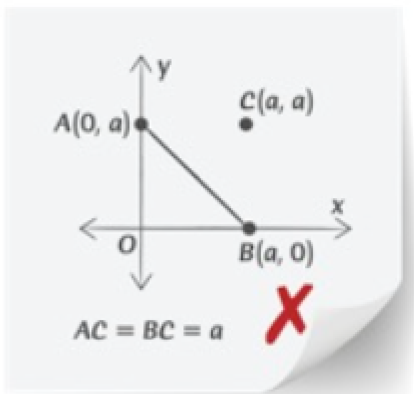
12. The area of a rhombus is half the product of the lengths of its diagonals.

PRACTICE & PROBLEM SOLVING

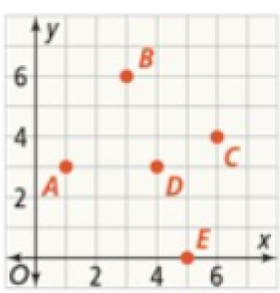
UNDERSTAND

13. Choose Efficient Methods What coordinates would you use to describe an equilateral triangle in the coordinate plane? Explain.

14. Error Analysis Tonya drew a diagram to prove the Perpendicular Bisector Theorem using coordinate geometry. What error did she make in her choice of coordinates?



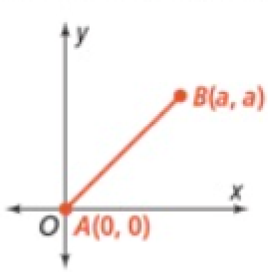
For Exercises 15 and 16, use the graph.



15. Analyze and Persevere Write a plan for a coordinate proof to show that $\triangle ABC$ is a right triangle.

16. Analyze and Persevere Describe how you would prove points B, D, and E are collinear.

17. Mathematical Connections What is the equation of the line containing the perpendicular bisector of $\overline{AB}$?



PRACTICE

For Exercises 18–21, write a plan for a coordinate proof. SEE EXAMPLE 1

18. The diagonals of a parallelogram that is not a rectangle are not congruent.

19. The length of a diameter of a circle is twice that of its radius.

20. The diagonals of a parallelogram bisect each other.

21. The area of $\triangle XYZ$ is twice the area of $\triangle XWZ$, where W is the midpoint of $\overline{YZ}$.

For Exercises 22–25, plan and write a coordinate proof. SEE EXAMPLES 2 AND 3

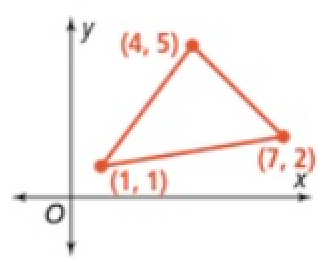
22. The length of a diagonal of a rectangle is the square root of the sum of the squares of the lengths of two adjacent sides.

23. All right triangles with one acute angle measuring 30° are similar.

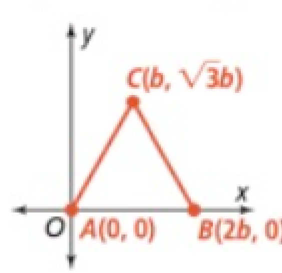
24. At least one diagonal of a kite bisects the other.

25. The length of the median to the hypotenuse of a right triangle is half the length of the hypotenuse.

26. Find the centroid of the triangle by finding the point two thirds of the distance from the vertex on one median. SEE EXAMPLE 4



27. Show that the centroid and circumcenter of an equilateral triangle are the same point.



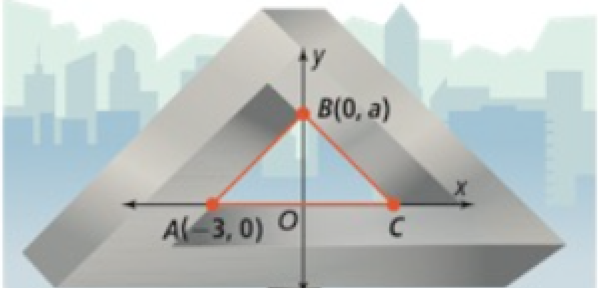
PRACTICE & PROBLEM SOLVING

APPLY

28. **Apply Math Models** Each student in a woodworking class inlays brass wire along the lines connecting the midpoints of adjacent sides of a trivet. A trivet is six square inches. If the wire costs \$0.54 per inch, what is the cost of the wire for making 12 trivets?

29. **Analyze and Persevere** The owner of an animal park wants a quadrilateral trail that connects the four sides of the iscosceles-trapezoid-shaped park, with all sides of the trail the same length. Deon says that if the trail connects the midpoint of each side to the midpoints of the adjacent sides, the trail will be a rhombus. Write a coordinate proof to show that Deon is correct.

30. **Higher Order Thinking** In the sculpture, $\triangle ABC$ is symmetric about the y-axis. Point A is located at $(-3, 0)$, and $\overline{AC}$ is the longest side of $\triangle ABC$. The perimeter of $\triangle ABC$ is 16. What is the value of a?



ASSESSMENT PRACTICE

31. An isosceles right triangle is placed on a coordinate plane. Which possible set of vertices makes it easiest to calculate the length of the sides of the triangle? GR.3.2

Ⓐ $(0, 0), (a, 0), \left(\frac{a}{2}, \frac{a}{2}\right)$

Ⓑ $(0, a), (0, -a), (a, a)$

Ⓒ $(0, 0), (a, 0), (0, a)$

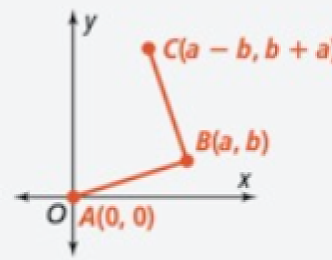
Ⓓ $(0, 0), (0, a), \left(\frac{a}{2}, \frac{a}{2}\right)$

32. **SAT/ACT** Which statements are true?

I. $AB = BC$

II. $\overline{AB} \perp \overline{BC}$

III. $AC = 2a^2 + 2b^2$



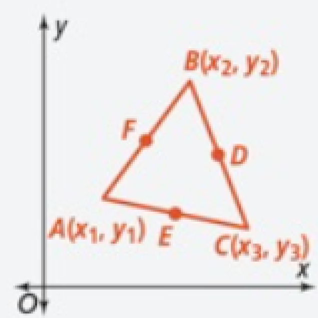
Ⓐ I only

Ⓑ II only

Ⓒ I and II only

Ⓓ I, II, and III

33. **Performance Task** Consider $\triangle ABC$ on the coordinate plane. Points D, E, and F are the midpoints of the sides.



Part A What are the coordinates of D, E, and F?

Part B Prove that the coordinates of the point of concurrency of the medians is the average of the x and y coordinates of the vertices.

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Hint: Apply the Concurrency of Medians Theorem and find the point $\frac{2}{3}$ of the way from A to D.

Part C Explain why it does not make sense to place one of the coordinates at the origin for this proof.

9-3 Circles in the Coordinate Plane

I CAN… use the equations and graphs of circles to solve problems.

MA.912.GR.7.2–Given a mathematical or real-world context, derive and create the equation of a circle using key features.
Also GR.7.3, GR.3.2
MA.K12.MTR.2.1, MTR.4.1, MTR.6.1

CONCEPTUAL UNDERSTANDING

MODEL & DISCUSS

Damian uses an app to find all pizza restaurants within a certain distance of his current location.

A. What is the shape of the region that the app uses to search for pizza restaurants? Explain how you know.

B. What information do you think the app needs to determine the area to search?

C. **Communicate and Justify** If Damian's friend is using the same app from a different location, could the app find the same pizza restaurant for both boys? Explain.

ESSENTIAL QUESTION

How is the equation of a circle determined in the coordinate plane?

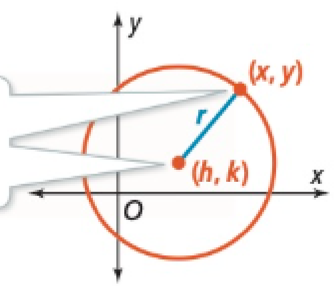
EXAMPLE 1 Derive the Equation of a Circle

What equation defines a circle in the coordinate plane?

Draw a circle with point (h, k) as the center of the circle. Then select any point (x, y) on the circle.

Use the Distance Formula to find the distance r between the two points.

Use variables that can apply to any circle on the coordinate plane.



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$(x - h)^2 + (y - k)^2 = r^2$$

Because the radius is the same from the center to any point (x, y) on the circle, this equation satisfies all points of the circle.

GENERALIZE
What other formula or formulas compares the sum of two squares to a third square? How do these formulas relate?

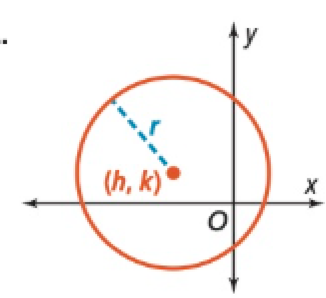
Try It! **1.** What are the radius and center of the circle with the equation $(x - 2)^2 + (y - 3)^2 = 25$?

THEOREM 9-1 Equation of a Circle

An equation of a circle with center (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

PROOF: SEE EXERCISE 13.

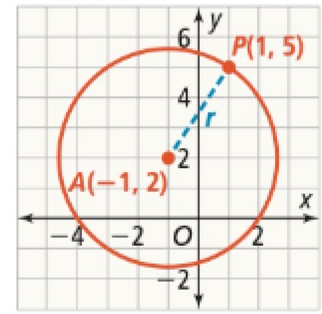
If...



Then... $(x - h)^2 + (y - k)^2 = r^2$

EXAMPLE 2 Write the Equation of a Circle

What is the equation for ⊙A?



The notation ⊙A means a circle with center at point A.

Step 1 Find the radius r.

The radius is the distance from P to A.

$$r = \sqrt{(-1-1)^2 + (2-5)^2} = \sqrt{13}$$

The radius of the circle is $\sqrt{13}$.

Step 2 Use the radius and center to write the equation.

$$(x - h)^2 + (y - k)^2 = r^2$$ Use the equation of a circle.

$$(x - (-1))^2 + (y - 2)^2 = (\sqrt{13})^2$$ Substitute values for h, k, and r.

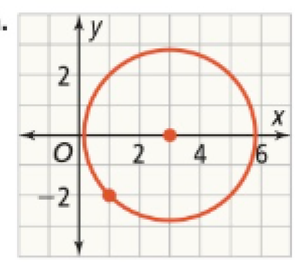
$$(x + 1)^2 + (y - 2)^2 = 13$$

The equation for ⊙A is $(x + 1)^2 + (y - 2)^2 = 13$.

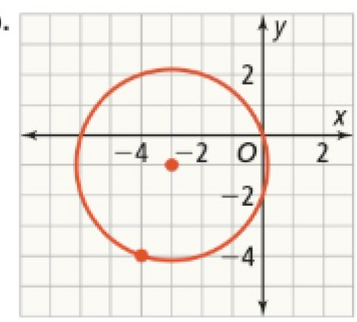
COMMON ERROR
Be careful with the signs of coordinates. Coordinates of the center are subtracted, so if a coordinate is negative, the expression will convert to addition.

Try It! **2.** What is the equation for each circle?

a.



b.



EXAMPLE 3 Determine Whether a Point Lies on a Circle

Circle Q has radius 7 and is centered at the origin. Does the point $(-3\sqrt{2}, 5)$ lie on ⊙Q?

Step 1 Write the equation for ⊙Q.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 0)^2 + (y - 0)^2 = 7^2$$

$$x^2 + y^2 = 49$$

Step 2 Test the point $(-3\sqrt{2}, 5)$ in the equation.

$$(-3\sqrt{2})^2 + 5^2 \stackrel{?}{=} 49$$

$$18 + 25 \stackrel{?}{=} 49$$

$$43 \neq 49$$

The point $(-3\sqrt{2}, 5)$ does not lie on ⊙Q.

STUDY TIP
Remember that to square an expression $a\sqrt{b}$, you square both factors: $(a\sqrt{b})^2 = a^2(\sqrt{b})^2$.

Try It! **3.** Determine whether each point lies on the given circle.

a. $(-3, \sqrt{11})$; circle with center at the origin and radius $2\sqrt{5}$

b. $(6, 3)$; circle with center at $(2, 4)$ and radius $3\sqrt{3}$

EXAMPLE 4 Complete the Square to Find the Center and Radius of a Circle

HAVE A GROWTH MINDSET
How can you take on challenges with positivity?

The equation $x^2 + y^2 - 6x + 8y + 16 = 0$ represents a circle. How can you rewrite the equation to identify the center and radius of the circle? Graph the circle.

Write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$ by using the method of completing the square. Start by grouping terms by variable.

$$(x^2 - 6x) + (y^2 + 8y) = -16$$

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = -16 + 9 + 16$$

Add 9 and 16 to each side to complete the squares.

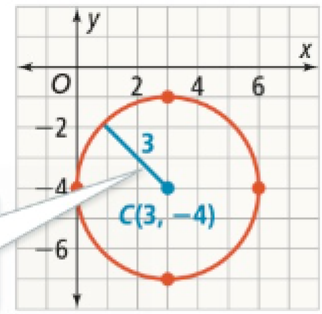
$$(x - 3)^2 + (y + 4)^2 = 9$$

$$(x - 3)^2 + (y - (-4))^2 = 3^2$$

The center (h, k) is $(3, -4)$. The radius r is 3.

Draw the circle.

Plot the center at (3, −4). Then plots points 3 units above, below, left, and right of the center and use the points to draw the circle.



Try It! 4. What is the graph of each circle?

a. $x^2 + 4x + y^2 = 21$

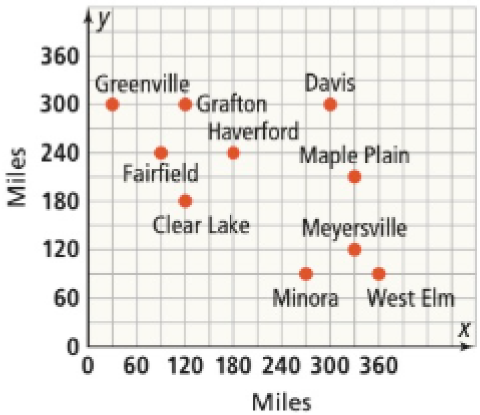
b. $x^2 + 2x + y^2 - 4y = -4$

APPLICATION

EXAMPLE 5 Use the Graph and Equation of a Circle to Solve Problems

Doppler radar detects precipitation within a 90-mile radius. Doppler radar gear in Grafton and Meyersville does not extend to Clear Lake or Davis.

Where can a third Doppler station be placed so all towns are covered?



Draw circles with a 90-mile radius with centers at Grafton and Meyersville.

A circle with a 90-mile radius has a diameter of 180 miles, so 180 miles is the farthest distance between two locations covered by the same radar.

CHECK FOR REASONABLENESS
Think about the parts of the equation for a circle. How could you write an inequality to determine whether a point is within a circle?

Use the Distance Formula to find the distance between Clear Lake and Davis.

$$\sqrt{(300 - 180)^2 + (300 - 120)^2} \approx 216$$

The towns are more than 180 miles apart. Adding one more Doppler radar will not cover all the towns.

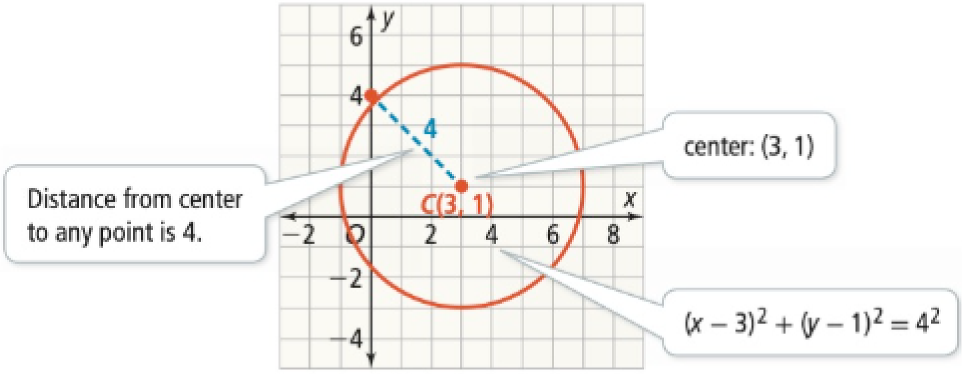
Try It! 5. If one or both of the existing radar stations could be moved to another town, would three radar stations be sufficient to cover all the towns? Explain.

CONCEPT SUMMARY Equations and Graphs of Circles

WORDS A circle is the set of points equidistant from a fixed point. The fixed point is the center.

ALGEBRA $(x - h)^2 + (y - k)^2 = r^2$ where (h, k) is the center and r is the radius.

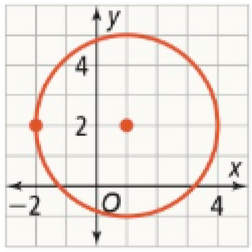
GRAPH



Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is the equation of a circle determined in the coordinate plane?

2. **Error Analysis** Leo says that the equation for the circle is $(x - 1)^2 + (y - 2)^2 = 3$. What is his error?



3. **Communicate and Justify** If you are given the coordinates of the center and one point on a circle, can you determine the equation of the circle? Explain.

4. **Analyze and Persevere** How could you write the equation of a circle given only the coordinates of the endpoints of its diameter?

Do You KNOW HOW?

5. What are the center and radius of the circle with equation $(x - 4)^2 + (y - 9)^2 = 1$?

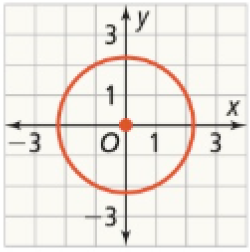
6. What is the equation for the circle with center (6, 2) and radius 8?

7. What are the center and radius of the circle with equation $(x + 7)^2 + (y - 1)^2 = 9$?

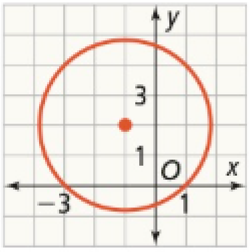
8. What is the equation for the circle with center (–9, 5) and radius 4?

For Exercises 9 and 10, write an equation for each circle shown.

9.



10.



11. Is point (5, –2) on the circle with radius 5 and center (8, 2)?

12. What is the equation for the circle with center (5, 11) that passes through (9, –2)?

PRACTICE & PROBLEM SOLVING

UNDERSTAND

13. Communicate and Justify Write a proof of Theorem 9-1.

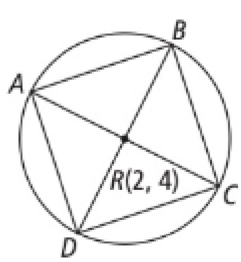
14. Mathematical Connections What are the point(s) of intersection of $x^2 + y^2 = 25$ and $y = 2x - 5$? Graph both equations to check your answer.

15. Error Analysis LaTanya was asked to determine if $(3\sqrt{5}, 4)$ lies on the circle with radius 7 centered at $(0, -2)$. What is her error?

> $x^2 + (y - 2)^2 = 49$
> $(3\sqrt{5})^2 + (4 - 2)^2 \stackrel{?}{=} 49$
> $45 + 4 = 49$
>
> The point $(3\sqrt{5}, 4)$ lies on the circle with radius 7 and center $(0, -2)$. ✗

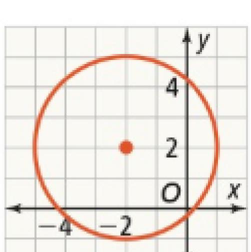
16. Communicate and Justify Describe the graph of $(x - a)^2 + (y - b)^2 = 0$.

17. Use Patterns and Structure If the area of square *ABCD* is 50, what is the equation for ⊙*R*?



18. Analyze and Persevere The points (a, b) and (c, d) are the endpoints of a diameter of a circle. What are the center and radius of the circle?

19. Higher Order Thinking Isabel says the graph shows the circle with center $(-2, 2)$ and radius 3. Nicky says the graph shows all possible centers for a circle that passes through $(-2, 2)$ with radius 3. Which student is correct? Explain.



PRACTICE

For Exercises 20–23, find the center and radius for each equation of a circle. SEE EXAMPLE 1

20. $(x - 4)^2 + (y + 3)^2 = 64$

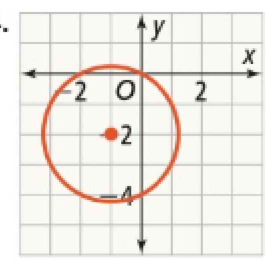
21. $(x + 2)^2 + y^2 = 13$

22. $(x + 5)^2 + (y + 11)^2 = 32$

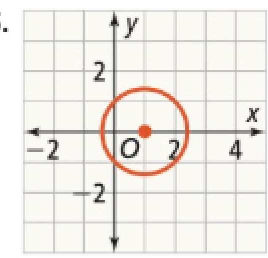
23. $(x - 8)^2 + (y - 12)^2 = 96$

For Exercises 24 and 25, write the equation for the circle shown in each graph. SEE EXAMPLE 2

24.



25.



For Exercises 26–29, write the equation for each circle with the given radius and center. SEE EXAMPLE 2

26. radius: 4, center: (5, 1)

27. radius: 9, center: (−3, 8)

28. radius: $5\sqrt{5}$, center: (2, −4)

29. radius: $\sqrt{13}$, center: (−5, −9)

For Exercises 30–32, determine whether each given point lies on the circle with the given radius and center. SEE EXAMPLE 3

30. (2, 4); radius: 4, center: (−1, 1)

31. $(\sqrt{17}, 8)$; radius: 9, center: (0, 0)

32. (2, 0); radius: $\sqrt{10}$, center: (3, −9)

For Exercises 33–35, graph each equation. SEE EXAMPLES 4 AND 5

33. $x^2 - 10x + y^2 + 2y = -22$

34. $x^2 + y^2 - 2y = 15$

35. $x^2 + 6x + y^2 + 8y = -16$

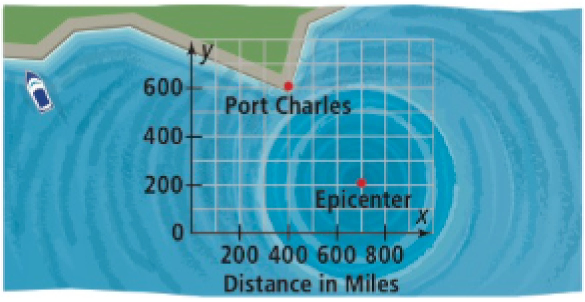
36. The point $(2, b)$ lies on the circle with radius 5 and center $(-1, -1)$. What are the possible values of b?

37. Is (7, 2) inside, outside, or on the circle $(x - 4)^2 + y^2 = 25$? Explain.

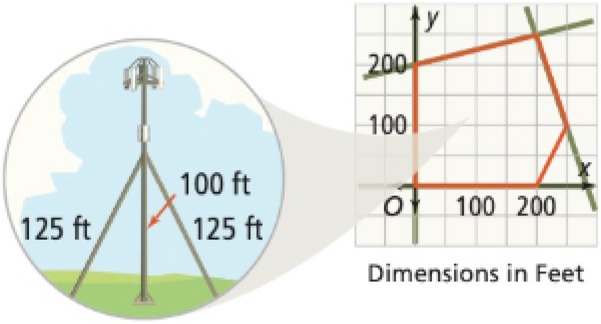
PRACTICE & PROBLEM SOLVING

APPLY

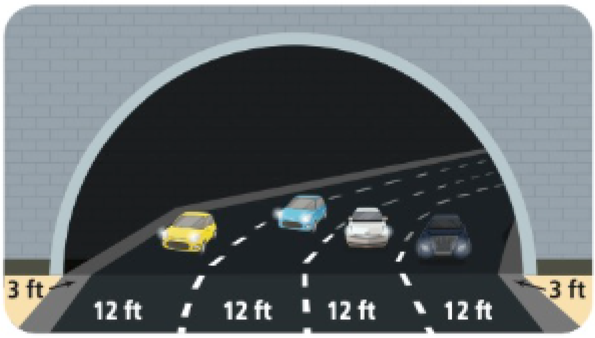
38. Apply Math Models After an earthquake, a circle-shaped tsunami travels outward from the epicenter at an average speed of 420 miles per hour. If the earthquake with the epicenter shown occurred at 5:48 A.M., at what time will the tsunami reach Port Charles? Justify your answer.



39. Analyze and Persevere A cell phone tower is attached to the ground as shown. A circular security fence must be placed around the tower 10 feet from where the guy wires are attached to the ground. Can a cell phone tower be placed in the region enclosed by the red border? If so, what are possible coordinates of the tower?



40. Represent and Connect Semitrailer trucks can be up to 14 feet tall. Should they be allowed in the outer lanes of the semicircular tunnel? Explain.



ASSESSMENT PRACTICE

41. A circle has center (0, 0) and passes through the point (−5, 2). Which other points lie on the circle? Select all that apply. GR.3.3

- ☐ **A.** (0, 6)
- ☐ **B.** $(\sqrt{11}, 3\sqrt{2})$
- ☐ **C.** (2, 5)
- ☐ **D.** (−5, −2)
- ☐ **E.** $(4, -\sqrt{13})$
- ☐ **F.** $(-\sqrt{29}, 0)$

42. SAT/ACT Which equation represents the circle with center (−3, 7) and radius 9?

Ⓐ $(x + 3)^2 + (y - 7)^2 = 3^2$

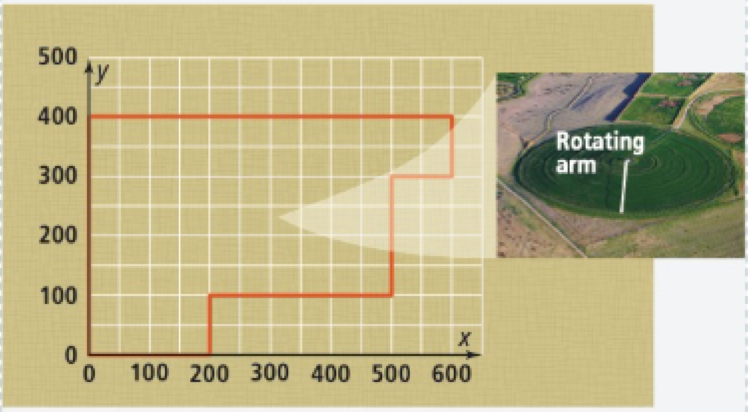
Ⓑ $(x - 3)^2 + (y + 7)^2 = 9^2$

Ⓒ $(x - 7)^2 + (y + 3)^2 = 9^2$

Ⓓ $(x + 3)^2 + (y - 7)^2 = 9^2$

Ⓔ $(x + 7)^2 + (y - 3)^2 = 3^2$

43. Performance Task A farmer can use up to four rotating sprinklers for the field shown. He has ten 50-meter sections that can be combined to form rotating arms with lengths from 50 m to 500 m. The irrigation circles cannot overlap and must not extend beyond the edges of the field. The distance between grid lines is 50 m.



Part A Design an irrigation system for the field that irrigates as much of the field as possible. Draw a sketch of your system. For each sprinkler, give the coordinates of the center of the sprinkler, the radius, the equation, the domain and range.

Part B What is the total area of the field? What is the total area irrigated by your system? What percent of the field does your system irrigate?

TOPIC 9

Topic Review

TOPIC ESSENTIAL QUESTION

1. How can geometric relationships be proven by applying algebraic properties to geometric figures represented in the coordinate plane?

Vocabulary Review

Choose the correct term to complete each sentence.

2. The __________ of a line is the ratio of the vertical change to the horizontal change.
3. A __________ is a set of points equidistant from a point.
4. The __________ of a function is the set of all the *x*-values.
5. Any segment with one endpoint on a circle and the other endpoint at the center of the circle is a __________.
6. The __________ of a function is the set of all the *y*-values.

- center
- circle
- domain
- radius
- range
- slope

Concepts & Skills Review

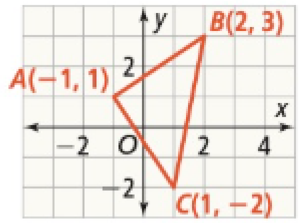
LESSON 9-1 Polygons in the Coordinate Plane

Quick Review

When a geometric figure is represented in a coordinate plane, you can use slope, distance, and midpoints to analyze properties of the figure.

Example

Is $\triangle ABC$ an isosceles triangle? Explain.



A triangle is isosceles if two sides are congruent. Use the Distance Formula to find the side lengths.

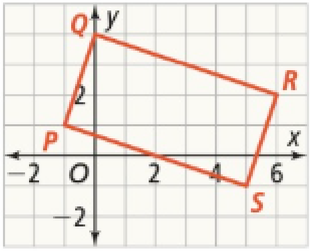
$$AB = \sqrt{(2-(-1))^2 + (3-1)^2} = \sqrt{13}$$
$$BC = \sqrt{(1-2)^2 + (-2-3)^2} = \sqrt{26}$$
$$CA = \sqrt{(1-(-1))^2 + (-2-1)^2} = \sqrt{13}$$

Since $AB = CA$, $\triangle ABC$ is isosceles.

Practice & Problem Solving

For Exercises 7–10, determine whether each figure is the given type of figure.

7. $F(-2, 4)$, $G(0, 0)$, $H(3, 1)$; right triangle
8. $A(7, 2)$, $B(3, -1)$, $C(3, 4)$; equilateral triangle
9. $J(-4, -4)$, $K(-7, 0)$, $L(-4, 4)$, $M(-1, 0)$; rhombus
10. What are the area and perimeter of *PQRS*?



11. **Analyze and Persevere** Parallelogram *WXYZ* has coordinates $W(a, b)$, $X(c, d)$, $Y(f, g)$, and $Z(h, j)$. What equation can you use to determine whether *WXYZ* is a rhombus? Explain.

LESSON 9-2 Proofs Using Coordinate Geometry

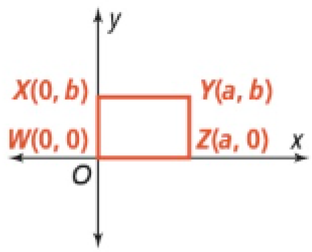
Quick Review

To prove theorems using coordinate geometry, place the figure on the coordinate plane. Use slope, midpoint, and distance to write an algebraic proof.

Example

Prove that the diagonals of a rectangle are congruent.

Place a rectangle on a coordinate plane with one vertex at the origin and two sides along the axes.



$$WY = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$
$$XZ = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2}$$

Since $WY = XZ$, the diagonals of a rectangle are congruent.

Practice & Problem Solving

For Exercises 10–12, give the coordinates of each missing vertex.

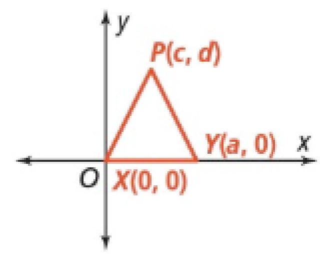
12. *ABCD* is a parallelogram; $A(0, 0)$, $B(p, q)$, $D(t, 0)$

13. *JKLM* is a kite; $J(0, 0)$, $K(a, b)$, $L(0, c)$

14. *WXYZ* is a rhombus; $W(0, 0)$, $Y(0, h)$, $Z(j, k)$

15. **Communicate and Justify** If you are given the coordinates of a quadrilateral, how can you prove that the quadrilateral is an isosceles trapezoid?

16. The diagram shows a fenced garden area, where $PX = PY$. The gardener is dividing the garden with a fence from *P* to the midpoint of $\overline{XY}$. Will the new fence be perpendicular to $\overline{XY}$? Use coordinate geometry to explain.



LESSON 9-3 Circles in the Coordinate Plane

Quick Review

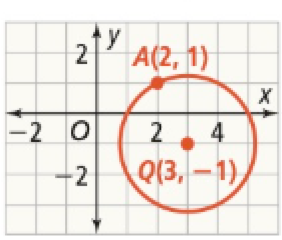
The equation of a circle in the coordinate plane is

$$(x-h)^2 + (y-k)^2 = r^2$$

where (h, k) is the center of the circle and r is the radius.

Example

What is the equation of ⊙*Q*?



The center of the circle is $(3, -1)$, and the radius is *QA*.

$$QA = \sqrt{(2-3)^2 + (1-(-1))^2} = \sqrt{5}$$

So, the equation of ⊙*Q* is $(x-3)^2 + (y+1)^2 = 5$.

Practice & Problem Solving

For Exercises 17–19, write the equation for the circle with the given center and radius.

17. center: $(0, 0)$, radius: 9

18. center: $(-2, 3)$, radius: 5

19. center: $(-5, -8)$, radius: $\sqrt{13}$

For Exercises 20 and 21, determine whether the given point lies on the circle with the given center and radius.

20. $(-3, 0)$; center: $(-5, 2)$, radius: $2\sqrt{2}$

21. $(11, -1)$; center: $(4, 4)$, radius: $6\sqrt{2}$

22. Suppose that $\overline{AB}$, with $A(1, 15)$ and $B(13, -1)$, and $\overline{CD}$, with $C(15, 13)$ and $D(-1, 1)$, are diameters of ⊙*T*. What is the equation of ⊙*T*?

23. **Communicate and Justify** Is it possible to write the equation of a circle given only two points on the circle? Explain.

TOPIC
10

Circles

TOPIC ESSENTIAL QUESTION

When a line or lines intersect a circle how are the figures formed related to the radius, circumference, and area of the circle?

Topic Overview

enVision® STEM Project:
Design Space Cities

10-1 Arcs and Sectors
GR.6.4, GR.6.2, MTR.5.1, MTR.3.1, MTR.6.1

10-2 Lines Tangent to a Circle
GR.6.1, GR.3.3, GR.5.5, MTR.5.1, MTR.4.1, MTR.6.1

Mathematical Modeling in 3 Acts:
Earth Watch
GR.6.1, MTR.7.1

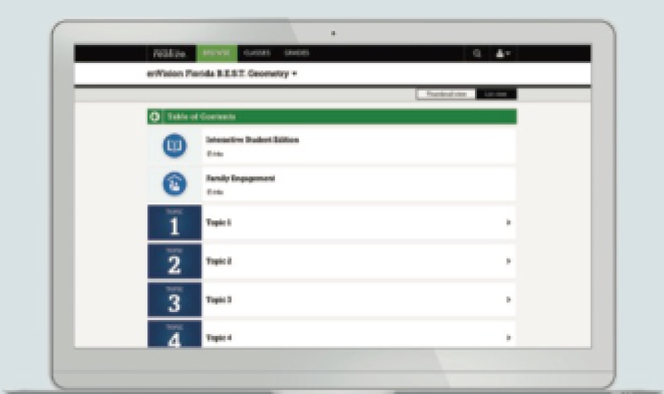
10-3 Chords
GR.6.1, GR.5.4, MTR.4.1, MTR.1.1, MTR.7.1

10-4 Inscribed Angles
GR.6.1, GR.6.2, GR.6.3, MTR.2.1, MTR.1.1, MTR.5.1

10-5 Secant Lines and Segments
GR.6.1, GR.6.2, MTR.1.1, MTR.4.1, MTR.5.1

Topic Vocabulary

- arc length
- central angle
- chord
- inscribed angle
- intercepted arc
- major arc
- minor arc
- point of tangency
- radian
- secant
- sector of a circle
- segment of a circle
- tangent to a circle



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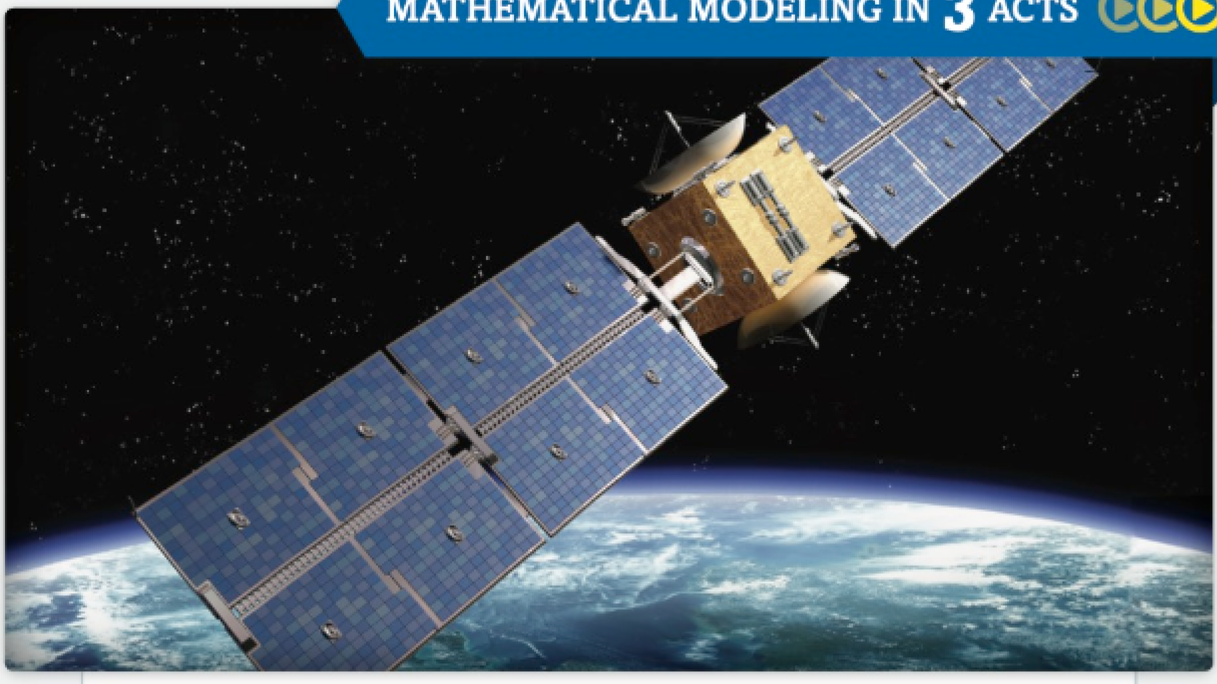
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

MATHEMATICAL MODELING IN 3 ACTS



Earth Watch

Scientists estimate that there are currently about 3,000 operational man-made satellites orbiting Earth. These satellites serve different purposes, from communication to navigation and global positioning. Some are weather satellites that collect environmental information.

The International Space Station is the largest man-made satellite that orbits Earth. It serves as a space environment research facility, and it also offers amazing views of Earth. Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision**® ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you.*

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

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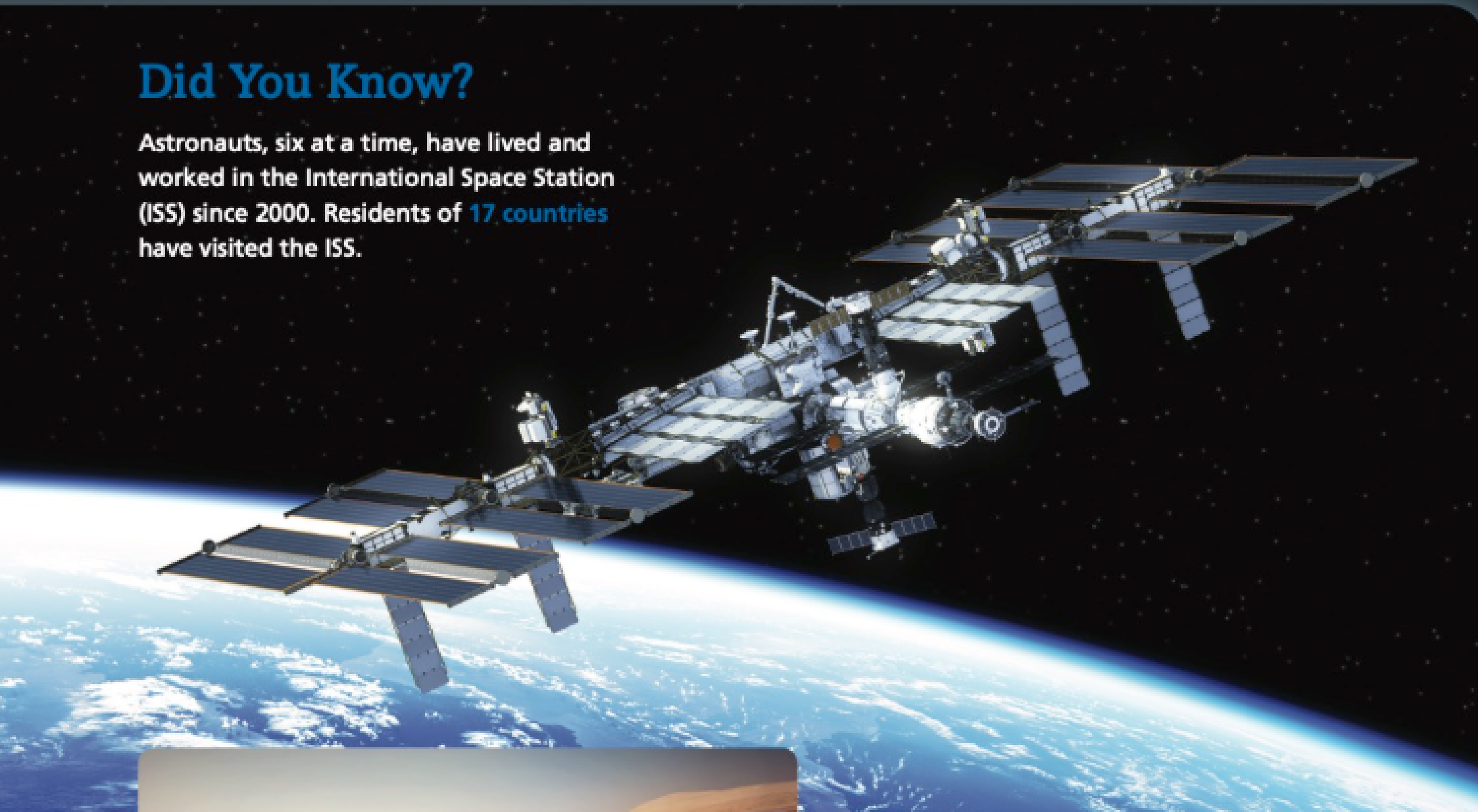
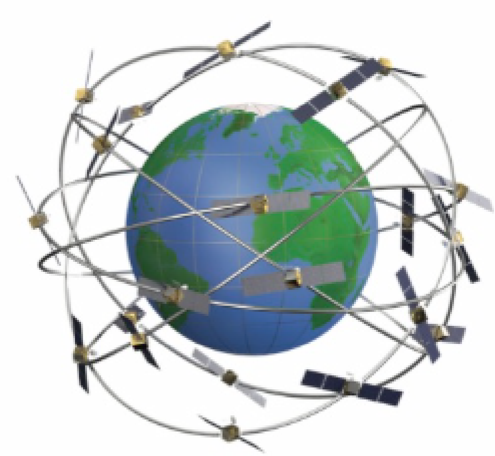
Did You Know?

Astronauts, six at a time, have lived and worked in the International Space Station (ISS) since 2000. Residents of **17 countries** have visited the ISS.



The size of a football field, ISS circles the Earth every 90 minutes at an **altitude of 248 miles** and a speed of about **17,500 miles per hour.**



At its closest, the planet Mars is **150 times as far from Earth** as the Moon is. Despite the distance, the United States and Russia have been landing spacecraft and scientific instruments on Mars for several decades.

Your Task: Design Space Cities

Suppose it's 500 years in the future. Space stations the size of small cities are journeying through space. Use trigonometry and the geometry of circles to calculate the measurements of two of these stations, then design, measure, and describe a group of three "space cities."

10-1 Arcs and Sectors

I CAN… find arc length and sector area of a circle and use them to solve problems.

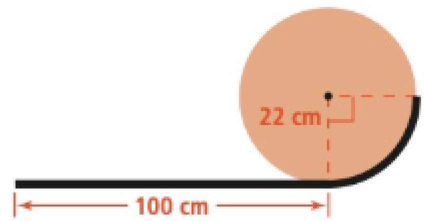
VOCABULARY

- arc length
- central angle
- intercepted arc
- major arc
- minor arc
- radian
- sector of a circle
- segment of a circle

MA.912.GR.6.4–Solve mathematical and real-world problems involving the arc length and area of a sector in a given circle. **Also GR.6.2**
MA.K12.MTR.5.1, MTR.3.1, MTR.6.1

EXPLORE & REASON

Darren bends a piece of wire using a circular disc to make the shape as shown.



A. How long does the piece of wire need to be to make the shape? Explain.

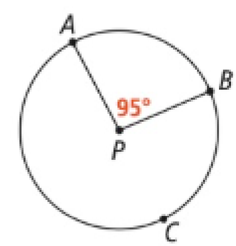
B. **Communicate and Justify** What information do you think is needed to find part of the circumference of a circle? Justify your answer.

ESSENTIAL QUESTION

How are arc length and sector area related to circumference and area of a circle?

EXAMPLE 1 Relate Central Angles and Arc Measures

What are $m\widehat{AB}$ and $m\widehat{ACB}$?

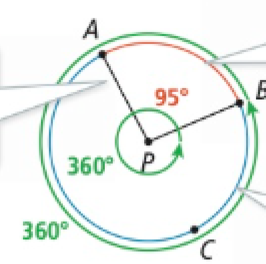


A **central angle** of a circle is an angle formed by two radii with the vertex at the center of the circle. Angle *APB* is a central angle.

A central angle creates two intercepted arcs. An **intercepted arc** is the part of a circle that lies between two segments, rays, or lines that intersect the circle.

A central angle and its intercepted minor arc have equal measure.

$\angle APB$ is a central angle, and $\widehat{AB}$ is its corresponding intercepted arc.



A **minor arc** of a circle is an arc that is smaller than a semicircle. $\widehat{AB}$ is a minor arc.

A **major arc** of a circle is an arc that is larger than a semicircle. $\widehat{ACB}$ is a major arc.

STUDY TIP

A minor arc may be written with just two letters, as in $\widehat{AB}$. Use the third point between the endpoints of an arc to name a major arc, as in $\widehat{ACB}$.

Find $m\widehat{AB}$.

$$m\widehat{AB} = m\angle APB = 95$$

The degree measure of $\widehat{AB}$ is equal to the measure of its corresponding central angle $\angle APB$.

Find $m\widehat{ACB}$.

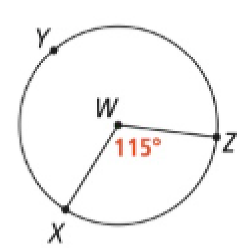
$$m\widehat{ACB} = 360 - 95 = 265$$

Try It!

1. Use $\odot W$.

a. What is $m\widehat{XZ}$?

b. What is $m\widehat{XYZ}$?



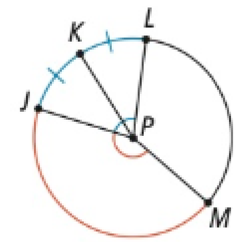
CONCEPT Arc Measure

The measure of an arc is equal to the measure of its corresponding central angle.

$m\widehat{JM} = m\angle JPM$

Congruent central angles intercept congruent arcs, and congruent arcs are intercepted by congruent central angles.

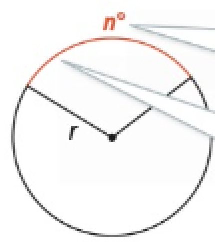
$\angle JPK \cong \angle KPL$ $\qquad \widehat{JK} \cong \widehat{KL}$



CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Relate Arc Length to Circumference

A. How do you find the length *s* of an arc measured in degrees?



The *measure* of an arc is a fraction of 360°

The **arc length** is a fraction of the circumference.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\text{arc measure (degrees)}}{360}$$

$$\frac{s}{2\pi r} = \frac{n}{360}$$

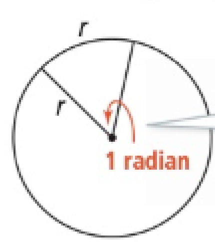
$$s = \frac{n}{360} \cdot 2\pi r$$

Use a proportion to represent the relationship between arc length s, radius r, and arc measure n.

The formula to find the length of an arc is $s = \frac{n}{360} \cdot 2\pi r$.

B. How do you find the length *s* of an arc measured in radians?

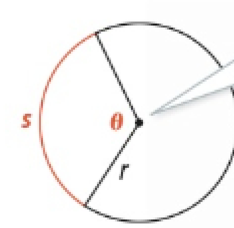
Besides degrees, angle measures can be expressed in *radians*.



A **radian** is equal to the measure of a central angle that intercepts an arc with length equal to the radius of the circle.

Circumference is $2\pi r$, which is 2π arcs of length r. Since each arc of length r corresponds to 1 radian, there are 2π radians in a circle. So, 2π radians is equivalent to 360°.

To find the arc length, use the following proportion.



The variable theta (θ) is often used for angles measured in radians.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\text{arc measure (radians)}}{2\pi}$$

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$

$$s = \frac{\theta}{2\pi} \cdot 2\pi r = \theta r$$

Arc length is proportional to the angle θ, with the radius r as the constant of proportionality.

To find the length of an arc measured in radians, use the formula $s = \theta r$.

STUDY TIP

Remember that the circumference measures the distance around all of the circle and the arc length is the distance around part of the circle.

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Try It! **2. a.** In a circle with radius 4, what is the length of an arc that has a measure of 80? Round to the nearest tenth.

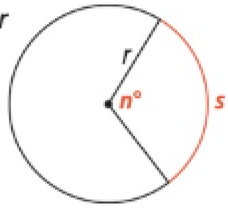
b. In a circle with radius 6, what is the length of an arc that has a measure of π radians? Round to the nearest tenth.

CONCEPT Arc Length

The length s of an arc of a circle is the product of the ratio relating the measure of the central angle in degrees to 360 and the circumference of the circle. The length of the arc is also the product of the radius and the central angle measure in radians.

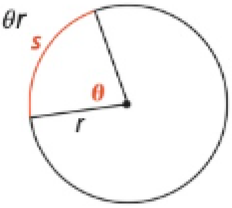
Central angle in degrees:

$s = \frac{n}{360} \cdot 2\pi r$



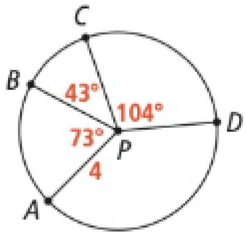
Central angle in radians:

$s = \theta r$



EXAMPLE 3 Apply Arc Length

What is the length of $\widehat{AD}$? Express the answer in terms of π.



Step 1 Find the arc measure.

$$m\widehat{AD} = 360 - m\widehat{AB} - m\widehat{BC} - m\widehat{CD}$$
$$= 360 - 73 - 43 - 104$$
$$= 140$$

Each arc measure is equal to the measure of the corresponding central angle.

Step 2 Find the arc length.

$$s = \frac{n}{360} \cdot 2\pi r$$
$$= \frac{140}{360} \cdot 2\pi(4) = \frac{28}{9}\pi$$

Use the formula for arc length for angles given in degrees.

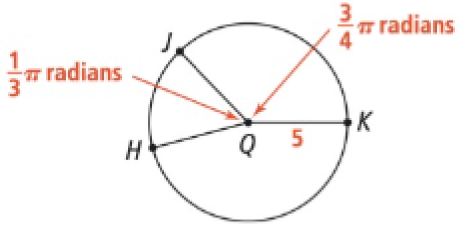
The length of $\widehat{AD}$ is $\frac{28}{9}\pi$.

ANALYZE AND PERSEVERE
Think about when you should express arc lengths in terms of π and when you should give approximate answers. How would you decide?

Try It! **3.** Use $\odot Q$. Express answers in terms of π.

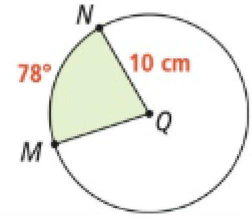
a. What is the length of $\widehat{JK}$?

b. What is the length of $\widehat{HK}$?



EXAMPLE 4 Relate the Area of a Circle to the Area of a Sector

A sector of a circle is the region bounded by two radii and the intercepted arc. What is the area of sector *MQN*?



> **GENERALIZE**
> Compare the formulas for arc length and sector area. What relationships do you see between the arc length and sector area for any given arc?

To find the area of the sector, find $\frac{78}{360}$ of the area of the circle.

$$A = \frac{78}{360} \cdot \pi r^2$$

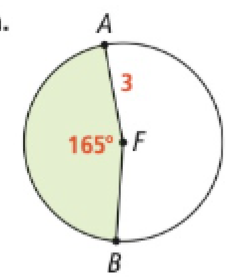
$$= \frac{78}{360} \cdot \pi(10)^2 = \frac{65}{3}\pi$$

In general, the area of a sector is $A = \frac{n}{360} \cdot \pi r^2$, where $n°$ is the measure of the intercepted arc and r is the radius of the circle.

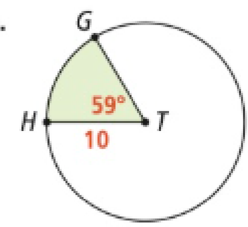
The area of sector *MQN* is $\frac{65}{3}\pi$ cm^2.

Try It! **4.** What is the area of each sector?

a.

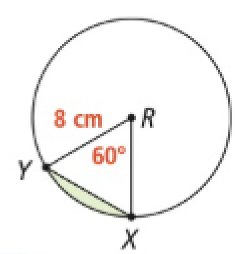


b.



EXAMPLE 5 Find the Area of a Segment of a Circle

A segment of a circle is the part of a circle bounded by an arc and the segment joining its endpoints. What is the area of the shaded region?



To find the area of the segment, subtract the area of the triangle from the area of the sector.

Step 1 Find the area of the sector.

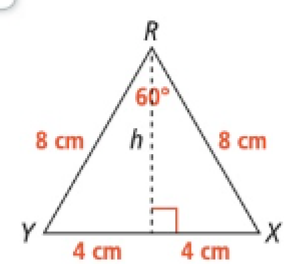
$$A = \frac{n}{360} \cdot \pi r^2 = \frac{32}{3}\pi$$

Use the formula for area of a sector.

> **STUDY TIP**
> To find areas of triangles in circles, you may need to apply trigonometric ratios to find the base and height.

Step 2 Find the area of the triangle.

Since $\overline{RX}$ and $\overline{RY}$ are both radii and the angle between them is 60°, $\triangle RYX$ is equilateral.



Use the Pythagorean Theorem to find h.

$$4^2 + h^2 = 8^2$$

$$h^2 = 8^2 - 4^2$$

$$h = \sqrt{48} = 4\sqrt{3}$$

Find the area of the triangle.

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}(8)(4\sqrt{3}) = 16\sqrt{3}$$

Step 3 Find the area of the segment.

area of segment = area of sector − area of triangle

$$= \frac{32}{3}\pi - 16\sqrt{3} \approx 5.8$$

The area of the shaded region is about 5.8 cm^2.

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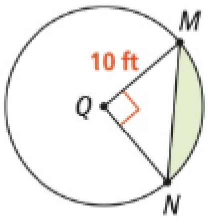
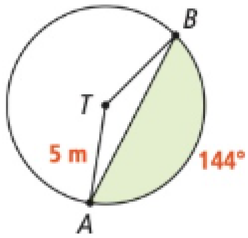
Try It! **5.** What is the area of each segment?

a.



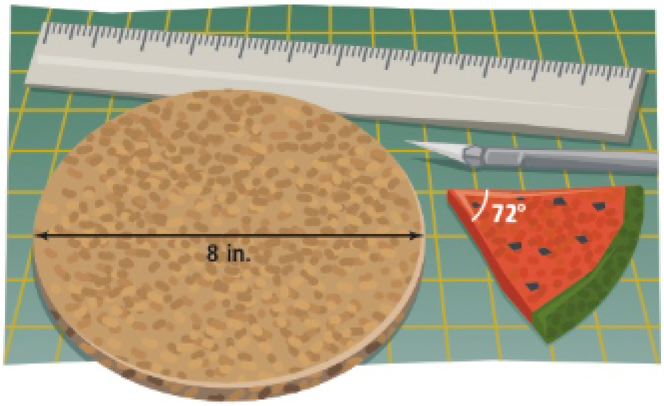
b.



APPLICATION

EXAMPLE 6 Solve Problems Involving Circles

Chen uses sectors cut from circular corkboards to make 18 watermelon coasters to sell at a craft fair.



A. He paints one side of each coaster with special paint. Each jar of paint covers 200 in.^{2}. Will one jar of paint be enough to paint all the coasters?

Find the area of one watermelon coaster. Use 3.14 for π.

$$A = \frac{n}{360} \cdot \pi r^2$$
$$= \frac{72}{360} \cdot \pi(4)^2$$
$$\approx 10.0$$

Use the formula for area of a sector.

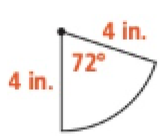
The area of one watermelon coaster is about 10 in.^{2}.

Chen can paint approximately $200 \div 10 = 20$ coasters with one jar of paint, so he has enough paint for 18 coasters.

B. He puts decorative tape around the edge of each coaster. How much tape does he need for each coaster?

The perimeter of the coaster consists of two radii and an arc.

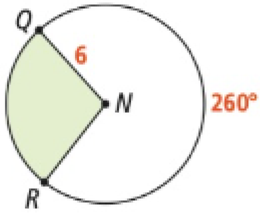
$$P = r + r + \frac{n}{360} \cdot 2\pi r$$
$$= 4 + 4 + \frac{72}{360} \cdot 2\pi(4)$$
$$\approx 13.0$$



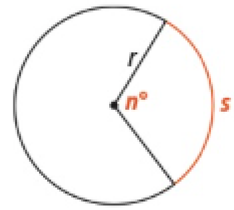
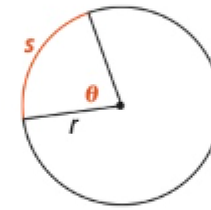
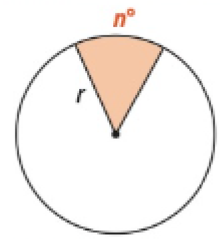
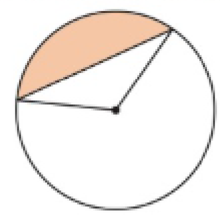
Chen needs about 13 inches of tape for each coaster.

COMMON ERROR
Be careful not to confuse the formula for the area of a sector with the formula for arc length. Remember that the area of a sector is proportional to the area of the circle, and the arc length is proportional to the circumference.

Try It! **6.** What is the area and perimeter of sector *QNR*? Round to the nearest tenth.



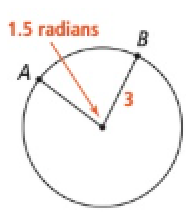
CONCEPT SUMMARY Arc Length and Sector Area

WORDS	DIAGRAMS
Arc Length The arc length is a fraction of the circumference.	**Degrees**  $s = \frac{n}{360} \cdot 2\pi r$ **Radians**  $s = \theta r$
Sector A sector of a circle is the region bounded by two radii and the intercepted arc.	 $A = \frac{n}{360} \cdot \pi r^2$
Segment A segment of a circle is the part of a circle bounded by an arc and the segment joining its endpoints.	 segment area = sector area − triangle area

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are arc length and sector area related to circumference and area of a circle?

2. **Error Analysis** Luke was asked to compute the length of $\overset{\frown}{AB}$. What is Luke's error?



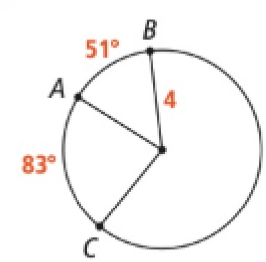
$$S = \frac{n}{360} \cdot 2\pi r$$
$$= \frac{1.5}{360} \cdot 2\pi(3)$$
$$= 0.0785$$

3. **Vocabulary** How can the word *segment* help you remember what a *segment of a circle* is?

4. **Check for Reasonableness** Mercedes says that she can find the area of a quarter of a circle using the formula $A = \frac{1}{4}\pi r^2$. Using the formula for the area of a sector, explain why Mercedes is correct.

Do You KNOW HOW?

For Exercises 5 and 6, find the measures and lengths of each arc. Express the answers in terms of π.

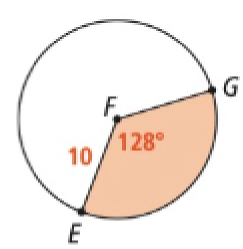


5. $\overset{\frown}{BC}$

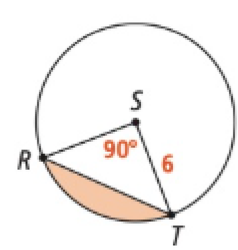
6. $\overset{\frown}{ABC}$

7. Circle P has radius 8. Points Q and R lie on circle P, and the length of $\overset{\frown}{QR}$ is 4π. What is $m\angle QPR$ in radians?

8. What is the area of sector EFG? Express the answer in terms of π.



9. What is the area of the segment? Express the answer in terms of π.

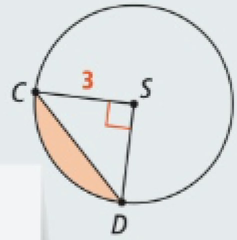


PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Generalize** Is it always true that two arcs with the same length have the same measure? Explain.

11. **Error Analysis** Steve is asked to compute the area of the shaded region. What is his error?



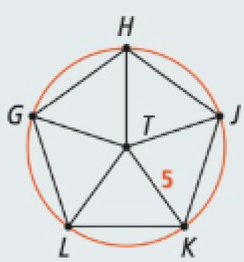
Segment area = sector area − triangle area

$= \frac{90}{360} \cdot 2\pi(3) - \frac{1}{2}(3)(3)$

≈ 0.21

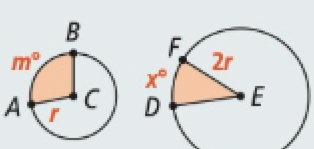
12. **Mathematical Connections** The equation $(x-2)^2 + (y-3)^2 = 25$ represents $\odot T$. Points $X(-2, 6)$ and $Y(-1, -1)$ lie on $\odot T$. What is $m\widehat{XY}$? Explain how you know.

13. **Choose Efficient Methods** Figure *GHJKL* is a regular pentagon. Rounded to the nearest tenth, what percent of the area of $\odot T$ is not part of the area of *GHJKL*? Explain.



14. **Use Patterns and Structure** Explain why the length of an arc with arc measure *a* is proportional to the radius of the circle.

15. **Higher Order Thinking** The areas of sectors *ACB* and *DEF* are equal. What expression gives the value of *x*? Show your work.



PRACTICE

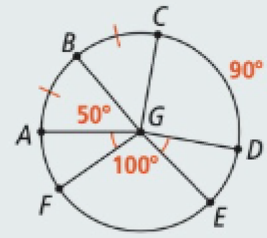
For Exercises 16–19, find each arc measure.
SEE EXAMPLE 1

16. $m\widehat{FE}$

17. $m\widehat{BC}$

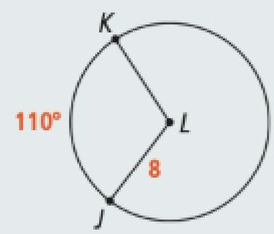
18. $m\widehat{CE}$

19. $m\widehat{CFE}$

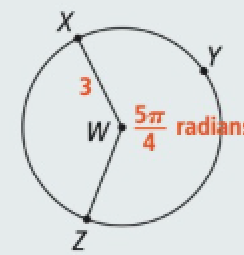


For Exercises 20 and 21, find each arc length in terms of π. SEE EXAMPLES 2 AND 3

20. length of $\widehat{JK}$

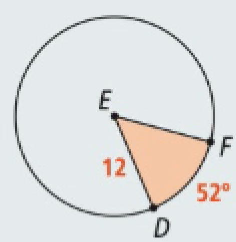


21. length of $\widehat{XYZ}$

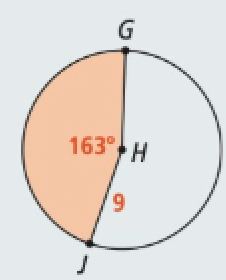


For Exercises 22 and 23, find the area of each sector. Round to the nearest tenth. SEE EXAMPLES 4 AND 6

22. sector *DEF*

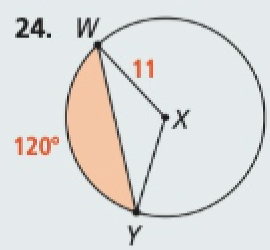


23. sector *GHJ*

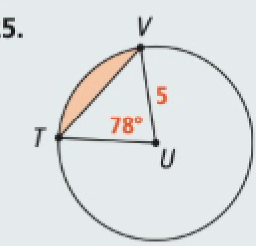


For Exercises 24 and 25, find the area of each segment. Round to the nearest tenth. SEE EXAMPLE 5

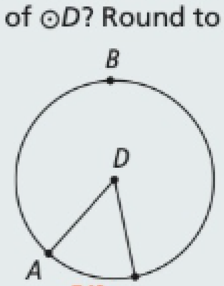
24.



25.



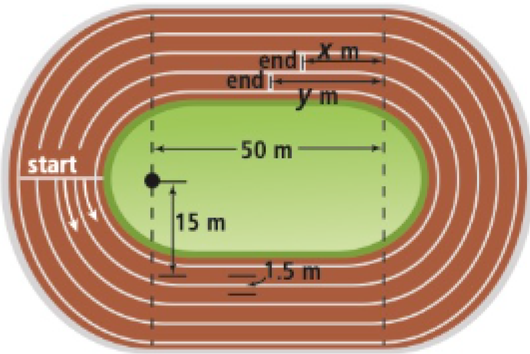
26. The length of $\widehat{ABC}$ is 110 ft. What is the radius of $\odot D$? Round to the nearest tenth.



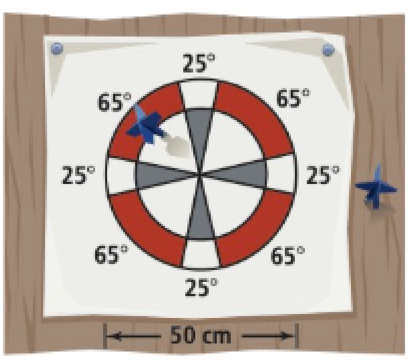
PRACTICE & PROBLEM SOLVING

APPLY

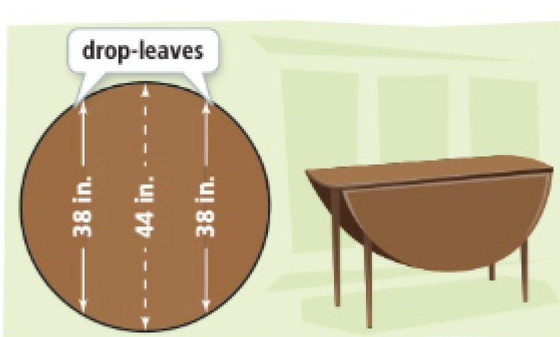
27. Analyze and Persevere Aubrey and Fatima will each run 150 m on the two inside lanes of the track, so the end markers need to be placed correctly. To the nearest hundredth, what are x and y?



28. Represent and Connect Charlie is designing a dart board and wants the red sections to be 25% of the total area. What should be the radius of the inner circle? Round to the nearest tenth.



29. Use Patterns and Structure Enrique is selling the drop-leaf table and wants to include the area of the table when the leaves are down in his ad. What is the area of the center section when the leaves are down? Round to the nearest square inch. Explain how you found your answer.



ASSESSMENT PRACTICE

30. Select all true statements relating to circles.
GR.6.4

☐ **A.** Arc length of a circle is the region bounded by two radii and the intercepted arc.

☐ **B.** A sector is a fraction of the circle circumference.

☐ **C.** The measure of an arc is equal to the measure of its corresponding central angle.

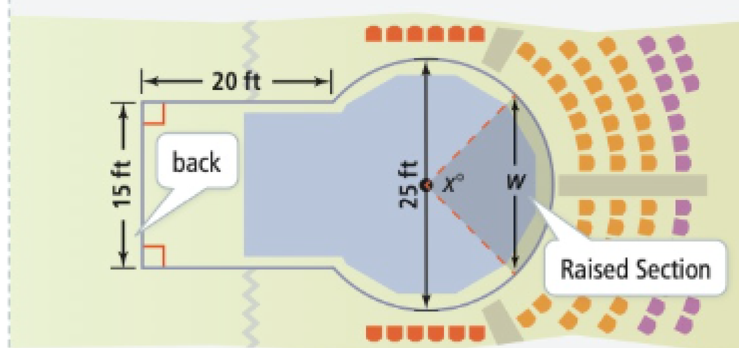
☐ **D.** A circle is the set of all points in a plane that are a given distance, the radius, from a given point, the center.

☐ **E.** A segment of a circle is the part of a circle bounded by an arc and the segment joining its endpoints.

31. SAT/ACT An arc has a central angle of $\frac{2}{5}\pi$ radians and a length of 6π. What is the circumference of the circle?

Ⓐ 12π Ⓑ 15π Ⓒ 30π Ⓓ 36π

32. Performance Task A carpenter is constructing the stage for a concert.



Part A What is the total amount of flooring needed to cover the stage? Round to the nearest square foot. Explain how you found your answer.

Part B A string of lights will be strung along the sides and front of the stage. What is the total length of light string needed? Show your work.

Part C One portion of the stage can be raised during the concert. The lift mechanism can lift a maximum area of 180 ft^2, but the band needs the width w of the raised area to be at least 20 ft. What could be the value of x? Justify your answer.

10-2 Lines Tangent to a Circle

I CAN… use properties of tangent lines to solve problems.

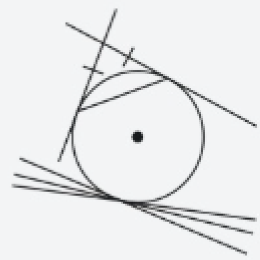
VOCABULARY
- point of tangency
- tangent to a circle

MA.912.GR.6.1–Solve mathematical and real-world problems involving the length of a secant, tangent, segment or chord in a given circle.
Also GR.3.3, GR.5.5
MA.K12.MTR.4.1, MTR.5.1, MTR.6.1

CRITIQUE & EXPLAIN

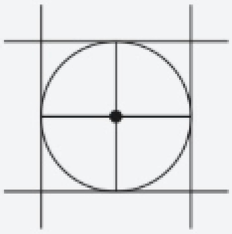
Alicia and Renaldo made conjectures about the lines that intersect a circle only once.

Alicia



- Many lines intersect the circle once at the same point.
- Two lines that intersect the circle once and the segment connecting the points form an isosceles triangle.

Renaldo



- Parallel lines intersect the circle at opposite ends of the same diameter.
- The lines intersecting the circle at one point are perpendicular to a diameter of the circle.

A. Check for Reasonableness Which of the four conjectures do you agree with? Which do you disagree with? Draw sketches to support your answers.

B. What other conjectures can you make about lines that intersect a circle at one point?

ESSENTIAL QUESTION How is a tangent line related to the radius of a circle at the point of tangency?

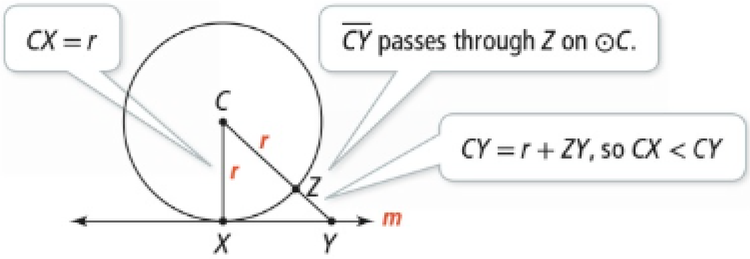
CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Understand Tangents to a Circle

What is the relationship between a circle and a tangent to the circle?

A **tangent to a circle** is a line in the plane of the circle that intersects the circle in exactly one point. That point is the **point of tangency**.

Circle C has tangent line m with point of tangency X. Point Y is any other point on m.



So, $\overline{CX}$ is the shortest segment from C to line m. Since the shortest segment from a point to a line is perpendicular to the line, $\overline{CX} \perp m$.

GENERALIZE
Point Y represents any point other than the point of tangency. Would the result be true no matter where point Y is located on m?

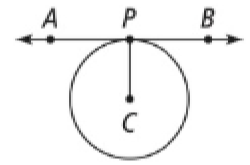
Try It! 1. Does Example 1 support Renaldo's conjecture that parallel lines intersect the circle at opposite ends of the same diameter? Explain.

THEOREM 10-1 AND THE CONVERSE

Theorem

If $\overleftrightarrow{AB}$ is tangent to $\odot C$ at P, then $\overleftrightarrow{AB}$ is perpendicular to $\overline{CP}$.

If...

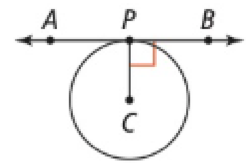


Then... $\overleftrightarrow{AB} \perp \overline{CP}$

Converse

If $\overleftrightarrow{AB}$ is perpendicular to radius $\overline{CP}$ at P, then $\overleftrightarrow{AB}$ is tangent to $\odot C$.

If...

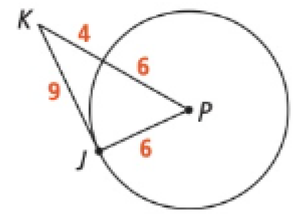


Then... $\overleftrightarrow{AB}$ is tangent to $\odot C$.

PROOF: SEE EXERCISES 12 AND 13.

EXAMPLE 2 Use Tangents to Solve Problems

A. Is $\overline{KJ}$ tangent to $\odot P$ at J?



A segment or ray that intersects a circle in one point is tangent to the circle if it is part of a tangent line.

If $\overline{KJ}$ is part of a line that is tangent to $\odot P$ at J, then $\overline{PJ} \perp \overline{JK}$ and $\triangle PJK$ is a right triangle.

$$9^2 + 6^2 \stackrel{?}{=} (4 + 6)^2$$
$$117 \neq 100$$

Use the Converse of the Pythagorean Theorem to determine whether $\triangle PJK$ is a right triangle.

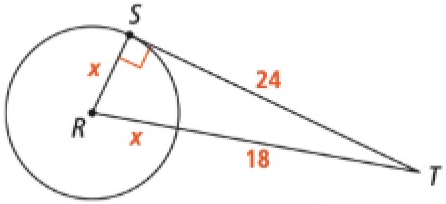
So, $\overline{PJ}$ is not perpendicular to $\overline{KJ}$.

Therefore, $\overline{KJ}$ is not tangent to $\odot P$ at J.

LEARN TOGETHER
What are ways to stay positive and work toward goals?

B. Segment ST is tangent to $\odot R$. What is the radius of $\odot R$?

Since $\overline{ST}$ is tangent to $\odot R$, $\triangle RST$ is a right triangle.



$$x^2 + 24^2 = (x + 18)^2$$
$$x^2 + 576 = x^2 + 36x + 324$$
$$252 = 36x$$
$$7 = x$$

Use the Pythagorean Theorem with length of the hypotenuse $x + 18$.

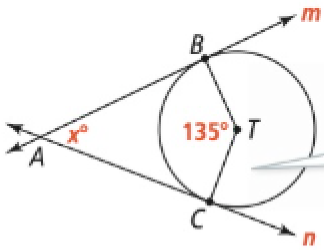
The radius of $\odot R$ is 7.

COMMON ERROR
You may incorrectly square just the terms x and 18. Recall how to square a binomial. It may be helpful to first write $(x + 18)^2$ as $(x + 18)(x + 18)$ and multiply.

CONTINUED ON THE NEXT PAGE

EXAMPLE 2 CONTINUED

C. Line *m* is tangent to ⊙*T* at *B*, and line *n* is tangent to ⊙*T* at *C*. What is the value of *x*?



Points *B* and *C* are points of tangency, so $m\angle ABT = 90$ and $m\angle ACT = 90$.

> **STUDY TIP**
> Remember that for any convex polygon, the sum of the interior angles is $(n - 2)180°$, where n is the number of sides.

Use the Polygon Angle-Sum Theorem to find *x*.

$$m\angle BAC + m\angle ACT + m\angle CTB + m\angle TBA = 360$$

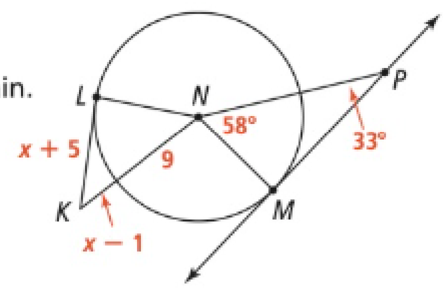
$$x + 90 + 135 + 90 = 360$$

$$x = 45$$

Try It! **2.** Use ⊙*N*.

a. Is $\overleftrightarrow{MP}$ tangent to ⊙*N*? Explain.

b. If $\overline{LK}$ is tangent to ⊙*N* at *L*, what is *KN*?



EXAMPLE 3 Find the Tangent Point

> **STUDY TIP**
> The tangent point is the same as the point of tangency.

The line $3x - 4y = 19$ is tangent to a circle whose center is located at (2, 3). What is the tangent point?

Step 1 Find the slope of the tangent line.

$y = \frac{3}{4}x - \frac{19}{4}$

Write the equation in slope-intercept form.

The slope of the given line is $\frac{3}{4}$.

Step 2 Write the equation of the line perpendicular to the tangent line through the point (2, 3).

$(y - 3) = -\frac{4}{3}(x - 2)$

Use point-slope form. The slope of the perpendicular line is $-\frac{4}{3}$.

$4x + 3y = 17$

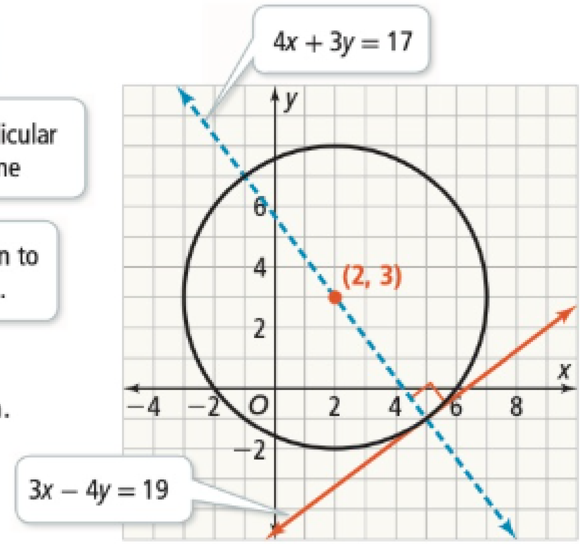
CONTINUED ON THE NEXT PAGE

ANALYZE AND PERSEVERE
The tangent point is at the intersection of the tangent line and the perpendicular line which includes a radius.

Step 3 Solve the system of equations to find the coordinates at the point of intersection.

$3x - 4y = 19$ — Tangent line
$4x + 3y = 17$ — The line perpendicular to the tangent line

$9x - 12y = 57$
$16x + 12y = 68$
$25x = 125$
$x = 5$
— Apply elimination to solve the system.

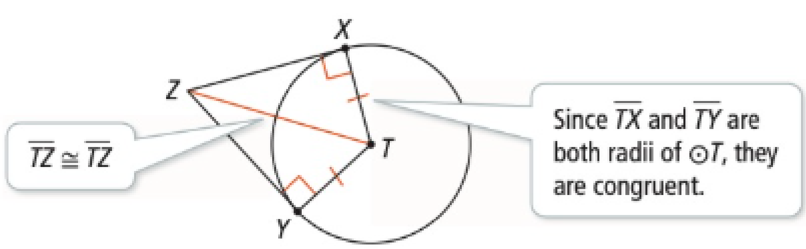


Substitute $x = 5$ into either equation.
$3x - 4y = 19$
$3(5) - 4y = 19$
$y = -1$
The tangent point is (5, −1).

Try It! **3.** Find a second tangent point on the circle with the same slope as $3x - 4y = 19$. (*Hint*: The two tangent lines are parallel, so you can use slope to find the second tangent point.)

EXAMPLE 4 Find Lengths of Segments Tangent to a Circle

What is the relationship between $\overline{YZ}$ and $\overline{XZ}$?



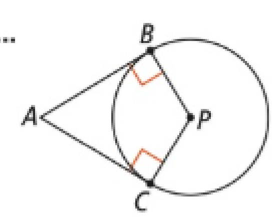
By HL, $\triangle TXZ \cong \triangle TYZ$, so $\overline{YZ} \cong \overline{XZ}$ by CPCTC.

Try It! **4.** If $TX = 12$ and $TZ = 20$, what are XZ and YZ?

THEOREM 10-2 Segments Tangent to a Circle Theorem

If two segments with a common endpoint exterior to a circle are tangent to the circle, then the segments are congruent.

If...



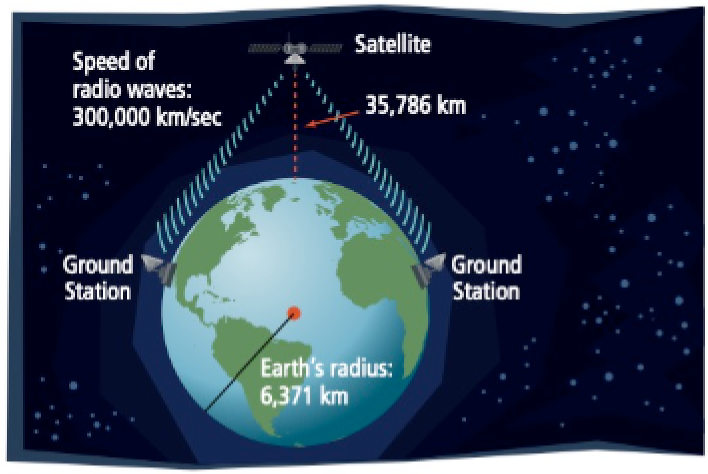
Then... $\overline{AB} \cong \overline{AC}$

PROOF: SEE EXERCISE 14.

APPLICATION

EXAMPLE 5 Find Measures Involving Tangent Lines

A satellite requires a line of sight for communication. Between the ground stations farthest from the satellite, what is the amount of time needed for a signal to go from one station up to the satellite, and then down to the other station?

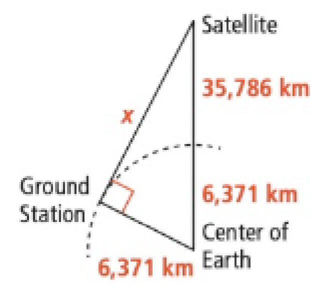


Formulate The lines from the satellite to the farthest ground stations are tangent to Earth's surface.

Use the Pythagorean Theorem to compute the distance to the ground stations. Then compute the time for radio waves to travel twice this distance.

Compute **Step 1** Find the distance from the farthest ground stations to the satellite.

$$x^2 + 6{,}371^2 = (6{,}371 + 35{,}786)^2$$
$$x^2 + 40{,}589{,}641 = 1{,}777{,}212{,}649$$
$$x^2 = 1{,}736{,}623{,}008$$
$$x \approx 41{,}673$$

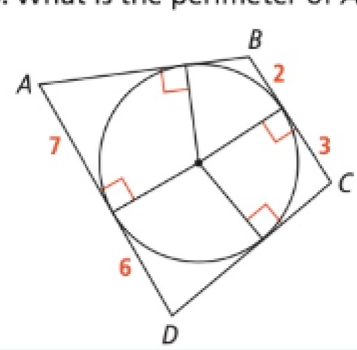


Step 2 Find the time for radio waves to travel this distance twice.

$$(41{,}673 \times 2) \text{ km} \div 300{,}000 \text{ km/sec} \approx 0.28 \text{ sec}$$

Interpret The amount of time for a signal to travel from one of the farthest ground stations to the satellite and back to the other ground station is about 0.28 second.

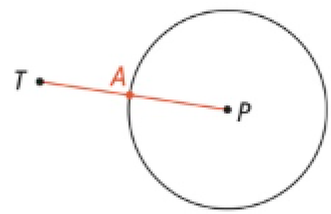
Try It! **5.** What is the perimeter of *ABCD*?



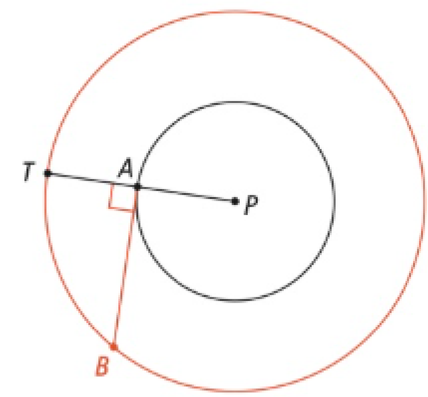
EXAMPLE 6 Construct Tangent Lines

How do you construct a tangent to $\odot P$ passing through point *T*?

Step 1 Use a straightedge to draw $\overline{PT}$. Label point *A* where $\overline{PT}$ intersects the circle.

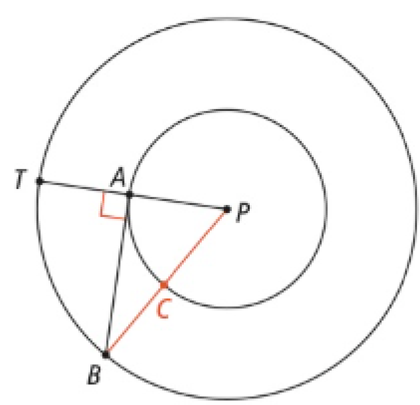


Step 2 Use a compass to construct a circle with center *P* and passing through *T*. Construct a perpendicular to $\overline{PT}$ at *A*. Label point *B* where the perpendicular intersects the outer circle.

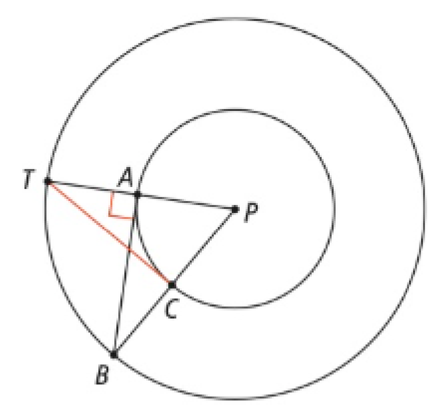


COMMON ERROR
You may think that *A* is the midpoint of $\overline{PT}$. However, the construction of a perpendicular line here is different from constructing the perpendicular bisector of $\overline{PT}$.

Step 3 Use a straightedge to construct $\overline{BP}$. Label point *C* where $\overline{BP}$ intersects the inner circle.



Step 4 Use a straightedge to construct $\overline{TC}$.

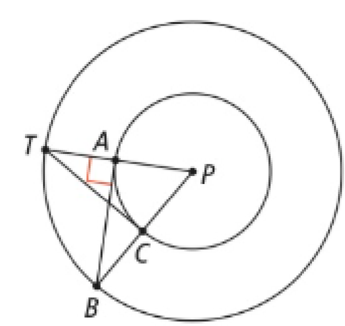


The tangent to $\odot P$ passing through *T* is $\overline{TC}$.

Try It! **6.** Prove that $\overline{TC}$ is tangent to $\odot P$.

Given: Two circles share center *P*, points *A* and *C* on the smaller circle, points *T* and *B* on the larger circle, $\overline{AB} \perp \overline{PT}$

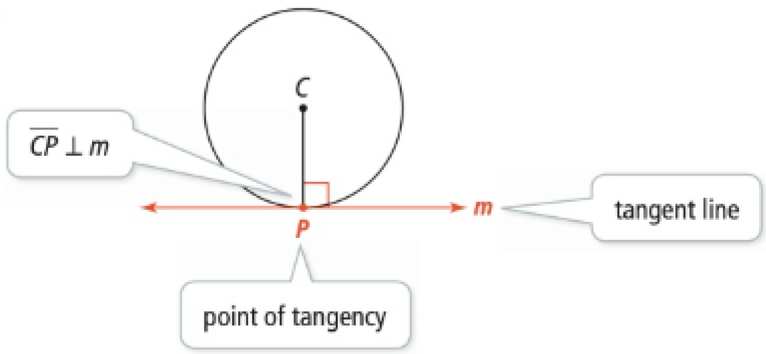
Prove: $\overline{TC}$ is tangent to $\odot P$ at *C*.



CONCEPT SUMMARY Tangents to a Circle

WORDS A tangent to a circle intersects the circle at exactly one point. The radius that contains the point of tangency is perpendicular to the tangent.

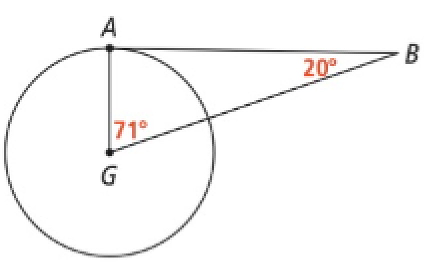
DIAGRAM



Do You UNDERSTAND?

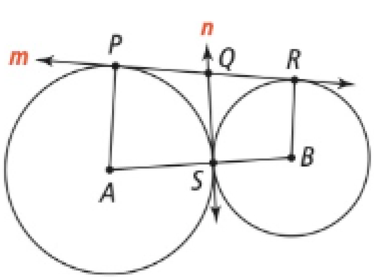
1. **ESSENTIAL QUESTION** How is a tangent line related to the radius of a circle at the point of tangency?

2. **Error Analysis** Kona looked at the figure shown and said that $\overline{AB}$ is tangent to ⊙G at A because it intersects ⊙G only at A. What was Kona's error?



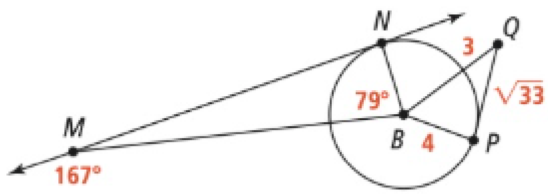
3. **Vocabulary** Can any point on a circle be a *point of tangency*? Explain.

4. **Use Patterns and Structure** Lines m and n are tangent to circles A and B. What are the relationships between ∠PAS, ∠PQS, ∠RQS, and ∠RBS? Explain.



Do You KNOW HOW?

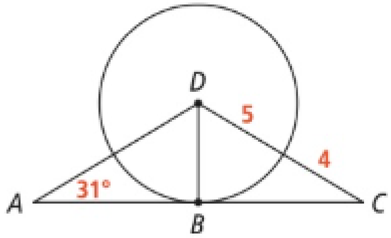
Tell whether each line or segment is a tangent to ⊙B.



5. $\overleftrightarrow{MN}$

6. $\overline{QP}$

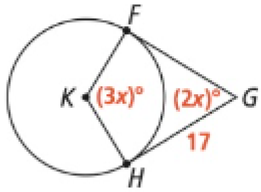
Segment AC is tangent to ⊙D at B. Find each value.



7. m∠ADB

8. BC

Segment FG is tangent to ⊙K at F and $\overline{HG}$ is tangent to ⊙K at H. Find each value.



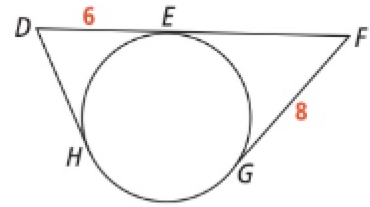
9. FG

10. m∠FGH

PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. Error Analysis Segments $\overline{DF}$, $\overline{DH}$, and $\overline{GF}$ are tangent to the circle. Andrew was asked to find DF. Explain Andrew's error.

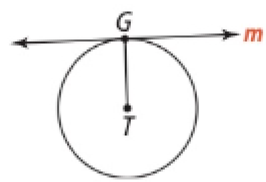


$DF = DE + EF$
By Theorem 10-2, $DE = EF$.
So, $DF = 6 + 6 = 12$. ✗

12. Communicate and Justify Use the following outline to write an indirect proof of Theorem 10-1.

Given: Line m is tangent to $\odot T$ at G.

Prove: $\overline{GT} \perp m$

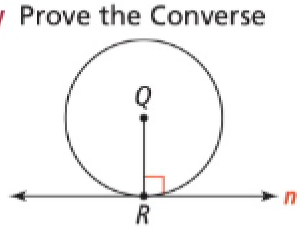


- Assume that $\overline{GT}$ is not perpendicular to m.
- Draw $\overline{HT}$ such that $\overline{HT} \perp m$.
- Use triangles to show that $GT > HT$.
- Show that this is a contradiction, since H is in the exterior of $\odot T$.

13. Communicate and Justify Prove the Converse of Theorem 10-1.

Given: $\overline{QR} \perp n$

Prove: n is tangent to $\odot Q$ at R

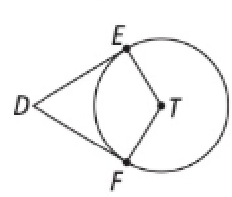


Hint: Select any other point S on line n. Show that $\overline{QS}$ is the hypotenuse of $\triangle QRS$, so $QS > QR$ and therefore S lies outside $\odot Q$.

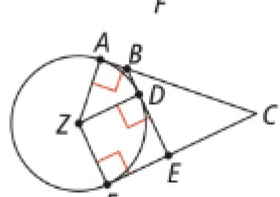
14. Communicate and Justify Prove Theorem 10-2.

Given: $\overline{DE}$ and $\overline{DF}$ are tangent to $\odot T$.

Prove: $\overline{DE} \cong \overline{DF}$



15. Higher Order Thinking If $AC = x$, what is the perimeter of $\triangle BCE$? Explain.

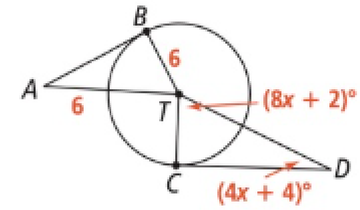


PRACTICE

The segments $\overline{AB}$ and $\overline{CD}$ are tangent to $\odot T$. Find each value. SEE EXAMPLES 1 AND 2

16. AB

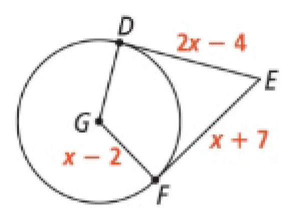
17. $m\angle TDC$



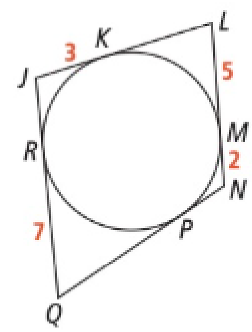
18. Find the line of tangency to the circle defined by $(x - 3)^2 + (y - 7)^2 = 169$ at the point (15, 2).
SEE EXAMPLE 3

For Exercises 19–21, the segments are tangent to the circle. Find each value. SEE EXAMPLES 4 AND 5

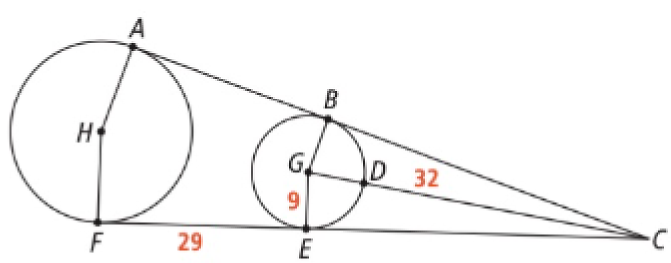
19. DG



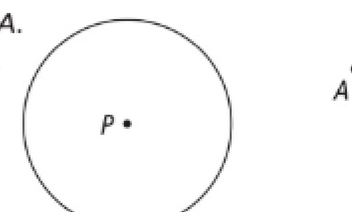
20. Perimeter of $JLNQ$



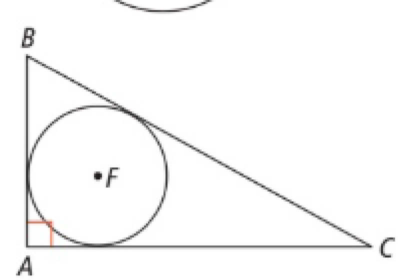
21. AC



22. Trace $\odot P$ and point A. Construct a tangent to $\odot P$ that passes through A.
SEE EXAMPLE 6



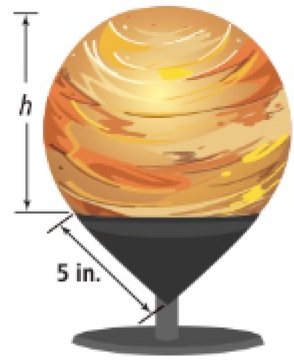
23. The diameter of $\odot F$ is 8; $AB = 10$; and $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are tangent to $\odot F$. What is the perimeter of $\triangle ABC$?



PRACTICE & PROBLEM SOLVING

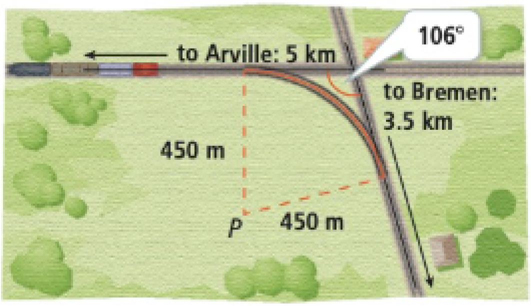
APPLY

24. Analyze and Persevere Yumiko is shopping for a stand for a decorative glass ball with an 8-inch diameter. She is considering the stand shown and wants to know the height h of the portion of the ball that will be visible if the sides of the stand are tangent to the sphere. What is the value of h?



25. Use Patterns and Structure Samantha is looking out from the observation deck of the lighthouse on a clear day. How far away is the horizon? Earth's radius is about 6,400 km.

26. Mathematical Connections Rail planners want to connect the two straight tracks with a curved track as shown. Any curves must have a radius of at least 450 m.



a. Explain how engineers can locate point P, the center of the curved section of track.

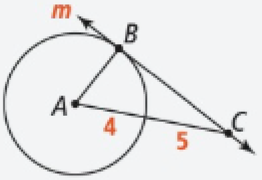
b. Once the curved track is constructed, what distance will trains travel between Arvillle and Bremen? Justify your answer.

ASSESSMENT PRACTICE

27. Circle P is described by the equation $(x + 3)^2 + (y - 2)^2 = 25$. Which of the following lines are tangent to $\odot P$? Select all that apply.
GR.6.1

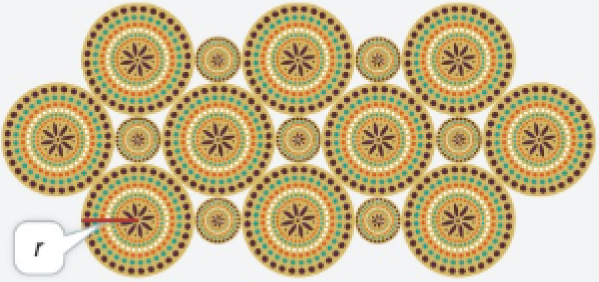
Ⓐ $y = x + 3$
Ⓑ $y = 5$
Ⓒ $y = x$
Ⓓ $x = 2$
Ⓔ $y = -3$
Ⓕ $y = x - 3$

28. SAT/ACT Line m is tangent to $\odot A$ at B. What is the area of $\triangle ABC$?



Ⓐ 10
Ⓑ 18
Ⓒ $2\sqrt{65}$
Ⓓ $\frac{5\sqrt{65}}{2}$

29. Performance Task The South African art design below is based on circles that are tangent to each other.



Part A If the radius of the larger circles is r, what is the radius of the smaller circles?

Part B Choose a value for the larger radius and draw the pattern. Measure the radii of the small and large circles. Are the values related in the way you described in Part A?

Part C In your diagram for Part B, mark the points where the small and large circles are tangent to each other. Add lines that are tangent to the circles at these points. Describe how the tangent lines you drew illustrate Theorems 10-1 and 10-2.

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.6.1–Solve mathematical and real-world problems involving the length of a secant, tangent, segment or chord in a given circle. Also GR.5.5

MA.K12.MTR.7.1

Earth Watch

Scientists estimate that there are currently about 3,000 operational man-made satellites orbiting Earth. These satellites serve different purposes, from communication to navigation and global positioning. Some are weather satellites that collect environmental information.

The International Space Station is the largest man-made satellite that orbits Earth. It serves as a space environment research facility, and it also offers amazing views of Earth. Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

10-3 Chords

I CAN... relate the length of a chord to its central angle and the arc it intercepts.

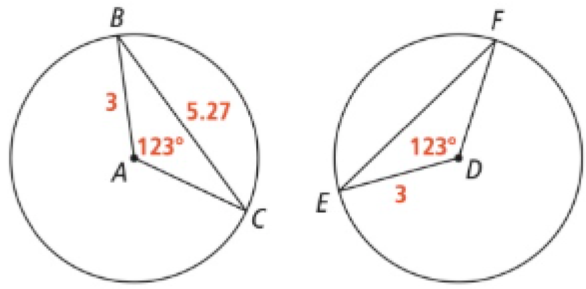
VOCABULARY

- chord

MA.912.GR.6.1–Solve mathematical and real-world problems involving the length of a secant, tangent, segment or chord in a given circle. **Also GR.5.4**
MA.K12.MTR.4.1, MTR.1.1, MTR.7.1

EXPLORE & REASON

Use the diagram to answer the questions.



A. What figures in the diagram are congruent? Explain.

B. **Use Patterns and Structure** How can you find EF?

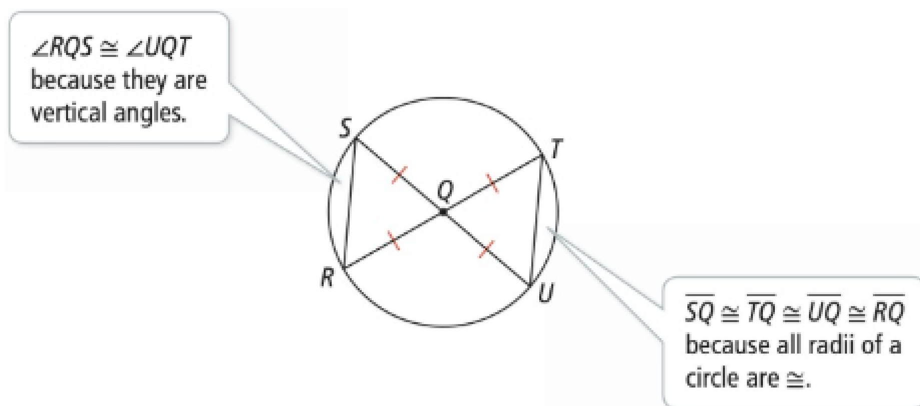
ESSENTIAL QUESTION

How are chords related to their central angles and intercepted arcs?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Relate Central Angles and Chords

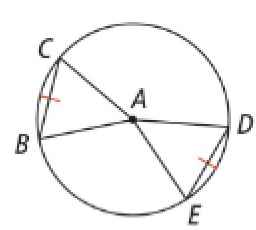
A chord is a segment whose endpoints are on a circle. Why is $\overline{RS} \cong \overline{UT}$?



STUDY TIP
Refer to the diagram as you read the proof. Note which parts of the triangles are congruent.

By the SAS Congruence Theorem, $\triangle QRS \cong \triangle QUT$. Therefore $\overline{RS} \cong \overline{UT}$ because they are corresponding parts of congruent triangles.

Try It! 1. Why is $\angle BAC \cong \angle DAE$?



THEOREM 10-3 AND THE CONVERSE

Theorem

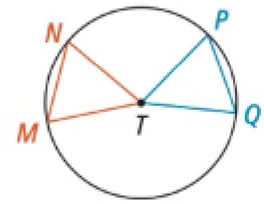
If two chords in a circle or in congruent circles are congruent, then their central angles are congruent.

Converse

If two central angles in a circle or in congruent circles are congruent, then their chords are congruent.

PROOF: SEE EXERCISES 12 AND 13.

If... $\overline{MN} \cong \overline{PQ}$
Then... $\angle MTN \cong \angle PTQ$



If... $\angle MTN \cong \angle PTQ$
Then... $\overline{MN} \cong \overline{PQ}$

THEOREM 10-4 AND THE CONVERSE

Theorem

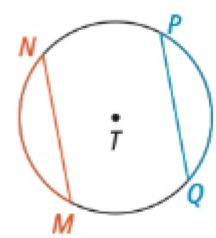
If two arcs in a circle or in congruent circles are congruent, then their chords are congruent.

Converse

If two chords in a circle or in congruent circles are congruent, then their arcs are congruent.

PROOF: SEE EXAMPLE 2 AND EXAMPLE 2 TRY IT.

If... $\widehat{MN} \cong \widehat{PQ}$
Then... $\overline{MN} \cong \overline{PQ}$



If... $\overline{MN} \cong \overline{PQ}$
Then... $\widehat{MN} \cong \widehat{PQ}$

PROOF

EXAMPLE 2 Relate Arcs and Chords

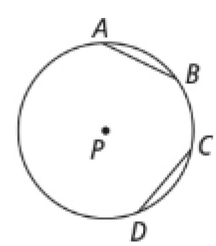
ANALYZE AND PERSEVERE
Think about other strategies you can use. How could you use congruent triangles to prove the relationship?

Write a proof of Theorem 10-4.

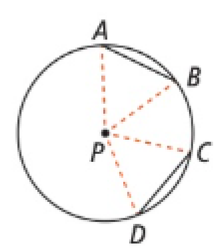
Given: $\widehat{AB} \cong \widehat{CD}$

Prove: $\overline{AB} \cong \overline{CD}$

Plan: Use the relationship between central angles and arcs by drawing the radii $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PD}$.



Proof: Since $\widehat{AB} \cong \widehat{CD}$, you know that $m\widehat{AB} = m\widehat{CD}$. And since the measure of a central angle is equal to the measure of its arc, $m\angle APB = m\widehat{AB}$ and $m\angle CPD = m\widehat{CD}$. By substitution $m\angle APB = m\angle CPD$ and $\angle APB \cong \angle CPD$. So, by the Converse of Theorem 10-3, $\overline{AB} \cong \overline{CD}$.



Try It! **2.** Write a flow proof of the Converse of Theorem 10-4.

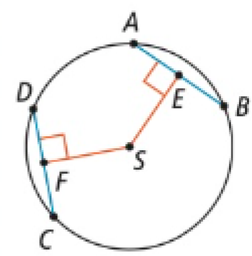
THEOREM 10-5 AND THE CONVERSE

Theorem

If chords are equidistant from the center of a circle or the centers of congruent circles, then they are congruent.

Converse

If chords in a circle or in congruent circles are congruent, then they are equidistant from the center or centers.



If... $\overline{SE} \cong \overline{SF}$, **Then...** $\overline{AB} \cong \overline{CD}$

If... $\overline{AB} \cong \overline{CD}$, **Then...** $\overline{SE} \cong \overline{SF}$

PROOF: SEE EXAMPLE 3 AND EXAMPLE 3 TRY IT.

PROOF

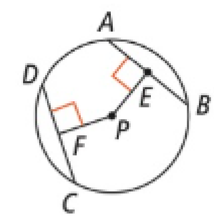
EXAMPLE 3 Relate Chords Equidistant from the Center

COMMON ERROR
Be sure to construct the triangles with corresponding parts that yield the desired conclusion.

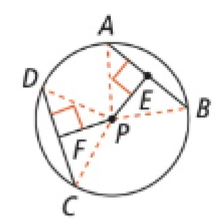
Write a proof of Theorem 10-5.

Given: $\odot P$ with $\overline{AB} \perp \overline{PE}$, $\overline{CD} \perp \overline{PF}$, $\overline{PE} \cong \overline{PF}$

Prove: $\overline{AB} \cong \overline{CD}$



Plan: Construct triangles by drawing the radii $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PD}$. Then show that the triangles are congruent in order to apply CPCTC.



Proof:

- $\overline{AB} \perp \overline{PE}$, $\overline{CD} \perp \overline{PF}$ — Given
- $\overline{PA} \cong \overline{PB} \cong \overline{PC} \cong \overline{PD}$ — All radii of a circle are ≅.
- $\angle AEP$, $\angle BEP$, $\angle CFP$, and $\angle DFP$ are right $\angle$s. — Def. of ⊥
- $\overline{PE} \cong \overline{PF}$ — Given
- $\triangle AEP$, $\triangle BEP$, $\triangle CFP$, and $\triangle DFP$ are right $\triangle$s. — Def. of right $\triangle$s
- $\triangle AEP \cong \triangle BEP \cong \triangle CFP \cong \triangle DFP$ — By HL
- $\overline{AE} \cong \overline{BE} \cong \overline{CF} \cong \overline{DF}$ — By CPCTC
- $AB = AE + EB$ and $CD = CF + FD$ — Seg. Addition Post.
- $AB = 2AE$ and $CD = 2CF$ — Substitution
- $AB = CD$ — Substitution
- $\overline{AB} \cong \overline{CD}$ — Def. of ≅ segments

Try It! **3.** Write a flow proof of the Converse of Theorem 10-5.

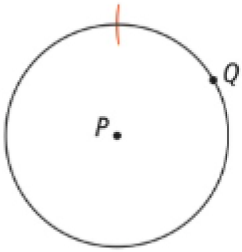
EXAMPLE 4 Construct a Regular Hexagon Inscribed in a Circle

STUDY TIP
Remember, a regular polygon is both equilateral and equiangular.

How do you draw a regular hexagon inscribed in ⊙*P*?

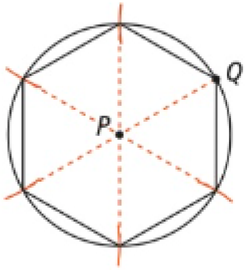
Step 1 Mark point *Q* on the circle.

Step 2 Set the compass to the radius of the circle. Place the compass point at *Q* and draw an arc through the circle.



Step 3 Keep the compass setting. Move the compass point to the intersection of the arc and the circle. Draw another arc through the circle. Each point of intersection is a vertex of the hexagon. Continue this way until you have five more arcs.

Step 4 Draw chords connecting consecutive points on the circle.



The side lengths of the resulting figure are all congruent because they have the same length as the radius of the circle.

Connecting the center of the circle with the six vertices of the inscribed polygon forms six equilateral triangles, so each angle measures 120. The figure is a regular hexagon.

Try It! **4.** Construct an equilateral triangle inscribed in a circle.

THEOREM 10-6 AND THE CONVERSE

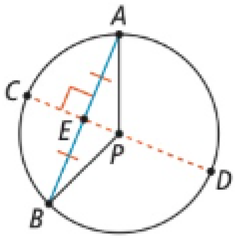
Theorem

If a diameter is perpendicular to a chord, then it bisects the chord.

Converse

If a diameter bisects a chord (that is not a diameter), then it is perpendicular to the chord.

PROOF: SEE EXERCISES 15 AND 16.



If... $\overline{CD}$ is a diameter, $\overline{AB} \perp \overline{CD}$
Then... $\overline{AE} \cong \overline{BE}$

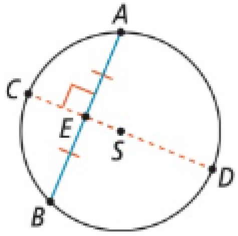
If... $\overline{CD}$ is a diameter, $\overline{AE} \cong \overline{BE}$
Then... $\overline{AB} \perp \overline{CD}$

THEOREM 10-7

The perpendicular bisector of a chord contains the center of the circle.

PROOF: SEE EXERCISE 28.

If...

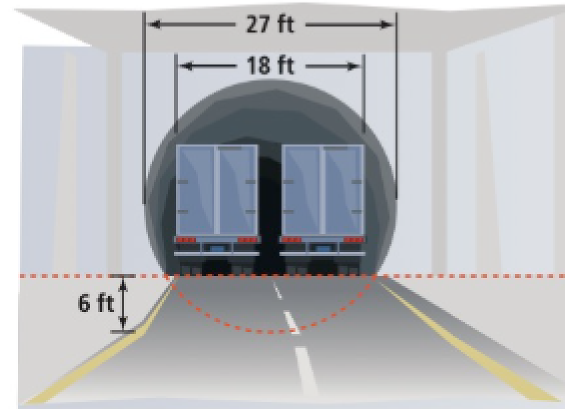


Then... *S* is on $\overline{CD}$

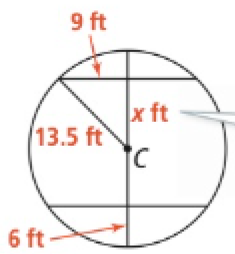
APPLICATION

EXAMPLE 5 Solve Problems Involving Chords of Circles

An engineer is designing a service tunnel to accommodate two trucks simultaneously. If the tunnel can accommodate a width of 18 ft, what is the greatest truck height that the tunnel can accommodate? Subtract 0.5 ft to account for fluctuations in pavement.



Formulate Draw and label a sketch to help solve the problem.



Let x be the distance from the center to the greatest height. The radius is 13.5 ft.

Compute Write and solve an equation for x.

$$9^2 + x^2 = 13.5^2$$

Use the Pythagorean Theorem.

$$x^2 = 13.5^2 - 9^2$$

$$x = \sqrt{13.5^2 - 9^2}$$

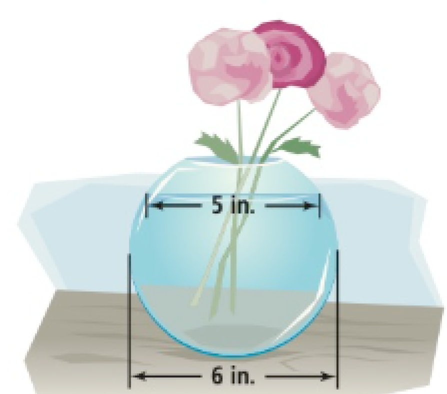
$$x \approx 10.06$$

Add the distance from the ground to the center $13.5 - 6 = 7.5$ to x and subtract 0.5 ft to account for fluctuations in pavement.

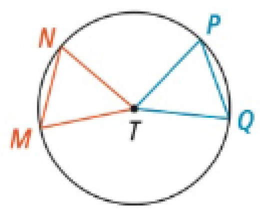
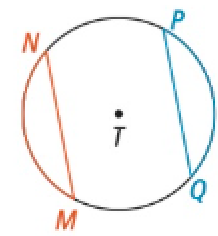
$$7.5 + 10.06 - 0.5 = 17.06$$

Interpret The greatest height that the tunnel can accommodate is about 17.06 ft.

Try It! **5.** Fresh cut flowers need to be in at least 4 inches of water. A spherical vase is filled until the surface of the water forms a circle 5 inches in diameter. Is the water deep enough for the flowers? Explain.



CONCEPT SUMMARY Chords

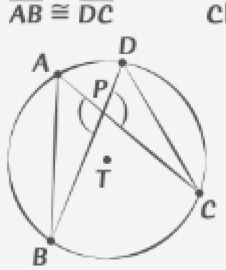
	Chords and Central Angles	Chords and Arcs
WORDS	Two chords in a circle or in congruent circles are congruent if and only if the central angles of the chords are congruent.	Two chords in a circle or in congruent circles are congruent if and only if the chords intercept congruent arcs.
DIAGRAMS	 In $\odot T$, $\angle MTN \cong \angle PTQ$ if and only if $\overline{MN} \cong \overline{PQ}$.	 $\widehat{MN} \cong \widehat{PQ}$ if and only if $\overline{MN} \cong \overline{PQ}$.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are chords related to their central angles and intercepted arcs?

2. **Error Analysis** Sasha writes a proof to show that two chords are congruent. What is her error?

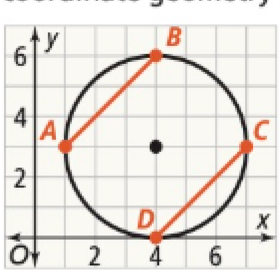
$\angle APB \cong \angle CPD$	Vert. $\angle$s $\cong$
$\widehat{AB} \cong \widehat{CD}$	Intercepted by $\cong$ $\angle$s
$\overline{AB} \cong \overline{DC}$	Chords intercept $\cong$ arcs



✗

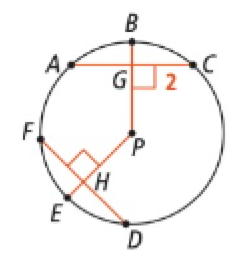
3. **Vocabulary** Explain why all diameters of circles are also chords of the circles.

4. **Choose Efficient Methods** How can you use coordinate geometry to show that $\widehat{AB} \cong \widehat{CD}$?



Do You KNOW HOW?

For Exercises 5–10, in $\odot P$, $m\widehat{AB} = 43$, and $AC = DF$. Find each measure.

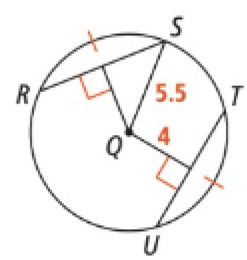


5. DF 6. $m\widehat{AC}$

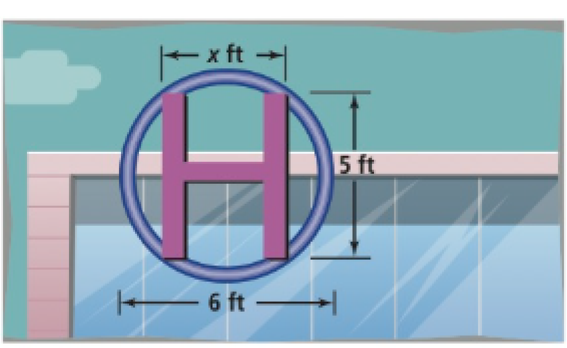
7. FH 8. $m\widehat{DE}$

9. AC 10. $m\widehat{DF}$

11. **Represent and Connect** Given $\widehat{RS} \cong \widehat{UT}$, find UT.



12. For the corporate headquarters, an executive wants to place a company logo that is six feet in diameter with the sides of the H five feet tall on the front wall. What is the width x of the crossbar for the H?



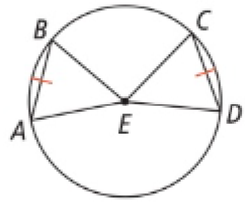
PRACTICE & PROBLEM SOLVING

UNDERSTAND

13. Communicate and Justify Write a paragraph proof of Theorem 10-3.

Given: $\overline{AB} \cong \overline{CD}$

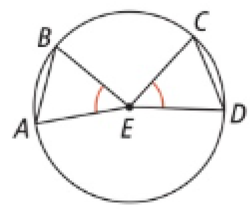
Prove: $\angle AEB \cong \angle CED$



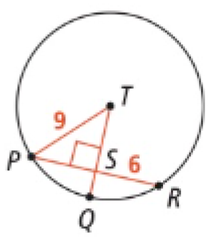
14. Communicate and Justify Write a two-column proof of the Converse of Theorem 10-3.

Given: $\angle AEB \cong \angle CED$

Prove: $\overline{AB} \cong \overline{CD}$



15. Error Analysis What is Ashton's error?

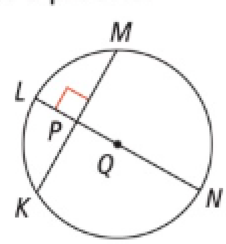


$TS = \sqrt{PR^2 - PS^2}$
$= \sqrt{12^2 - 9^2}$
≈ 7.9

16. Communicate and Justify Write a proof of Theorem 10-6.

Given: $\overline{LN}$ is a diameter of $\odot Q$; $\overline{LN} \perp \overline{KM}$

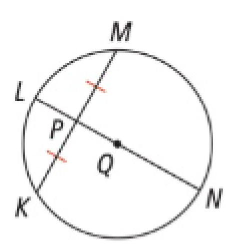
Prove: $\overline{KP} \cong \overline{MP}$



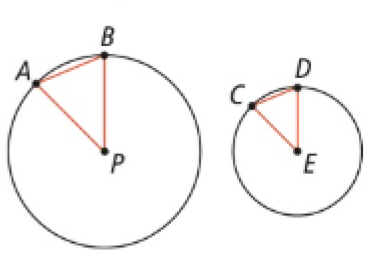
17. Communicate and Justify Write a proof of the Converse of Theorem 10-6.

Given: $\overline{LN}$ is a diameter of $\odot Q$; $\overline{KP} \cong \overline{MP}$

Prove: $\overline{LN} \perp \overline{KM}$

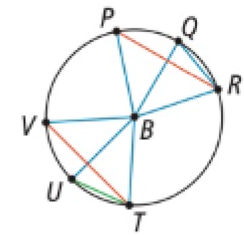


18. Higher Order Thinking $\triangle ABP \sim \triangle CDE$. How do you show that $\widehat{AB} \cong \widehat{CD}$?



PRACTICE

For Exercises 19–22, in $\odot B$, $m\angle VBT = m\widehat{PR} = 90$, and $QR = TU$. SEE EXAMPLES 1 AND 2



19. Find $m\angle PBR$.

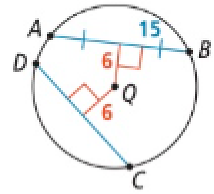
20. Find $m\widehat{TV}$.

21. Which angle is congruent to $\angle QBR$?

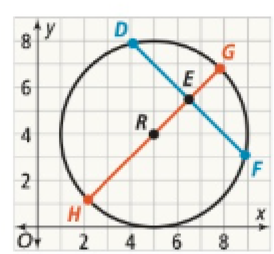
22. Which segment is congruent to $\overline{TV}$?

23. Construct a square inscribed in a circle. How is drawing an inscribed square different from drawing an inscribed hexagon or triangle? SEE EXAMPLE 4

24. Find CD. SEE EXAMPLE 3



25. Use coordinate geometry to determine if the diameter is perpendicular to the chord. Does the diameter bisect the chord? Explain. SEE EXAMPLE 5



26. A chord is 12 cm long. It is 30 cm from the center of the circle. What is the radius of the circle? SEE EXAMPLE 5

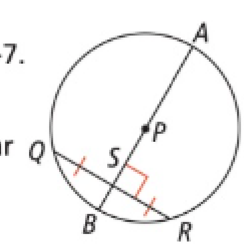
27. The diameter of a circle is 39 inches. The circle has two chords of length 8 inches. What is the distance from each chord to the center of the circle?

28. A chord is 4 units from the center of a circle. The radius of the circle is 5 units. What is the length of the chord?

29. Write a proof of Theorem 10-7.

Given: $\overline{QR}$ is a chord in $\odot P$; $\overline{AB}$ is the perpendicular bisector of $\overline{QR}$.

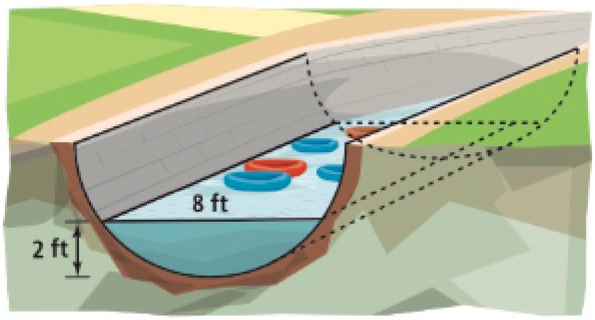
Prove: $\overline{AB}$ contains P.



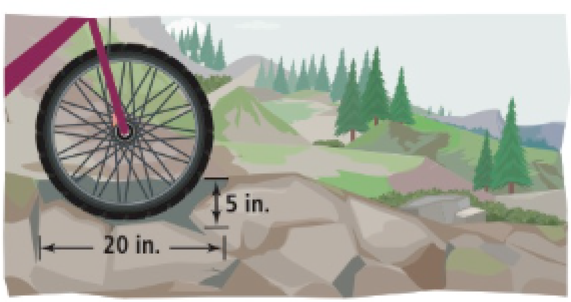
PRACTICE & PROBLEM SOLVING

APPLY

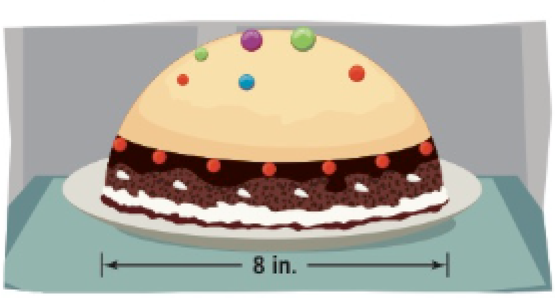
30. Mathematical Connections Nadia designs a water ride and wants to use a half-cylindrical pipe in the construction. If she wants the waterway to be 8 ft wide when the water is 2 ft deep, what is the diameter of the pipe?



31. Apply Math Models A bike trail has holes up to 20 in. wide and 5 in. deep. If the diameter of the wheels of Anna's bike is 26 in., can she ride her bike without the wheels hitting the bottom of the holes? Explain.



32. Analyze and Persevere The bottom of a hemispherical cake has diameter 8 in.

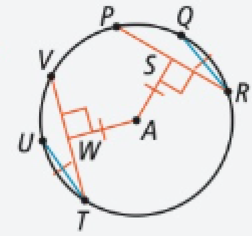


a. If the cake is sliced horizontally in half so each piece has the same height, would the top half fit on a plate with diameter 6 in.? Explain.

b. If the cake is sliced horizontally in thirds so each piece has the same height, would the top third fit on a plate with diameter 5 in.? Explain.

ASSESSMENT PRACTICE

33. Which must be true? Select all that apply. GR.6.1



☐ **A.** $\widehat{QR} \cong \widehat{TU}$

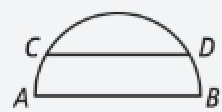
☐ **B.** $PR = TV$

☐ **C.** $VW = AS$

☐ **D.** $PS = SR$

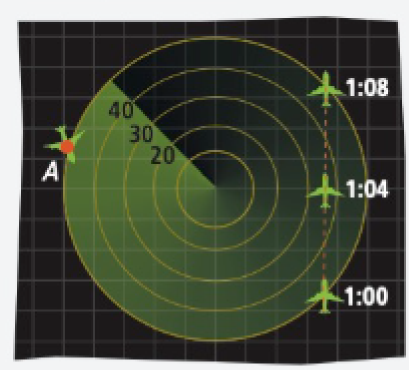
☐ **E.** $\widehat{PR} \cong \widehat{TV}$

34. SAT/ACT The radius of the semicircle is r, and $CD = \frac{3}{4} \cdot AB$. What is the distance from the chord to the diameter?



Ⓐ $\frac{5}{4}r$ Ⓑ $\frac{\sqrt{7}}{4}r$ Ⓒ $\frac{\sqrt{7}}{4}\pi r$ Ⓓ $\frac{5}{4}\pi r$

35. Performance Task The radius of the range of a radar is 50 miles. At 1:00 P.M., a plane enters the radar screen flying due north. At 1:04 P.M. the aircraft is due east of the radar. At 1:08 P.M., the aircraft leaves the screen. The plane is moving at 8 miles per minute.



Part A What distance does the plane fly on the controller's screen?

Part B What is the distance of the plane from the radar at 1:04 P.M.?

Part C Another plane enters the screen at point A at 1:12 P.M. and flies in a straight line at 9 miles per minute. If it gets no closer than 40 miles from the radar, at what time does it leave the screen? Explain.

10-4 Inscribed Angles

I CAN… use the relationships between angles and arcs in circles to find their measures.

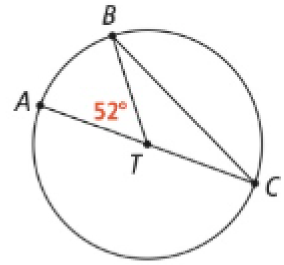
VOCABULARY

- inscribed angle

MA.912.GR.6.1–Solve mathematical and real-world problems involving the length of a secant, tangent, segment or chord in a given circle. Also GR.6.2, GR.6.3
MA.K12.MTR.2.1, MTR.1.1, MTR.5.1

EXPLORE & REASON

Consider ⊙T.



A. Analyze and Persevere List at least seven things you can conclude about the figure.

B. How is $\angle ACB$ related to $\angle ATB$? Explain.

ESSENTIAL QUESTION

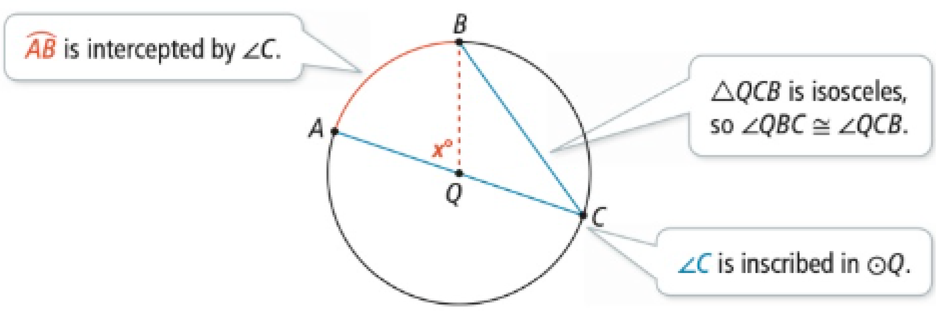
How is the measure of an inscribed angle related to its intercepted arc?

EXAMPLE 1 Relate Inscribed Angles to Intercepted Arcs

CONCEPTUAL UNDERSTANDING

What is the relationship between $\widehat{AB}$ and $\angle ACB$?

An **inscribed angle** has its vertex on a circle and its side along chords of the circle.



Draw radius $\overline{QB}$ to form $\triangle QCB$ and central angle $\angle AQB$.

$$m\angle QBC + m\angle QCB = x$$
Apply the Triangle Exterior Angle Theorem.

$$2(m\angle QCB) = x$$
$m\angle QBC = m\angle QCB$ since $\angle QBC \cong \angle QCB$.

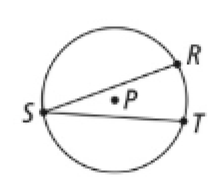
$$m\angle QCB = \frac{1}{2}x$$

$$m\angle ACB = \frac{1}{2}m\widehat{AB}$$
$m\angle QCB = m\angle ACB$ and $x = m\widehat{AB}$.

The measure of an inscribed angle $\angle ACB$ is half the measure of the intercepted arc $\widehat{AB}$.

STUDY TIP
There are an infinite number of inscribed angles that intercept the arc. These inscribed angles all have the same angle measure.

Try It! 1. Given ⊙P with inscribed angle $\angle S$, if $m\widehat{RT} = 47$, what is $m\angle S$?



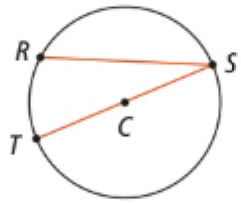
THEOREM 10-8 Inscribed Angles Theorem

The measure of an inscribed angle is half the measure of its intercepted arc.

Case 1

The center is on one side of the angle.

If...

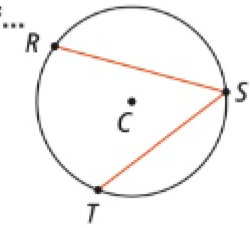


Then... $m\angle S = \frac{1}{2}m\widehat{RT}$

Case 2

The center is inside the angle.

If...

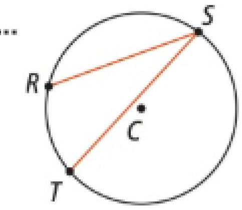


Then... $m\angle S = \frac{1}{2}m\widehat{RT}$

Case 3

The center is outside the angle.

If...

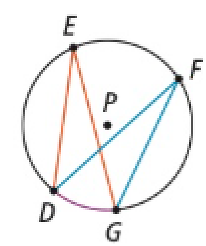


Then... $m\angle S = \frac{1}{2}m\widehat{RT}$

PROOF: SEE EXERCISES 19, 32, AND 33.

EXAMPLE 2 Use the Inscribed Angles Theorem

A. If $m\widehat{DG} = 45.6$, what are $m\angle E$ and $m\angle F$?



$m\angle E = \frac{1}{2}m\widehat{DG}$

$= \frac{1}{2}(45.6) = 22.8$

$m\angle F = \frac{1}{2}m\widehat{DG}$

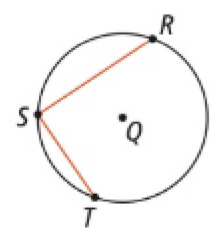
$= \frac{1}{2}(45.6) = 22.8$

USE PATTERNS AND STRUCTURE
The diameter of a circle is a straight angle. What is the measure of the arc intercepted by a diameter?

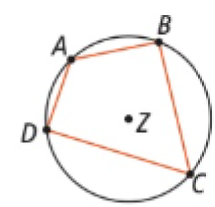
B. If $\widehat{RT}$ is a semicircle, what is $m\angle RST$?

$m\angle S = \frac{1}{2}m\widehat{RT}$

$= \frac{1}{2}(180) = 90$



C. If $m\widehat{ABC} = 184$ and $m\widehat{BCD} = 242$, what are the measures of the angles of quadrilateral $ABCD$?



$m\angle A = \frac{1}{2}m\widehat{BCD}$

$= \frac{1}{2}(242) = 121$

$m\angle B = \frac{1}{2}m\widehat{ADC}$

$= \frac{1}{2}(360 - 184) = 88$

$m\angle D = \frac{1}{2}m\widehat{ABC}$

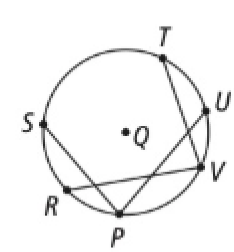
$= \frac{1}{2}(184) = 92$

$m\angle C = 360 - (121 + 88 + 92)$

$= 59$

Try It! **2. a.** If $m\widehat{RST} = 164$, what is $m\angle RVT$?

b. If $m\angle SPU = 79$, what is $m\widehat{STU}$?

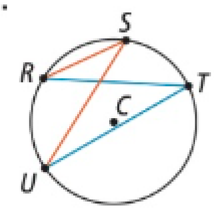


COROLLARIES TO THE INSCRIBED ANGLES THEOREM

Corollary 1

Two inscribed angles that intercept the same arc are congruent.

If...

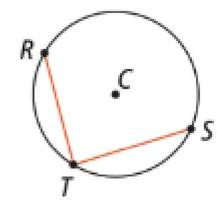


Then... $\angle S \cong \angle T$

Corollary 2

An angle inscribed in a semicircle is a right angle.

If... $m\widehat{RS} = 180$

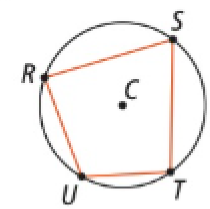


Then... $m\angle T = 90$

Corollary 3

The opposite angles of an inscribed quadrilateral are supplementary.

If...



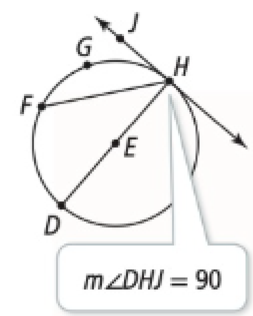
Then...

$m\angle R + m\angle T = 180$

$m\angle S + m\angle U = 180$

EXAMPLE 3 Explore Angles Formed by a Tangent and a Chord

Given chord $\overline{FH}$ and $\overleftrightarrow{HJ}$ tangent to $\odot E$ at point H, what is the relationship between $\angle FHJ$ and $\widehat{FGH}$?



Consider the angles and arcs formed by the chord, tangent line, and diameter.

Let $m\angle FHJ = x$, so $m\angle FHD = 90 - x$.

$$m\angle FHD = \frac{1}{2}m\widehat{DF}$$

$$90 - x = \frac{1}{2}m\widehat{DF}$$

$$m\widehat{DF} = 180 - 2x$$

Use the Inscribed Angles Theorem.

COMMON ERROR
Be careful not to assume arc measure relationships such as assuming $m\widehat{DF} = m\widehat{FH}$. Think about concepts and theorems you can apply when writing mathematical statements.

Since $\overline{DH}$ is a diameter, $m\widehat{DFH} = 180$.

$$m\widehat{DF} + m\widehat{FGH} = m\widehat{DFH}$$

$$180 - 2x + m\widehat{FGH} = 180$$

$$m\widehat{FGH} = 2x$$

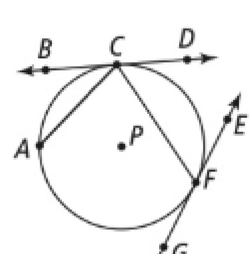
$$m\widehat{FGH} = 2m\angle FHJ$$

$$m\angle FHJ = \frac{1}{2}m\widehat{FGH}$$

Try It!

3. a. Given $\overleftrightarrow{BD}$ tangent to $\odot P$ at point C, if $m\widehat{AC} = 88$, what is $m\angle ACB$?

b. Given $\overleftrightarrow{EG}$ tangent to $\odot P$ at point F, if $m\angle GFC = 115$, what is $m\widehat{FAC}$?

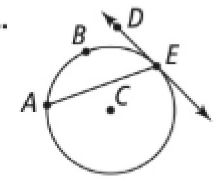


THEOREM 10-9

The measure of an angle formed by a tangent and a chord is half the measure of its intercepted arc.

PROOF: SEE EXERCISE 34.

If...

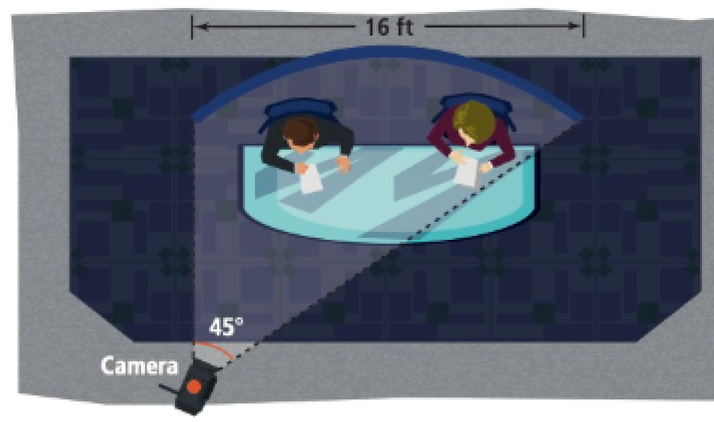


Then... $m\angle AED = \frac{1}{2}m\widehat{ABE}$

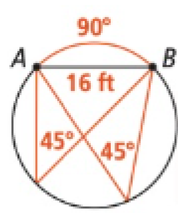
APPLICATION

EXAMPLE 4 Use Arc Measure to Solve a Problem

A director wants to position two cameras to capture an entire circular backdrop behind two newscasters. Where should he position the cameras?



Formulate ◀ Represent the set as a chord $\overline{AB}$ of a circle that intercepts an arc measuring 90.

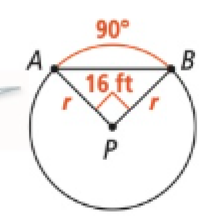


Any point on the major arc $\widehat{AB}$ is the vertex of a 45° angle that intercepts arc $\widehat{AB}$.

To find the size of the circle, find the radius of the circle.

Compute ◀ Let P be the center of the circle.

$m\angle APB = m\widehat{AB} = 90$



Find r.

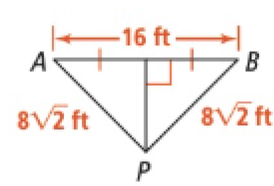
$$\sqrt{2} \cdot r = 16$$

$$r = \frac{16}{\sqrt{2}}$$

$$r = 8\sqrt{2}$$

$\triangle APB$ is a 45°-45°-90° triangle, and the length of the hypotenuse is 16.

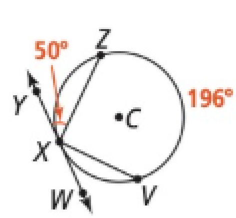
Center P is on the perpendicular bisector of $\overline{AB}$, so P is 8 ft from the midpoint of $\overline{AB}$.



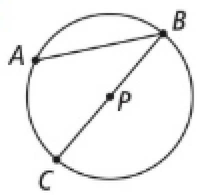
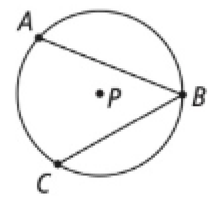
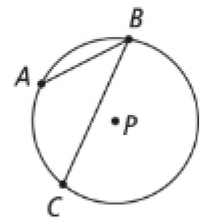
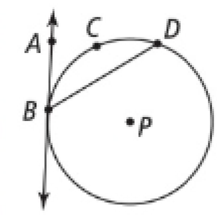
Interpret ◀ Position a camera on any point of circle with radius $8\sqrt{2}$ ft and center 8 ft from the midpoint of the set.

Try It! 4. a. Given $\overleftrightarrow{WY}$ tangent to ⊙C at point X, what is $m\widehat{XZ}$?

b. What is $m\angle VXW$?



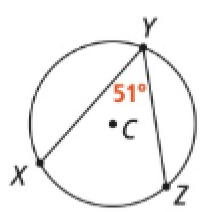
CONCEPT SUMMARY Inscribed Angles and Intercepted Arcs

	Inscribed Angles	Angles Formed by a Tangent and a Chord
WORDS	The measure of an inscribed angle is one-half the measure of its intercepted arc.	The measure of an angle formed by a tangent and a chord is one-half the measure of its intercepted arc.
DIAGRAMS	  	
SYMBOLS	$m\angle ABC = \frac{1}{2}m\widehat{AC}$	$m\angle ABD = \frac{1}{2}m\widehat{BCD}$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How is the measure of an inscribed angle related to its intercepted arc?

2. **Error Analysis** Darren is asked to find $m\widehat{XZ}$. What is his error?



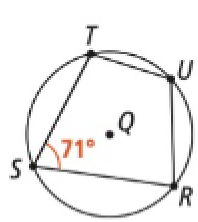
$m\widehat{XZ} = \frac{1}{2}\, m\angle XYZ$

$= \frac{1}{2}(51)$ ✗

$= 25.5$

3. **Represent and Connect** Can the measure of an inscribed angle be greater than the measure of the intercepted arc? Explain.

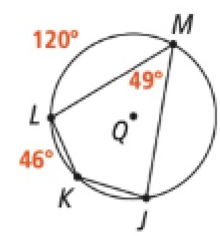
4. **Analyze and Persevere** Is there enough information in the diagram to find $m\widehat{RST}$? Explain.



Do You KNOW HOW?

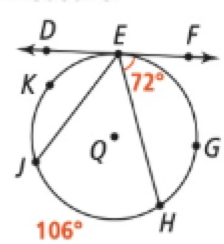
For Exercises 5–8, find each measure in ⊙*Q*.

5. $m\widehat{JL}$
6. $m\widehat{MJ}$
7. $m\angle KJM$
8. $m\angle KLM$



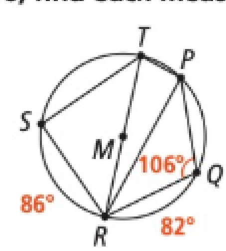
For Exercises 9–12, $\overleftrightarrow{DF}$ is tangent to ⊙*Q* at point *E*. Find each measure.

9. $m\widehat{EH}$
10. $m\widehat{EJ}$
11. $m\angle HEJ$
12. $m\angle DEJ$



For Exercises 13–16, find each measure in ⊙*M*.

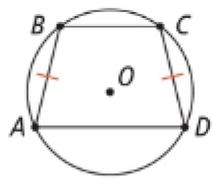
13. $m\angle PRQ$
14. $m\angle PTR$
15. $m\angle RST$
16. $m\angle SRT$



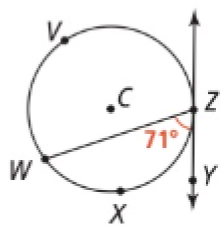
PRACTICE & PROBLEM SOLVING

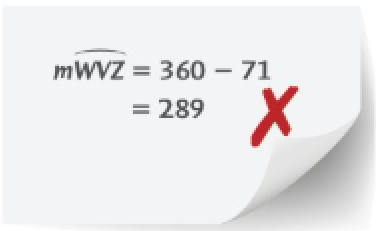
UNDERSTAND

17. Mathematical Connections Given $m\widehat{ABC} = x$, what is an expression for $m\widehat{DAB}$ in terms of x? Explain.

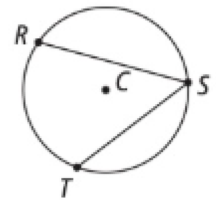


18. Error Analysis Casey is asked to find $m\widehat{WVZ}$. What is Casey's error?





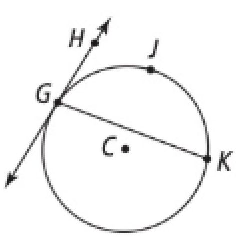
19. Higher Order Thinking Write a proof of the Inscribed Angles Theorem, Case 2.



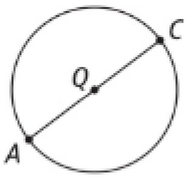
Given: Center C is inside $\angle RST$.

Prove: $m\angle RST = \frac{1}{2}m\widehat{RT}$

20. Communicate and Justify Margaret measures $\angle HGK$ with a protractor and says that it is 98°. Is Margaret's answer reasonable? Explain.



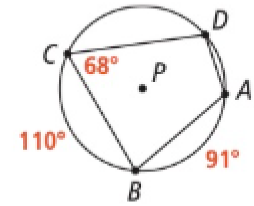
21. Use Patterns and Structure Given $\odot Q$ with diameter $\overline{AC}$, if point B is located on $\odot Q$, can $\angle ABC$ ever be less than 90°? Can it ever be greater than 90°? Explain.



PRACTICE

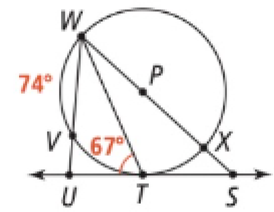
For Exercises 22–25, find each measure in $\odot P$.
SEE EXAMPLES 1 AND 2

22. $m\widehat{AD}$
23. $m\widehat{BDC}$
24. $m\angle ADC$
25. $m\angle BAD$



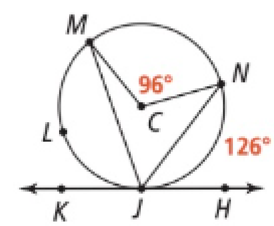
For Exercises 26–28, $\overleftrightarrow{SU}$ is tangent to $\odot P$ at point T. Find each measure. SEE EXAMPLES 2 AND 3

26. $m\widehat{TW}$
27. $m\angle TWX$
28. $m\angle TWV$



For Exercises 29–31, $\overleftrightarrow{HK}$ is tangent to $\odot C$ at point J. Find each measure. SEE EXAMPLES 3 AND 4

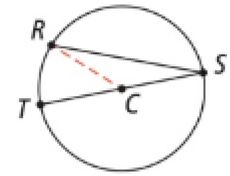
29. $m\angle KJM$
30. $m\angle MJN$
31. $m\angle HJN$



32. Write a proof of the Inscribed Angles Theorem, Case 1.

Given: Center C is on $\overline{ST}$.

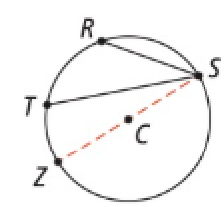
Prove: $m\angle RST = \frac{1}{2}m\widehat{RT}$



33. Write a proof of the Inscribed Angles Theorem, Case 3.

Given: Center C is outside $\angle RST$.

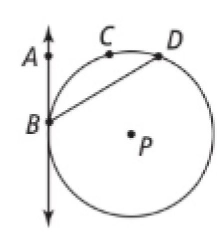
Prove: $m\angle RST = \frac{1}{2}m\widehat{RT}$



34. Write a two-column proof of Theorem 10-9.

Given: $\overleftrightarrow{AB}$ tangent to $\odot P$ at point B.

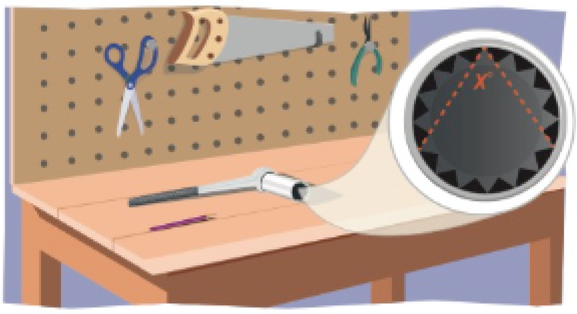
Prove: $m\angle ABD = \frac{1}{2}m\widehat{BCD}$



PRACTICE & PROBLEM SOLVING

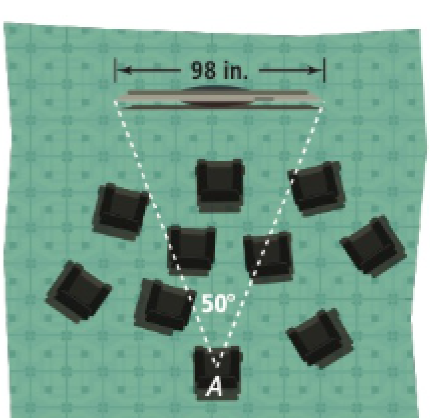
APPLY

35. Communicate and Justify Deondra needs to know the angle measure for each notch in the 16-notch socket wrench she is designing. The notches will be the same size. What is the angle measure?



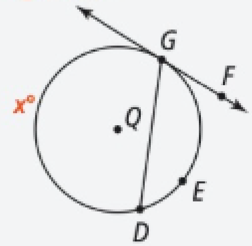
36. Use Patterns and Structure Cheyenne wants to make a replica of an antique sundial using the fragment of the sundial she acquired. Is there enough information for her to determine the diameter of the sundial? Explain.

37. Represent and Connect Malcom sets up chairs for a home theater showing on his television. His optimal viewing angle is 50°. Besides at chair A, where else could he sit with the same viewing angle? Draw a diagram and explain.

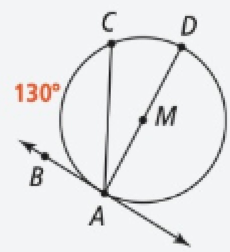


ASSESSMENT PRACTICE

38. Write an expression that represents $m\angle DGF$.
GR.6.1



39. SAT/ACT Segment AB is tangent to $\odot M$ at Point A. What is $m\angle DAC$?



Ⓐ 25

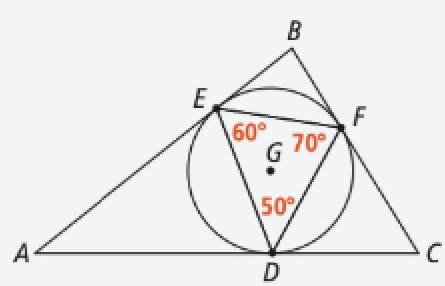
Ⓑ 65

Ⓒ 50

Ⓓ 90

Ⓔ 100

40. Performance Task Triangle DEF is inscribed in $\odot G$, and $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are tangent to $\odot G$.



Part A Are there any isosceles triangles in the diagram? If so, explain why the triangles are isosceles. If not, explain why not.

Part B Are $\triangle ABC$ and $\triangle DEF$ similar? Explain.

10-5 Secant Lines and Segments

I CAN… use angle measures and segment lengths formed by intersecting lines and circles to solve problems.

VOCABULARY

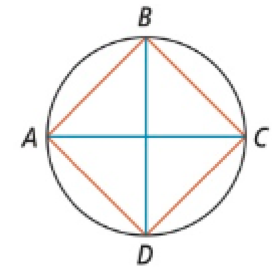
- secant

MA.912.GR.6.1–Solve mathematical and real-world problems involving the length of a secant, tangent, segment or chord in a given circle. Also GR.6.2
MA.K12.MTR.1.1, MTR.4.1, MTR.5.1

CONCEPTUAL UNDERSTANDING

EXPLORE & REASON

Skyler made the design shown. Points *A*, *B*, *C*, and *D* are spaced evenly around the circle.



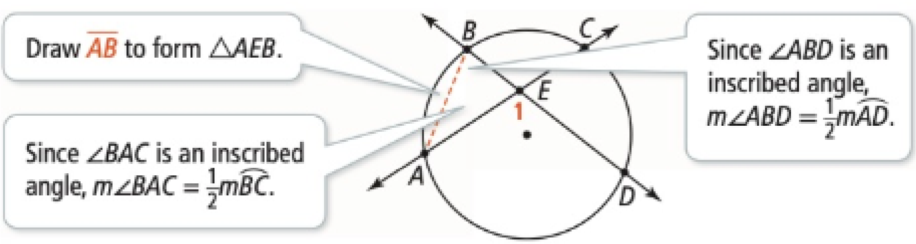
A. Using points *A*, *B*, *C*, and *D* as vertices, what congruent angles can you find? How can you justify that they are congruent?

B. Analyze and Persevere What strategy did you use to make sure you found all congruent angles?

ESSENTIAL QUESTION How are the measures of angles, arcs, and segments formed by intersecting secant lines related?

EXAMPLE 1 Relate Secants and Angle Measures

A secant is a line, ray, or segment that intersects a circle at two points. Secants $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect to form ∠1. How can you use arc measures to find $m\angle 1$?



Apply the Triangle Exterior Angle Theorem.

$$m\angle 1 = m\angle ABD + m\angle BAC$$
$$= \frac{1}{2}m\widehat{AD} + \frac{1}{2}m\widehat{BC}$$

So the measure of the angle is half the sum of the measures of the two intercepted arcs.

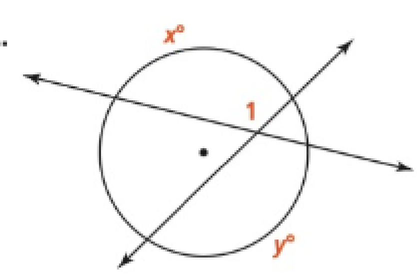
ANALYZE AND PERSEVERE Consider other relationships in the diagram. What is an alternate plan you could use to solve the problem?

Try It! 1. If $m\widehat{AD} = 155$ and $m\widehat{BC} = 61$, what is $m\angle 1$?

THEOREM 10-10

The measure of an angle formed by two secant lines that intersect inside a circle is half the sum of the measures of the intercepted arcs.

If…



Then… $m\angle 1 = \frac{1}{2}(x + y)$

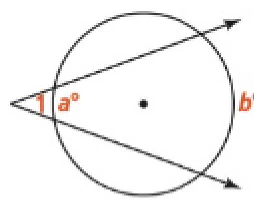
PROOF: SEE EXERCISE 18.

THEOREM 10-11

The measure of an angle formed by two lines that intersect outside a circle is half the difference of the measures of the intercepted arcs.

Case 1

If...

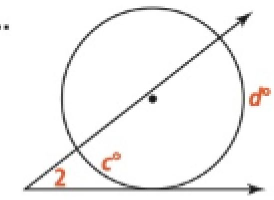


Then...

$m\angle 1 = \frac{1}{2}(b - a)$

Case 2

If...

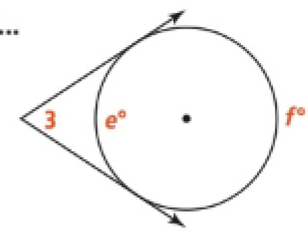


Then...

$m\angle 2 = \frac{1}{2}(d - c)$

Case 3

If...



Then...

$m\angle 3 = \frac{1}{2}(f - e)$

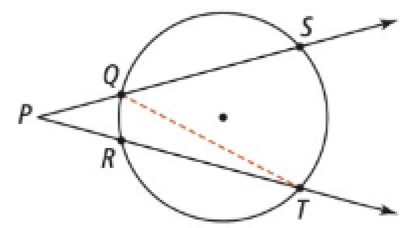
PROOF: SEE EXAMPLE 2, TRY IT 2, AND EXERCISE 19.

EXAMPLE 2 Prove Theorem 10-11, Case 1

Write a proof for Theorem 10-11, Case 1.

Given: Secants $\overrightarrow{PS}$ and $\overrightarrow{PT}$

Prove: $m\angle P = \frac{1}{2}(m\widehat{ST} - m\widehat{QR})$

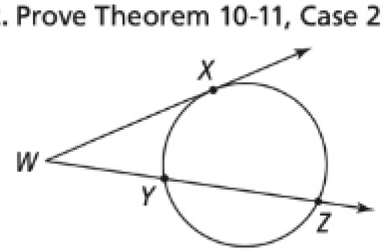


Proof:

Statement	Reason
1) $\overrightarrow{PS}$ and $\overrightarrow{PT}$ are secants.	**1)** Given
2) Draw $\overline{QT}$.	**2)** Two points determine a segment.
3) $m\angle QTP = \frac{1}{2}m\widehat{QR}$	**3)** Inscribed Angles Theorem
4) $m\angle SQT = \frac{1}{2}m\widehat{ST}$	**4)** Inscribed Angles Theorem
5) $m\angle SQT = m\angle P + m\angle QTP$	**5)** Triangle Exterior Angle Theorem
6) $m\angle P = m\angle SQT - m\angle QTP$	**6)** Subtraction Property of Equality
7) $m\angle P = \frac{1}{2}m\widehat{ST} - \frac{1}{2}m\widehat{QR}$	**7)** Substitution
8) $m\angle P = \frac{1}{2}(m\widehat{ST} - m\widehat{QR})$	**8)** Distributive Property

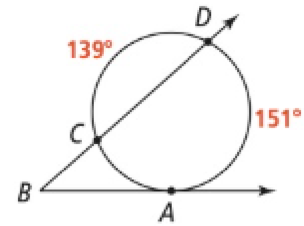
STUDY TIP
Remember to look for helpful relationships that you can draw on the given figure when completing a proof. Drawing $\overline{QT}$ forms inscribed angles, which are needed for this proof.

Try It! **2.** Prove Theorem 10-11, Case 2.



EXAMPLE 3 Use Secants and Tangents to Solve Problems

A. What is $m\angle ABD$?



Step 1 Find $m\widehat{AC}$.

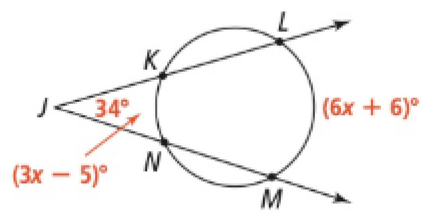
$$m\widehat{AC} = 360 - m\widehat{AD} - m\widehat{CD}$$
$$= 360 - 151 - 139$$
$$= 70$$

Step 2 Find $m\angle ABD$.

$$m\angle ABD = \frac{1}{2}(m\widehat{AD} - m\widehat{AC})$$
$$= \frac{1}{2}(151 - 70)$$
$$= 40.5$$

Since the angle is formed outside the circle by a secant and a tangent, apply Theorem 10-11, Case 2.

B. What is $m\widehat{LM}$?



COMMON ERROR
Remember to add the arc measures when the vertex is inside the circle and to subtract them when it is outside the circle.

Step 1 Find x.

$$m\angle LJM = \frac{1}{2}(m\widehat{LM} - m\widehat{KN})$$
$$34 = \frac{1}{2}((6x + 6) - (3x - 5))$$
$$34 = \frac{1}{2}(3x + 11)$$
$$68 = 3x + 11$$
$$19 = x$$

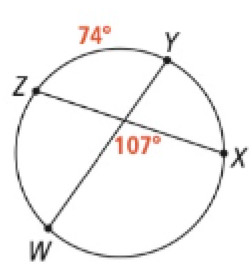
Since the angle is formed outside the circle by two secants, apply Theorem 10-11, Case 1.

Step 2 Find $m\widehat{LM}$.

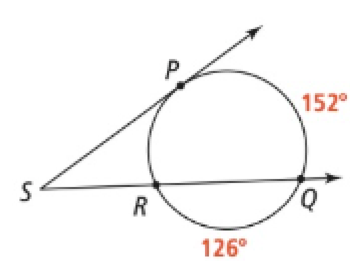
$$m\widehat{LM} = 6x + 6$$
$$= 6(19) + 6$$
$$= 120$$

Substitute the value of x found in Step 1.

Try It! 3. a. What is $m\widehat{WX}$?

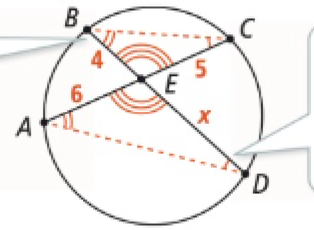


b. What is $m\angle PSQ$?



EXAMPLE 4 Develop Chord Length Relationships

What is the value of x?



By the Angle-Angle Similarity Theorem, $\triangle AED \sim \triangle BEC$.

$$\frac{ED}{EC} = \frac{EA}{EB}$$

The ratios of corresponding sides of similar triangles are equal.

$$\frac{x}{5} = \frac{6}{4}$$

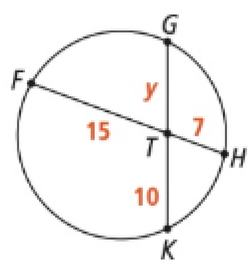
$$4 \bullet x = 6 \bullet 5$$

$$x = 7.5$$

The value of x is 7.5.

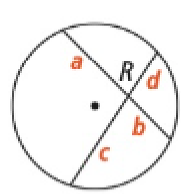
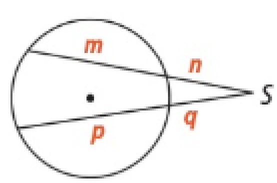
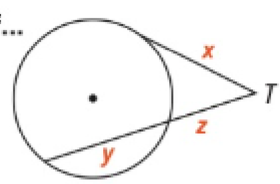
HAVE A GROWTH MINDSET
In what ways can you be inquisitive and open to learning new things?

Try It! **4.** What is the value of y?



THEOREM 10-12

For a given point and circle, the product of the lengths of the two segments from the point to the circle is constant along any line through the point and circle.

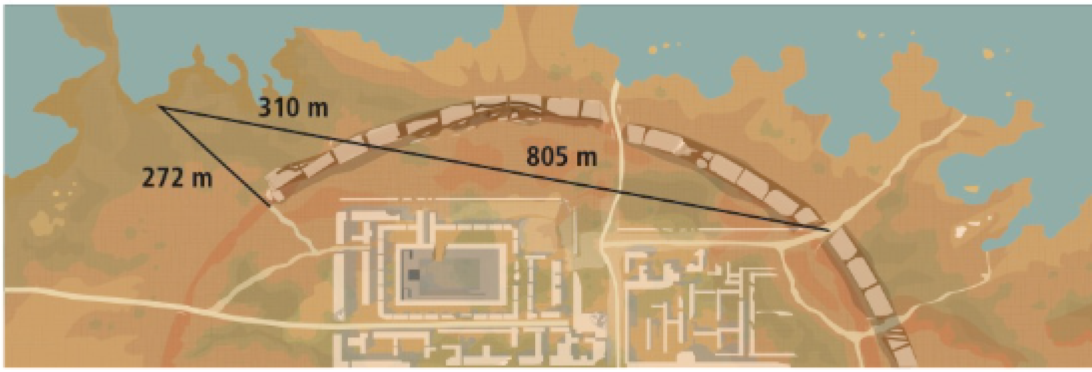
Case 1	Case 2	Case 3
If... 	If... 	If... 
Then... $ab = cd$	Then... $(n + m)n = (q + p)q$	Then... $x^2 = (z + y)z$

PROOF: SEE EXERCISES 11, 23, and 24.

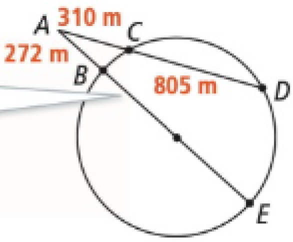
APPLICATION

EXAMPLE 5 Use Segment Relationships to Find Lengths

Archaeologists found part of the circular wall that surrounds an ancient city. They measure the distances shown. The 272-m segment lies on a line through the center of the circular wall. What was the diameter of the circular wall?



Draw a diagram to represent the situation.



The line through $\overline{AB}$ continues to the center of the circle, so $\overline{BE}$ is a diameter.

ANALYZE AND PERSEVERE
Are there other measurements that archaeologists could have taken to help find the diameter using another method?

Write an equation to relate the segment lengths.

$$(AB + BE)AB = (AC + CD)AC$$

$$(272 + BE)(272) = (310 + 805)(310)$$

Apply Theorem 10-12 and substitute known segment lengths.

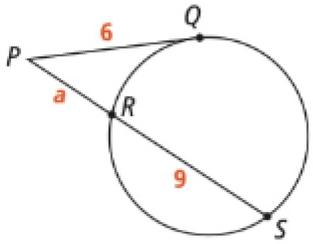
$$73{,}984 + 272 \cdot BE = 345{,}650$$

$$272 \cdot BE = 271{,}666$$

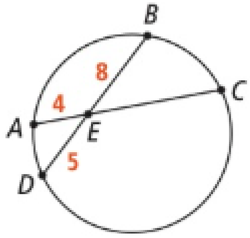
$$BE \approx 998.8$$

The diameter of the circular wall was about 998.8 meters.

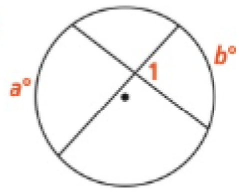
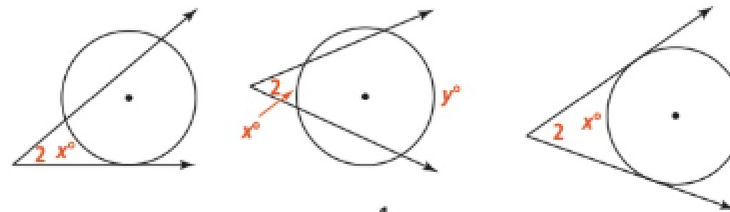
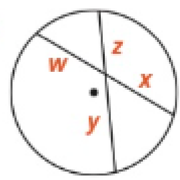
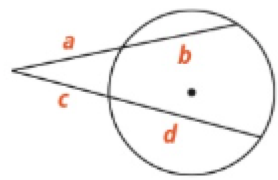
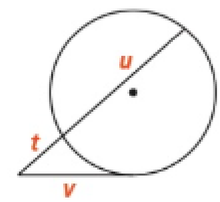
Try It! **5. a.** What is the value of a?



b. What is EC?



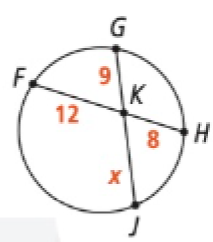
CONCEPT SUMMARY Angle and Segment Relationships in a Circle

	Vertex Inside the Circle	Vertex Outside the Circle
ANGLES	 $m\angle 1 = \frac{1}{2}(a + b)$	 $m\angle 2 = \frac{1}{2}(y - x)$
SEGMENTS	 $wx = yz$	 $(a + b)a = (c + d)c$  $(t + u)t = v^2$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the measures of angles, arcs, and segments formed by intersecting secant lines related?

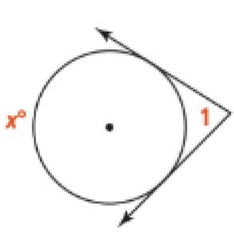
2. **Error Analysis** Derek is asked to find the value of *x*. What is his error?



$GK \cdot FK = HK \cdot JK$
$12 \cdot 9 = 8 \cdot x$
$x = 13\frac{1}{2}$ ✗

3. **Vocabulary** How are *secants* and *tangents* to a circle alike and different?

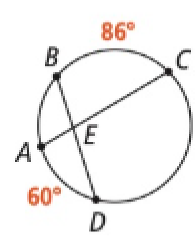
4. **Communicate and Justify** The rays shown are tangent to the circle. Show that $m\angle 1 = (x - 180)$.



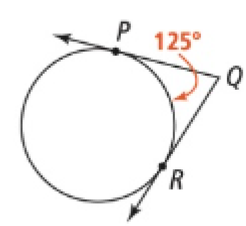
Do You KNOW HOW?

For Exercises 5 and 6, find each angle measure. Rays *QP* and *QR* are tangent to the circle in Exercise 6.

5. $m\angle BEC$

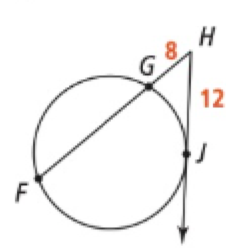


6. $m\angle PQR$

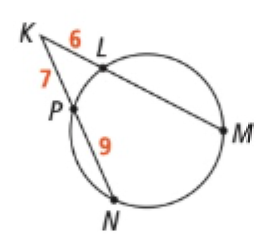


For Exercises 7 and 8, find each length. Ray *HJ* is tangent to the circle in Exercise 7.

7. *GF*



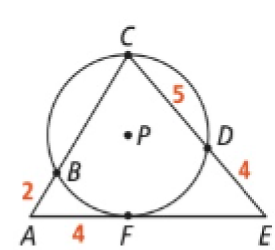
8. *LM*



For Exercises 9 and 10, $\overline{AE}$ is tangent to ⊙*P*. Find each length.

9. *BC*

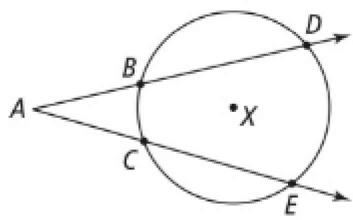
10. *EF*



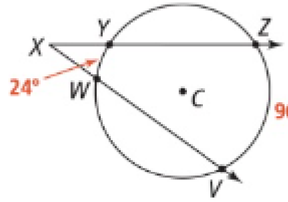
PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. Communicate and Justify Given ⊙X, write a two-column proof of Theorem 10-12, Case 2.



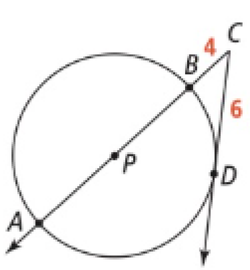
12. Error Analysis Cindy is asked to find $m\angle VXZ$. What is her error?



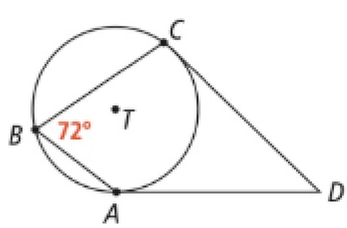
$$m\angle VXZ = \frac{1}{2}(m\widehat{WY} + m\widehat{VZ})$$
$$= \frac{1}{2}(24 + 96)$$
$$= 60$$

✗

13. Mathematical Connections Given ⊙P, secant $\overrightarrow{CA}$, and tangent $\overrightarrow{CD}$, what is the area of ⊙P?



14. Higher Order Thinking Given ⊙T, and tangents $\overline{AD}$ and $\overline{CD}$, what is the measure of $\angle ADC$?



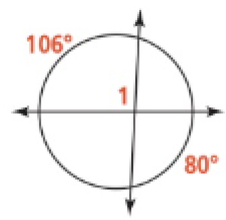
15. Represent and Connect How would you describe each case of Theorem 10-11?

PRACTICE

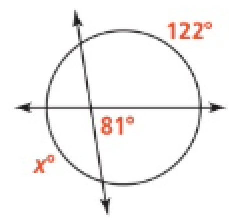
For Exercises 16 and 17, find each measure.

SEE EXERCISE 1

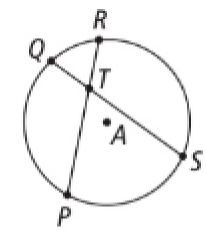
16. $m\angle 1$



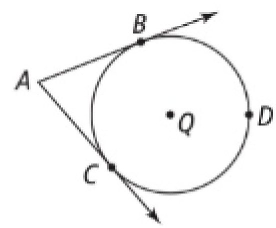
17. x



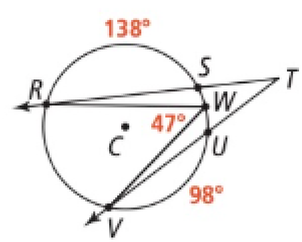
18. Given ⊙A and secants $\overline{PR}$ and $\overline{QS}$, write a paragraph proof of Theorem 10-10. SEE EXAMPLE 1



19. Given ⊙Q and tangents $\overrightarrow{AB}$ and $\overrightarrow{AC}$, write a two-column proof of Theorem 10-11, Case 3. SEE EXAMPLE 2



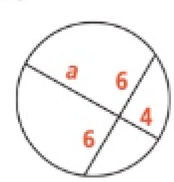
20. Given ⊙C, inscribed angle $\angle RWV$, and secants $\overrightarrow{TR}$ and $\overrightarrow{TV}$, what is the measure of $\angle RTV$? SEE EXAMPLE 3



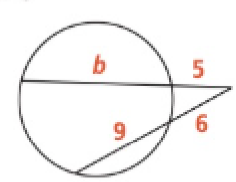
For Exercises 21 and 22, find each length.

SEE EXERCISE 4

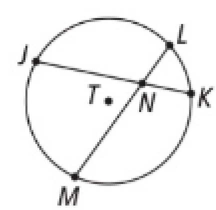
21. a



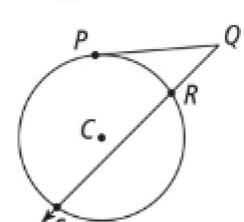
22. b



23. Given ⊙T and secants $\overline{JK}$ and $\overline{LM}$ intersecting at point N, write a paragraph proof of Theorem 10-12, Case 1. SEE EXAMPLE 4



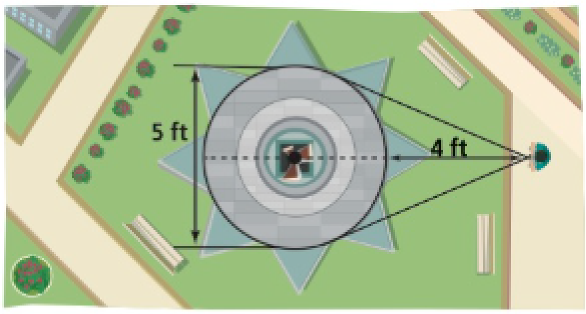
24. Given ⊙C, secant $\overrightarrow{QS}$ and tangent $\overline{PQ}$, write a two-column proof of Theorem 10–12, Case 3. SEE EXAMPLE 5



PRACTICE & PROBLEM SOLVING

APPLY

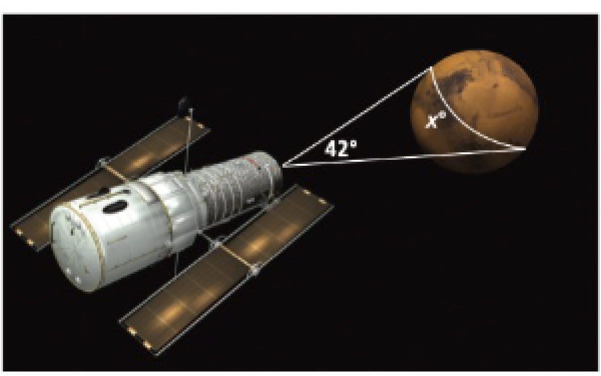
25. **Use Patterns and Structure** Chris stands in the position shown to take a picture of a sculpture with a circular base.



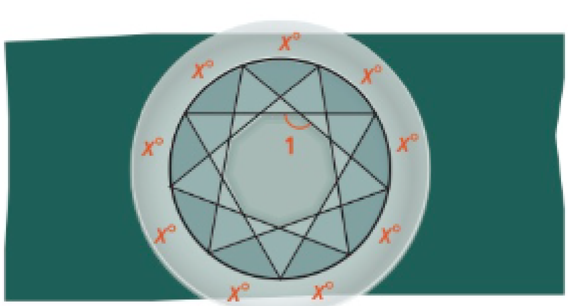
a. Chris is deciding on which lens to use. What is the minimum view angle from where he stands so he can get as much of the base as possible in his picture?

b. If Chris uses a lens with a view angle of 40°, what is the shortest distance he could stand from the sculpture?

26. **Generalize** A satellite orbits above the equator of Mars as shown and transmits images back to a scientist in the control room. What percent of the equator is the scientist able to see? Explain.



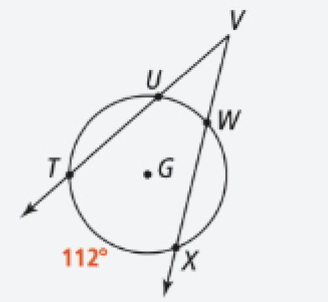
27. **Use Patterns and Structure** Carolina wants to etch the design shown onto a circular piece of glass. At what measure should she cut ∠1? Explain.



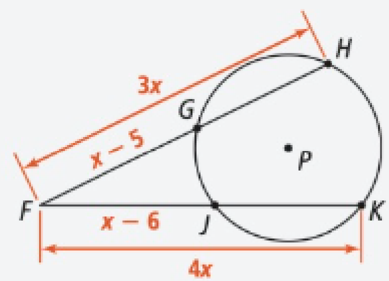
ASSESSMENT PRACTICE

28. For what measure of $\widehat{UW}$ does $m\angle TVX = 34$?
GR.6.1

$m\widehat{UW} =$ ________

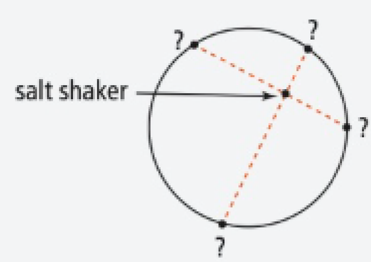


29. **SAT/ACT** Given ⊙P and secants $\overline{FH}$ and $\overline{FK}$, what is FG?



Ⓐ 3 Ⓑ 4 Ⓒ 9 Ⓓ 27 Ⓔ 36

30. **Performance Task** Alberto, Benson, Charles, and Deon sit at a round lunch table with diameter 54 inches. The salt shaker is 27 inches from Charles, 18 inches from Benson, 20 inches from Deon, and 30 inches from Alberto.



Part A In what order around the table are they seated? Explain.

Part B Alberto, Benson, Charles, and Deon change the positions of their seats and sit evenly spaced around the table. If the location of the salt shaker does not change, what is the closest that one of them could be from the salt shaker? What is the farthest?

TOPIC 10

Topic Review

TOPIC ESSENTIAL QUESTION

1. When a line or lines intersect a circle how are the figures formed related to the radius, circumference, and area of the circle?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A(n) __________ is a region of a circle with two radii and an arc of the circle as borders.
3. A(n) __________ is an angle with its vertex on the circle.
4. A(n) __________ and the corresponding circle have exactly one point in common.
5. When both rays of an angle intersect a circle, the __________ is the portion of the circle between the rays.

- central angle
- chord
- inscribed angle
- intercepted arc
- secant
- sector of a circle
- segment of a circle
- tangent to a circle

Concepts & Skills Review

LESSON 10-1 Arcs and Sectors

Quick Review

Arc length and the area of a **sector of a circle** are proportional to the corresponding central angle.

The length of an arc is $\frac{n}{360}$ of the circumference, where n is the measure of the central angle in degrees, and the area of a sector is $\frac{n}{360}$ of the area of the circle.

Example

Circle J has a radius of 6 cm. What is the area of a sector with a central angle of 80°?

Write the formula for the area of a sector:

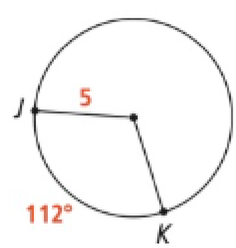
$$A = \frac{n}{360} \cdot \pi r^2$$
$$= \frac{80}{360} \cdot \pi(6)^2$$
$$\approx 25.1$$

The area of the sector is about 25.1 cm^2.

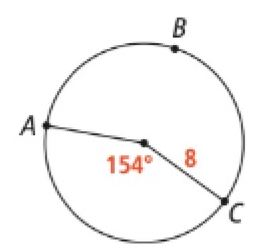
Practice & Problem Solving

Find each arc length in terms of π.

6. $\widehat{JK}$

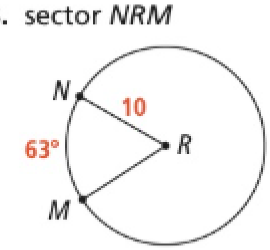


7. $\widehat{ABC}$

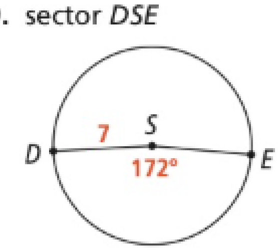


Find the area of each sector in terms of π.

8. sector NRM



9. sector DSE



10. **Represent and Connect** If you know the circumference of a circle and the area of a sector of the circle, how could you determine the central angle of the sector? Explain.

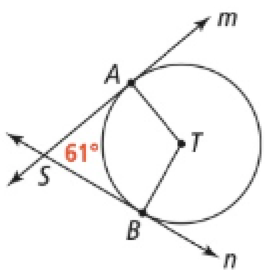
LESSON 10-2 Lines Tangent to a Circle

Quick Review

A **tangent to a circle** is perpendicular to the radius of the circle at the **point of tangency**. You can use properties of right triangles to solve problems involving tangents.

Example

Lines m and n are tangent to $\odot T$. What is $m\angle ATB$?



Since lines m and n are tangent lines, $m\angle SAT = m\angle SBT = 90$.

Points A, T, B, and S form a quadrilateral, so use the angle sum of a quadrilateral to solve the problem.

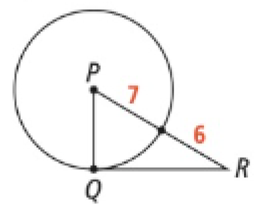
$$m\angle SAT + m\angle ASB + m\angle SBT + m\angle ATB = 360$$
$$90 + 61 + 90 + m\angle ATB = 360$$
$$m\angle ATB = 119$$

So, $m\angle ATB = 119$.

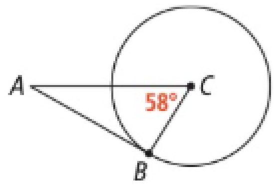
Practice & Problem Solving

For Exercises 11–12, $\overline{QR}$ and $\overline{AB}$ are tangent to the circle. Find each value.

11. QR

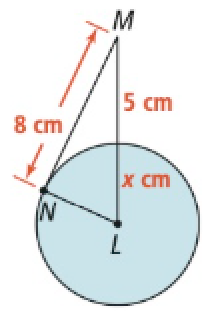


12. $m\angle CAB$



13. **Communicate and Justify** If $\overline{GH}$ is a diameter of $\odot T$, is it possible to draw tangents to G and H from the same point external to $\odot T$? Explain.

14. Segment MN is tangent to $\odot L$. What is the radius of the circle? Explain.



15. The line $y = -0.75x + 1.25$ is tangent to a circle whose center is located at (2, 6). Find the tangent point and a second tangent point of a line with the same slope as the given line.

LESSON 10-3 Chords

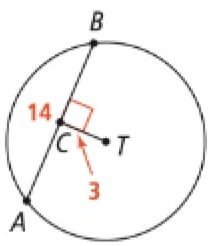
Quick Review

Chords in a circle have the following properties:

- Two chords in the same circle with the same length have congruent central angles, have congruent arcs, and are equidistant from the center of the circle.
- Chords are bisected by the diameter of the circle that is perpendicular to the chord.

Example

What is the radius of ⊙*T*?



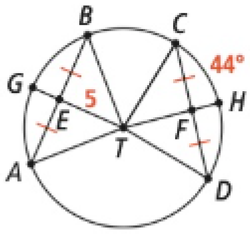
Since $\overline{CT} \perp \overline{AB}$, $\overline{CT}$ bisects $\overline{AB}$. So, $CB = 7$. Use the Pythagorean Theorem to find the radius.

$$(CT)^2 + (CB)^2 = (BT)^2$$
$$3^2 + 7^2 = (BT)^2$$
$$BT \approx 7.6$$

The radius of ⊙*T* is about 7.6.

Practice & Problem Solving

For Exercises 16–18, the radius of ⊙*T* is 7. Find each value. Round to the nearest tenth.



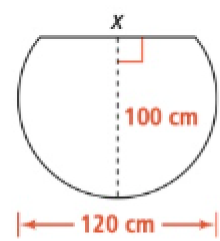
16. *FH*

17. *CD*

18. $m\angle BTA$

19. **Use Patterns and Structure** Circles *T* and *S* intersect at points *A* and *B*. What is the relationship between $\overline{AB}$ and $\overleftrightarrow{TS}$? Explain.

20. A contractor cuts off part of a circular countertop so that it fits against a wall. What should be the length *x* of the cut? Round to the nearest tenth.



LESSON 10-4 Inscribed Angles

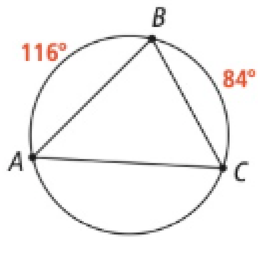
Quick Review

The measure of an **inscribed angle** is half the measure of its intercepted arc. As a result:

- Opposite angles of an inscribed quadrilateral are supplementary.
- The measure of an angle formed by a tangent and chord is half the measure of the intercepted arc.

Example

What are the angle measures of $\triangle ABC$?



Use inscribed angles:

$m\angle BAC = \frac{1}{2}(84) = 42$

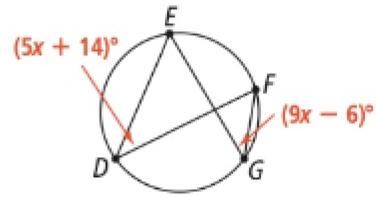
$m\angle BCA = \frac{1}{2}(116) = 58$

$m\angle ABC = 180 - 42 - 58 = 80$

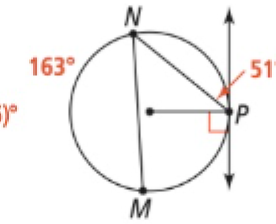
Practice & Problem Solving

Find each value.

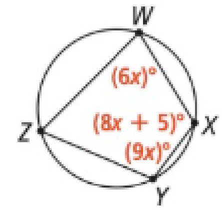
21. $m\widehat{EF}$



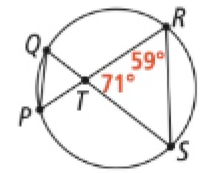
22. $m\angle MNP$



23. $m\angle WZY$



24. $m\angle QPR$



25. **Generalize** If a rectangle is inscribed in a circle, what must be true about the diagonals of the rectangle? Explain.

LESSON 10-5 Secant Lines and Segments

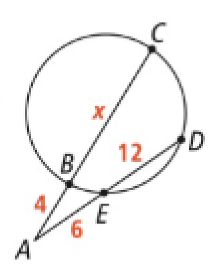
Quick Review

Secant lines form angles with special relationships:

- The measure of an angle formed by secants intersecting inside a circle is half the sum of the measures of the intercepted arcs.
- The measure of an angle formed by secants intersecting outside a circle is half the difference of the measures of the intercepted arcs.

Example

What is the value of x?



Use secant segment relationships.

$(AE)(AE + ED) = (AB)(AB + BC)$

$6(6 + 12) = 4(4 + x)$

$108 = 16 + 4x$

$92 = 4x$

$23 = x$

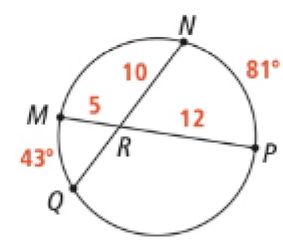
The value of x is 23.

Practice & Problem Solving

For Exercises 26 and 27, find each value in the figure shown.

26. QR

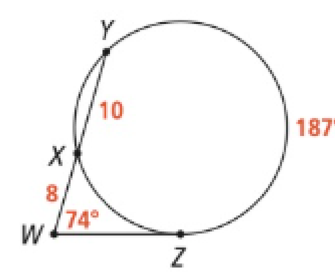
27. $m\angle NRP$



For Exercises 28 and 29, find each value in the figure shown. The segment $\overline{WZ}$ is tangent to the circle.

28. WZ

29. $m\widehat{XZ}$



30. **Communicate and Justify** A student said that if $\angle A$ is formed by two secants intersecting outside of a circle, then $m\angle A < 90$. Do you agree? Explain.

TOPIC
11

Two- and Three-Dimensional Models

TOPIC ESSENTIAL QUESTION

How is Cavalieri's Principle helpful in understanding the volume formulas for solids?

Topic Overview

enVision® STEM Project:
Design a Rigid Package

11-1 Three-Dimensional Figures and Cross Sections
GR.4.1, GR.4.2, MTR.1.1, MTR.2.1, MTR.3.1

11-2 Surface Area
GR.4.6, GR.4.3, MTR.2.1, MTR.4.1, MTR.5.1

11-3 Volumes of Prisms and Cylinders
GR.4.5, GR.4.3, GR.4.4, MTR.2.1, MTR.6.1, MTR.7.1

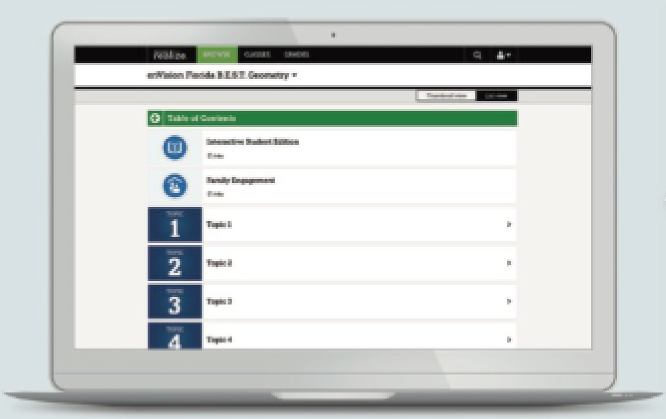
Mathematical Modeling in 3 Acts:
Box 'Em Up
GR.4.5, GR.4.6, MTR.7.1

11-4 Volumes of Pyramids and Cones
GR.4.5, GR.4.4, MTR.5.1, MTR.1.1, MTR.4.1

11-5 Volumes of Spheres
GR.4.5, MTR.1.1, MTR.3.1, MTR.7.1

Topic Vocabulary

- Cavalieri's Principle
- hemisphere
- oblique cylinder
- oblique prism



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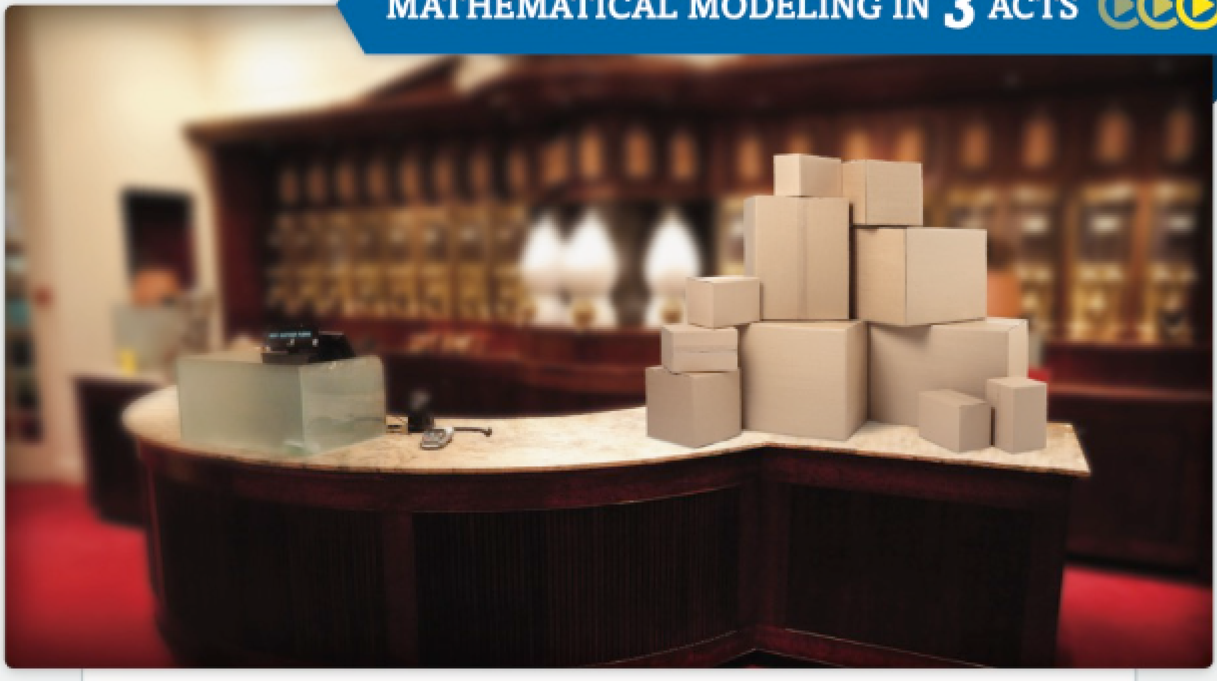
ANIMATION View and interact with real-world applications.

PRACTICE Practice what you've learned.

TOPIC

MATHEMATICAL MODELING IN 3 ACTS



Box 'Em Up

With so many people and businesses shopping online, retailers, and especially e-retailers, ship more and more packages every day. Some of the products people order have unusual sizes and shapes and need custom packaging. Imagine how you might package a surfboard, or a snow blower, or even live crawfish to ship to someone's house!

Think about this during the Mathematical Modeling in 3 Acts lesson.

VIDEOS Watch clips to support ***Mathematical Modeling in 3 Acts Lessons*** and **enVision®** ***STEM Projects***.

ADAPTIVE PRACTICE Practice that is *just right* and *just for you*.

GLOSSARY Read and listen to English and Spanish definitions.

CONCEPT SUMMARY Review key lesson content through multiple representations.

ASSESSMENT Show what you've learned.

TUTORIALS Get help from ***Virtual Nerd***, right when you need it.

MATH TOOLS Explore math with digital tools and manipulatives.

DESMOS Use Anytime and as embedded Interactives in Lesson content.

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Did You Know?

Cardboard boxes look simple, but the machines that make them are not. This **cartoning machine** is capable of making **15,000 boxes per day**.

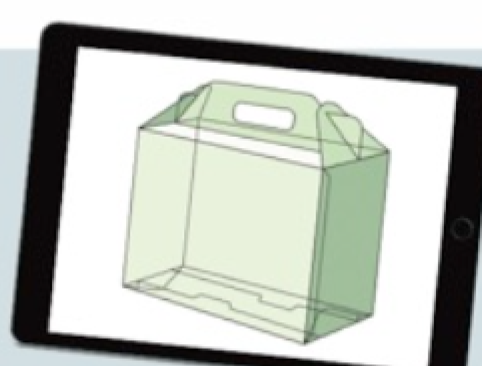
Packages come in many shapes. The familiar milk carton shape is called a **gable-top carton** because of its resemblance to a house gable. **Cylindrical** packaging is often used for sugar, tea and grains that don't have rigid shapes.

Manufacturers consider many factors when designing a package.

- ☑ Marketing appeal
- ☑ Cost of materials
- ☑ Simplicity
- ☑ Safety
- ☑ Recyclability

Your Task: Design a Rigid Package

You will design a rigid package for a product of your choice. Your design will address factors such as attractiveness, protection for the product, and cost. You will then draw two- and three-dimensional representations of your package and build a prototype.



11-1 Three-Dimensional Figures and Cross Sections

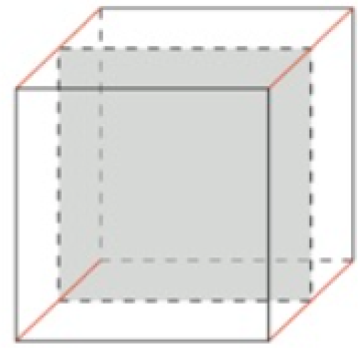
I CAN… identify three-dimensional figures and their relationships with polygons to solve problems.

MA.912.GR.4.1–Identify the shapes of two-dimensional cross-sections of three-dimensional figures. Also GR.4.2
MA.K12.MTR.1.1, MTR.2.1, MTR.3.1

CONCEPTUAL UNDERSTANDING

EXPLORE & REASON

Consider a cube of cheese. If you slice straight down through the midpoints of four parallel edges of the cube, the outline of the newly exposed surface is a square.



A. How would you slice the cube to expose a triangular surface?

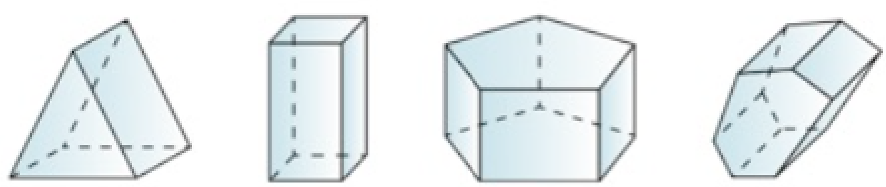
B. Choose Efficient Methods How would you slice the cube to expose a triangular surface with the greatest possible area?

ESSENTIAL QUESTION

How are three-dimensional figures and polygons related?

EXAMPLE 1 Develop Euler's Formula

How many faces, vertices, and edges does each prism contain? Do you notice any patterns in these quantities?



Make a table of the number of vertices, edges, and faces for each prism. Look for patterns and relationships.

Type of Prism	Faces (*F*)	Vertices (*V*)	Edges (*E*)
triangular	5	6	9
rectangular	6	8	12
pentagonal	7	10	15
hexagonal	8	12	18

For each additional face on the prism, the prism gains 2 vertices and 3 edges.

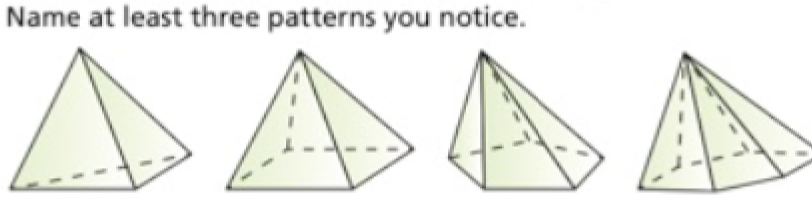
REPRESENT AND CONNECT
Look at the relationships between the number of vertices, faces, and edges. How might you represent these relationships in an equation?

Look at the sums of the faces and vertices. Compare it to the number of edges.

$5 + 6 = 9 + 2$
$6 + 8 = 12 + 2$
$7 + 10 = 15 + 2$
$8 + 12 = 18 + 2$

The sum of the faces and vertices is always 2 more than the number of edges.

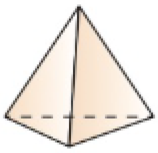
Try It! 1. How many faces, vertices, and edges do the pyramids have? Name at least three patterns you notice.



CONCEPT Euler's Formula

The sum of the number of faces (F) and vertices (V) of a polyhedron is 2 more than the number of its edges (E).

$$F + V = E + 2$$



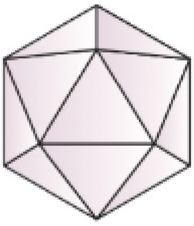
$$F + V = E + 2$$
$$4 + 4 = E + 2$$
$$E = 6$$

APPLICATION

COMMON ERROR
Remember to add the number of faces and vertices on one side of the equation and to add the number of edges plus 2 on the other side of the equation.

EXAMPLE 2 Apply Euler's Formula

To make polyhedron-shaped game pieces using a 3D printer, Juanita enters the number of faces, edges, and vertices into a program. If she wants a game piece with 20 faces and 30 edges, how many vertices does the piece have?



$$F + V = E + 2$$
$$20 + V = 30 + 2$$
$$V = 12$$

Apply Euler's Formula.

The game piece has 12 vertices.

Try It! **2. a.** A polyhedron has 12 faces and 30 edges. How many vertices does it have?

b. Can a polyhedron have 4 faces, 5 vertices, and 8 edges? Explain.

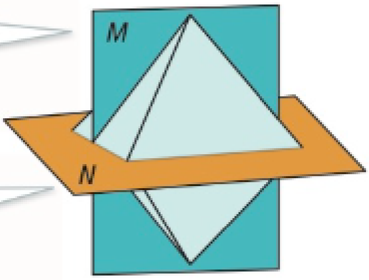
STUDY TIP
Recall that a *cross section* is the intersection of a solid and a plane.

EXAMPLE 3 Describe a Cross Section

Plane *M* and plane *N* intersect the regular octahedron as shown. What is the shape of each cross section?

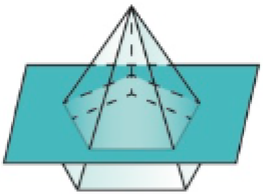
Plane *M* slices the octahedron in half through the top and bottom vertices. The cross section is a rhombus.

Plane *N* slices horizontally between the bases of two square pyramids. The cross section is a square.



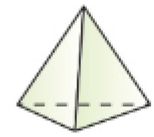
Try It! **3. a.** What shape is the cross section shown?

b. What shape is the cross section if the plane is perpendicular to the base and passes through the vertex of the pyramid?

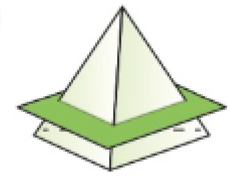


EXAMPLE 4 Draw a Cross Section

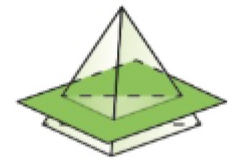
A plane intersects a tetrahedron parallel to the base. How do you draw the cross section?



Step 1 Visualize the plane intersecting the tetrahedron.

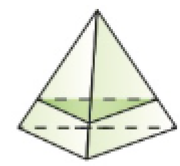


Step 2 Draw lines where the plane cuts the surface of the polyhedron.



STUDY TIP
It is not possible for a polyhedron with *n* faces to have a cross section with more than *n* sides.

Step 3 Shade the cross section.



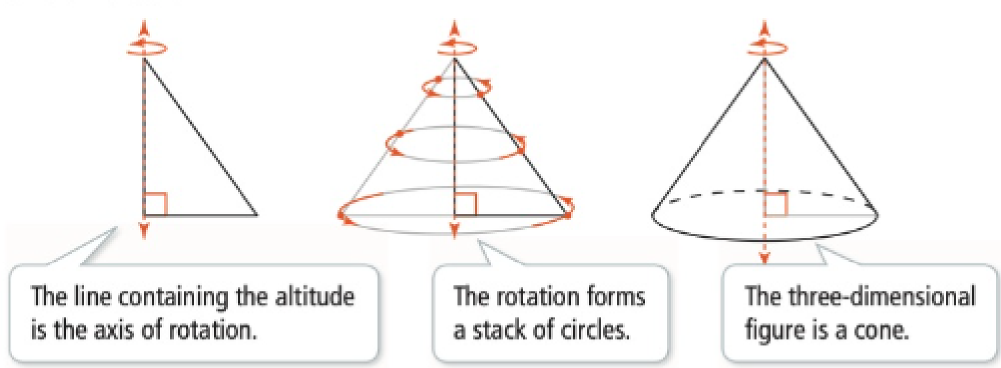
Try It! **4. a.** Draw the cross section of a plane intersecting the tetrahedron through the top vertex and perpendicular to the base.

b. Draw the cross section of a plane intersecting a hexagonal prism perpendicular to the base.

EXAMPLE 5 Rotate a Polygon to Form a Three-Dimensional Figure

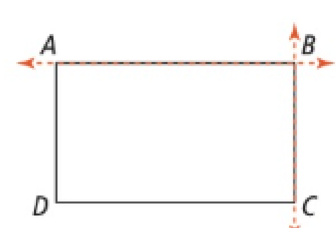
If you rotate a right triangle about the altitude, what three-dimensional figure does the triangle form?

As the triangle rotates, each point on the sides traces out a circle about the axis of rotation.

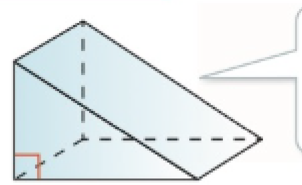
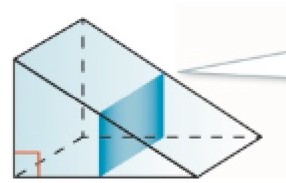
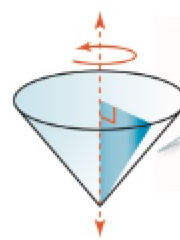


Try It! **5. a.** What three-dimensional figure is formed by rotating rectangle *ABCD* about $\overline{BC}$?

b. What three-dimensional figure is formed by rotating rectangle *ABCD* about $\overline{CD}$?



CONCEPT SUMMARY Polyhedrons, Cross Sections, and Rotating a Polygon

WORDS	DIAGRAMS
Euler's Formula The faces of a polyhedron are polygons. The sum of the number of faces F and vertices V of a polyhedron is 2 more than the number of its edges E. $F + V = E + 2$	 $F + V = E + 2$ $5 + 6 \stackrel{?}{=} 9 + 2$ $11 = 11$
Cross Sections A cross section is the intersection of a plane and a solid. The cross section of a plane and a convex polyhedron is a polygon.	 A cross section perpendicular to the base of the triangular prism is a rectangle.
Rotation of Polygons Rotating a polygon about an axis forms a three-dimensional figure with at least one circular cross section.	 The right triangle is rotated about the line containing a leg to form a right cone.

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are three-dimensional figures and polygons related?

2. **Error Analysis** Nicholas drew a figure to find a cross section of an icosahedron, a polyhedron with 20 faces. What is Nicholas's error?

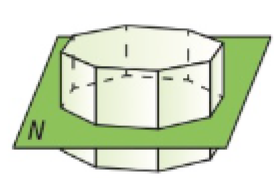
3. **Analyze and Persevere** Can a polyhedron have 3 faces, 4 vertices, and 5 edges? Explain.

Do You KNOW HOW?

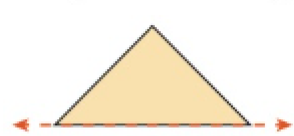
For Exercises 4–7, copy and complete the table.

	Faces	Vertices	Edges
4.	5	6	
5.	8		18
6.		12	20
7.	22	44	

8. What polygon is formed by the intersection of plane N and the octagonal prism shown?



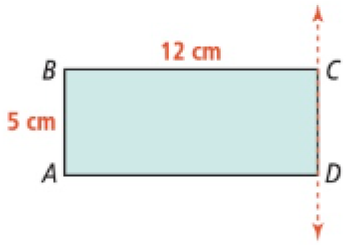
9. Describe the three-dimensional figure that is formed from rotating the isosceles right triangle about the hypotenuse.



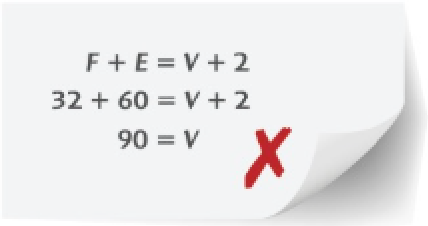
PRACTICE & PROBLEM SOLVING

UNDERSTAND

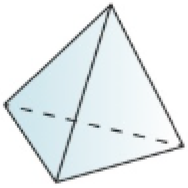
10. **Mathematical Connections** If you rotate rectangle *ABCD* about $\overleftrightarrow{CD}$, what is the volume of the resulting three-dimensional figure?



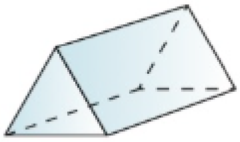
11. **Error Analysis** Philip was asked to find the number of vertices of a polyhedron with 32 faces and 60 edges. What is his error?



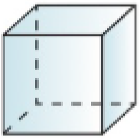
12. **Analyze and Persevere** A tetrahedron is a polyhedron with four triangular faces. Can a plane intersect the tetrahedron shown to form a cross section with four sides? Explain.



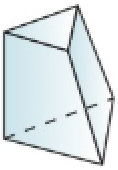
13. **Represent and Connect** Can the intersection of a plane and a triangular prism produce a rectangular cross section? Draw a diagram to explain.



14. **Communicate and Justify** Is it possible to rotate a polygon to form a cube? Explain.



15. **Higher Order Thinking** Does the figure have a cross section with five sides? Copy the figure and draw the cross section or explain why not.



PRACTICE

For Exercises 16–20, find the missing number for each polyhedron. SEE EXAMPLES 1 AND 2

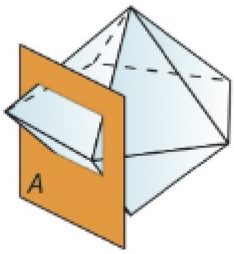
16. A polyhedron has 24 edges and 12 vertices. How many faces does it have?

17. A polyhedron has 20 faces and 12 vertices. How many edges does it have?

18. A polyhedron has 8 faces and 15 edges. How many vertices does it have?

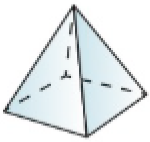
19. A polyhedron has 16 edges and 10 vertices. How many faces does it have?

20. Draw the cross section formed by the intersection of plane *A* and the polyhedron shown. What type of polygon is the cross section? SEE EXAMPLE 3



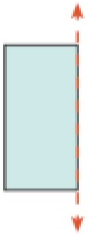
For Exercises 20 and 21, use the square pyramid shown. SEE EXAMPLE 4

21. Visualize a plane intersecting the square pyramid parallel to the base. Describe the cross section.

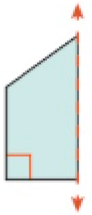


22. Visualize a plane intersecting the square pyramid through the vertex and perpendicular to opposite edges of the base. Describe the cross section.

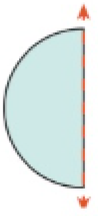
23. Describe the three-dimensional figure that is formed from by rotating the rectangle about the side. SEE EXAMPLE 5



24. Describe the three-dimensional figure that is formed by rotating the trapezoid about the line shown. SEE EXAMPLE 5



25. Describe the three-dimensional figure that is formed by rotating the semicircle about the straight edge.



PRACTICE & PROBLEM SOLVING

APPLY

26. **Represent and Connect** Parker cuts 12 pentagons and 20 hexagons out of fabric to make the pillow shown. The pillow has 60 vertices. If it takes 20 inches of thread per seam to connect the edges of the polygons, how many inches of thread does Parker need to make the pillow?

27. **Use Patterns and Structure** A gem cutter cuts a polyhedral crystal from a garnet gemstone. The crystal has 10 fewer vertices than edges and twice as many edges as faces. How many faces, vertices, and edges does the crystal have?

28. **Communicate and Justify** Rebecca wants to install a safety mat under the path of a revolving door. What shape should she make the mat? Explain.

ASSESSMENT PRACTICE

29. Select all true statements regarding rotated two-dimensional figures. GR.4.1

☐ **A.** A circle rotated about its diameter forms a cylinder.

☐ **B.** A right triangle rotated about its height forms a cone.

☐ **C.** A rectangle rotated about its edge forms a cylinder.

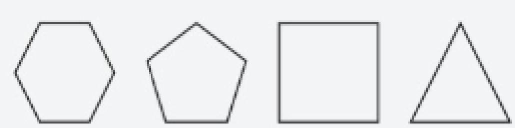
☐ **D.** A semicircle rotated about its straight edge forms a sphere.

☐ **E.** A right triangle rotated about its hypotenuse forms a square pyramid.

30. **SAT/ACT** Which best describes the cross section of plane X and the polyhedron shown?

Ⓐ hexagon
Ⓑ pentagon
Ⓒ rectangle
Ⓓ trapezoid
Ⓔ triangle

31. **Performance Task** Draw a polyhedron with the fewest possible faces, vertices, and edges. Choose the faces from the polygons shown. You may use a polygon more than once.



Part A Explain why you chose the polygon or polygons that you chose.

Part B How do you know that there is no polyhedron with fewer faces, vertices, or edges than the one you drew?

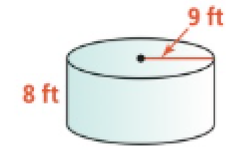
11-2
Surface Area

I CAN… find the surface areas of three-dimensional figures.

MA.912.GR.4.6 Solve mathematical and real-world problems involving the surface area of three-dimensional figures limited to cylinders, pyramids, prisms, cones, and spheres.
Also GR.4.3
MA.K12.MTR.2.1, MTR.4.1, MTR.5.1

CRITIQUE & EXPLAIN

Janelle and Dante find the surface area of a cylinder.



Janelle

Find the area of the curved surface
$A = bh = (8)(18) \approx 144$ ft^2

Then find the area of each base
$A = \pi r^2 = \pi(9)^2 \approx 254.4$ ft^2

Total surface area = 144 + 2(254.4) ≈ 652.8 ft^2

Dante

Find the area of the curved surface
$A = Ph = (18\pi)(8) \approx 452.4$ ft^2

Then find the area of each base
$A = \pi r^2 = \pi(9)^2 \approx 254.4$ ft^2

Total surface area = 452.4 + 2(254.4) ≈ 961.2 ft^2

A. **Communicate and Justify** Is either student correct? Explain.

B. **Choose Efficient Methods** Which method would you use to find the surface are of the cylinder? Is it more efficient than the method used by Janelle and Dante?

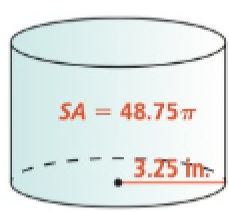
ESSENTIAL QUESTION **How can you find the surface area of a prism, pyramid, cylinder, cone and sphere?**

EXAMPLE 1 Solve Problems Involving the Surface Area of a Cylinder

A packaging engineer designs a cylindrical can with a diameter of 3.25 inches. To make the can, 48.75π cubic inches of aluminum are needed. A focus group participant asked if the can would fit on a shelf that has a maximum height of 6 inches. How should the engineer respond?

Draw a cylinder to represent the can. Then use the formula $SA = 2B + Ph$ to find the height of the can.

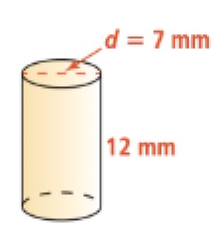
STUDY TIP
Recall, a cylinder has two circular bases that have the same area, B. The area of the curved surface is Ph, where P is the Perimeter, or circumference, of the circular base and h is height of the cylinder.



$$SA = 2B + Ph$$
$$SA = 2\pi r^2 + 2\pi r \cdot h$$
$$48.75\pi = 2\pi \cdot (3.25)^2 + (2\pi \cdot 3.25) \cdot h$$
$$4.25 = h$$

The height of the can is 4.25 inches, so the can will fit in the shelf.

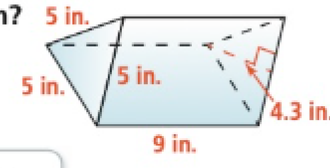
Try It! **1.** What is the surface area of the cylinder? Use 3.14 for π.



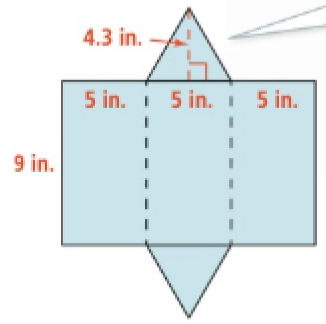
CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Find the Surface Area of a Prism with a Triangular Base

A. What is the surface area of the triangular prism?



Method 1 Draw a net of the triangular prism and find the areas of the faces and bases.



The two triangular bases are congruent.

$A = \frac{1}{2}bh$

$= \frac{1}{2}(5)(4.3)$

$= 10.75 \text{ in.}^2$

The area of each rectangular face is:
$A = bh = 5(9) = 45 \text{ in.}^2$

The total area of the rectangular faces is:
$5(9) + 5(9) + 5(9) = 135 \text{ in.}^2$

SA = Base Area + Base Area + Area of Rectangular Faces

$= 10.75 + 10.75 + 135$

$= 156.5 \text{ in.}^2$

REPRESENT AND CONNECT
How do you know that the area of the faces of a prism is equal to the product of the perimeter of the base P and h, the height of the prism?

Method 2 Apply a formula to find the surface area.

$SA = 2B + Ph$

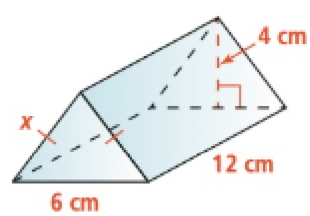
$= 2(10.75) + (5 + 5 + 5)(9)$

$= 21.5 + 135$

$= 156.5 \text{ in.}^2$

B = the area of the base
P = the perimeter of the base
h = the height of the prism

B. The surface area of the triangular prism is 216 in.^{2}. What is the side length x?



$SA = 2B + Ph$

$216 = 2(\frac{1}{2} \cdot 6 \cdot 4) + (2x + 6)(12)$

$216 = 24 + 24x + 72$

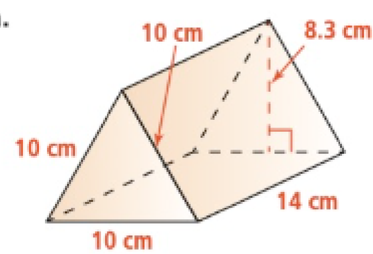
$120 = 24x$

$5 = x$

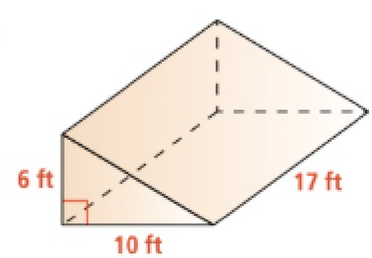
The side length x is 5 cm.

Try It! **2.** What is the surface area of each triangular prism?

a.



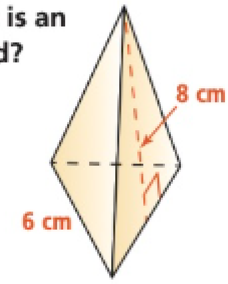
b.



EXAMPLE 3 Find the Surface Area of a Regular Pyramid with a Triangular Base

COMMON ERROR
A *regular pyramid* has a regular polygonal base. The slant height h_s is only defined for a regular pyramid. To find the surface area of a non-regular pyramid, find the area of each triangular face, the area of the base, and add them.

A. Nandi is building a cardboard pyramid with a base that is an equilateral triangle. How much cardboard does he need?



Draw a net of the pyramid and find the areas of the faces and the base

Step 1 Find the area of the congruent triangular faces.

$\frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh$

$= \frac{1}{2}(b + b + b) \cdot h$

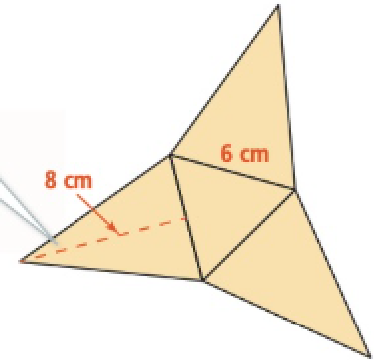
Distributive Property

$= \frac{1}{2}Ph_s$

$= \frac{1}{2}(18)(8)$

$= 72 \text{ cm}^2$

For a regular pyramid, the **slant height** h_s is the height of the triangular faces.



Step 2 Find the area of the triangular base.

$A = \frac{1}{2}bh$

$= \frac{1}{2}(6)(5.2)$

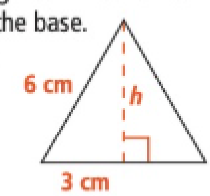
$= 15.6 \text{ cm}^2$

Use the Pythagorean Theorem to find the height of the base.

$h^2 + 3^2 = 6^2$

$h^2 = 27$

$h = 5.2$ cm



Step 3 Add the area of the base and the areas of the faces to find the surface area of the pyramid.

$SA = B + \frac{1}{2}Ph_s$

$= 15.6 + 72$

$= 87.6 \text{ cm}^2$

Nandi needs 87.6 cm^2 of cardboard.

GENERALIZE
When a three-dimensional figure is dilated by scale factor k, the surface area of the dilated figure is k^2 times the original figure.

B. Nandi dilates the dimensions of the pyramid by a scale factor of 2 to build a larger pyramid. How are the surface areas of the similar pyramids related?

Calculate the surface area of the dilated pyramid.

Use the lengths of the dilated side lengths and slant height.

$SA = B + \frac{1}{2}Ph_s$

$= (12)(10.4) + \frac{1}{2}(36)(16)$

$= 350.4 \text{ cm}^2$

The ratio of the surface areas of the pyramids is $\frac{350.4}{87.6} = \frac{4}{1}$.

The surface area of the dilated pyramid is $2^2 = 4$ times the surface area of the original pyramid.

Use the ratio.

$SA = B + \frac{1}{2}Ph_s$

$= \frac{1}{2}(2b)(2h) + \frac{1}{2}(2P)(2h_s)$

$= (2 \cdot 2)(\frac{1}{2}bh + \frac{1}{2}Ph_s)$

$= 2^2(\frac{1}{2}bh + \frac{1}{2}Ph_s)$

$= 2^2(87.6)$

$= 350.4 \text{ cm}^2$

The pyramid dilated by scale factor $k = 2$ has side lengths twice the length of the original pyramid.

The surface area of the dilated pyramid is k^2 times the surface area of the original pyramid.

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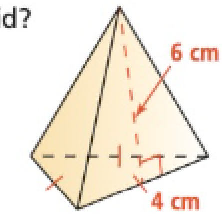
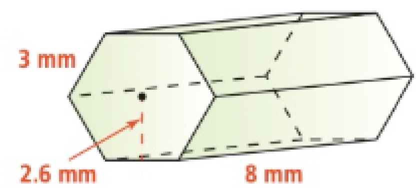
Try It! **3. a.** What is the surface area of the pyramid?

b. If the pyramid is dilated by a scale factor of 3, what is the surface area of the dilated pyramid?



EXAMPLE 4 Find the Surface Areas of Prisms with a Regular Polygonal Base

Alex is painting the wooden toy in the shape of a regular hexagonal prism. How much paint does Alex need?

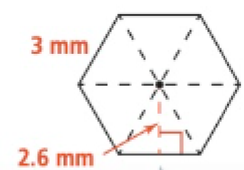


You can apply the formula for the surface area of a prism.

$SA = 2B + Ph$

Step 1 Find the area of the base.

The base is the regular hexagon. The regular hexagon can be decomposed into 6 congruent triangles.



$B = 6 \cdot (\frac{1}{2}bh)$

$= 6 \cdot \frac{1}{2} \cdot b \cdot a$

$= \frac{1}{2} \cdot 6 \cdot b \cdot a$

$= \frac{1}{2} \cdot P \cdot a$

$= \frac{1}{2} \cdot 18 \cdot 2.6$

$= 23.4 \text{ mm}^2$

The apothem *a* is the height of each triangle.

The perimeter *P* of the regular pentagon is $6 \cdot b$.

The **apothem** *a* of a regular polygon is the distance from the center to a side.

GENERALIZE
The formula for the area of a regular polygon is $A = \frac{1}{2}Pa$.

Step 2 Apply the formula to find the surface area.

$SA = 2B + Ph$

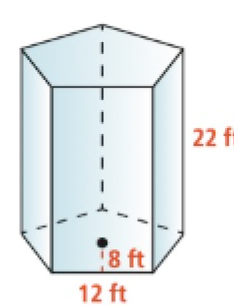
$= 2(\frac{1}{2}Pa) + Ph$

$= 2(23.4) + 18(8)$

$= 190.8 \text{ mm}^2$

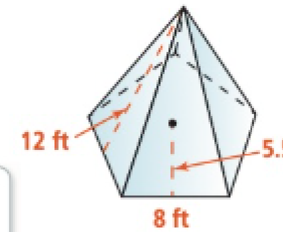
Alex needs 190.8 mm^2 of paint for the wood toy.

Try It! **4.** What is the surface area of the prism?



EXAMPLE 5 Find the Surface Area of a Pyramid with a Regular Polygonal Base

What is the surface area of the regular pentagonal pyramid?



$SA = B + \frac{1}{2}Ph_s$

$= \frac{1}{2}Pa + \frac{1}{2}Ph_s$

$= \frac{1}{2}(8 \cdot 5)(5.5) + \frac{1}{2}(8 \cdot 5)(12)$

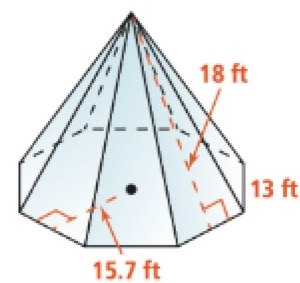
$= 350 \text{ ft}^2$

The apothem a is 8 ft and the slant height h_s is 12 ft

The surface area of pyramid is 350 ft^2.

COMMON ERROR
Be careful not to confuse the apothem, which is the distance of from the center to a side of a regular polygon, and the slant height, which is the height of the triangular faces.

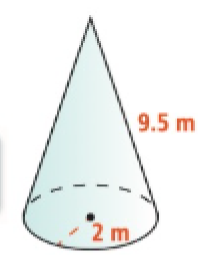
Try It! **5.** Find the surface area of the regular octagonal pyramid.



EXAMPLE 6 Find the Surface Area of Cones and Spheres

A. What is the surface area of the cone? Use 3.14 for π.

You can think of a cone as a pyramid with a circular base and an infinite number of faces.



$SA = B + \frac{1}{2}Ph_s$

Apply the formula for a pyramid to develop the formula for the surface area of a cone.

$SA = B + \frac{1}{2}(2\pi r)h_s$

$SA = B + \pi r h_s$

Surface area formula for a cone.

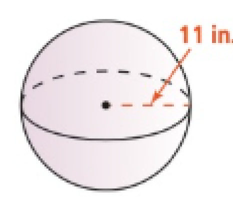
$= \pi(2)^2 + \pi(2)(9.5)$

$= 72.2 \text{ m}^2$

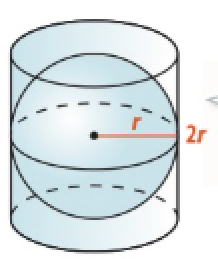
The surface area of the cone is 72.2 m^2.

USE PATTERNS AND STRUCTURE
How can the curved surface of a cone relate to the faces of a pyramid?

B. What is the surface area of the sphere? Use $\pi = 3.14$.



You can relate the sphere to an open cylinder with no top or bottom surface. An open cylinder that has the same radius as the sphere and a height of $2r$ has the same surface area as the sphere.



$SA = 2\pi r \cdot h$
$= 2\pi r \cdot 2r$
$= 4\pi r^2$

Apply the formula for the surface area of a sphere.

$SA = 4\pi r^2$

$= 4\pi(11)^2$

$= 1519.76 \text{ in.}^2$

The surface area of the sphere is 1519.76 in.^{2}

GENERALIZE
The formula for the surface area of a sphere is $SA = 4\pi r^2$.

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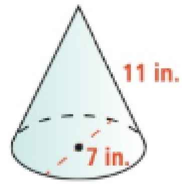
Try It! **6.** Find the surface area of each figure.

a.



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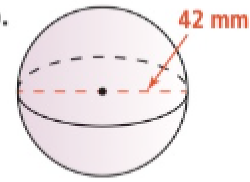
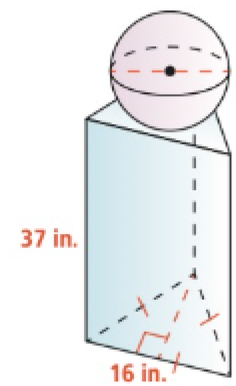


APPLICATION

EXAMPLE 7 Solve Problems Involving Surface Area of Composite Figures

Dominic makes characters from shapes using sheet metal. He is designing his next character and decides he wants the head to be a sphere and the body a triangular prism. He wants the diameter of the sphere to be equal to the height of the triangular base of the prism. How much sheet metal will Dominic need to make the character?



Step 1 Use the Pythagorean theorem to find the height of the triangle.

$$a^2 + b^2 = c^2$$
$$8^2 + b^2 = 16^2$$
$$b^2 = 192$$
$$b = 13.9$$

The height of the base of the figure is 13.9 in.

Step 2 Find the surface area of the triangular prism.

$$SA = 2B + Ph$$
$$= 2(\tfrac{1}{2} \cdot 16 \cdot 13.9) + (16 \cdot 3)(37)$$
$$= 1998.4$$

The surface area of the prism is 1998.4 in.^{2}

Step 3 Find the surface area of the sphere.

$$SA = 4\pi r^2$$
$$= 4\pi(6.95)^2$$
$$= 606.7 \text{ in.}^2$$

The surface area of the sphere is 606.7 in.^{2}

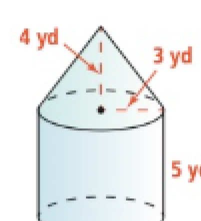
APPLY MATH MODELS
The figures do not share a face or base so the surface areas of each figure are added to find the total surface area of the character.

Step 4 Find the total surface area of the metal character.

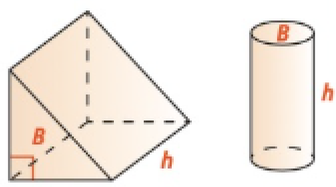
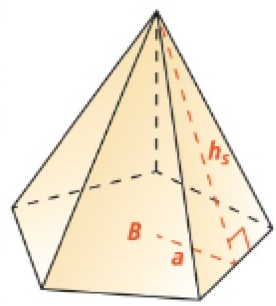
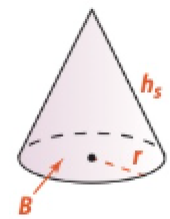
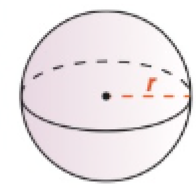
Total SA = (SA prism) + (SA of sphere)
= 1998.4 + 606.7
= 2605.1 in.^{2}

Dominic will need 2605.1 in.^{2} of metal to make the character.

Try It! **7.** A student group is building acastle pillar as part of their homecoming float. They plan to wrap the pillar in a fabric that is printed with the school mascot. How much of the fabric do they need to cover the pillar?



CONCEPT SUMMARY Surface Area

	Prisms and Cylinders	Pyramids	Cones	Sphere
WORDS	The surface area of a prisms or a cylinder is the sum of the areas of the face(s) and the area of the two bases.	The surface area of a pyramid is the sum of the area of base and the areas of the triangular faces.	The surface area of a cone is the sum of the area of the base and the area of the curved surface.	The surface area of a sphere is the area of the curved surface.
DIAGRAM				
FORMULA	$SA = 2B + Ph$	$SA = B + \frac{1}{2}Ph_s$	$SA = B + \pi rh_s$	$SA = 4\pi r^2$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How can you find the surface area of a prism, pyramid, cylinder, cone and sphere?

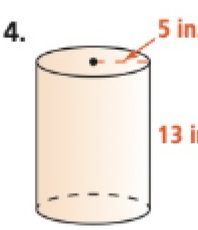
2. **Error Analysis** Miguel is finding the surface area of the cone. What is his error?

$$SA = 2B + \pi rh_s$$
$$= 2\pi(2)^2 + \pi(4)(5)$$
$$= 87.92 \text{ cm}^2$$

3. **Construct Arguments** Tanika is calculating the amount of foil wrap needed to cover a cheese wedge. She slices the cheese wedge horizontally along a side of the wedge and says that she needs the same amount of foil for each of the new wedges. Do you agree with Tanika? Explain.

Do You KNOW HOW?

Find the surface area of each figure. Use 3.14 for π.

4.


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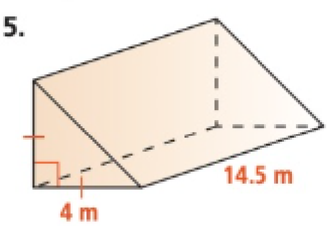
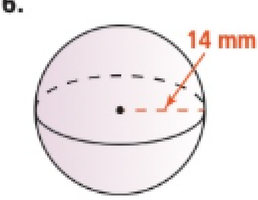


6.


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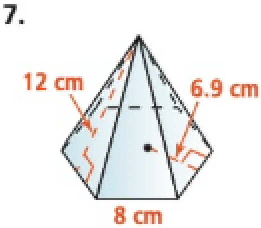

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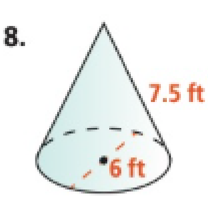

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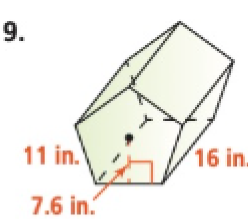


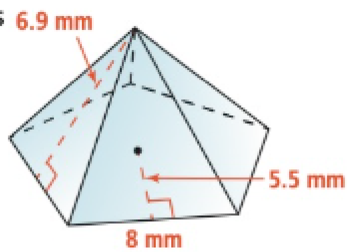
PRACTICE & PROBLEM SOLVING

UNDERSTAND

10. **Mathematical Connection** The largest cross-section of a sphere has a circumference of 94.25 m. What is the surface area of the sphere?

11. **Communicate and Justify** Gabrielle says that you find the surface areas of cylinders and prisms in the same way. Do you agree? How are cylinders and prisms alike and how are they different?

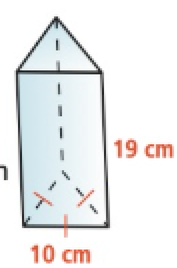
12. **Error Analysis** Sam is finding the surface area of the regular pentagonal pyramid shown. Explain the error.



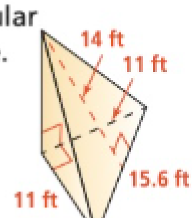
$$SA = B + Ph_s$$
$$= \frac{1}{2}Pa + \frac{1}{2}Ph_s$$
$$= \frac{1}{2}(40)(6.9) + \frac{1}{2}(40)(5.5)$$
$$= 110 + 276$$
$$= 386 \text{ mm}^2$$

13. **Use Patterns and Structure** The surface area of a cylinder is dilated at a scale factor of $\frac{1}{3}$. How is the surface area of the dilated cylinder related to the surface area of the original cylinder?

14. **Choose Efficient Methods** Tanya and Darien are finding the surface are of this triangular prism. Tanya says that you need to apply the Pythagorean Theorem to find the height of the triangular base to find the areas of the bases. Darien says you can apply the formula $SA = 2B + Ph$. Who is correct? Explain.



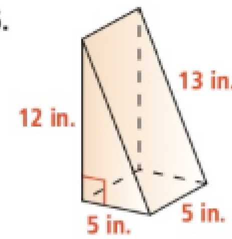
15. **Higher Order Thinking** The triangular pyramid has a right triangular base. What is the surface area of the pyramid? Explain why you cannot apply the formula for a *regular pyramid*.



PRACTICE

Find the surface area for each figure. Use 3.14 for. SEE EXAMPLES 2 AND 4

16.



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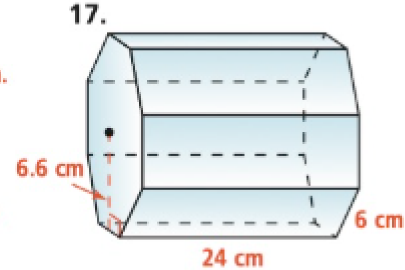
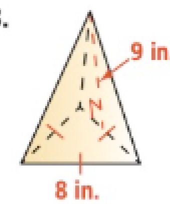


Find the surface area of each pyramid. SEE EXAMPLES 3 AND 5

18.



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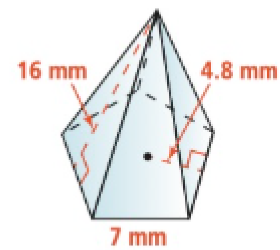
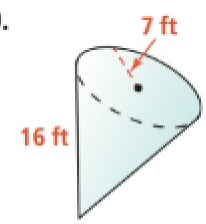


Find the surface area of each cone. Use 3.14 for π. SEE EXAMPLE 6

20.



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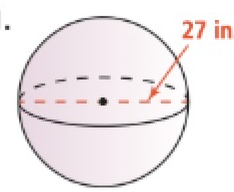
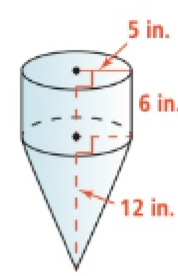
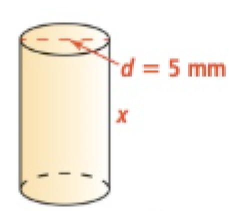


22. What is the surface area of the composite figure? SEE EXAMPLE 7



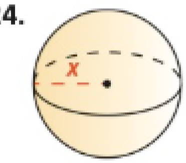
Find the values of x. SEE EXAMPLES 1 AND 6

23.



$SA = 227.77 \text{ mm}^2$

24.



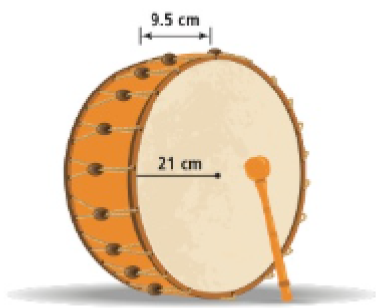
$SA = 640 \text{ cm}^2$

25. A pentagonal pyramid has a surface area 534 in.^{2}. If the pyramid is dilated by a scale factor of 3, what is the surface area of the dilated figure? SEE EXAMPLE 3

PRACTICE & PROBLEM SOLVING

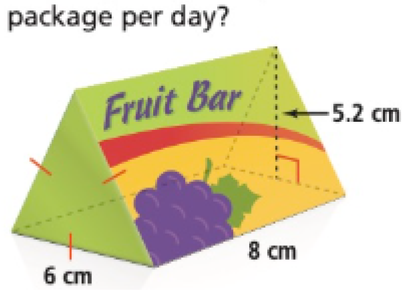
APPLY

26. **Apply Math Models** A dohol is a traditional Iranian cylindrical drum. The circular drum heads on each end are made of leathers and the drum body is usually made of wood, cane or bamboo. What is the minimum amount of leather needed to create both drum heads? What is the total area of the wood used to make the body of the drum? Use 3.14 for π

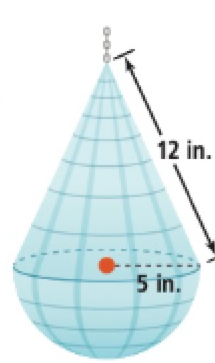


27. **Apply Math Models** Michaela received an order to hand-paint a decorative box with bases that are regular pentagons. She will first paint the outside of the box solid blue. What is the area that she needs to cover with blue paint?

28. **Analyze and Persevere** A manufacturing company packages fruit bars. Each fruit bar is placed in a cardboard package sketched below. The company receives 10,000 cm^2 of cardboard each day. How many fruit bars can they package per day?



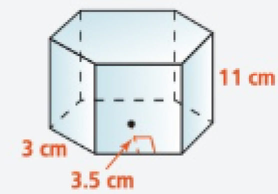
29. **Represent and Connect** The drama class is making a paper lantern prop. They have a roll of mulberry paper that will cover a surface up to 325 in.^{2}. Do they have enough paper to cover the prop? Explain.



ASSESSMENT PRACTICE

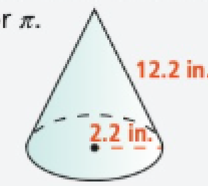
30. What is the surface area of the prism?
GR.4.6

Ⓐ 231.5 cm^2
Ⓑ 244.8 cm^2
Ⓒ 260.0 cm^2
Ⓓ 261.0 cm^2



31. **SAT/ACT** What is the surface area of the cone in square inches? Use 3.14 for π.

Ⓐ 90.2 in.^{2}
Ⓑ 99.5 in.^{2}
Ⓒ 100.89 in.^{2}
Ⓓ 237.5 in.^{2}



32. A farmer needs two steel silos for his farm. He decides on the dimensions of one silo.

Part A How can you determine the amount of steel needed to build the silo?

Part B How much steel is needed to build the silo?

Part C The farmer decides the dimensions of the second silo will be twice as big as the first. How much steel will be needed to build the second silo?

11-3 Volumes of Prisms and Cylinders

I CAN… use the properties of prisms and cylinders to calculate their volumes.

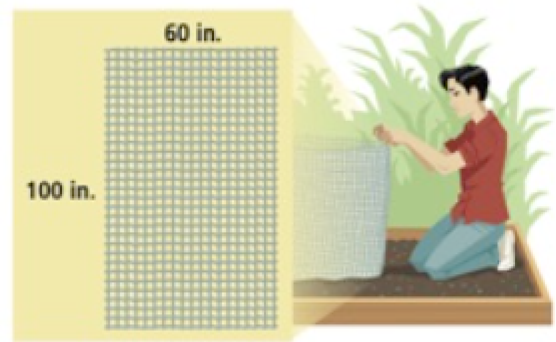
VOCABULARY
- Cavalieri's Principle
- oblique cylinder
- oblique prism

MA.912.GR.4.5–Solve mathematical and real-world problems involving the volume of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres. Also GR.4.3, GR.4.4
MA.K12.MTR.2.1, MTR.6.1, MTR.7.1

CONCEPTUAL UNDERSTANDING

MODEL & DISCUSS

The Environmental Club has a piece of wire mesh that they want to form into an open-bottom and open-top compost bin.



A. Using one side as the height, describe how you can form a compost bin in the shape of a rectangular prism using all of the mesh with no overlap.

B. **Communicate and Justify** Which height would result in the largest volume? Explain.

C. Suppose you formed a cylinder using the same height as a rectangular prism. How would the volumes compare?

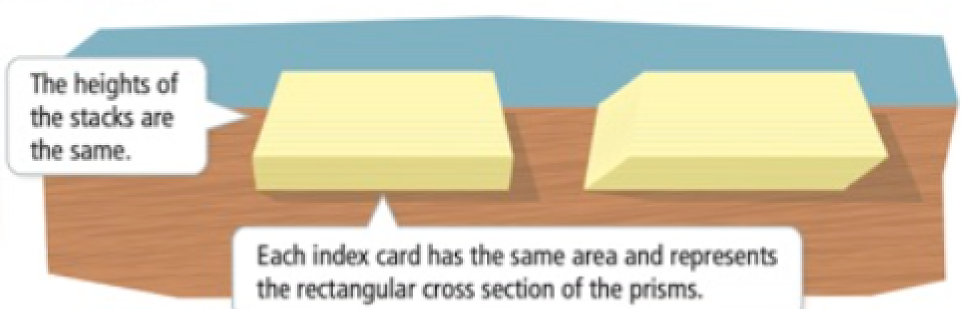
ESSENTIAL QUESTION

How does the volume of a prism or cylinder relate to a cross section parallel to its base?

EXAMPLE 1 Develop Cavalieri's Principle

How are the volumes of the two different stacks of index cards related?

The first stack forms a right prism. The second stack forms an *oblique prism*. An **oblique prism** is a prism such that some or all of the lateral faces are nonrectangular.



VOCABULARY
Remember that in a right prism, the sides are perpendicular to the bases. In an *oblique prism*, one or more sides are not perpendicular to the bases.

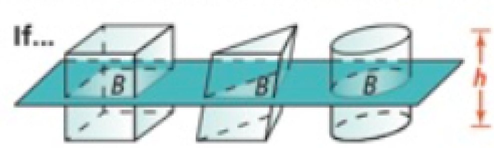
The volumes of the two stacks are the same because the sums of the areas of the cards are the same.

Try It! 1. Do you think that right and oblique cylinders that have the same height and cross-sectional area also have equal volume? Explain.

CONCEPT Cavalieri's Principle

Cavalieri's Principle states that if two three-dimensional figures have the same height and the same cross-sectional area at every level, then they have the same volume.

If...

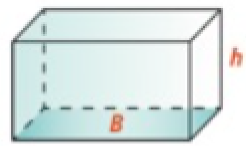


Then... the volumes are equal.

CONCEPT Volumes of Prisms and Cylinders

The volume of a prism is the product of the area of the base and the height of the prism.

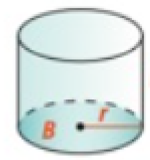
$V = Bh$



The volume of a cylinder is the product of the area of the base and the height of the cylinder.

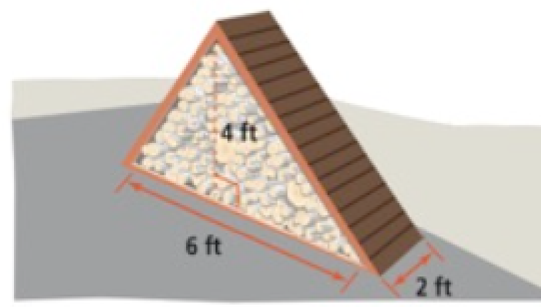
$V = Bh$

$V = \pi r^2 h$



EXAMPLE 2 Find the Volumes of Prisms and Cylinders

A. Lonzell needs to store 20 ft^3 of firewood. Could he use the storage rack shown?

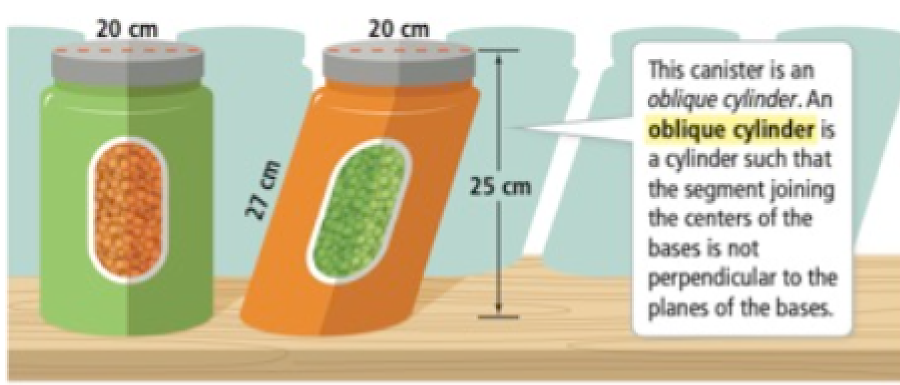


The rack is a triangular prism.

$$V = Bh$$
$$= \left[\frac{1}{2}(4)(6)\right](2) = 24$$

The volume of the storage rack is 24 ft^3, so Lonzell can store his firewood in the rack.

B. Keisha is deciding between the two canisters shown. Which canister holds more? What is the volume of the larger canister?



COMMON ERROR
The height of an oblique cylinder or prism is the length perpendicular to the bases, not the length of the sides of the figure.

The canisters have the same cross-sectional area at every height. So, by Cavalieri's Principle, the canisters have the same volume.

Apply the volume formula to find the volume of the canister on the left.

$$V = \pi r^2 h$$
$$= \pi(10)^2(25) \approx 7{,}854$$

The diameter is 20 cm, so the radius is 10 cm.

The volume of both canisters is about 7,854 cm^3.

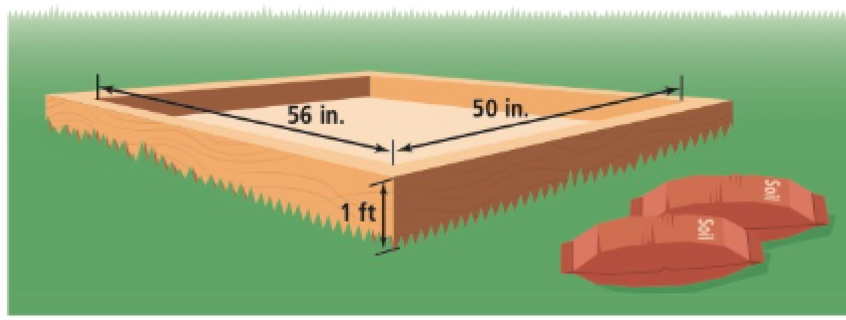
Try It! 2. a. How would the volume of the storage shed change if the length of the triangular base is reduced by half?

b. How would the volume of the canisters change if the diameter is doubled?

APPLICATION

EXAMPLE 3 Apply the Volumes of Prisms to Solve Problems

Marta is repurposing a sandbox to make a raised garden.



A. How much soil will Marta need to fill the frame?

Compute the volume of the sandbox in cubic inches.

$$V = Bh$$
$$= (56 \cdot 50)(12)$$
$$= 33{,}600$$

Use 1 ft = 12 in.

Marta needs 33,600 in.^{3} of soil.

B. Marta decides she wants to reduce the size of the garden by a scale factor of $\frac{3}{4}$. How much soil does she need now?

Use the lengths of the dilated side lengths and slant height.

$$V = Bh$$
$$= \left[\tfrac{3}{4}(56) \cdot \tfrac{3}{4}(50)\right]\left[\tfrac{3}{4}(12)\right]$$
$$= [42 \cdot 37.5][9]$$
$$= 14{,}175 \text{ cm}^3$$

The ratio of the volume of the similar prisms is $\frac{14175}{33600} = \frac{27}{64}$.

The volume of the raised garden frame is $\frac{3^3}{4^3} = \frac{27}{64}$ times the volume of the sandbox.

Use the ratio.

$$V = Bh$$
$$= \left(\tfrac{3}{4}l \cdot \tfrac{3}{4}w\right)\left(\tfrac{3}{4}h\right)$$
$$= \left(\tfrac{3}{4} \cdot \tfrac{3}{4} \cdot \tfrac{3}{4}\right)(lwh)$$
$$= \tfrac{3^3}{4^3}(lwh)$$
$$= \tfrac{3^3}{4^3}(33{,}600)$$
$$= 14{,}175 \text{ cm}^3$$

The raised garden frame dilated by scale factor $k = \frac{3}{4}$ has side lengths three-fourths the length of the sandbox.

The volume of the dilated prism has a volume that is k^3 times the volume of the sandbox.

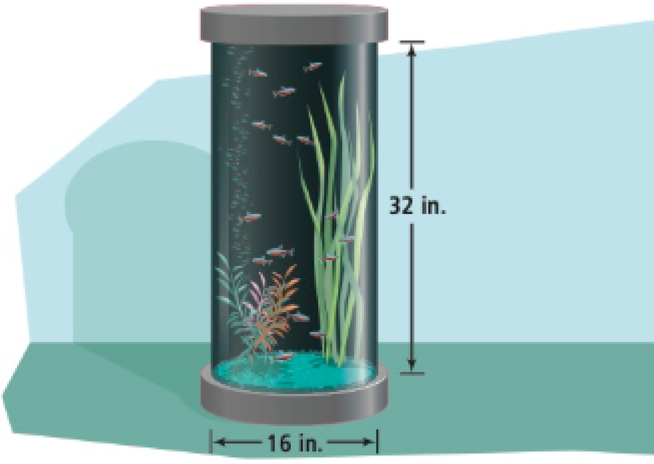
Marta will need 14,175 in^3 of soil for the smaller garden box.

GENERALIZE
If the scale factor between two similar figures is k, what is the ratio of their volumes?

Try It! **3.** The scale factor between two similar pyramids is 3. If the volume of the smaller pyramid is 75 ft^3, what is the volume of the larger pyramid?

EXAMPLE 4 Solve Density Problems

Benito has 15 neon tetras in his aquarium. Each neon tetra requires at least 2 gallons of water. What is the maximum number of neon tetras that Benito should have in his aquarium?
(*Hint:* 1 gal = 231 in.^{3})



You can solve this problem using a unit rate called *density*, a measure of one quantity, in this case the number of tetras, per unit of area or volume.

Step 1 Compute the volume of water in cubic inches.

$$V = \pi r^2 h$$
$$= \pi(8)^2(32)$$
$$\approx 6{,}434$$

The radius is half the diameter.

The volume of the water in the aquarium is about 6,434 in.^{3}.

Step 2 Find the volume of water in gallons.

$$6{,}434 \text{ in.}^3 \cdot \frac{1 \text{ gal}}{231 \text{ in.}^3} \approx 27.85 \text{ gal}$$

The volume of the water in the aquarium is about 27.85 gal.

STUDY TIP
You can also find the number of tetras by solving the following proportion.
$$\frac{x \text{ tetras}}{27.85 \text{ gal}} = \frac{1 \text{ tetra}}{2 \text{ gal}}$$

Step 3 Find the density of tetras per gallon.

$$\frac{1 \text{ tetras}}{2 \text{ gal}} = 0.5 \text{ tetras/gal}$$

Step 4 Computer the number of neon tetras that Benito's tank should hold.

Multiply the number of gallons in the aquarium by the density of tetras per gallon.

$$(27.85 \text{ gal}) \cdot (0.5 \text{ tetras/gal}) = 13.925 \text{ tetras}$$

Benito should have no more than 13 neon tetras in his aquarium.

Try It! **4.** Benito is considering the aquarium shown. What is the maximum number of neon tetras that this aquarium can hold?

APPLICATION

EXAMPLE 5 Determine Whether Volume or Surface Area Best Describes Size

A forester surveys giant sequoias by gathering data about the heights and circumference of the trees.

	A	B	C
Height	270 ft	258 ft	248 ft
Circumference (measured at 4.5 ft)	101 ft	109 ft	106 ft

A. Should the forester use surface area or volume to describe the sizes of the sequoias? Explain.

The amount of wood in a tree is represented by its volume, so she should use volume to determine the size of a giant sequoia.

LEARN TOGETHER
Do you seek help when needed? Do you offer help and support others?

B. What are the sizes of the sequoias shown? Rank them in order by size from largest to smallest.

Although the sequoias have branches and the trunk tapers gradually toward the top of the tree, each tree can be modeled as a cylinder. Find the volume of each cylinder to estimate the volume of each tree.

	Radius (ft)	Volume of Trunk (ft^3)
Tree A 270 ft	$r = \frac{101}{2\pi} \approx 16.1$	$V = \pi r^2 h$ $= \pi(16.1)^2(270)$ $\approx 219{,}870$
Tree B 258 ft	$r = \frac{109}{2\pi} \approx 17.3$	$V = \pi r^2 h$ $= \pi(17.3)^2(258)$ $\approx 242{,}584$
Tree C 248 ft	$r = \frac{106}{2\pi} \approx 16.9$	$V = \pi r^2 h$ $= \pi(16.9)^2(248)$ $\approx 222{,}523$

In order from largest to smallest, the three trees are: Tree B, Tree C, Tree A.

Try It! 5. Describe a situation when surface area might be a better measure of size than volume.

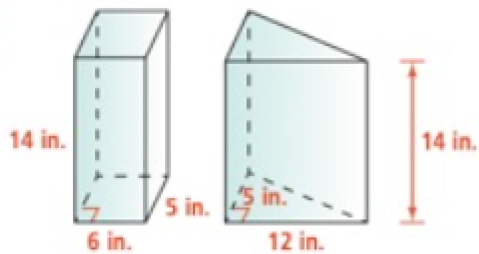
CONCEPT SUMMARY Volumes of Prisms and Cylinders

WORDS

Cavalieri's Principle Figures with the same height and same cross-sectional area at every level have the same volume.

As a result, right and oblique prisms and cylinders with the same base area and height have the same volume.

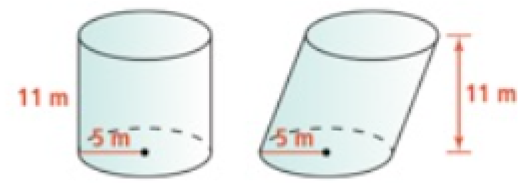
DIAGRAMS



$V = Bh$

$V = 30 \cdot 14$

$V = 420$ cubic inches



$V = Bh$

$V = \pi r^2 h$

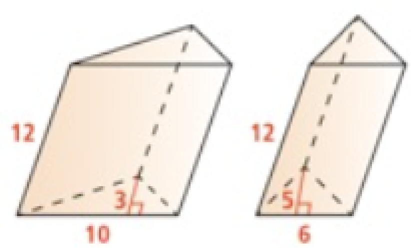
$= \pi \cdot 5^2 \cdot 11$

$= 863.9$ cubic meters

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does the volume of a prism or cylinder relate to a cross section parallel to its base?

2. **Error Analysis** Sawyer says that Cavalieri's Principle proves that the two prisms shown have the same volume. Explain Sawyer's error.



3. **Vocabulary** How are an oblique prism and an oblique cylinder alike and different?

4. **Represent and Connect** The circumference of the base of a cylinder is x, and the height of the cylinder is x. What expression gives the volume of the cylinder?

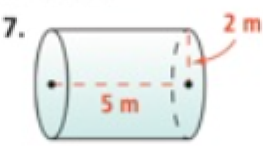
5. **Communicate and Justify** Denzel kicks a large dent into a trash can and says that the volume does not change because of Cavalieri's Principle. Do you agree with Denzel? Explain.

Do You KNOW HOW?

For Exercises 6–11, find the volume of each figure. Round to the nearest tenth.

6.

7.



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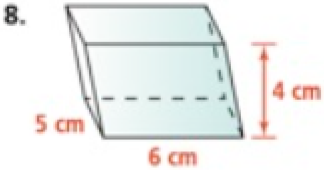

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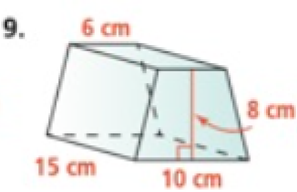

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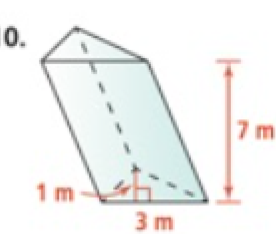

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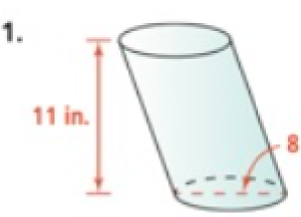
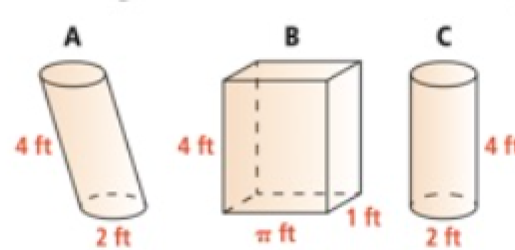


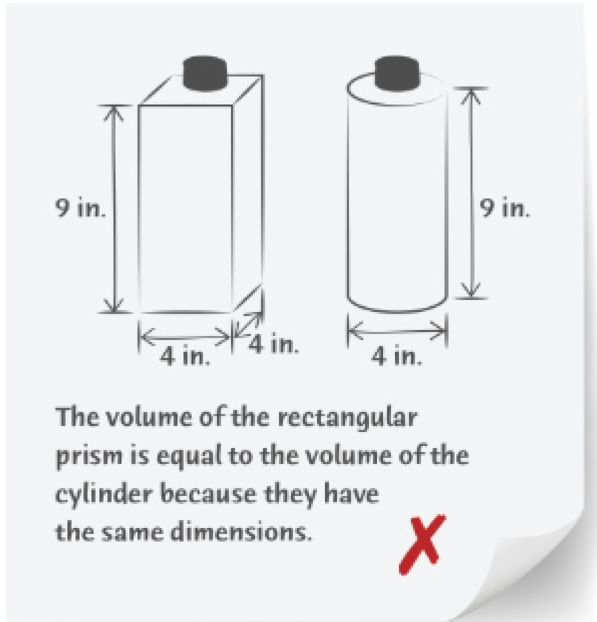
12. Which figures have the same volume? Explain.



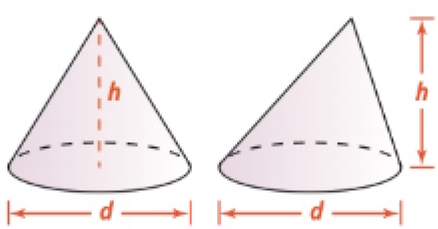
PRACTICE & PROBLEM SOLVING

UNDERSTAND

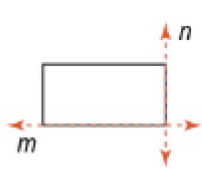
13. Error Analysis Dylan compares the volumes of two bottles. What is Dylan's error?



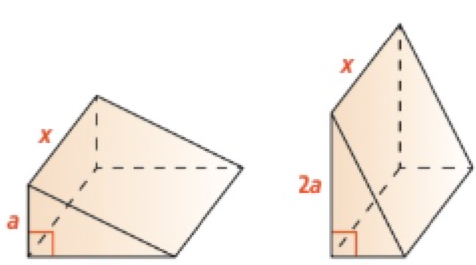
14. Higher Order Thinking Does Cavalieri's Principle apply to the volumes of the cones shown? Explain.



15. Mathematical Connections Does rotating the rectangle about line *m* result in a cylinder with the same volume as rotating the rectangle about line *n*? Explain.

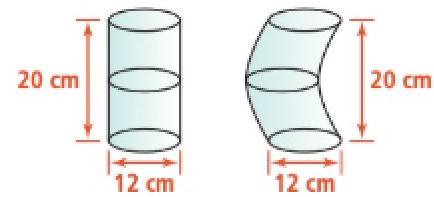


16. Represent and Connect Do the prisms shown have equivalent volumes? Explain.

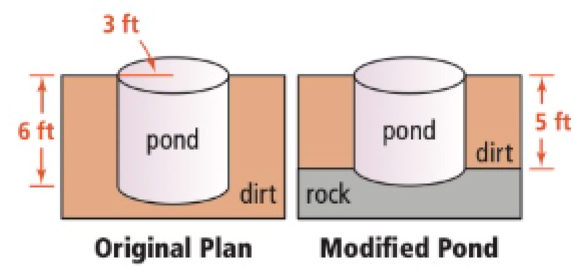


PRACTICE

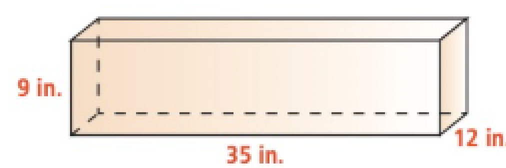
17. Katrina buys the two vases shown. How do the volumes of the vases compare? Explain. SEE EXAMPLE 1



18. Talisa plans a 6-foot deep pond. While digging, she hits rock 5 feet down. How can Talisa modify the radius to maintain the original volume of the pond? SEE EXAMPLE 2



19. The instructions for plant food say to use 0.25 gram per cubic inch of soil. How many grams of plant food should Jordan use if the planter box shown is full of soil? SEE EXAMPLE 3

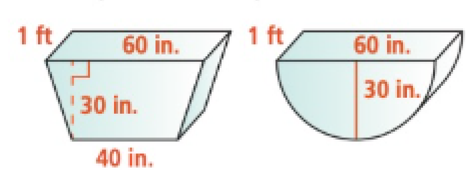


20. If a stack of 40 nickels fits snugly in the coin wrapper shown, how thick is 1 nickel? Round to the nearest hundredth. SEE EXAMPLE 4

nickel
$d = 21.21$ mm

wrapper
$V = 27{,}560$ mm^3

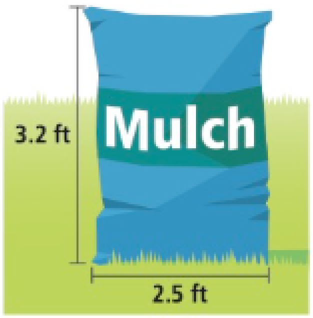
21. Sections of two flood-control ditches are shown. Which one holds the greater volume of water per foot? Explain. SEE EXAMPLE 5



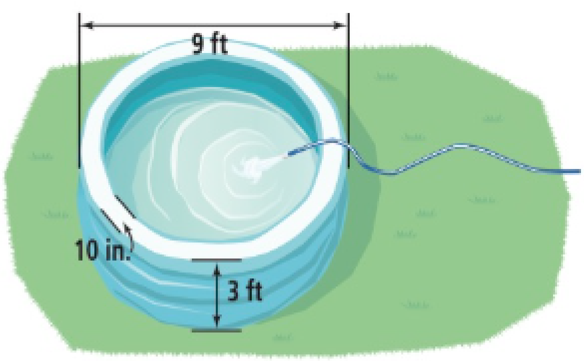
PRACTICE & PROBLEM SOLVING

APPLY

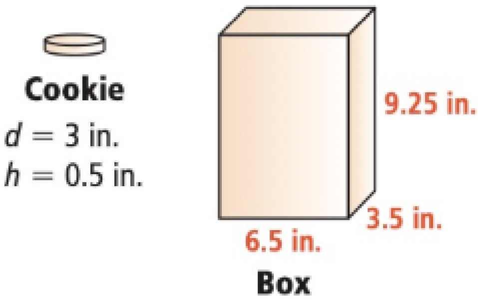
22. Represent and Connect How many 3-inch-thick bags of mulch should Noemi buy to cover 100 square feet at a depth of 4 inches?



23. Apply Math Models Ines's younger brother will be home in a half hour. If her garden hose flows at a rate 24 gal/min, does she have enough time to fill the pool before he gets home? Explain. (*Hint:* 1 $ft^3 = 7.48$ gal)



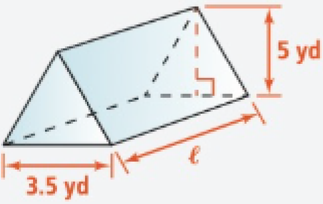
24. Check for Reasonableness The ABC Cookie Company wants to promise an average of "12 chocolate chips per cookie." Assuming that the cookies fill about 80% of the box by volume, will 600 chocolate chips for each box of cookies be sufficient to make the claim? Explain.



ASSESSMENT PRACTICE

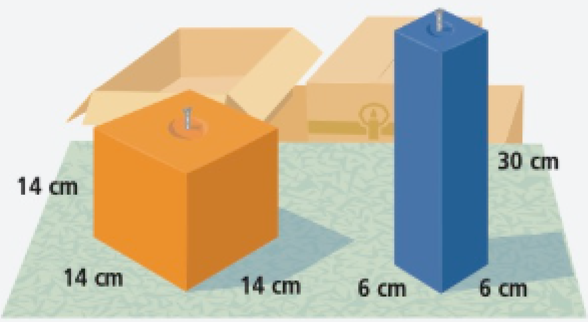
25. A paint company wants to change the height of its cans of paint to 15 cm tall, so when the cans sit on a shelf in a store they are more visible to customers. If the volume of the can must remain at 3,818 cubic cm, what will be the radius of the can to the nearest whole cm? GR.4.5

26. SAT/ACT If the volume of the prism shown is 70 cubic yards, what is its length?



Ⓐ 4 yd Ⓑ 8 yd Ⓒ 16 yd Ⓓ 28 yd

27. Performance Task A candle company receives an order for an overnight delivery of 8 short candles and 6 tall candles. The overnight service has a weight limit of 23 kg.



Part A The density of the wax used to make each candle is 0.0009 kg/cm^3. What is the weight of the order? Can the order be filled and shipped for delivery?

Part B If no tall candles are included in the order, what is the greatest number of short candles that can be delivered?

Part C What combination of tall and short candles can be delivered if the total number of candles delivered is 10?

MATHEMATICAL MODELING IN 3 ACTS

MA.912.GR.4.5–Solve mathematical and real-world problems involving the volume of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres. Also GR.4.6

MA.K12.MTR.7.1

Box 'Em Up

With so many people and businesses shopping online, retailers, and especially e-retailers, ship more and more packages every day. Some of the products people order have unusual sizes and shapes and need custom packaging. Imagine how you might package a surfboard, or a snow blower, or even live crawfish to ship to someone's house!

Think about this during the Mathematical Modeling in 3 Acts lesson.

ACT 1 Identify the Problem

1. What is the first question that comes to mind after watching the video?
2. Write down the main question you will answer about what you saw in the video.
3. Make an initial conjecture that answers this main question.
4. Explain how you arrived at your conjecture.
5. What information will be useful to know to answer the main question? How can you get it? How will you use that information?

ACT 2 Develop a Model

6. Use the math that you have learned in this Topic to refine your conjecture.

ACT 3 Interpret the Results

7. Did your refined conjecture match the actual answer exactly? If not, what might explain the difference?

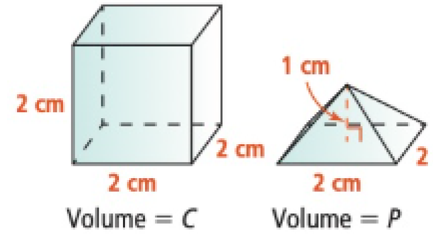
11-4 Volumes of Pyramids and Cones

I CAN… use the volumes of right and oblique pyramids and cones to solve problems.

MA.912.GR.4.5–Solve mathematical and real-world problems involving the volume of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres. **Also GR.4.4**
MA.K12.MTR.5.1, MTR.1.1, MTR.4.1

EXPLORE & REASON

Consider the cube and pyramid.



A. How many pyramids could you fit inside the cube? Explain.

B. Write an equation that shows the relationship between *C* and *P*.

C. **Use Patterns and Structure** Make a conjecture about the volume of any pyramid. Explain your reasoning.

ESSENTIAL QUESTION

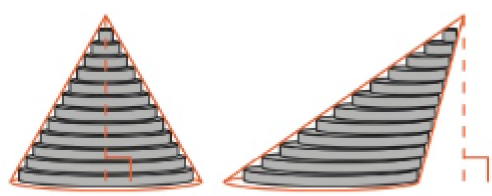
How are the formulas for volume of a pyramid and volume of a cone alike?

CONCEPTUAL UNDERSTANDING

EXAMPLE 1 Apply Cavalieri's Principle to Pyramids and Cones

How are the volumes of pyramids and cones with the same base area and height related?

Imagine a set of cardboard discs, each with a slightly smaller radius than the previous disc. You can stack the discs in different ways.

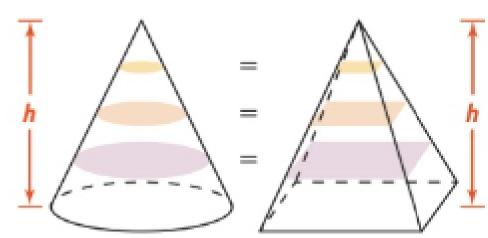


The heights of the stacks are the same, and the area at each level is the same. The total volume of cardboard in each stack is the same.

GENERALIZE
Think about the shape formed by the stacks. What would happen if the number of discs increases while the difference in the radii and the thickness of each disc decreases?

The stacks approximate cones. You can apply Cavalieri's Principle to cones and pyramids.

If two figures have the same height and equal area at every cross section, they have equal volumes.

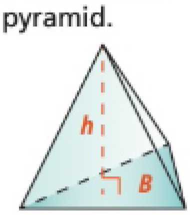


Try It! **1.** Is it possible to use only Cavalieri's Principle to show that a cone and a cylinder have equal volumes? Explain.

CONCEPT Volumes of Pyramids and Cones

The volume of a pyramid is one-third the product of the area of the base and the height of the pyramid.

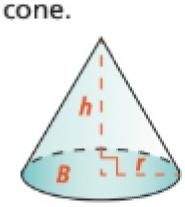
$V = \frac{1}{3}Bh$



The volume of a cone is one-third the product of the area of the base and the height of the cone.

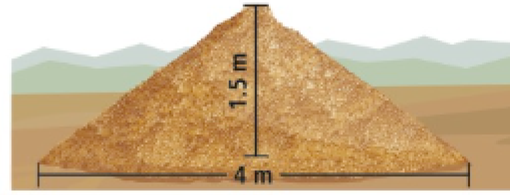
$V = \frac{1}{3}Bh$

$V = \frac{1}{3}\pi r^2 h$



EXAMPLE 2 Find the Volumes of Pyramids and Cones

A. Kyle's truck can haul 1.75 tons of corn per load. One cubic meter of corn weighs 0.8 ton. How many loads will Kyle haul to move this pile of corn?



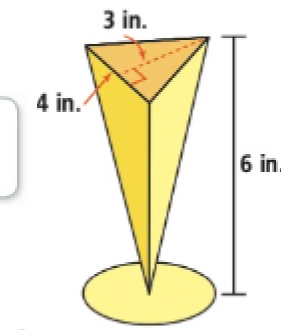
$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(2)^2(1.5) \approx 6.3 \text{ m}^3$

The pile is shaped like a cone, so apply the volume formula for cones.

Since $6.3 \text{ m}^3 \cdot 0.8 \text{ ton/m}^3 = 5.04$ tons, Kyle will need to haul $5.04 \div 1.75 = 2.88$ or 3 loads.

B. Jason is using the mold to make 12 candles. How many cubic inches of wax does he need?



$V = \frac{1}{3}Bh$

$= \frac{1}{3}\left[\frac{1}{2}(4)(3)\right](6) = 12 \text{ in.}^3$

Apply the volume formula for a pyramid.

Apply the area formula for a triangle to find the base.

For 12 candles, Jason needs $12 \text{ in.}^3 \cdot 12$, or 144 in.^3 of wax.

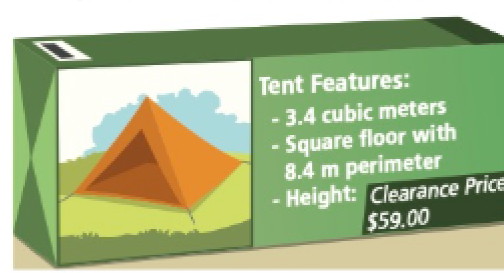
STUDY TIP
The base of a pyramid can be any polygon, so the formula you apply to determine the area of the base *B* depends on the shape of the base.

Try It! 2. a. What is the volume of a cone with base diameter 14 and height 16?

b. What is the volume of a pyramid with base area 10 and height 7?

EXAMPLE 3 Apply the Volumes of Pyramids to Solve Problems

Dyani is 1.8 m tall and wants to be able to stand inside her new tent. Should she buy this tent?



Step 1 Draw and label a square pyramid to represent the tent.

Since the perimeter of the square floor is 8.4 m, $s = 2.1$ m.

Step 2 Find the height of the pyramid.

$V = \frac{1}{3}Bh$

$3.4 = \frac{1}{3}(2.1)^2 h$

$h \approx 2.3$

The height of the pyramid is approximately 2.3 m.

Dyani will be able to stand in the tent, so she should buy this tent.

HAVE A GROWTH MINDSET
After receiving constructive feedback, how do you use it as an opportunity to improve?

CONTINUED ON THE NEXT PAGE

Try It! **3.** A rectangular pyramid has a base that is three times as long as it is wide. The volume of the pyramid is 75 ft^3 and the height is 3 ft. What is the perimeter of the base?

APPLICATION

EXAMPLE 4 Apply the Volumes of Cones to Solve Problems

A restaurant sells smoothies in two sizes. Which size is a better deal?

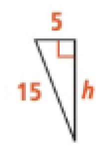
Formulate ◀ Compare the prices by determining the cost per cubic centimeter for each size.

The volume of fruit smoothie in each glass can be approximated as the volume of a cone.

Compute ◀ **Step 1** Calculate the height of each cone using the Pythagorean Theorem.

Large

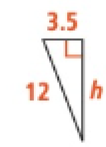
$$5^2 + h^2 = 15^2$$
$$h^2 = 200$$
$$h \approx 14.1$$



The height of the large cone is approximately 14.1 cm.

Small

$$3.5^2 + h^2 = 12^2$$
$$h^2 = 131.75$$
$$h \approx 11.5$$



The height of the small cone is approximately 11.5 cm.

Step 2 Calculate the volume of each cone.

Large

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(5)^2(14.1)$$
$$\approx 369.1$$

The volume of the large cone is approximately 369.1 cm^3.

Small

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(3.5)^2(11.5)$$
$$\approx 147.5$$

The volume of the small cone is approximately 147.5 cm^3.

Step 3 Calculate the cost per cubic centimeter for each size.

Large

$$\frac{\$5.89}{369.1 \text{ cm}^3} \approx \$0.016 \text{ per cm}^3$$

Small

$$\frac{\$3.49}{147.5 \text{ cm}^3} \approx \$0.024 \text{ per cm}^3$$

Interpret ◀ The large size smoothie costs less per cubic centimeter, so the large size smoothie is a better deal.

CONTINUED ON THE NEXT PAGE

Try It! **4.** A cone has a volume of 144π and a height of 12.

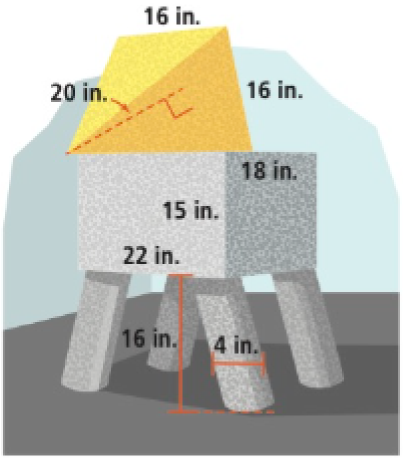
a. What is the radius of the base?

b. If the radius of the cone is tripled, what is the new volume? What is the relationship between the volumes of the two cones?

APPLICATION

EXAMPLE 5 Measure a Composite Figure

Kaitlyn is making a concrete animal sculpture. Each bag of concrete mix makes 0.6 ft^3 of concrete. How many bags of concrete mix does Kaitlyn need?



Calculate the volume of each part.

Step 1 Calculate the volume of one of the legs.

$$V = \pi r^2 h$$
$$= \pi(2)^2(16)$$
$$\approx 201 \text{ in.}^3$$

The legs are oblique cylinders.

Step 2 Calculate the volume of the body.

$$V = Bh$$
$$= (22 \cdot 18)(15)$$
$$= 5{,}940 \text{ in.}^3$$

The body is a rectangular prism.

Step 3 Calculate the volume of the head.

$$V = \frac{1}{3}Bh$$
$$= \frac{1}{3}(16 \cdot 16)(20)$$
$$\approx 1{,}707 \text{ in.}^3$$

The head is a square pyramid.

COMMON ERROR
When multiple parts of a composite figure have the same volume, make sure you account for each part in your total.

Step 4 Calculate the total volume.

$$V = 4(201) + 5{,}940 + 1{,}707$$
$$= 8{,}451$$

Add the volumes of the parts of the sculpture.

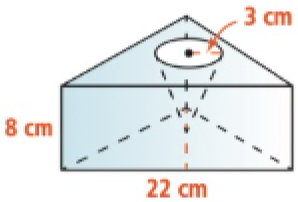
The total volume is 8,451 in.^{3}. Convert to cubic feet to determine the amount of concrete needed.

$$8{,}451 \cdot \frac{1}{1{,}728} = 4.9$$

$1 \text{ ft}^3 = (12 \text{ in.})^3 = 1{,}728 \text{ in.}^3$

To make the sculpture, 4.9 ft^3 of concrete is needed. Kaitlyn needs $4.9 \div 0.6 \approx 8.2$ or 9 bags of concrete mix.

Try It! **5.** A cone-shaped hole is drilled in a prism. The height of the triangular base is 12 cm. What is the volume of the remaining figure? Round to the nearest tenth.



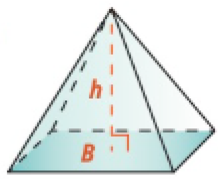
CONCEPT SUMMARY Volumes of Pyramids and Cones

WORDS

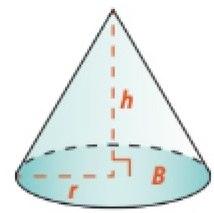
The volume of a pyramid is one-third the volume of a prism with the same base area and height.

The volume of a cone is one-third the volume of a cylinder with the same base area and height.

DIAGRAMS



$V = \frac{1}{3}Bh$



$V = \frac{1}{3}Bh$ or $V = \frac{1}{3}\pi r^2 h$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How are the formulas for volume of a pyramid and volume of a cone alike?

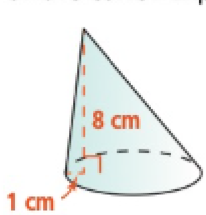
2. **Error Analysis** Zhang is finding the height of a square pyramid with a base side length of 9 and a volume of 162. What is his error?

$$V = Bh$$
$$162 = 9^2(h)$$
$$h = 2$$

✗

3. **Analyze and Persevere** A cone and cylinder have the same radius and volume. If the height of the cone is *h*, what is the height of the cylinder?

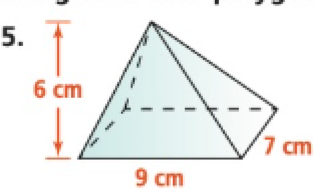
4. **Communicate and Justify** Do you have enough information to compute the volume of the cone? Explain.



Do You KNOW HOW?

For Exercises 5–10, find the volume of each figure. Round to the nearest tenth. Assume that all angles in each polygonal base are congruent.

5.



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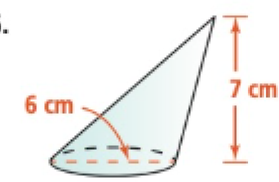

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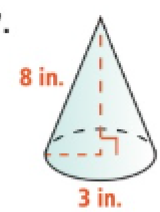

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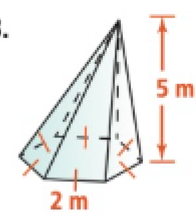

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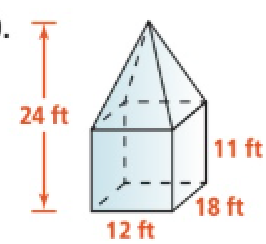
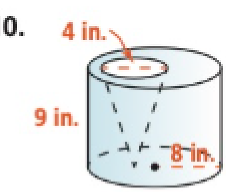


10.



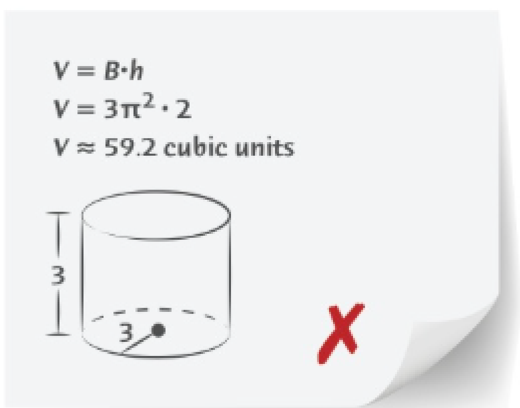
11. A solid metal square pyramid with a base side length of 6 in. and height of 9 in. is melted down and recast as a square pyramid with a height of 4 in. What is the base side length of the new pyramid?

PRACTICE & PROBLEM SOLVING

UNDERSTAND

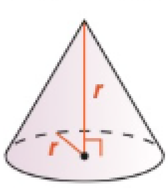
12. Communicate and Justify A stack of 39 pennies is exactly as tall as a stack of 31 nickels. Do the two stacks have the same volume? Explain.

13. Error Analysis Jacob is finding the volume of the cylinder. What is his error?

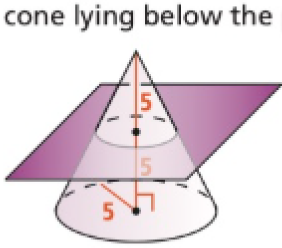


14. Communicate and Justify How would you find the volume of a right square pyramid with a base side length of 10 cm, and the altitude of a triangular side is 13 cm? Explain.

15. Mathematical Connections In terms of the radius r, what is the volume of a cone whose height is equal to its radius?

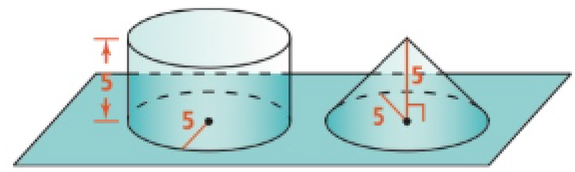


16. Higher Order Thinking A plane slices a cone parallel to the base at one-half of the height of the cone. What is the volume of the part of the cone lying below the plane?



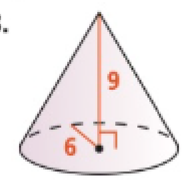
PRACTICE

17. The plane intersects sections of equal area in the two solids. Are the volumes equal?
SEE EXAMPLE 1

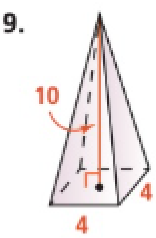


For Exercises 18–21, find the volume of each solid. Assume that all angles in each polygonal base are congruent. SEE EXAMPLE 2

18.



19.



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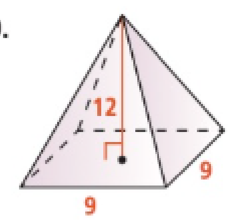
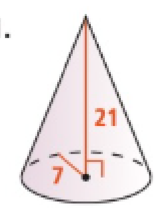
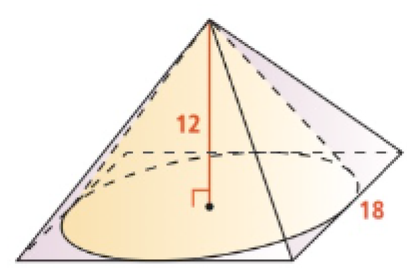


21.

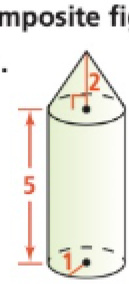


22. A cone is inscribed in a right square pyramid. What is the remaining volume if the cone is removed? SEE EXAMPLES 3 AND 4

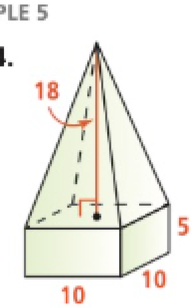


For Exercises 23 and 24, find the volume of each composite figure. SEE EXAMPLE 5

23.



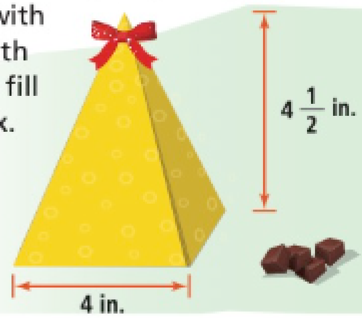
24.



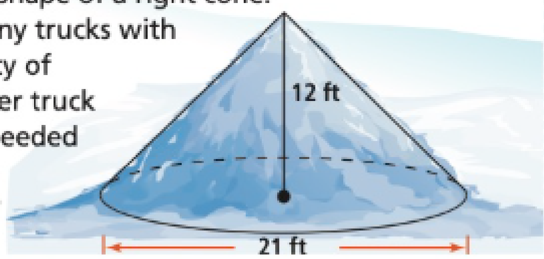
PRACTICE & PROBLEM SOLVING

APPLY

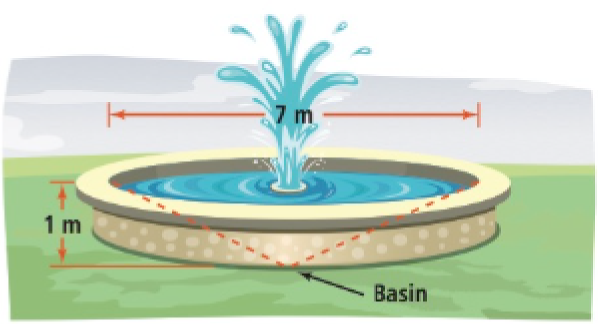
25. **Analyze and Persevere** Chiang makes gift boxes in the shape of a right square pyramid. She fills each box with chocolate cubes with $\frac{7}{8}$-in. sides. She can fill about 75% of a box. How many pieces can she fit in each box?



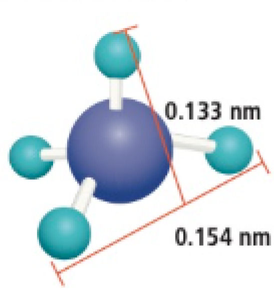
26. **Apply Math Models** A pile of snow is plowed into the shape of a right cone. How many trucks with a capacity of 10 yd^3 per truck will be needed to move the pile?



27. **Use Patterns and Structure** The basin beneath a fountain is a right cone that is 7 m across and 1 m deep at the center. After the fountain is cleaned, the pool is refilled at a rate of 300 L/min. One cubic meter is 1,000 L. How long does it take to refill the pool?



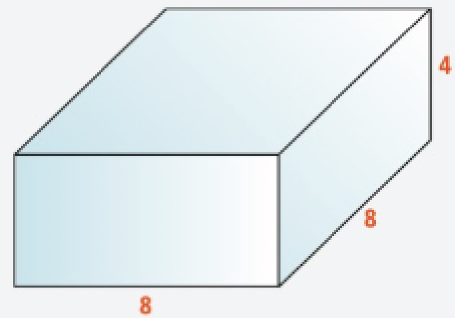
28. **Analyze and Persevere** A physicist wants to know what percentage of gas is empty space. A molecule of methane can be modeled by a regular tetrahedron with side length 0.154 nm (1 nm = 1×10^{-9} m). The altitude of each triangular side is 0.133 nm. If 6.022×10^{23} molecules make up 0.0224 m^3 of gas, how does the volume of the molecules compare to the volume of the gas?



ASSESSMENT PRACTICE

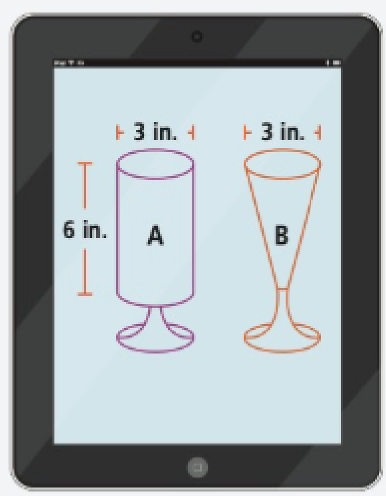
29. Mario makes candles in the shape of right cones that are 12 inches tall and have a radius of 2 inches. He wants to use the same volume of wax to make candles that are 6 inches tall. Find the radius of the shorter candles and round your answer to the nearest inch. GR.4.5

30. **SAT/ACT** Which is the volume of the largest cone that will fit entirely within the right square prism?



Ⓐ $\frac{16\pi}{3}$ Ⓑ $\frac{32\pi}{3}$ Ⓒ $\frac{64\pi}{3}$ Ⓓ $\frac{128\pi}{3}$

31. **Performance Task** A designer is working on a design for two goblets. Design A is based on a cylinder and design B is based on a cone. The client wants both goblets to be the same height and width.



Part A The client wants the smaller goblet to hold at least 10 fl oz. One fluid ounce is 1.8 $in.^3$. Will design B be large enough to meet the client's requirements? Explain.

Part B The client wants the larger goblet to hold 20 fl oz. Does design A meet the client's requirement? Explain.

Part C How could design B be changed if the client wants the smaller goblet to hold at least 12 fl oz?

11-5 Volumes of Spheres

I CAN… calculate the volume of a sphere and solve problems involving the volumes of spheres.

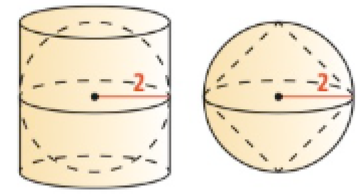
VOCABULARY

- hemisphere

MA.912.GR.4.5–Solve mathematical and real-world problems involving the volume of three-dimensional figures limited to cylinders, pyramids, prisms, cones and spheres.
MA.K12.MTR.1.1, MTR.3.1, MTR.7.1

CRITIQUE & EXPLAIN

Ricardo estimates the volume of a sphere with radius 2 by placing the sphere inside a cylinder and placing two cones inside the sphere. He says that the volume of the sphere is less than 16π and greater than $\frac{16}{3}\pi$.



A. Do you agree with Ricardo? Explain.

B. Use Patterns and Structure How might you estimate the volume of the sphere?

ESSENTIAL QUESTION

How does the volume of a sphere relate to the volumes of other solids?

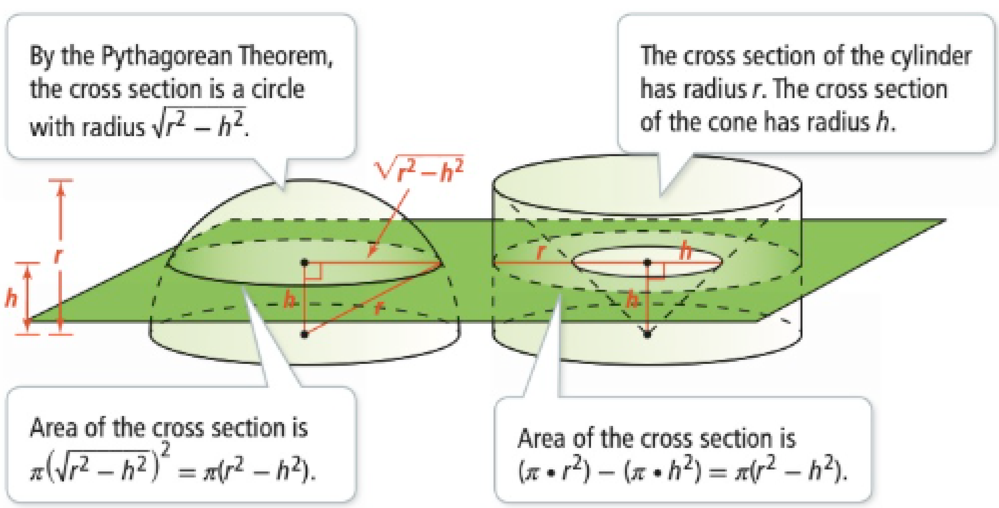
EXAMPLE 1 Explore the Volume of a Sphere

CONCEPTUAL UNDERSTANDING

What is the volume of a sphere? Why does the volume formula for a sphere make sense?

A plane, parallel to the bases, intersects half of a sphere with radius r and a cylinder with radius r and height r. The cylinder has a cone with radius r and height r removed from its center.

ANALYZE AND PERSEVERE
Think about how you can draw the section of the cylinder with the cone removed. What does the cross section look like?



Once the cone is removed, the areas of the cross sections of the solids are equal at any height. Therefore, by Cavalieri's Principle, the two solids have the same volume.

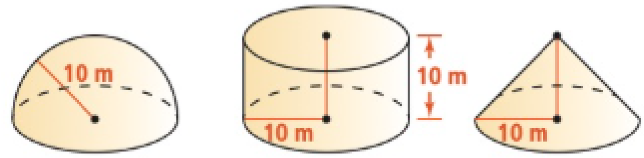
$$\begin{aligned}\text{volume of half a sphere} &= \text{volume of cylinder} - \text{volume of cone}\\ &= \pi r^2 \cdot r - \frac{1}{3}\pi r^2 \cdot r\\ &= \frac{2}{3}\pi r^3\end{aligned}$$

The volume of a sphere is twice the volume of half of the sphere, so the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

CONTINUED ON THE NEXT PAGE

EXAMPLE 1 CONTINUED

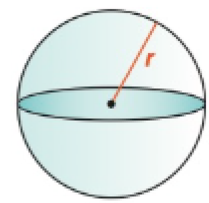
Try It! **1.** Find the volumes of the three solids. What do you notice?



CONCEPT Volume of a Sphere

The volume of a sphere is four-thirds of the product of π and the cube of the radius of the sphere.

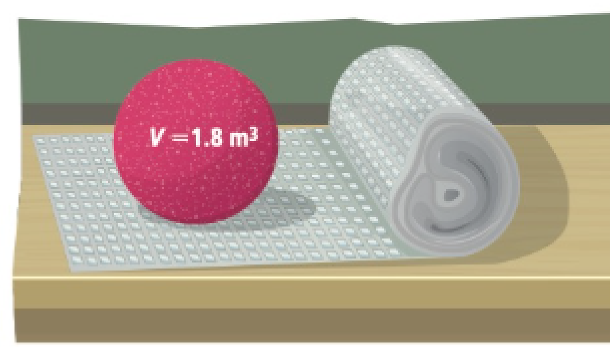
$V = \frac{4}{3}\pi r^3$



APPLICATION

EXAMPLE 2 Use the Volumes of Spheres to Solve Problems

The drama club makes a big ball from foam to hang above the stage for a play. They plan to cover the surface of the ball with metallic fabric. What is the minimum number of square meters of fabric that the club needs?



Apply the volume formula to determine the radius of the ball. Then use the surface area formula of a sphere.

First find r from the volume of the ball.

$V = \frac{4}{3}\pi r^3$ — Apply the volume formula.

$1.8 = \frac{4}{3}\pi r^3$

$r^3 = \frac{1.35}{\pi}$ — Use a calculator to find the cube root.

$r \approx 0.75$

STUDY TIP
Remember that the surface area of a sphere is four times the area of a circle with the same radius.

The radius of the ball is about 0.75 m. Next, calculate the surface area.

The surface area of a sphere with radius r is $\text{S.A.} = 4\pi r^2$.

$\text{S.A.} = 4\pi r^2$

$\text{S.A.} = 4\pi(0.75)^2$ — Substitute the radius into the surface area formula.

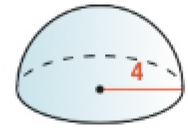
$\text{S.A.} \approx 7.1$

The club needs at least 7.1 m^2 of fabric.

Try It! **2.** What is the largest volume a sphere can have if it is covered by 6 m^2 of fabric?

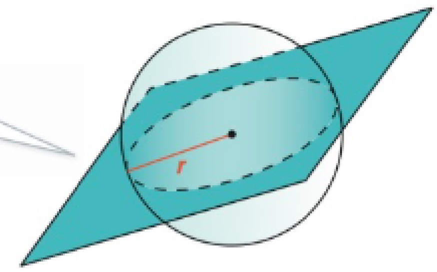
EXAMPLE 3 Find the Volumes of Hemispheres

What is the volume of the hemisphere?



A *great circle* is the intersection of a sphere and a plane containing the center of the sphere.

A great circle divides a sphere into two **hemispheres**.



VOCABULARY
The prefix *hemi* is from the Greek and means *half*. Thus, hemisphere means half-sphere.

The volume of a hemisphere is one-half the volume of a sphere with the same radius.

$V = \frac{2}{3}\pi r^3$

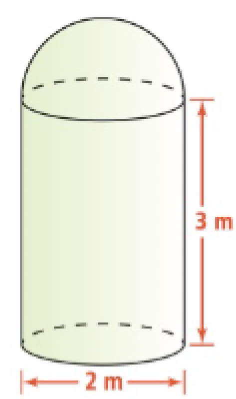
$V = \frac{2}{3}\pi \cdot 4^3 \approx 134.04$

Try It! **3. a.** What is the volume of a hemisphere with radius 3 ft?

b. What is the volume of a hemisphere with diameter 13 cm?

EXAMPLE 4 Find the Volumes of Composite Figures

A solid is composed of a right cylinder and a hemisphere as shown. If the density of the solid is 100 kg/m^3, what is the mass of the solid?



Find the volume of the solid.

$$\begin{aligned}\text{volume of solid} &= \text{volume of cylinder} \\ &\quad + \text{volume of hemisphere} \\ &= \pi r^2 h + \frac{2}{3}\pi r^3 \\ &= \pi(1)^2(3) + \frac{2}{3}\pi(1)^3 \\ &\approx 11.5\end{aligned}$$

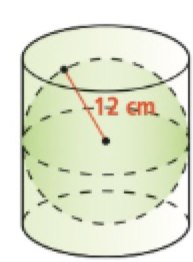
COMMON ERROR
As you break a composite figure into figures or parts of figures you are familiar with, be careful not to mix up the measurements of each figure.

The volume of the solid is about 11.5 m^3. Next, find the mass of the solid.

$11.5 \cdot 100 = 1,150$

The mass of the solid is about 1,150 kg.

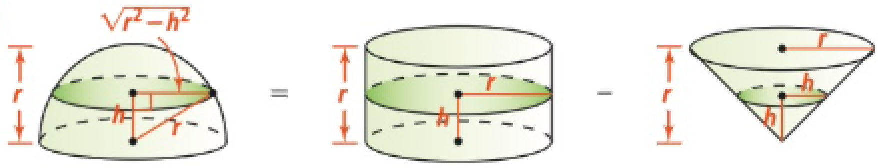
Try It! **4.** What is the volume of the space between the sphere and the cylinder?



CONCEPT SUMMARY Volumes of Spheres

WORDS Cavalieri's Principle can be used to show how the volume of the sphere is related to the volumes of a cylinder and cone. The area of a cross section of a hemisphere is the same as the area of a cross section of a cylinder with height equal to the radius minus the cross section of a cone with height equal to the radius.

DIAGRAMS

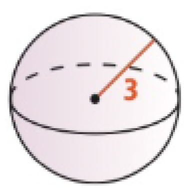


$$\frac{2}{3}\pi r^3 = \pi r^2 \cdot r - \frac{1}{3}\pi r^2 \cdot r$$

Do You UNDERSTAND?

1. **ESSENTIAL QUESTION** How does the volume of a sphere relate to the volumes of other solids?

2. **Error Analysis** Reagan is finding the volume of the sphere. What is her error?



S.A. $= \frac{4}{3}\pi r^3$

S.A. $= \frac{4}{3} \cdot \pi \cdot 3^3$

S.A. ≈ 113.1 square units

✗

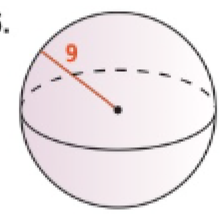
3. **Vocabulary** How does a great circle define a hemisphere?

4. **Analyze and Persevere** The radius of a sphere, the base radius of a cylinder, and the base radius of a cone are *r*. What is the height of the cylinder if the volume of the cylinder is equal to the volume of the sphere? What is the height of the cone if the volume of the cone is equal to the volume of the sphere?

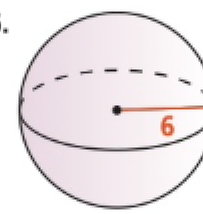
Do You KNOW HOW?

For Exercises 5 and 6, find the surface area of each solid.

5.

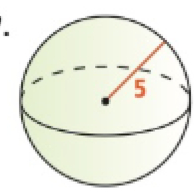


6.

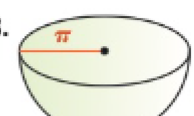


For Exercises 7 and 8, find the volume of each solid.

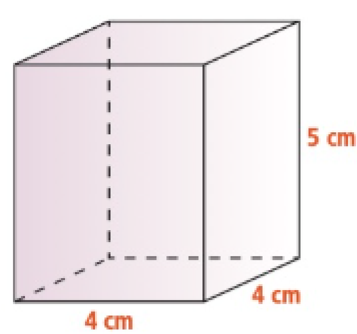
7.



8.



9. Find the volume of the largest sphere that can fit entirely in the rectangular prism.



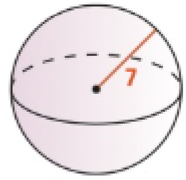
10. Find the volume and surface area of a sphere with radius 1.

PRACTICE & PROBLEM SOLVING

UNDERSTAND

11. **Communicate and Justify** How does Cavalieri's Principle apply to finding the volume of a hemisphere? Explain.

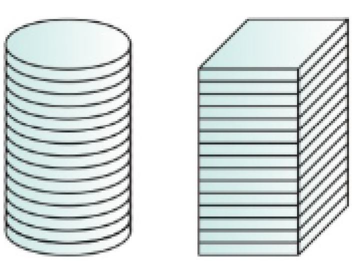
12. **Error Analysis** Kayden is finding the surface area of the sphere. What is her error?



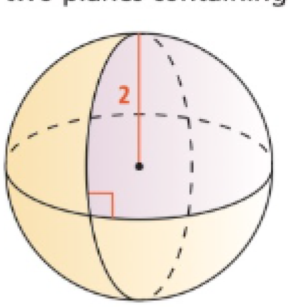
S.A. = $4\pi r^2$
S.A. = $4 \cdot \pi \cdot 14^2$
S.A. ≈ 2,463.0 square units ✗

13. **Mathematical Connections** Given the surface area of a sphere, write a formula for the volume of a sphere in terms of the surface area.

14. **Communicate and Justify** Fifteen cylinders and 15 rectangular prisms are stacked. Each cylinder has the same top surface area and height as each rectangular prism. What can you determine about the volumes of the two stacks of 15 solids? Explain.



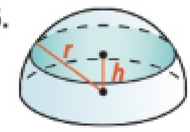
15. **Analyze and Persevere** A sphere is divided by two great circles that are perpendicular to each other. How would you find the surface area and volume of each part of the sphere between the two planes containing the great circles? Explain.



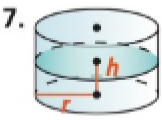
PRACTICE

For Exercises 16–18, find the area of each cross section. SEE EXAMPLE 1

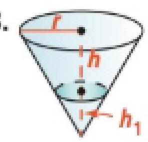
16.



17.

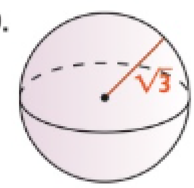


18.

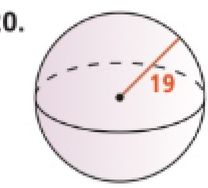


For Exercises 19–22, find the surface area of each solid to the nearest tenth. SEE EXAMPLE 2

19.



20.

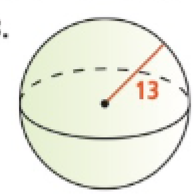


21. sphere with volume 35 cm^3

22. sphere with volume 100 $in.^3$

For Exercises 23–26, find the volume of each solid to the nearest tenth. SEE EXAMPLES 2 AND 3

23.



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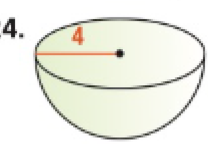
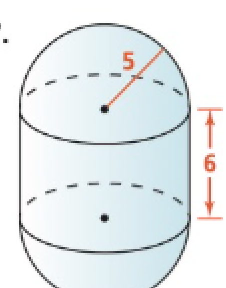


25. hemisphere with radius 12 ft

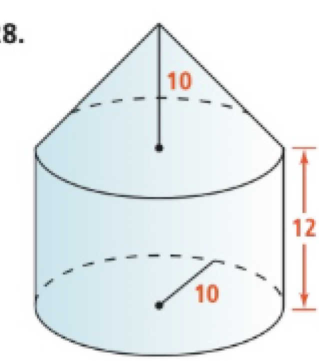
26. sphere with radius 25 m

For Exercises 27 and 28, find the volume of each composite figure to the nearest tenth.
SEE EXAMPLE 4

27.



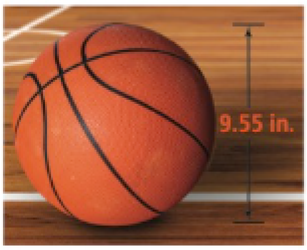
28.



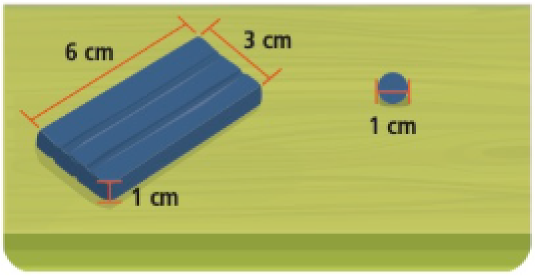
PRACTICE & PROBLEM SOLVING

APPLY

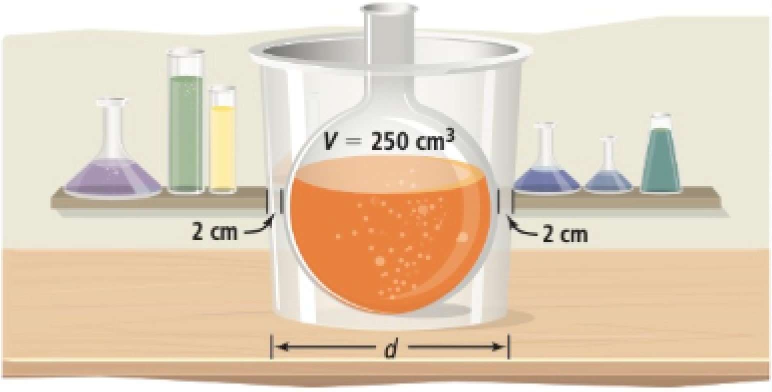
29. **Analyze and Persevere** To reach the regulation pressure for a game ball, the amount of air pumped into a ball is 1.54 times the volume of the ball. A referee adds 15 in.^{3} of air for each pump of air. How many pumps of air will it take the referee to fill an empty ball?



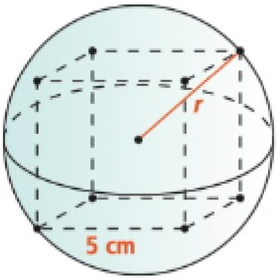
30. **Apply Math Models** Jeffery uses a block of clay to make round beads. How many beads can he make from the block?



31. **Choose Efficient Methods** Felipe places a spherical round-bottom flask in a cylindrical beaker containing hot water. The flask must fit into the beaker with 2 cm of space around the flask. What is the minimum diameter *d* of the beaker?



32. **Higher Order Thinking** A company packs each wireless cube-shaped speaker in a spherical shell protected by foam. How much foam does the company use for each speaker?



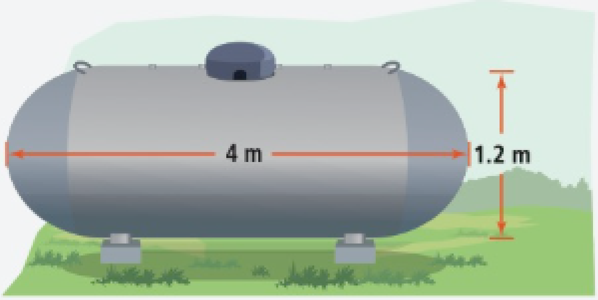
ASSESSMENT PRACTICE

33. Emily is designing a baseball to commemorate her school winning the Florida state high school baseball championships. She wants to use orange leather to make the baseball, and needs to calculate the amount of leather needed if the diameter of the baseball is 2.9 inches. GR.4.5

34. **SAT/ACT** The surface area of a sphere is 64π ft^2. What is the radius of the sphere?

Ⓐ 64 ft
Ⓑ 16 ft
Ⓒ 8 ft
Ⓓ 4 ft

35. **Performance Task** Jayesh is to fill the tank shown with liquid propane.



Part A The liquid propane expands and contracts as the temperature changes, so a propane tank is never filled to more than 80% capacity with liquid propane. How much liquid propane should Jayesh put in the tank?

Part B If Jayesh has 20 m^3 of liquid propane to fill another tank, what are the dimensions of the tank if the length of the cylindrical part is three times the diameter?

TOPIC
11

Topic Review

TOPIC ESSENTIAL QUESTION

1. How is Cavalieri's Principle helpful in understanding the volume formulas for solids?

Vocabulary Review

Choose the correct term to complete each sentence.

2. A prism is ________ if one or more faces are not perpendicular to the bases.
3. ________ describes the relationship between the volumes of three-dimensional figures that have the same height and the same cross sectional area at every level.
4. A great circle divides a sphere into two ________.

- Cavalieri's Principle
- cones
- cylinders
- hemispheres
- oblique
- right
- spheres

Concepts & Skills Review

LESSON 11-1 Three-Dimensional Figures and Cross Sections

Quick Review

The faces of a polyhedron are polygons. Euler's Formula states that the relationship between the number of faces F, number of vertices V, and number of edges E is

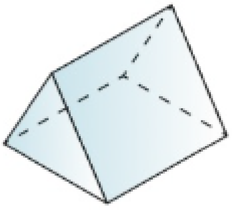
$$F + V = E + 2$$

The cross section of a plane and a convex polyhedron is a polygon.

Rotating a polygon about an axis forms a three-dimensional figure.

Example

A triangular prism has 5 faces and 9 edges. How many vertices does it have?



$$F + V = E + 2$$

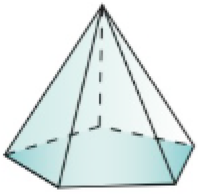
$$5 + V = 9 + 2$$

$$V = 6$$

The prism has 6 vertices.

Practice & Problem Solving

For Exercises 5 and 6, use the pyramid shown.



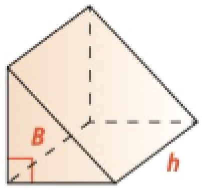
5. The pyramid has 6 vertices and 10 edges. How many faces does it have?
6. Visualize a plane intersecting the pyramid parallel to the base. Describe the cross section.
7. Describe the three-dimensional figure that is formed by rotating the rectangle about the line shown.

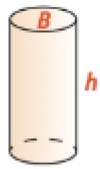
8. **Communicate and Justify** Can a polyhedron have the same number of faces, edges, and vertices? Explain.

LESSON 11-2 Surface Area

Quick Review

The surface area of a prism or cylinder is the sum of the areas of the face(s) and two bases.

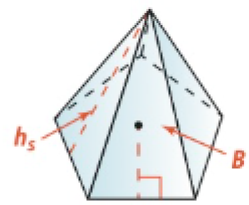




$SA = 2B + Ph$

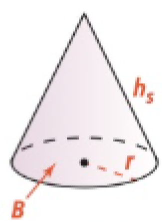
The surface area of a pyramid is the sum of the area of the base and the areas of the triangular faces.

$SA = B + \frac{1}{2}Ph_s$



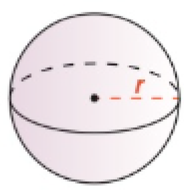
The surface area of a cone is the sum of the areas of the base and the curved surface.

$SA = B + \pi r h_s$



The surface area of a sphere is the area of the curved surface.

$SA = 4\pi r^2$



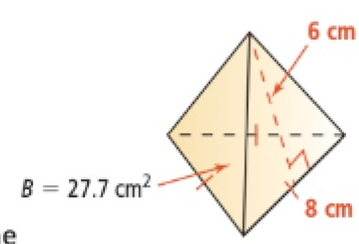
Example

What is the surface area of the regular triangular pyramid?

$SA = B + \frac{1}{2}Ph_s$

$= 27.7 + \frac{1}{2}(24)(6)$

$= 99.7 \text{ cm}^2$

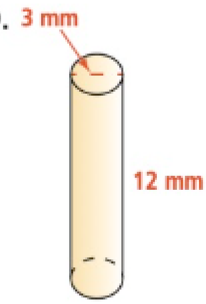


The surface area of the pyramid is 99.7 cm^2.

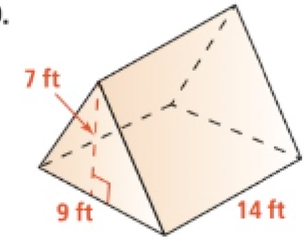
Practice & Problem Solving

Find the surface area of each figure to the nearest tenth.

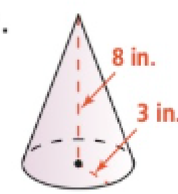
9.



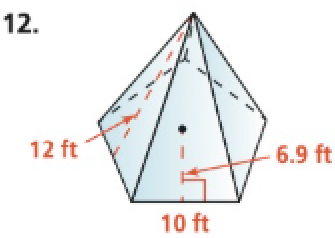
10.



11.



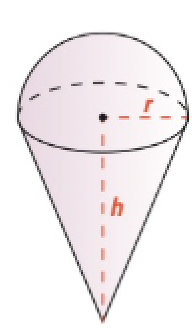
12.



13. Find the radius of the sphere to the nearest tenth.

$SA = 64 \text{ m}^3$

14. **Analyze and Persevere** The composite figure consists of a cone and a hemisphere. How do you find the surface area of the figure?

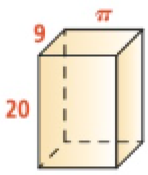


15. The diameter of Earth is approximately 7,917.5 mi. What is the approximate surface area of the earth to the nearest square mile?

LESSON 11-3 Volumes of Prisms and Cylinders

Quick Review

Cavalieri's Principle states that figures with the same height and same area at every horizontal cross section have the same volume.

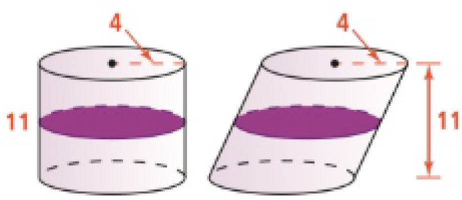


$V = Bh$
$V = (\ell \cdot w)h$
$V = (9\pi)(20)$
$V = 180\pi$

$V = Bh$
$V = (\pi r^2)h$
$V = (9\pi)(20)$
$V = 180\pi$

Example

Do the cylinders have the same volume? Explain.

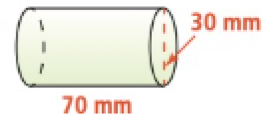


Yes, cylinders with the same base area and height have the same volume. The height of each cylinder is 11. The area of the base of each cylinder is $\pi(4)^2$, or 16π.

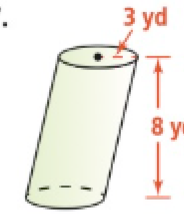
Practice & Problem Solving

For Exercises 16 and 17, find the volume of each figure. Round to the nearest tenth.

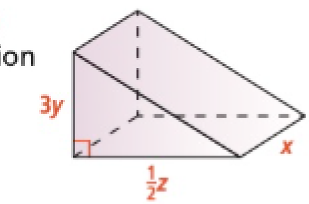
16.



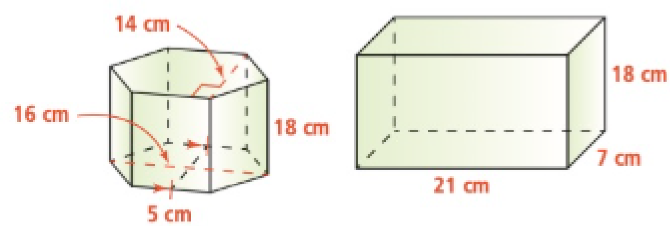
17.



18. Analyze and Persevere What does the expression $\frac{3}{4}xyz$ represent for the prism shown?



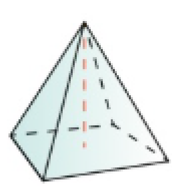
19. Malia pours wax in molds to make candles. Compare the amount of wax each mold holds.



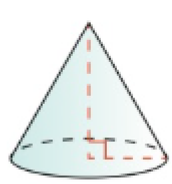
LESSON 11-4 Volumes of Pyramids and Cones

Quick Review

The volume of a pyramid is one-third the volume of a prism with the same base area and height. $V = \frac{1}{3}Bh$



The volume of a cone is one-third the volume of a cylinder with the same base area and height. $V = \frac{1}{3}Bh$

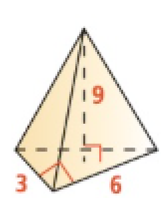


Example

What is the volume?

$V = \frac{1}{3}Bh = \frac{1}{3} \cdot \frac{1}{2}(6 \cdot 3)(9)$
$= 27$

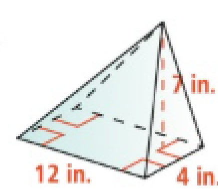
The volume of the pyramid is 27 cubic units.



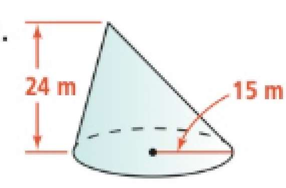
Practice & Problem Solving

For Exercises 20 and 21, find the volume of each figure. Round to the nearest tenth.

20.

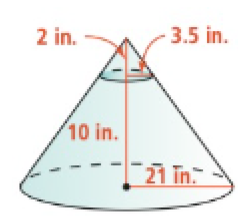


21.



22. A sculptor cuts a pyramid from a marble cube with volume t^3 ft^3. The pyramid is t ft tall. The area of the base is t^2 ft^2. Write an expression for the volume of marble removed.

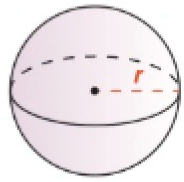
23. A company cuts 2 in. from the tops of the solid plastic cones. How much less plastic is used in the new design?



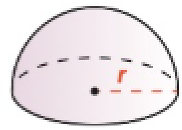
LESSON 11-5 Volumes of Spheres

Quick Review

The volume of a sphere is $V = \frac{4}{3}\pi r^3$.



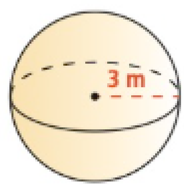
The volume of a hemisphere is one-half the volume of a sphere with the same radius, $V = \frac{2}{3}\pi r^3$.



Example

What is the volume of the sphere shown?

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(3)^3 = 36\pi$$

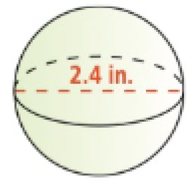


The volume of the sphere is 36π m^3.

Practice & Problem Solving

For Exercises 24 and 25, find the volume of each figure. Round to the nearest tenth.

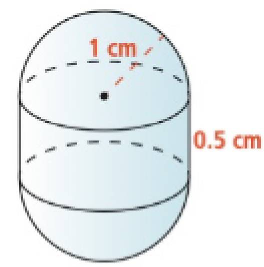
24.



25.

26. **Use Patterns and Structure** A golf ball has a radius of *r* centimeters. What is the least possible volume of a rectangular box that can hold 2 golf balls?

27. A capsule of liquid cold medicine is shown. If 1 dose is about 23 ml, how many capsules make up 1 dose? (*Hint:* 1 ml = 1 cm^3)



Visual Glossary

English / Spanish

A

Acute angle An acute angle is an angle whose measure is between 0 and 90.

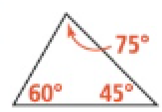
Ángulo agudo Un ángulo agudo es un ángulo que mide entre 0 y 90.

Example

Acute triangle An acute triangle has three acute angles.

Triángulo acutángulo Un triángulo acutángulo tiene los tres ángulos agudos.

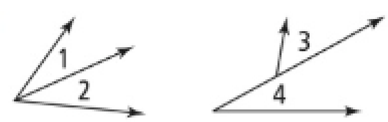
Example



Adjacent angles Adjacent angles are two coplanar angles that have a common side and a common vertex but no common interior points.

Ángulos adyacentes Los ángulos adyacentes son dos ángulos coplanarios que tienen un lado común y el mismo vértice, pero no tienen puntos interiores comunes.

Example

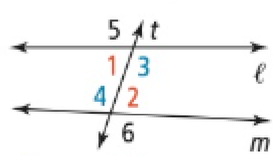


∠1 and ∠2 are adjacent. ∠3 and ∠4 are *not* adjacent.

Alternate interior (exterior) angles Alternate interior (exterior) angles are nonadjacent interior (exterior) angles that lie on opposite sides of the transversal.

Ángulos alternos internos (externos) Los ángulos alternos internos (externos) son ángulos internos (externos) no adyacentes situados en lados opuestos de la transversal.

Example



∠1 and ∠2 are alternate interior angles, as are ∠3 and ∠4. ∠5 and ∠6 are alternate exterior angles.

Altitude *See* **cone; cylinder; parallelogram; prism; pyramid; trapezoid; triangle.**

Altura *Ver* **cone; cylinder; parallelogram; prism; pyramid; trapezoid; triangle.**

Altitude of a triangle An altitude of a triangle is the perpendicular segment from a vertex to the line containing the side opposite that vertex.

Altura de un triángulo Una altura de un triángulo es el segmento perpendicular que va desde un vértice hasta la recta que contiene el lado opuesto a ese vértice.

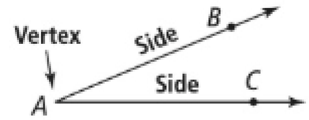
Example

English | Spanish

Angle An angle is formed by two rays with the same endpoint. The rays are the *sides* of the angle and the common endpoint is the *vertex* of the angle.

Ángulo Un ángulo está formado por dos semirrectas que convergen en un mismo extremo. Las semirrectas son los *lados* del ángulo y los extremos en común son el *vértice*.

Example

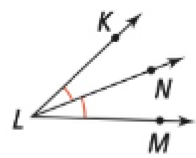


This angle could be named $\angle A$, $\angle BAC$, or $\angle CAB$.

Angle bisector An angle bisector is a ray that divides an angle into two congruent angles.

Bisectriz de un ángulo La bisectriz de un ángulo es una semirrecta que divide al ángulo en dos ángulos congruentes.

Example

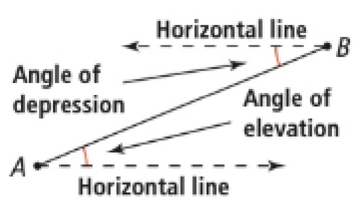


$\overrightarrow{LN}$ bisects $\angle KLM$.
$\angle KLN \cong \angle NLM$.

Angle of elevation or depression An angle of elevation (depression) is the angle formed by a horizontal line and the line of sight to an object above (below) the horizontal line.

Ángulo de elevación o depresión Un ángulo de elevación (depresión) es el ángulo formado por una línea horizontal y la recta que va de esa línea a un objeto situado arriba (debajo) de ella.

Example



Angle of rotation *See* **rotation.**

Ángulo de rotación *Ver* **rotation.**

Apothem *See* **regular polygon.**

Apotema *Ver* **polígono regular.**

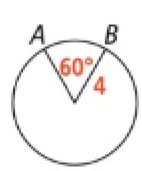
Arc *See* **major arc; minor arc.** *See also* **arc length; measure of an arc.**

Arco *Ver* **major arc; minor arc.** *Ver también* **arc length; measure of an arc.**

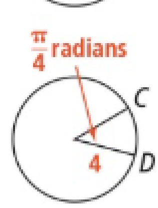
Arc length The length of an arc of a circle is the product of the circumference of the circle and the ratio of the corresponding central angle measure in degrees and 360. The length of the arc is also the product of the radius and central angle measure in radians.

Longitud de un arco La longitud del arco de un círculo es el producto de la circunferencia del círculo y la razón de la medida del ángulo central correspondiente en grados y 360. La longitud del arco es también el producto del radio y de la medida del ángulo central en radianes.

Example



$s = \frac{60}{360} \cdot 2\pi(4) = \frac{4\pi}{3}$

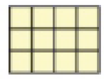


$s = \frac{\pi}{4}(4) = \pi$

English | Spanish

Area The area of a plane figure is the number of square units enclosed by the figure.

Área El área de una figura plana es la cantidad de unidades cuadradas que contiene la figura.

Example  The area of the rectangle is 12 square units, or 12 units^2.

Axes *See* **coordinate plane.**

Ejes *Ver* **coordinate plane.**

B

Base(s) *See* **cone; cylinder; isosceles triangle; parallelogram; prism; pyramid; trapezoid; triangle**.

Base(s) *Ver* **cone; cylinder; isosceles triangle; parallelogram; prism; pyramid; trapezoid; triangle.**

Base angles *See* **trapezoid; isosceles triangle.**

Ángulos de base *Ver* **trapezoid; isosceles triangle.**

Biconditional A biconditional statement is the combination of a conditional statement, $p \rightarrow q$, and its converse, $q \rightarrow p$. A biconditional contains the words "if and only if."

Bicondicional Un enunciado bicondicional es la combinación de un enunciado condicional, $p \rightarrow q$, y su recíproco, $q \rightarrow p$. El enunciado bicondicional incluye las palabras "si y solo si".

Example This biconditional statement is true: Two angles are congruent *if and only if* they have the same measure.

Bisector *See* **segment bisector; angle bisector.**

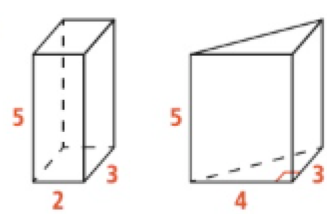
Bisectriz *Ver* **segment bisector; angle bisector.**

C

Cavalieri's Principle If two space figures have the same height and the same cross-sectional area at every level, then they have the same volume.

Principio de Cavalieri Si dos figuras sólidas tienen la misma altura y la misma área transversal en todos los niveles, entonces también tienen el mismo volumen.

Example



Both figures are prisms with height 5 units and horizontal cross-sectional area 6 square units.

$V = B \cdot h = (6)(5) = 30$ cubic units

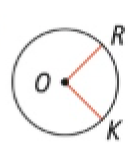
Center *See* **circle; dilation; regular polygon; rotation; sphere.**

Centro *Ver* **circle; dilation; regular polygon; rotation; sphere.**

Central angle of a circle A central angle of a circle is an angle formed by two radii with the vertex at the center of the circle.

Ángulo central de un círculo Un ángulo central de un círculo es un ángulo formado por dos radios que tienen el vértice en el centro del círculo.

Example



$\angle ROK$ is a central angle of $\odot O$.

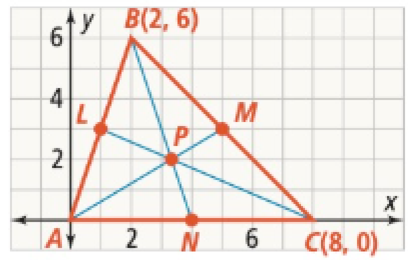
English

Spanish

Centroid of a triangle The centroid of a triangle is the point of concurrency of the medians of the triangle.

Centroide de un triángulo El centroide de un triángulo es el punto de intersección de sus medianas.

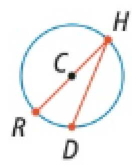
Example P is the centroid of $\triangle ABC$.



Chord A chord of a circle is a segment whose endpoints are on the circle.

Cuerda Una cuerda de un círculo es un segmento cuyos extremos son dos puntos del círculo.

Example

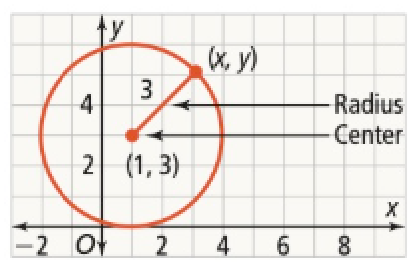


$\overline{HD}$ and $\overline{HR}$ are chords of $\odot C$.

Circle A circle is the set of all points in a plane that are a given distance, the radius, from a given point, the center. The standard form for an equation of a circle with center (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

Círculo Un círculo es el conjunto de todos los puntos de un plano situados a una distancia dada, el radio, de un punto dado, el centro. La fórmula normal de la ecuación de un círculo con centro (h, k) y radio r es $(x - h)^2 + (y - k)^2 = r^2$.

Example



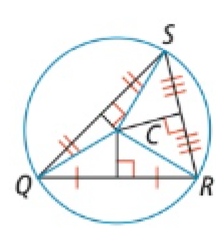
The equation of the circle whose center is (1, 3) and whose radius is 3 is $(x - 1)^2 + (y - 3)^2 = 9$.

Circumcenter of a triangle The circumcenter of a triangle is the point of concurrency of the perpendicular bisectors of the sides of the triangle.

Circuncentro de un triángulo El circuncentro de un triángulo es el punto de intersección de las bisectrices perpendiculares de los lados del triángulo.

Example

$QC = SC = RC$



C is the circumcenter.

Circumference The circumference of a circle is the distance around the circle. Given the radius r of a circle, you can find its circumference C by using the formula $C = 2\pi r$.

Circunferencia La circunferencia de un círculo es la distancia alrededor del círculo. Dado el radio r de un círculo, se puede hallar la circunferencia C usando la fórmula $C = 2\pi r$.

Example

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi(4) \\ &= 8\pi \end{aligned}$$

Circumference is the distance around the circle.

English | Spanish

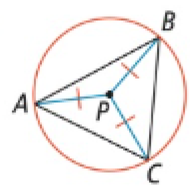
Circumference of a sphere *See* **sphere**.

Circunferencia de una esfera Ver sphere.

Circumscribed A circumscribed circle of a triangle is the circle that contains the three vertices of the triangle.

Circunscrito El círculo circunscrito de un triángulo es el círculo que contiene los tres vértices del triángulo.

Example

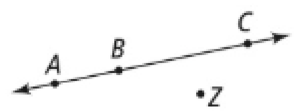


$\odot P$ is the circumscribed circle of $\triangle ABC$.

Collinear points Collinear points lie on the same line.

Puntos colineales Los puntos colineales son los que están sobre la misma recta.

Example



Points A, B, and C are collinear, but points A, B, and Z are noncollinear.

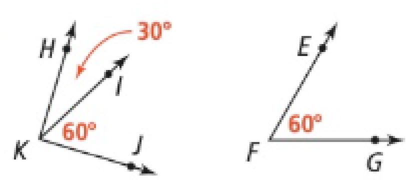
Compass A compass is a tool for drawing arcs and circles of different sizes and can be used to copy lengths.

Compás El compás es un instrumento que se usa para dibujar arcos y círculos de diferentes tamaños, y que se puede usar para copiar longitudes.

Complementary angles Two angles are complementary angles if the sum of their measures is 90.

Ángulos complementarios Dos ángulos son complementarios si la suma de sus medidas es igual a 90.

Example

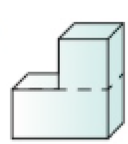


$\angle HKI$ and $\angle IKJ$ are complementary angles, as are $\angle HKI$ and $\angle EFG$.

Composite space figures A composite space figure is the combination of two or more figures into one object.

Figuras geométricas compuestas Una figura geométrica compuesta es la combinación de dos o más figuras en un mismo objeto.

Example



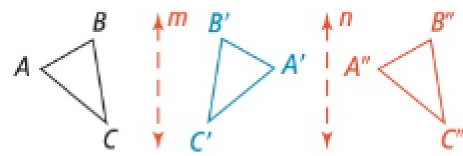
English

Spanish

Composition of rigid motions A composition of rigid motions is a transformation with two or more rigid motions in which the second rigid motion is performed on the image of the first rigid motion.

Composición de movimientos rígidos Una composición de movimientos rígidos es una transformación de dos o más movimientos rígidos en la que el segundo movimiento rígido se realiza sobre la imagen del primer movimiento rígido.

Example



If you reflect $\triangle ABC$ across line *m* to get $\triangle A'B'C'$ and then reflect $\triangle A'B'C'$ across line *n* to get $\triangle A''B''C''$, you perform a composition of rigid motions.

Concave polygon *See* **polygon.**

Polígono cóncavo *Ver* **polygon.**

Concentric circles Concentric circles lie in the same plane and have the same center.

Círculos concéntricos Los círculos concéntricos están en el mismo plano y tienen el mismo centro.

Example

The two circles both have center *D* and are therefore concentric.

Conclusion The conclusion is the part of an *if-then* statement (conditional) that follows *then*.

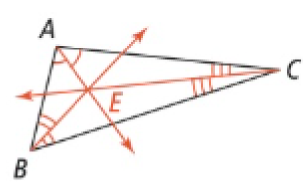
Conclusión La conclusión es lo que sigue a la palabra entonces en un enunciado (condicional), *si . . ., entonces. . . .*

Example In the statement, "If it rains, then I will go outside," the conclusion is "I will go outside."

Concurrent Three or more lines are concurrent if they intersect at one point. The point at which they intersect is the *point of concurrency*.

Concurrente Tres o más rectas son concurrentes si se intersecan en un punto. El punto en el que se intersecan es el *punto de concurrencia*.

Example



Point *E* is the point of concurrency of the bisectors of the angles of $\triangle ABC$. The bisectors are concurrent.

Conditional A conditional is an *if-then* statement that relates a hypothesis, the part that follows if, to a conclusion, the part that follows then.

Condicional Un enunciado condicional es del tipo *si . . ., entonces. . . .*

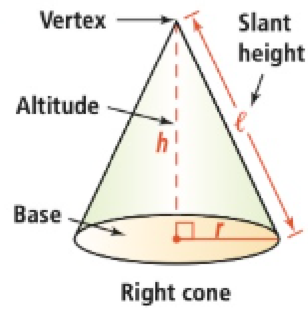
Example *If* you act politely, *then* you will earn respect.

English / Spanish

Cone A cone is a three-dimensional figure that has a circular *base*, a *vertex* not in the plane of the circle, and a curved lateral surface, as shown in the diagram. The *altitude* of a cone is the perpendicular segment from the vertex to the plane of the base. The *height* is the length of the altitude. In a *right* cone, the altitude contains the center of the base. The *slant height* of a right cone is the distance from the vertex to the edge of the base.

Cono Un cono es una figura tridimensional que tiene una *base* circular, un *vértice* que no está en el plano del círculo y una superficie lateral curvada (indicada en el diagrama). La *altura* de un cono es el segmento perpendicular desde el vértice hasta el plano de la base. La *altura*, por extensión, es la longitude de la altura. Un *cono recto* es un cono cuya altura contiene el centro de la base. La *longitude de la generatriz* de un cono recto es la distancia desde el vértice hasta el borde de la base.

Example



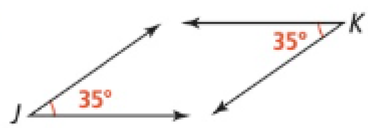
Congruence transformation A congruence transformation maps a figure to a congruent figure. A rigid motion is sometimes called a congruence transformation.

Transformación de congruencia En una transformación de congruencia, una figura es la imagen de otra figura congruente. Los movimientos rígidos son llamados a veces transformaciones de congruencia.

Congruent angles Congruent angles are angles that have the same measure.

Ángulos congruentes Los ángulos congruentes son ángulos que tienen la misma medida.

Example

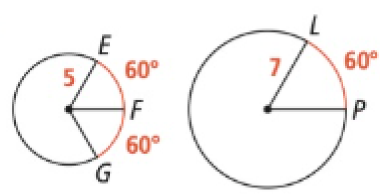


$m\angle J = m\angle K$, so $\angle J \cong \angle K$.

Congruent arcs Congruent arcs are arcs that have the same measure and are in the same circle or congruent circles.

Arcos congruentes Arcos congruentes son arcos que tienen la misma medida y están en el mismo círculo o en círculos congruentes.

Example

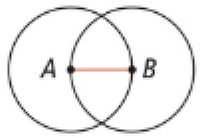


$\overset{\frown}{EF} \cong \overset{\frown}{FG}$ $\overset{\frown}{EF} \ncong \overset{\frown}{LP}$

Congruent circles Congruent circles are circles whose radii are congruent.

Círculos congruentes Los círculos congruentes son círculos cuyos radios son congruentes.

Example



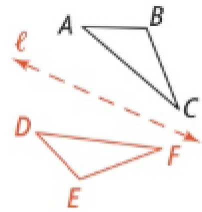
$\odot A$ and $\odot B$ have the same radius, so $\odot A \cong \odot B$.

English	Spanish

Congruent figures Two figures are congruent if there is a rigid motion that maps one figure to the other.

Figuras congruentes Dos figuras son congruentes si hay un movimiento rígido en el que una figura es imagen de la otra.

Example

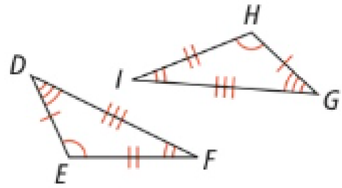


Since the reflection across line ℓ maps $\triangle ABC$ to $\triangle DEF$, $\triangle ABC$ and $\triangle DEF$ are congruent figures.

Congruent polygons Congruent polygons are polygons that have corresponding sides congruent and corresponding angles congruent.

Polígonos congruentes Los polígonos congruentes son polígonos cuyos lados correspondientes son congruentes y cuyos ángulos correspondientes son congruentes.

Example

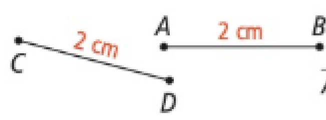


$\triangle DEF \cong \triangle GHI$

Congruent segments Congruent segments are segments that have the same length.

Segmentos congruentes Los segmentos congruentes son segmentos que tienen la misma longitud.

Example



$\overline{AB} \cong \overline{CD}$

Conjecture A conjecture is an unproven statement or rule that is based on inductive reasoning.

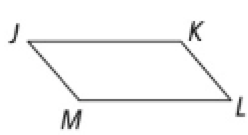
Conjetura Una conjetura es una afirmación o regla no demostrada, que está fundamentada en un razonamiento inductivo.

Example As you walk down the street, you see many people holding unopened umbrellas. You make the conjecture that the forecast must call for rain.

Consecutive angles Consecutive angles of a polygon share a common side.

Ángulos consecutivos Los ángulos consecutivos de un polígono tienen un lado común.

Example



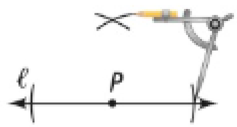
In ▱$JKLM$, $\angle J$ and $\angle M$ are consecutive angles, as are $\angle J$ and $\angle K$. $\angle J$ and $\angle L$ are *not* consecutive.

English	Spanish

Construction A construction is a geometric figure made with only a straightedge and compass.

Construcción Una construcción es una figura geométrica trazada solamente con una regla sin graduación y un compás.

Example



The diagram shows the construction (in progress) of a line perpendicular to a line ℓ through a point P on ℓ.

Contrapositive The contrapositive is obtained by negating and reversing the hypothesis and the conclusion of a conditional. The contrapositive of the conditional "if *p*, then *q*" is the conditional "if not *q*, then not *p*." A conditional and its contrapositive always have the same truth value.

Contrapositivo El contrapositivo se obtiene al negar e intercambiar la hipótesis y la conclusión de un condicional. El contrapositivo del condicional "si *p*, entonces *q*" es el condicional "si no *q*, entonces no *p*". Un condicional y su contrapositivo siempre tienen el mismo valor verdadero.

Example **Conditional:** If a figure is a triangle, then it is a polygon.
Contrapositive: If a figure is not a polygon, then it is not a triangle.

Converse The converse reverses the hypothesis and conclusion of a conditional.

Expresión recíproca La expresión recíproca intercambia la hipótesis y la conclusión de un condicional.

Example The converse of "If I was born in Houston, then I am a Texan" would be "If I am a Texan, then I am born in Houston."

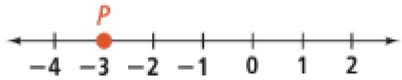
Convex polygon *See* **polygon.**

Polígono convexo *Ver* **polygon.**

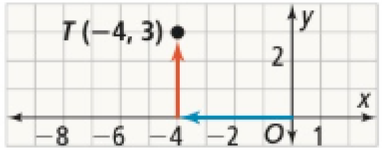
Coordinate(s) of a point The coordinate of a point is its distance and direction from the origin of a number line. The coordinates of a point on a coordinate plane are in the form (*x*, *y*), where *x* is the *x*-coordinate and *y* is the *y*-coordinate.

Coordenada(s) de un punto La coordenada de un punto es su distancia y dirección desde el origen en una recta numérica. Las coordenadas de un punto en un plano de coordenadas se expresan como (*x*, *y*), donde *x* es la coordenada *x*, e *y* es la coordenada *y*.

Example



The coordinate of P is -3.



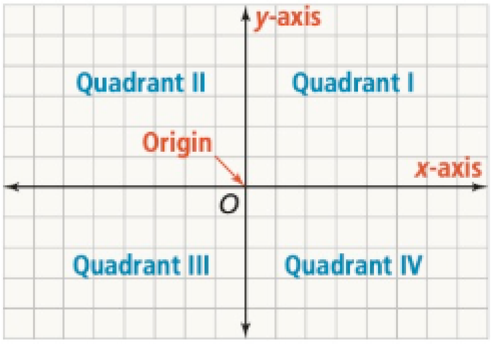
The coordinates of T are $(-4, 3)$.

English	Spanish

Coordinate plane The coordinate plane is formed by two number lines, called the axes, intersecting at right angles. The x-axis is the horizontal axis, and the y-axis is the vertical axis. The two axes meet at the origin, $O(0, 0)$. The axes divide the plane into four *quadrants*.

Plano de coordenadas El plano de coordenadas se forma con dos rectas numéricas, llamadas *ejes*, que se cortan en ángulos rectos. El eje x es el eje horizontal y el eje y es el eje vertical. Los dos ejes se unen en el *origen*, $O(0, 0)$. Los ejes dividen el plano de coordenadas en cuatro *cuadrantes*.

Example



Coordinate proof *See* **proof.**

Prueba de coordenadas *Ver* **proof.**

Corollary A corollary is a theorem that can be proved easily using another theorem.

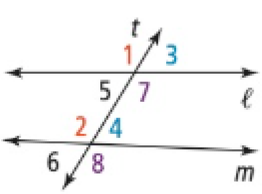
Corolario Un corolario es un teorema que se puede probar fácilmente usando otro teorema.

Example **Theorem:** If two sides of a triangle are congruent, then the angles opposite those sides are congruent.
Corollary: If a triangle is equilateral, then it is equiangular.

Corresponding angles Corresponding angles lie on the same side of the transversal t and in corresponding positions relative to ℓ and m.

Ángulos correspondientes Los ángulos correspondientes están en el mismo lado de la transversal t y en las correspondientes posiciones relativas a ℓ y m.

Example



$\angle 1$ and $\angle 2$ are corresponding angles, as are $\angle 3$ and $\angle 4$, $\angle 5$ and $\angle 6$, and $\angle 7$ and $\angle 8$.

Cosine ratio See **trigonometric ratios.**

Razón coseno *Ver* **trigonometric ratios.**

Counterexample A counterexample is an example that shows a statement or conjecture is false.

Contraejemplo Un contraejemplo es un ejemplo que demuestra que una afirmación o conjetura es falsa.

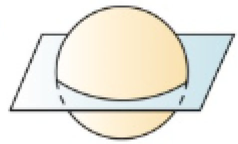
Example **Statement:** All apples are red.
Counterexample: A Granny Smith Apple is green.

English	Spanish

Cross section A cross section is the intersection of a solid and a plane.

Sección de corte Una sección de corte es la intersección de un plano y un cuerpo geométrico.

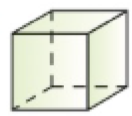
Example



Cube A cube is a polyhedron with six faces, each of which is a square.

Cubo Un cubo es un poliedro de seis caras, cada una de las caras es un cuadrado.

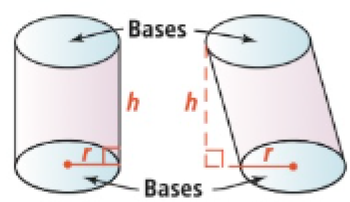
Example



Cylinder A cylinder is a three-dimensional figure with two congruent circular bases that lie in parallel planes. An *altitude* of a cylinder is a perpendicular segment that joins the planes of the bases. Its length is the *height* of the cylinder. In a *right cylinder*, the segment joining the centers of the bases is an altitude. In an *oblique cylinder*, the segment joining the centers of the bases is not perpendicular to the planes containing the bases.

Cilindro Un cilindro es una figura tridimensional con dos bases congruentes circulares en planos paralelos. Una *altura* de un cilindro es un segmento perpendicular que une los planos de las bases. Su longitud es, por extensión, la *altura* del cilindro. En un *cilindro* recto, el segmento que une los centros de las bases es una altura. En un *cilindro oblicuo*, el segmento que une los centros de las bases no es perpendicular a los planos que contienen las bases.

Example

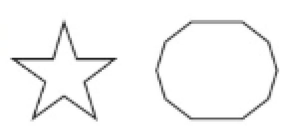


D

Decagon A decagon is a polygon with ten sides.

Decágono Un decágono es un polígono de diez lados.

Example



Deductive reasoning Deductive reasoning is a process of reasoning using given and previously known facts to reach a logical conclusion.

Razonamiento deductivo El razonamiento deductivo es un proceso de razonamiento en el que se usan hechos dados y previamente conocidos para llegar a una conclusión lógica.

Example Based on the fact that the sum of any two even numbers is even, you can deduce that the product of any whole number and any even number is even.

Diagonal *See* **polygon.**

Diagonal *Ver* **polygon.**

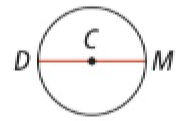
English

Spanish

Diameter of a circle A diameter of a circle is a segment that contains the center of the circle and whose endpoints are on the circle. The term *diameter* can also mean the length of this segment.

Diámetro de un círculo Un diámetro de un círculo es un segmento que contiene el centro del círculo y cuyos extremos están en el círculo. El término *diámetro* también puede referirse a la longitud de este segmento.

Example



$\overline{DM}$ is a diameter of $\odot C$.

Diameter of a sphere The diameter of a sphere is a segment passing through the center, with endpoints on the sphere.

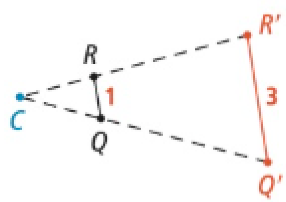
Diámetro de una esfera El diámetro de una esfera es un segmento que contiene el centro de la esfera y cuyos extremos están en la esfera.

Example

Dilation A dilation is a transformation that has *center C* and *scale factor n*, where $n > 0$, and maps a point R to R' in such a way that R' is on $\overrightarrow{CR}$ and $CR' = n \cdot CR$. The center of a dilation is its own image. If $n > 1$, the dilation is an *enlargement*, and if $0 < n < 1$, the dilation is a *reduction*.

Dilatación Una dilatación, o *transformación de semejanza*, tiene *centro C* y *factor de escala n* para $n > 0$, y asocia un punto R a R' de tal modo que R' está en $\overrightarrow{CR}$ y $CR' = n \cdot CR$. El centro de una dilatación es su propia imagen. Si $n > 1$, la dilatación es un aumento, y si $0 < n < 1$, la dilatación es una reducción.

Example



$\overline{R'Q'}$ is the image of $\overline{RQ}$ under a dilation with center C and scale factor 3.

Distance between two points on a line The distance between two points on a line is the absolute value of the difference of the coordinates of the points.

Distancia entre dos puntos de una línea La distancia entre dos puntos de una línea es el valor absoluto de la diferencia de las coordenadas de los puntos.

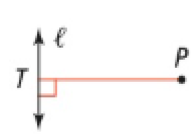
Example

$AB = |a - b|$

Distance from a point to a line The distance from a point to a line is the length of the perpendicular segment from the point to the line.

Distancia desde un punto hasta una recta La distancia desde un punto hasta una recta es la longitud del segmento perpendicular que va desde el punto hasta la recta.

Example



The distance from point P to a line ℓ is PT.

English / Spanish

E

Edge *See* **polyhedron.**

Arista *Ver* **polyhedron.**

Endpoint *See* **ray; segment.**

Extremo *Ver* **ray; segment.**

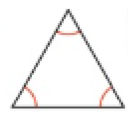
Enlargement *See* **dilation.**

Aumento *Ver* **dilation.**

Equiangular triangle An equiangular triangle is a triangle whose angles are all congruent.

Triángulo equiángulo Un triángulo equiángulo es un triángulo cuyos ángulos son todos congruentes.

Example

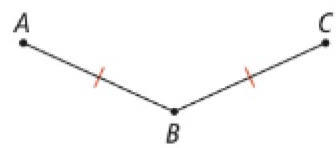


Each angle of the triangle is a 60° angle.

Equidistant A point is equidistant from two objects if it is the same distance from the objects.

Equidistante Un punto es equidistante de dos objetos si la distancia entre el punto y los objetos es igual.

Example

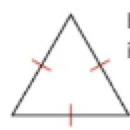


Point B is equidistant from points A and C.

Equilateral triangle An equilateral triangle is a triangle whose sides are all congruent.

Triángulo equilátero Un triángulo equilátero es un triángulo cuyos lados son todos congruentes.

Example



Each side of the triangle is 1.5 cm long.

Equivalent statements Equivalent statements are statements with the same truth value.

Enunciados equivalentes Los enunciados equivalentes son enunciados con el mismo valor verdadero.

Example The following statements are equivalent:
If a figure is a square, then it is a rectangle.
If a figure is not a rectangle, then it is not a square.

Extended proportion *See* **proportion.**

Proporción extendida *Ver* **proportion.**

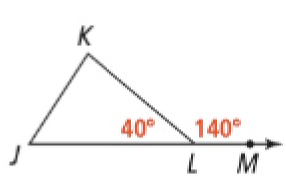
Extended ratio *See* **ratio.**

Razón extendida *Ver* **ratio.**

Exterior angle of a polygon An exterior angle of a polygon is an angle formed by a side and an extension of an adjacent side.

Ángulo exterior de un polígono El ángulo exterior de un polígono es un ángulo formado por un lado y una extensión de un lado adyacente.

Example



$\angle KLM$ is an exterior angle of $\triangle JKL$.

F

Flow proof *See* **proof.**

Prueba de flujo *Ver* **proof.**

English | Spanish

G

Geometric mean The geometric mean is the number x such that $\frac{a}{x} = \frac{x}{b}$, where a, b, and x are positive numbers.

Media geométrica La media geométrica es el número x tanto que $\frac{a}{x} = \frac{x}{b}$, donde a, b y x son números positivos.

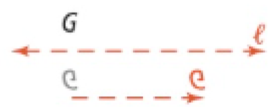
Example The geometric mean of 6 and 24 is 12.

$$\frac{6}{x} = \frac{x}{24}$$
$$x^2 = 144$$
$$x = 12$$

Glide reflection A glide reflection is the composition of a reflection followed by a translation in a direction parallel to the line of reflection.

Reflexión deslizada Una reflexión por deslizamiento es la composición de una reflexión seguida de una traslación en una dirección paralela a la recta de reflexión.

Example

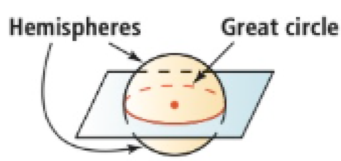


The red G in the diagram is a glide reflection image of the black G.

Great circle A great circle is the intersection of a sphere and a plane containing the center of the sphere. A great circle divides a sphere into two *hemispheres*.

Círculo máximo Un círculo máximo es la intersección de una esfera y un plano que contiene el centro de la esfera. Un círculo máximo divide una esfera en dos *hemisferios*.

Example



H

Height *See* **cone; cylinder; parallelogram; prism; pyramid; trapezoid; triangle.**

Altura *Ver* **cone; cylinder; parallelogram; prism; pyramid; trapezoid; triangle.**

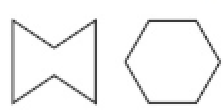
Hemisphere *See* **great circle.**

Hemisferio *Ver* **great circle.**

Hexagon A hexagon is a polygon with six sides.

Hexágono Un hexágono es un polígono de seis lados.

Example



Hypotenuse *See* **right triangle.**

Hipotenusa *Ver* **right triangle.**

Hypothesis In an *if-then* statement (conditional) the hypothesis is the part that follows *if*.

Hipótesis En un enunciado *si . . . entonces . . .* (condicional), la hipótesis es la parte del enunciado que sigue el *si*.

Example In the conditional "If an animal has four legs, then it is a horse," the hypothesis is "an animal has four legs."

I

Image *See* **transformation.**

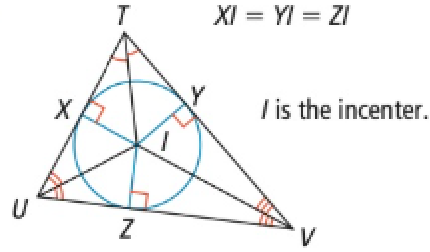
Imagen *Ver* **transformation.**

English | Spanish

Incenter of a triangle The incenter of a triangle is the point of concurrency of the angle bisectors of the triangle.

Incentro de un triángulo El incentro de un triángulo es el punto donde concurren las tres bisectrices de los ángulos del triángulo.

Example



Indirect proof *See* **indirect reasoning; proof.**

Prueba indirecta *Ver* **indirect reasoning; proof.**

Indirect reasoning Indirect reasoning is a type of reasoning in which all possiblities are considered and then all but one are proved false. The remaining possibility must be true.

Razonamiento indirecto Razonamiento indirecto es un tipo de razonamiento en el que se consideran todas las posibilidades y se prueba que todas son falsas, a excepción de una. La posibilidad restante debe ser verdadera.

Example Eduardo spent more than \$60 on two books at a store. Prove that at least one book costs more than \$30.
Proof: Suppose neither costs more than \$30. Then he spent no more than \$60 at the store. Since this contradicts the given information, at least one book costs more than \$30.

Inductive reasoning Inductive reasoning is a type of reasoning that reaches conclusions based on a pattern of specific examples or past events.

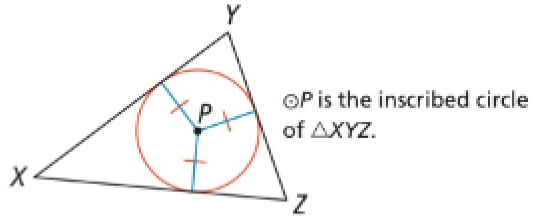
Razonamiento inductivo El razonamiento inductivo es un tipo de razonamiento en el cual se llega a conclusiones con base en un patrón de ejemplos específicos o sucesos pasados.

Example You see four people walk into a building. Each person emerges with a small bag containing food. You use inductive reasoning to conclude that this building contains a restaurant.

Inscribed The inscribed circle of a triangle is the circle that intersects each side of the triangle at exactly one point.

Inscrito El círculo inscrito de un triángulo es el círculo que interseca cada lado del triángulo en exactamente un punto..

Example



Inscribed angle An angle is inscribed in a circle if the vertex of the angle is on the circle and the sides of the angle are chords of the circle.

Ángulo inscrito Un ángulo está inscrito en un círculo si el vértice del ángulo está en el círculo y los lados del ángulo son cuerdas del círculo.

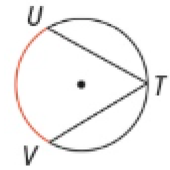
Example

English | Spanish

Intercepted arc An intercepted arc is the part of a circle that lies between two segments that intersect the circle.

Arco interceptor Un arco interceptor es la parte de un círculo que yace entre dos segmentos de recta que intersecan al círculo.

Example

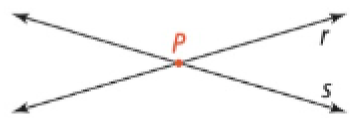


$\overset{\frown}{UV}$ is the intercepted arc of inscribed $\angle T$.

Intersection The intersection of two or more geometric figures is the set of points the figures have in common.

Intersección La intersección de dos o más figuras geométricas es el conjunto de puntos que las figuras tienen en común.

Example



The intersection of lines r and s is point P.

Inverse The inverse is obtained by negating both the hypothesis and the conclusion of a conditional. The inverse of the conditional "if p, then q" is the conditional "if not p, then not q."

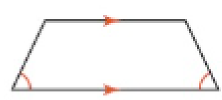
Inverso El inverso es la negación de la hipótesis y de la conclusión de un condicional. El inverso del condicional "si p, entonces q" es el condicional "si no p, entonces no q".

Example **Conditional:** If a figure is a square, then it is a parallelogram.
Inverse: If a figure is not a square, then it is not a parallelogram.

Isosceles trapezoid An isosceles trapezoid is a trapezoid whose base angles are congruent.

Trapecio isósceles Un trapecio isosceles es un trapecio cuyas bases tienen ángulos congruentes.

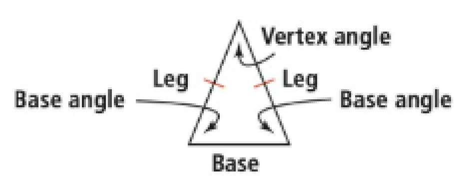
Example



Isosceles triangle An isosceles triangle is a triangle that has at least two congruent sides. If there are two congruent sides, they are called *legs*. The *vertex angle* is between them. The third side is called the *base* and the other two angles are called the *base angles*.

Triángulo isósceles Un triángulo isósceles es un triángulo que tiene por lo menos dos lados congruentes. Si tiene dos lados congruentes, éstos se llaman *catetos*. Entre ellos se encuentra el *ángulo del vértice*. El tercer lado se llama *base* y los otros dos ángulos se llaman *ángulos de base*.

Example

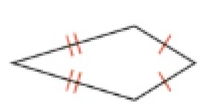


K

Kite A kite is a quadrilateral with two pairs of congruent consecutive sides.

Cometa Una cometa es un cuadrilátero con dos pares de lados consecutivos congruentes.

Example



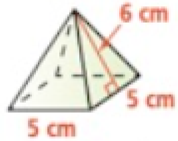
English	Spanish

L

Lateral area The lateral area of a prism or pyramid is the sum of the areas of the lateral faces. The lateral area of a cylinder or cone is the area of the curved surface.

Área lateral El área lateral de un prisma o pirámide es la suma de las áreas de sus caras laterals. El área lateral de un cilindro o de un cono es el área de la superficie curvada.

Example



$$\text{L.A. of pyramid} = \frac{1}{2}p\ell = \frac{1}{2}(20)(6) = 60\text{ cm}^2$$

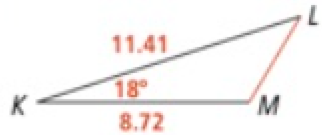
Lateral face *See* **prism; pyramid.**

Cara lateral *Ver* **prism; pyramid.**

Law of Cosines In $\triangle ABC$, let a, b, and c represent the lengths of the sides opposite $\angle A$, $\angle B$, and $\angle C$, respectively. Then
$a^2 = b^2 + c^2 - 2bc\cos A$,
$b^2 = a^2 + c^2 - 2ac\cos B$, and
$c^2 = a^2 + b^2 - 2ab\cos C$

Ley de cosenos En $\triangle ABC$, sean a, b y c las longitudes de los lados opuestos a $\angle A$, $\angle B$, y $\angle C$, respectivamente. Entonces
$a^2 = b^2 + c^2 - 2bc\cos A$,
$b^2 = a^2 + c^2 - 2ac\cos B$, y
$c^2 = a^2 + b^2 - 2ab\cos C$

Example

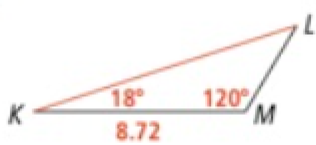


$$LM^2 = 11.41^2 + 8.72^2 - 2(11.42)(8.72)\cos 18°$$
$$LM^2 = 16.9754$$
$$LM = 4.12$$

Law of Sines In $\triangle ABC$, let a, b, and c represent the lengths of the sides opposite $\angle A$, $\angle B$, and $\angle C$, respectively. Then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Ley de senos En $\triangle ABC$, sean a, b y c las longitudes de los lados opuestos a $\angle A$, $\angle B$ y $\angle C$, respectivamente. Entonces $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

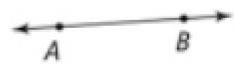
Example



$$m\angle L = 180 - (120 + 18)$$
$$= \frac{KL}{\sin 120°} = \frac{872}{\sin 42°}$$
$$KL = \frac{872\sin 120°}{\sin 42°}$$
$$KL = 11.26$$

English	Spanish
Leg *See* **isosceles triangle; right triangle; trapezoid.**	**Cateto** *Ver* **isosceles triangle; right triangle; trapezoid.**
Line A line is undefined. You can think of a line as a straight path that extends in two opposite directions without end and has no thickness. A line contains infinitely many points. In spherical geometry, you can think of a line as a great circle of a sphere.	**Recta** Una recta es indefinida. Se puede pensar en una recta como un camino derecho que se extiende en direcciones opuestas sin fin ni grosor. Una recta tiene un número infinito de puntos. En la geometría esférica, se puede pensar en una recta como un gran círculo de una esfera.

Example



English	Spanish
Linear pair A linear pair is a pair of adjacent angles whose noncommon sides are opposite rays.	**Par lineal** Un par lineal es un par de ángulos adjuntos cuyos lados no comunes son semirrectas opuestas.

Example

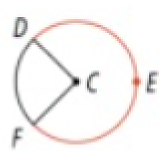
$\angle 1$ and $\angle 2$ are a linear pair.

English	Spanish
Line of reflection *See* **reflection.**	**Eje de reflexión** *Ver* **reflection.**
Line of symmetry *See* **reflectional symmetry.**	**Eje de simetría** *Ver* **reflectional symmetry.**

M

English	Spanish
Major arc A major arc of a circle is an arc that is larger than a semicircle.	**Arco mayor** Un arco mayor de un círculo es cualquier arco más grande que un semicírculo.

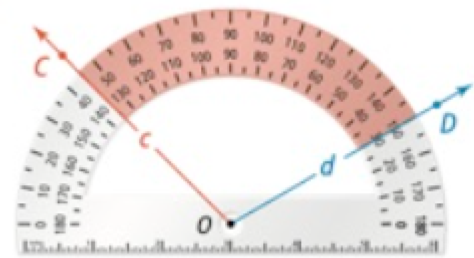
Example



$\overset{\frown}{DEF}$ is a major arc of $\odot C$.

English	Spanish
Map *See* **transformation.**	**Trazar** *Ver* **transformation.**
Measure of an angle Consider $\overrightarrow{OD}$ and a point C on one side of $\overleftrightarrow{OD}$. Every ray of the form $\overrightarrow{OC}$ can be paired one to one with a real number from 0 to 180. The measure of $\angle COD$ is the absolute value of the difference of the real numbers paired with $\overrightarrow{OC}$ and $\overrightarrow{OD}$.	**Medida de un ángulo** Toma en cuenta $\overrightarrow{OD}$ y un punto C a un lado de $\overleftrightarrow{OD}$. Cada semirrecta de la forma $\overrightarrow{OC}$ puede ser emparejada exactamente con un número real de 0 a 180. La medida de $\angle COD$ es el valor absoluto de la diferencia de los números reales emparejados con $\overrightarrow{OC}$ y $\overrightarrow{OD}$

Example



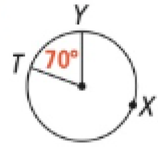
$m\angle COD = 105$

English	Spanish

Measure of an arc The measure of a minor arc is the measure of its central angle. The measure of a major arc is 360° minus the measure of its related minor arc.

Medida de un arco La medida de un arco menor es la medida de su ángulo central. La medida de un arco mayor es 360° menos la medida en grados de su arco menor correspondiente.

Example

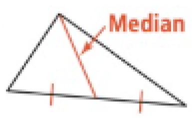


$m\widehat{TY} = 70$
$m\widehat{TXY} = 290$

Median of a triangle A median of a triangle is a segment that has as its endpoints at a vertex of the triangle and the midpoint of the opposite side.

Mediana de un triángulo Una mediana de un triángulo es un segmento que tiene en sus extremos el vértice del triángulo y el punto medio del lado opuesto.

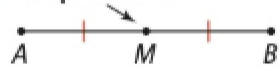
Example



Midpoint of a segment A midpoint of a segment is the point that divides the segment into two congruent segments.

Punto medio de un segmento El punto medio de un segmento es el punto que divide el segmento en dos segmentos congruentes.

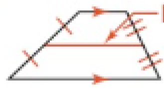
Example



Midsegment of a trapezoid The midsegment of a trapezoid is the segment that joins the midpoints of the nonparallel opposite sides of a trapezoid.

Segmento medio de un trapecio El segmento medio de trapecio es el segmento que une los puntos medios de los lados no paralelos de un trapecio.

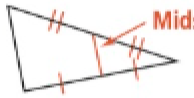
Example



Midsegment of a triangle A midsegment of a triangle is a segment that joins the midpoints of two sides of the triangle.

Segmento medio de un triángulo Un segmento medio de un triángulo es un segmento que une los puntos medios de dos lados del triángulo.

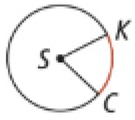
Example



Minor arc A minor arc is an arc that is smaller than a semicircle.

Arco menor Un arco menor de un círculo es un arco más corto que un semicírculo.

Example



$\widehat{KC}$ is a minor arc of $\odot S$.

English — Spanish

N

Negation The negation of a statement has the opposite meaning of the original statement.

Negación La negación de un enunciado tiene el sentido opuesto del enunciado original.

Example **Statement:** The angle is obtuse.
Negation: The angle is not obtuse.

n-gon An n-gon is a polygon with n sides.

n-ágono Un n-ágono es un polígono de n lados.

Example A polygon with 25 sides is a 25-gon.

Nonagon A nonagon is a polygon with nine sides.

Nonágono Un nonágono es un polígono de nueve lados.

Example

O

Oblique cylinder or prism *See* **cylinder; prism.**

Cilindro oblicuo o prisma *Ver* **cylinder; prism.**

Obtuse angle An obtuse angle is an angle whose measure is between 90 and 180.

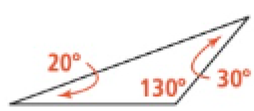
Ángulo obtuso Un ángulo obtuso es un ángulo que mide entre 90 y 180.

Example

Obtuse triangle An obtuse triangle has one obtuse angle.

Triángulo obtusángulo Un triángulo obtusángulo tiene un ángulo obtuso.

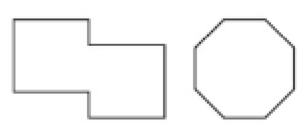
Example



Octagon An octagon is a polygon with eight sides.

Octágono Un octágono es un polígono de ocho lados.

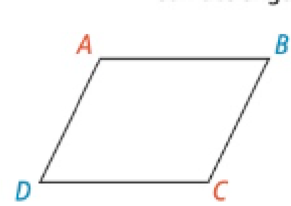
Example



Opposite angles Opposite angles of a quadrilateral are two angles that do not share a side.

Ángulos opuestos Los ángulos opuestos de un cuadrilátero son dos ángulos que no comparten lados.

Example



$\angle A$ and $\angle C$ are opposite angles, as are $\angle B$ and $\angle D$.

English | Spanish

Opposite rays Opposite rays are collinear rays with the same endpoint. They form a line.

Semirrectas opuestas Las semirrectas opuestos son semirrectas colineales con el mismo extremo. Forman una recta.

Example

$\overrightarrow{UT}$ and $\overrightarrow{UN}$ are opposite rays.

Opposite sides Opposite sides of a quadrilateral are two sides that do not share a vertex.

Lados opuestos Los lados opuestos de un cuadrilátero son dos lados que no tienen un vértice en común.

Example

$\overline{PQ}$ and $\overline{SR}$ are opposite sides, as are $\overline{PS}$ and $\overline{QR}$.

Orientation Two figures have opposite orientation if a reflection is needed to map one onto the other. If a reflection is not needed to map one figure onto the other, the figures have the same orientation.

Orientación Dos figuras tienen orientación opuesta si una reflexión es necesaria para trazar una sobre la otra. Si una reflexión no es necesaria para trazar una figura sobre la otra, las figuras tienen la misma orientación.

Example R Я The two R's have opposite orientation.

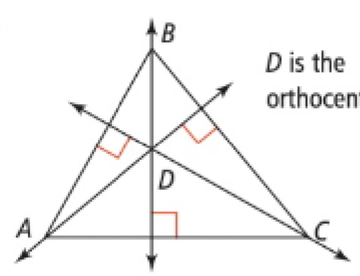
Origin *See* **coordinate plane**.

Origen *Ver* **coordinate plane**.

Orthocenter of a triangle The orthocenter of a triangle is the point of concurrency of the lines containing the altitudes of the triangle.

Ortocentro de un triángulo El ortocentro de un triángulo es el punto donde se intersecan las alturas de un triángulo.

Example



D is the orthocenter.

P

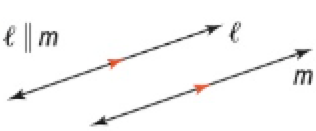
Pragraph proof *See* **proof**.

Prueba de párrafo *Ver* **proof**.

Parallel lines Two lines are parallel if they lie in the same plane and do not intersect. The symbol ∥ means "is parallel to."

Rectas paralelas Dos rectas son paralelas si están en el mismo plano y no se cortan. El símbolo ∥ significa "es paralelo a".

Example $\ell \parallel m$



The red symbols indicate parallel lines.

English	Spanish

Parallelogram A parallelogram is a quadrilateral with two pairs of parallel sides. You can choose any side to be the *base*. An *altitude* is any segment perpendicular to the line containing the base drawn from the side opposite the base. The *height* is the length of an altitude.

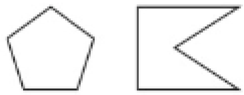
Paralelogramo Un paralelogramo es un cuadrilátero con dos pares de lados paralelos. Se puede escoger cualquier lado como la *base*. Una *altura* es un segmento perpendicular a la recta que contiene la base, trazada desde el lado opuesto a la base. La *altura*, por extensión, es la longitud de una altura.

Example

Pentagon A pentagon is a polygon with five sides.

Pentágono Un pentágono es un polígono de cinco lados.

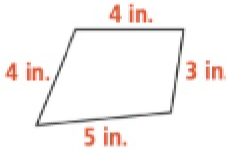
Example



Perimeter of a polygon The perimeter of a polygon is the sum of the lengths of its sides.

Perímetro de un polígono El perímetro de un polígono es la suma de las longitudes de sus lados

Example

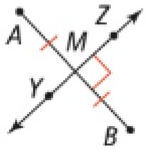


$$P = 4 + 4 + 5 + 3 = 16 \text{ in.}$$

Perpendicular bisector The perpendicular bisector of a segment is a line, segment, or ray that is perpendicular to the segment and divides the segment into two congruent segments.

Mediatriz La mediatriz de un segmento es una recta, segmento o semirrecta que es perpendicular al segmento y que divide al segmento en dos segmentos congruentes.

Example

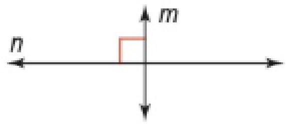


$\overleftrightarrow{YZ}$ is the perpendicular bisector of $\overline{AB}$. It is perpendicular to $\overline{AB}$ and intersects $\overline{AB}$ at midpoint M.

Perpendicular lines Perpendicular lines are lines that intersect and form right angles. The symbol ⊥ means "is perpendicular to."

Rectas perpendiculares Las rectas perpendiculares son rectas que se cortan y forman ángulos rectos. El símbolo ⊥ significa "es perpendicular a".

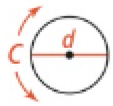
Example



$m \perp n$

English	Spanish
Pi Pi (π) is the ratio of the circumference of any circle to its diameter. The number π is irrational and is approximately 3.14159.	**Pi** Pi (π) es la razón de la circunferencia de cualquier írculo a su diámetro. El número π es irracional y se aproxima a $\pi \approx 3.14159$.

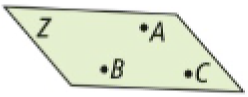
Example



$$\pi = \frac{C}{d}$$

Plane A plane is undefined. You can think of a plane as a flat surface that extends without end and has no thickness. A plane contains infinitely many lines.	**Plano** Un plano es indefinido. Se puede pensar en un plano como una superficie plana sin fin, ni grosor. Un plano tiene un número infinito de rectas.

Example



Plane *ABC* or plane *Z*

Point A point is undefined. You can think of a point as a location. A point has no size.	**Punto** Un punto es indefinido. Puedes imaginarte a un punto como un lugar. Un punto no tiene dimensión.

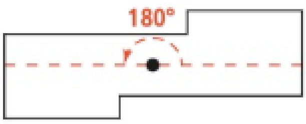
Example • *P*

Point of concurrency *See* **concurrent lines**.	**Punto de concurrencia** *Ver* **concurrent lines**.
Point of tangency *See* **tangent to a circle**.	**Punto de tangencia** *Ver* **tangent to a circle**.
Point-slope form The point-slope form for a nonvertical line with slope *m* and through point (x_1, y_1) is $y - y_1 = m(x - x_1)$.	**Forma punto-pendiente** La forma punto-pendiente para una recta no vertical con pendiente *m* y que pasa por el punto (x_1, y_1) es $y - y_1 = m(x - x_1)$.

Example $y + 1 = 3(x - 4)$
In this equation, the slope is 3 and (x_1, y_1) is $(4, -1)$.

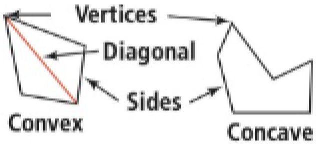
Point symmetry Point symmetry is the type of symmetry for which there is a rotation of 180° that maps a figure onto itself.	**Simetría central** La simetría central es un tipo de simetría en la que una figura se ha rotado 180° sobre sí misma.

Example



Polygon A polygon is a closed plane figure formed by three or more segments. Each segment intersects exactly two other segments, but only at their endpoints, and no two segments with a common endpoint are collinear. The *vertices* of the polygon are the endpoints of the sides. A *diagonal* is a segment that connects two nonconsecutive vertices. A polygon is *convex* if no diagonal contains points outside the polygon. A polygon is *concave* if a diagonal contains points outside the polygon.	**Polígono** Un polígono es una figura plana compuesta por tres o más segmentos. Cada segmento interseca los otros dos segmentos exactamente, pero únicamente en sus puntos extremos y ningúno de los segmentos con extremos comunes son colineales. Los *vértices* del polígono son los extremos de los lados. Una *diagonal* es un segmento que conecta dos vértices no consecutivos. Un polígono es *convexo* si ninguna diagonal tiene puntos fuera del polígono. Un polígono es *cóncavo* si una diagonal tiene puntos fuera del polígono.

Example



English / Spanish

Polyhedron A polyhedron is a three-dimensional figure whose surfaces, or *faces*, are polygons. The vertices of the polygons are the *vertices* of the polyhedron. The intersections of the faces are the *edges* of the polyhedron.

Poliedro Un poliedro es una figura tridimensional cuyas superficies, o *caras*, son polígonos. Los vértices de los polígonos son los *vértices* del poliedro. Las intersecciones de las caras son las *aristas* del poliedro.

Example

Postulate A postulate is an accepted statement of fact.

Postulado Un postulado es un enunciado que se acepta como un hecho.

Example **Postulate:** Through any two points there is exactly one line.

Preimage *See* **transformation**.

Preimagen *Ver* **transformation**.

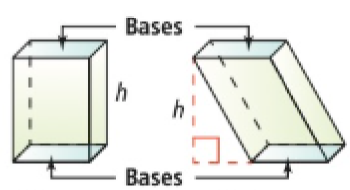
Prime notation *See* **transformation**.

Notación prima *Ver* **transformation**.

Prism A prism is a polyhedron with two congruent and parallel faces, which are called the *bases*. The other faces, which are parallelograms, are called the *lateral faces*. An *altitude* of a prism is a perpendicular segment that joins the planes of the bases. Its length is the *height* of the prism. A *right prism* is one whose lateral faces are rectangular regions and a lateral edge is an altitude. In an *oblique prism*, some or all of the lateral faces are nonrectangular.

Prisma Un prisma es un poliedro con dos caras congruentes paralelas llamadas *bases*. Las otras caras son paralelogramos llamados *caras laterales*. La *altura* de un prisma es un segmento perpendicular que une los planos de las bases. Su longitud es también la *altura* del prisma. En un *prisma rectangular*, las caras laterales son rectangulares y una de las aristas laterales es la altura. En un *prisma oblicuo*, algunas o todas las caras laterales no son rectangulares.

Example



Right prism Oblique prism

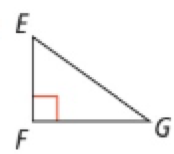
English

Spanish

Proof A proof is a convincing argument that uses deductive reasoning. A proof can be written in many forms. In a two-column proof, the statements and reasons are aligned in columns. In a paragraph proof, the statements and reasons are connected in sentences. In a flow proof, arrows show the logical connections between the statements. In a coordinate proof, a figure is drawn on a coordinate plane and the formulas for slope, midpoint, and distance are used to prove properties of the figure. An indirect proof involves the use of indirect reasoning.

Prueba Una prueba es un argumento convincente en el cual se usa el razonamiento deductivo. Una prueba se puede escribir de varias maneras. En una *prueba de dos columnas*, los enunciados y las razones se alinean en columnas. En una *prueba de párrafo*, los enunciados y razones están unidos en oraciones. En una *prueba de flujo*, hay flechas que indican las conexiones lógicas entre enunciados. En una *prueba de coordenadas*, se dibuja una figura en un plano de coordenadas y se usan las fórmulas de la pendiente, punto medio y distancia para probar las propiedades de la figura. Una *prueba indirecta* incluye el uso de razonamiento indirecto.

Example



Given: $\triangle EFG$, with right angle $\angle F$
Prove: $\angle E$ and $\angle G$ are complementary.

Paragraph Proof: Because $\angle F$ is a right angle, $m\angle F = 90$. By the Triangle Angle-Sum Theorem, $m\angle E + m\angle F + m\angle G = 180$. By substitution, $m\angle E + 90 + m\angle G = 180$. Subtracting 90 from each side yields $m\angle E + m\angle G = 90$. $\angle E$ and $\angle G$ are complementary by definition.

Proportion A proportion is a statement that two ratios are equal. An *extended proportion* is a statement that three or more ratios are equal.

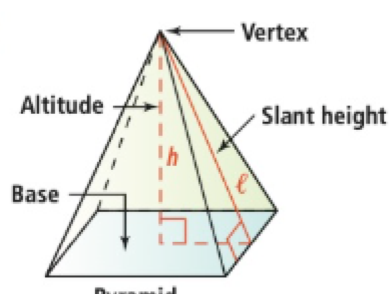
Proporción Una proporción es un enunciado en el cual dos razones son iguales. Una *proporción extendida* es un enunciado que dice que tres razones o más son iguales.

Example $\frac{x}{5} = \frac{3}{4}$ is a proportion.
$\frac{9}{27} = \frac{3}{9} = \frac{1}{3}$ is an extended proportion.

Pyramid A pyramid is a polyhedron in which one face, the *base*, is a polygon and the other faces, the *lateral faces*, are triangles with a common vertex, called the *vertex* of the pyramid. An *altitude* of a pyramid is the perpendicular segment from the vertex to the plane of the base. Its length is the *height* of the pyramid. The *slant height* of a regular pyramid is the length of an altitude of a lateral face.

Pirámide Una pirámide es un poliedro en donde una cara, la *base*, es un polígono y las otras caras, las *caras laterales*, son triángulos con un vértice común, llamado el *vértice* de la pirámide. Una *altura* de una pirámide es el segmento perpendicular que va del vértice hasta el plano de la base. Su longitude es, por extensión, la *altura* de la pirámide. La *apotema* de una pirámide regular es la longitude de la altura de la cara lateral.

Example



English | Spanish

Pythagorean triple A Pythagorean triple is a set of three nonzero whole numbers *a*, *b*, and *c*, that satisfy the equation $a^2 + b^2 = c^2$.

Tripleta de Pitágoras Una tripleta de Pitágoras es un conjunto de tres números enteros positivos *a*, *b*, and *c* que satisfacen la ecuación $a^2 + b^2 = c^2$.

Example The numbers 5, 12, and 13 form a Pythagorean triple because $5^2 + 12^2 = 13^2 = 169$.

Q

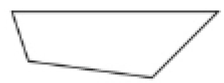
Quadrant *See* **coordinate plane.**

Cuadrante *Ver* **coordinate plane**.

Quadrilateral A quadrilateral is a polygon with four sides.

Cuadrilátero Un cuadrilátero es un polígono de cuatro lados.

Example

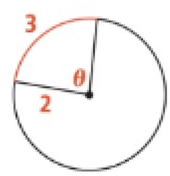


R

Radian A radian is equal to the measure of a central angle that intercepts an arc with length equal to the radius of the circle.

Radián Un radián es igual a la medida de un ángulo central que interseca a un ángulo de la misma longitud que el radio del círculo.

Example



$s = \theta r$

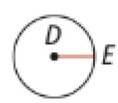
$\theta = \frac{s}{r}$

$\theta = \frac{3}{2} = 1.5$ radians

Radius of a circle A radius of a circle is any segment with one endpoint on the circle and the other endpoint at the center of the circle. *Radius* can also mean the length of this segment.

Radio de un círculo Un radio de un círculo es cualquier segmento con extremo en el círculo y el otro extremo en el centro del círculo. *Radio* también se refiere a la longitude de este segmento.

Example



$\overline{DE}$ is a radius of $\odot D$.

Radius of a sphere The radius of a sphere is a segment that has one endpoint at the center and the other endpoint on the sphere.

Radio de una esfera El radio de una esfera es un segmento con un extremo en el centro y otro en la esfera.

Example

Ratio A ratio is a comparison of two quantities by division. An *extended ratio* is a comparison of three or more quantities by division.

Razón Una razón es una comparación de dos cantidades usando la división. Una *razón extendida* es una comparación de tres o más cantidades usando la división.

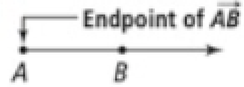
Example 5 to 7, 5 : 7, and $\frac{5}{7}$ are ratios.
3 : 5 : 6 is an extended ratio.

English | Spanish

Ray A ray is the part of a line that consists of one *endpoint* and all the points of the line on one side of the endpoint.

Semirrecta Una semirrecta es la parte de una recta que tiene un *extremo* de donde parten todos los puntos de la recta.

Example



Rectangle A rectangle is a parallelogram with four right angles.

Rectángulo Un rectángulo es un paralelogramo con cuatro ángulos rectos.

Example

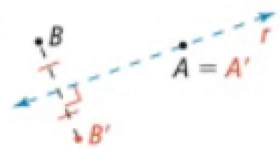
Reduction *See* **dilation**.

Reducción *Ver* **dilation**.

Reflection A reflection across line r, called the *line of reflection*, is a transformation such that if a point A is on line r, then the image of A is itself, and if a point B is not on line r, then its image B' is the point such that r is the perpendicular bisector of $\overline{BB'}$. A reflection is a rigid motion.

Reflexión Una reflexión a través de una línea r, llamada el *eje de reflexión*, es una transformación en la que si un punto A es parte de la línea r, la imagen de A es sí misma, y si un punto B no está en la línea r, su imagen B' es el punto en el cual la línea r es la bisectriz perpendicular de $\overline{BB'}$. Una reflexión es un movimiento rígido.

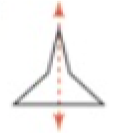
Example



Reflectional symmetry Reflectional symmetry is the type of symmetry for which there is a reflection that maps a figure onto itself. The reflection line is the *line of symmetry*. The line of symmetry divides a figure with reflectional symmetry into two congruent halves.

Simetría reflexiva Simetría reflexiva es el tipo de simetría donde hay una reflexión que ubica una figura en sí misma. El eje de reflexión es *el eje de simetría*. El eje de simetría divide una figura con simetría reflexiva en dos mitades congruentes.

Example

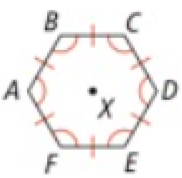


A reflection across the given line maps the figure onto itself.

Regular polygon A regular polygon is a polygon that is both equilateral and equiangular. Its *center* is the point that is equidistant from its vertices. The *apothem* is the distance from the center to a side.

Polígono regular Un polígono regular es un polígono que es equilateral y equiangular. Su *centro* es el punto equidistante de sus vértices. La *apotema* es la distancia desde el centro hasta un lado.

Example



ABCDEF is a regular hexagon. Point *X* is its center.

English | Spanish

Remote interior angles Remote interior angles are the two nonadjacent interior angles corresponding to each exterior angle of a triangle.

Ángulos interiores remotos Los ángulos interiores remotos son los dos ángulos interiores no adyacentes que corresponden a cada ángulo exterior de un triángulo.

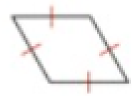
Example

$\angle 1$ and $\angle 2$ are remote interior angles of $\angle 3$.

Rhombus A rhombus is a parallelogram with four congruent sides.

Rombo Un rombo es un paralelogramo de cuatro lados congruentes.

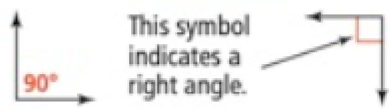
Example



Right angle A right angle is an angle whose measure is 90.

Ángulo recto Un ángulo recto es un ángulo que mide 90.

Example



Right cone *See* **cone**.

Cono recto *Ver* **cone**.

Right cylinder *See* **cylinder**.

Cilindro recto *Ver* **cylinder**.

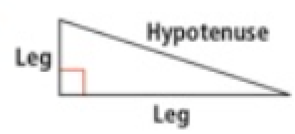
Right prism *See* **prism**.

Prisma rectangular *Ver* **prism**.

Right triangle A right triangle contains one right angle. The side opposite the right angle is the *hypotenuse* and the other two sides are the *legs*.

Triángulo rectángulo Un triángulo rectángulo contiene un ángulo recto. El lado opuesto del ángulo recto es la *hipotenusa* y los otros dos lados son los *catetos*.

Example



Rigid motion A rigid motion is a transformation that preserves distance and angle measure. A rigid motion is sometimes called a *congruence transformation*.

Movimiento rígido Un movimiento rígido es una transformación en el plano que no cambia la distancia ni la medida del ángulo. Los movimientos rígidos se conocen a veces como *transformaciones de congruencia*.

Example The four rigid motions are reflections, translations, rotations, and glide reflections.

Rotation A rotation of x about a point R, called the *center of rotation*, is a transformation such that for any point V, its image is the point V', where $RV = RV'$ and $m\angle VRV' = x$. The image of R is itself. The positive number of degrees x that a figure rotates is the *angle of rotation*. A rotation is a rigid motion.

Rotación Una rotación de x sobre un punto R, llamado el *centro de rotación*, es una transformación en la que para cualquier punto V, su imagen es el punto V', donde $RV = RV'$ y $m\angle VRV' = x$. La imagen de R es sí misma. El número positivo de grados x que una figura rota es el *ángulo de rotación*. Una rotación es un movimiento rígido.

Example

English

Spanish

Rotational symmetry Rotational symmetry is the type of symmetry for which there is a rotation of 360° or less that maps a figure onto itself.

Simetría rotacional La simetría rotacional es un tipo de simetría en la que una rotación de 360° o menos vuelve a trazar una figura sobre sí misma.

Example

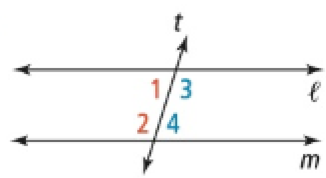
The figure has 120° rotational symmetry.

S

Same-side interior angles Same-side interior angles lie on the same side of the transversal t and between ℓ and m.

Ángulos internos del mismo lado Los ángulos internos del mismo lado están situados en el mismo lado de la transversal t y dentro de ℓ y m.

Example



$\angle 1$ and $\angle 2$ are same-side interior angles, as are $\angle 3$ and $\angle 4$.

Scale A scale is the ratio of any length in a scale drawing to the corresponding actual length. The lengths may be in different units.

Escala Una escala es la razón de cualquier longitud en un dibujo a escala en relación a la longitud verdadera correspondiente. Las longitudes pueden expresarse en distintas unidades.

Example 1 cm to 1 ft
1 cm = 1 ft
1 cm : 1 ft

Scale drawing A scale drawing is a drawing in which all lengths are proportional to corresponding actual lengths.

Dibujo a escala Un dibujo a escala es un dibujo en el que todas las longitudes son proporcionales a las longitudes verdaderas correspondientes.

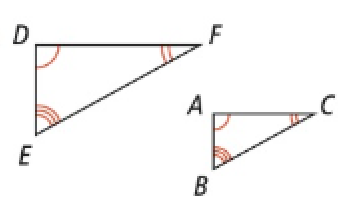
Example

Scale:
1 in. = 30 ft

Scale factor A scale factor is the ratio of corresponding linear measurements of two similar figures.

Factor de escala El factor de escala es la razón de las medidas lineales correspondientes de dos figuras semejantes.

Example



$\triangle ABC \sim \triangle DEF$

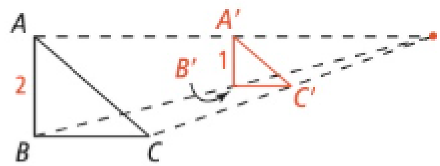
$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

English	Spanish

Scale factor of a dilation The scale factor of a dilation is the ratio of the distances from the center of dilation to an image point and to its preimage point.

Factor de escala de dilatación El factor de escala de dilatación es la razón de las distancias desde el centro de dilatación hasta un punto de la imagen y hasta un punto de la preimagen.

Example



The scale factor of the dilation that maps $\triangle ABC$ to $\triangle A'B'C'$ is $\frac{1}{2}$.

Scalene triangle A scalene triangle has no congruent sides.

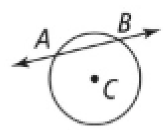
Triángulo escaleno Un triángulo escaleno no tiene lados congruentes.

Example

Secant A secant is a line, ray, or segment that intersects a circle at two points.

Secante Una secante es una recta, semirrecta o segmento que corta un círculo en dos puntos.

Example

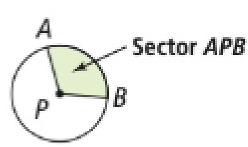


$\overleftrightarrow{AB}$ is a secant of $\odot C$.

Sector of a circle A sector of a circle is the region bounded by two radii and the intercepted arc.

Sector de un círculo Un sector de un círculo es la región limitada por dos radios y el arco abarcado por ellos.

Example



Segment A segment is the part of a line that consists of two points, called *endpoints*, and all points between them.

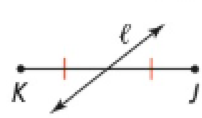
Segmento Un segmento es la parte de una recta que tiene dos puntos, llamados *extremos*, entre los cuales están todos los puntos de esa recta.

Example

Segment bisector A segment bisector is a line, segment, ray, or plane that intersects a segment at the midpoint.

Bisectriz de un segmento La bisectriz de un segmento es una recta, segmento, semirrecta o plano que corta un segmento en el punto medio.

Example



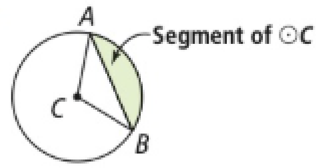
ℓ bisects $\overline{KJ}$.

English | Spanish

Segment of a circle A segment of a circle is the part of a circle bounded by an arc and the segment joining its endpoints.

Segmento de un círculo Un segmento de un círculo es la parte de un círculo bordeada por un arco y el segmento que une sus extremos.

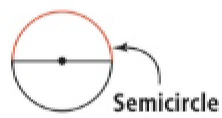
Example



Semicircle A semicircle is half a circle.

Semicírculo Un semicírculo es la mitad de un círculo.

Example



Side *See* **angle; polygon**.

Lado *Ver* **angle; polygon**.

Similar figures Similar figures are two figures that have the same shape, but not necessarily the same size. Two figures are similar if there is a similarity transformation that maps one figure to the other.

Figuras semejantes Los figuras semejantes son dos figuras que tienen la misma forma pero no necesariamente el mismo tamaño. Dos figuras son semejantes si hay una transformación de semajanza en la que una figura es la imagen de la otra.

Example

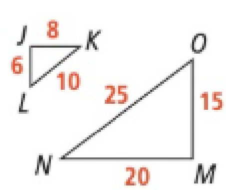
Similarity transformation A composition of one or more rigid motions and a dilation.

Transformación de semejanza Una composición de uno o más movimientos rígidos y una dilatación.

Similar polygons Similar polygons are polygons having corresponding angles congruent and the lengths of corresponding sides proportional. Similarity is denoted by ~.

Polígonos semejantes Los polígonos semejantes son polígonos cuyos ángulos correspondientes son congruentes y las longitudes de los lados correspondientes son proporcionales. El símbolo ~ significa "es semejante a".

Example



$\triangle JKL \sim \triangle MNO$

Scale factor $= \frac{2}{5}$

Sine ratio *See* **trigonometric ratios**.

Razón seno *Ver* **trigonometric ratios**.

Slant height *See* **cone; pyramid**.

Generatriz (cono) o apotema (pirámide) *Ver* **cone; pyramid**.

Slope-intercept form The slope-intercept form of a linear equation is $y = mx + b$, where m is the slope of the line and b is the y-intercept.

Forma pendiente-intercepto La forma pendiente-intercepto es la ecuación lineal $y = mx + b$, en la que m es la pendiente de la recta y b es el punto de intersección de esa recta con el eje y.

Example $y = \frac{1}{2}x - 3$

In this equation, the slope is $\frac{1}{2}$ and the y-intercept is -3.

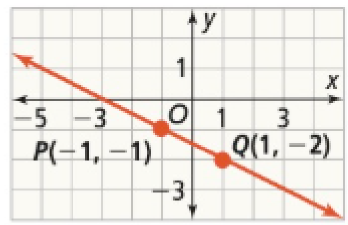
English

Spanish

Slope of a line The slope of a line is the ratio of its vertical change in the coordinate plane to the corresponding horizontal change. If (x_1, y_1) and (x_2, y_2) are points on a nonvertical line, then the slope is $\frac{y_2 - y_1}{x_2 - x_1}$. The slope of a horizontal line is 0, and the slope of a vertical line is undefined.

Pendiente de una recta La pendiente de una recta es la razón del cambio vertical en el plano de coordenadas en relación al cambio horizontal correspondiente. Si (x_1, y_1) y (x_2, y_2) son puntos en una recta no vertical, entonces la pendiente es $\frac{y_2 - y_1}{x_2 - x_1}$. La pendiente de una recta horizontal es 0, y la pendiente de una recta vertical es indefinida.

Example

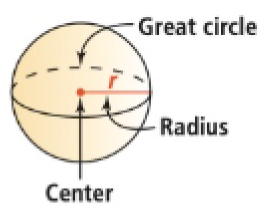


The line containing $P(-1, -1)$ and $Q(1, -2)$ has slope $\frac{-2-(-1)}{1-(-1)} = \frac{-1}{2} = -\frac{1}{2}$.

Sphere A sphere is the set of all points in space that are a given distance *r*, the *radius*, from a given point *C*, the *center*. A *great circle* is the intersection of a sphere with a plane containing the center of the sphere. The *circumference* of a sphere is the circumference of any great circle of the sphere.

Esfera Una esfera es el conjunto de los puntos del espacio que están a una distancia dada *r*, el *radio*, de un punto dado *C*, el *centro*. Un *círculo máximo* es la intersección de una esfera y un plano que contiene el centro de la esfera. La *circunferencia* de una esfera es la circunferencia de cualquier círculo máximo de la esfera.

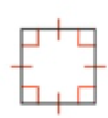
Example



Square A square is a parallelogram with four congruent sides and four right angles.

Cuadrado Un cuadrado es un paralelogramo con cuatro lados congruentes y cuatro ángulos rectos.

Example



Standard form of the equation of a circle The standard form of the equation of a circle with center (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

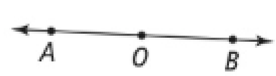
Forma normal de la ecuación de un círculo La forma normal de la ecuación de un círculo con centro en (h, k) y un radio r es $(x - h)^2 + (y - k)^2 = r^2$.

Example $(x - 3)^2 + (y - 4)^2 = 4$

Straight angle A straight angle is an angle whose measure is 180.

Ángulo llano Un ángulo llano es un ángulo que mide 180.

Example



$m\angle AOB = 180°$

English | Spanish

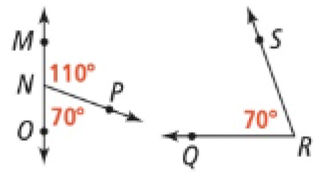
Straightedge A straightedge is a tool for drawing straight lines.

Regla sin graduación Una regla sin graduación es un instrumento para dibujar líneas rectas.

Supplementary angles Two angles are supplementary if the sum of their measures is 180.

Ángulos suplementarios Dos ángulos son suplementarios cuando sus medidas suman 180.

Example

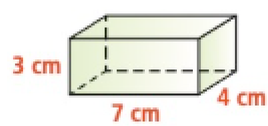


$\angle MNP$ and $\angle ONP$ are supplementary, as are $\angle MNP$ and $\angle QRS$.

Surface area The surface area of a prism, cylinder, pyramid, or cone is the sum of the lateral area and the areas of the bases. The surface area of a sphere is four times the area of a great circle.

Área El área de un prisma, pirámide, cilindro o cono es la suma del área lateral y las áreas de las bases. El área de una esfera es igual a cuatro veces el área de un círculo máximo.

Example

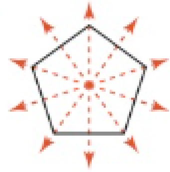


$$\begin{aligned}\text{S.A. of prism} &= \text{L.A.} + 2B \\ &= 66 + 2(28) \\ &= 122\ \text{cm}^2\end{aligned}$$

Symmetry A figure has symmetry if there is a rigid motion that maps the figure onto itself. *See also* **point symmetry; reflectional symmetry; rotational symmetry**.

Simetría Una figura tiene simetría si hay un movimiento rígido que traza la figura sobre sí misma. *Ver también* **point symmetry; reflectional symmetry; rotational symmetry**.

Example



A regular pentagon has reflectional symmetry and 72° rotational symmetry.

T

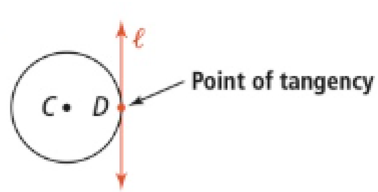
Tangent ratio *See* **trigonometric ratios.**

Razón tangente *Ver* **trigonometric ratios.**

Tangent to a circle A tangent to a circle is a line in the plane of the circle that intersects the circle in exactly one point. That point is the *point of tangency*.

Tangente de un círculo Una tangente de un círculo es una recta en el plano del círculo que corta el círculo en exactamente un punto. Ese punto es el *punto de tangencia*.

Example



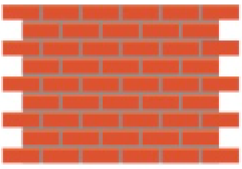
Line ℓ is tangent to $\odot C$. Point D is the point of tangency.

English | Spanish

Tessellation. A tessellation is a pattern in a plane consisting of a shape that is repeated over the entire plane with no overlaps or gaps.

Teselación. Una teselación es un patrón repetido de formas congruentes que cubre completamente un plano, sin espacios o sobreposiciones.

Example



Theorem A theorem is a conjecture that is proven.

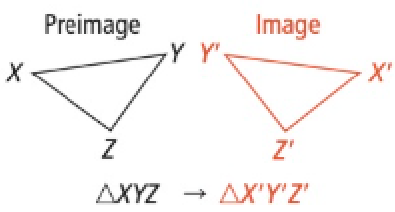
Teorema Un teorema es una conjetura que se demuestra.

Example The theorem "Vertical angles are congruent" can be proven by using postulates, definitions, properties, and previously stated theorems.

Transformation A transformation is a change in the position, size, or shape of a geometric figure. The given figure is called the *preimage* and the resulting figure is called the *image*. A transformation *maps* a figure onto its image. *Prime notation* is sometimes used to identify image points. In the diagram, X' (read "X prime") is the image of X.

Transformación Una transformación es un cambio en la posición, tamaño o forma de una figura. La figura dada se llama la *preimagen* y la figura resultante se llama la *imagen*. Una transformación *traza* la figura sobre su propia imagen. La *notación prima* a veces se utilize para identificar los puntos de la imagen. En el diagrama de la derecha, X' (leído X prima) es la imagen de X.

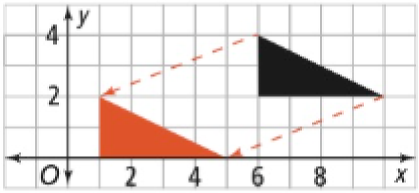
Example



Translation A translation is a transformation that moves points the same distance and in the same direction. A translation is a rigid motion.

Traslación Una traslación es una transformación en la que se mueven puntos la misma distancia en la misma dirección. Una traslación es un movimiento rígido.

Example

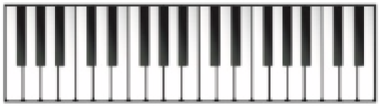


The blue triangle is the image of the black triangle under the translation $\langle -5, -2 \rangle$.

Translational symmetry A figure has translational symmetry if a translation maps the figure onto itself.

Simetría traslacional Una figura tiene simetría traslacional si la traslación sobrepone la figura sobre sí misma.

Example

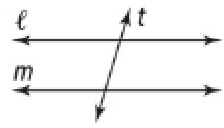


English	Spanish

Transversal A transversal is a line that intersects two or more lines at distinct points.

Transversal Una transversal es una línea que interseca dos o más líneas en puntos precisos.

Example

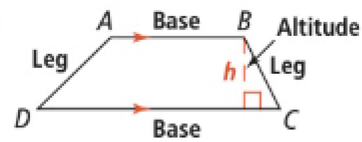


t is a transversal of ℓ and m.

Trapezoid A trapezoid is a quadrilateral with at least one pair of parallel sides, called *bases*. The other pair of sides are *legs*. An *altitude* of a trapezoid is a perpendicular segment from one base to the line containing the other base. Its length is a *height* of the trapezoid.

Trapecio Un trapecio es un cuadrilátero que tiene por lo menos un par de lados paralelos, llamados *bases*. Los otros dos lados se llaman *lados*. La *altura* de un trapecio es un segmento perpendicular desde una base hasta la linea que contiene la otra base. La *longitud* de ese segmento es la altura del trapecio.

Example

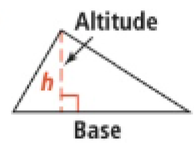


In trapezoid $ABCD$, $\angle ADC$ and $\angle BCD$ are one pair of base angles, and $\angle DAB$ and $\angle ABC$ are the other.

Triangle A triangle is a polygon with three sides. You can choose any side to be a *base*. The *height* is the length of the altitude drawn to the line containing that base.

Triángulo Un triángulo es un polígono con tres lados. Se puede escoger cualquier lado como *base*. La *altura*, entonces, es la longitud de la altura trazada hasta la recta que contiene la base.

Example



Trigonometric ratios The trigonometric ratios, or functions, relate the side lengths of a right triangle to its acute angles. In right $\triangle ABC$ with acute $\angle A$,

sine $\angle A = \sin A = \dfrac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}}$

cosine $\angle A = \cos A = \dfrac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}}$

tangent $\angle A = \tan A = \dfrac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$

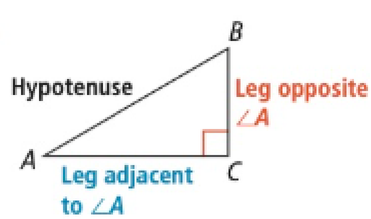
Razones trigonométricas Las razones trigonométricas, o funciones, relacionan las longitudes de lado de un triángulo rectángulo con sus ángulos agudos. En un triángulo rectángulo $\triangle ABC$ con ángulo agudo $\angle A$,

seno $\angle A = \text{sen } A = \dfrac{\text{cateto opuesto a } \angle A}{\text{hipotenusa}}$

coseno $\angle A = \cos A = \dfrac{\text{cateto adyecente a } \angle A}{\text{hipotenusa}}$

tangente $\angle A = \tan A = \dfrac{\text{cateto opuesto a } \angle A}{\text{cateto adyecente a } \angle A}$

Example



English | Spanish

Truth table A truth table is a table that lists all the possible combinations of truth values for two or more statements.

Tabla de verdad Una tabla de verdad es una tabla que muestra todas las combinaciones posibles de valores de verdad de dos o más enunciados.

Example

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth value The truth value of a statement is "true" (T) or "false" (F) according to whether the statement is true or false, respectively.

alor verdadero El valor verdadero de un enunciado es "verdadero" (T) o "falso" (F) según el enunciado sea verdadero o falso, respectivamente.

Two-column proof *See* **proof**.

Prueba de dos columnas *Ver* **proof**.

V

Vertex *See* **angle; cone; polygon; polyhedron; pyramid.** The plural form of *vertex* is *vertices*.

Vértice *Ver* **angle; cone; polygon; polyhedron; pyramid.**

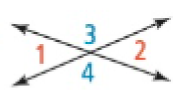
Vertex angle *See* **isosceles triangle**.

Ángulo del vértice *Ver* **isosceles triangle**.

Vertical angles Vertical angles are two angles whose sides form two pairs of opposite rays.

Ángulos opuestos por el vértice Dos ángulos son ángulos opuestos por el vértice si sus lados son semirrectas opuestas.

Example



$\angle 1$ and $\angle 2$ are vertical angles, as are $\angle 3$ and $\angle 4$.

Volume Volume is a measure of the space a figure occupies.

Volumen El volumen es una medida del espacio que ocupa una figura.

Weighted average A weighted average is an average that favors numbers, or points, with greater weight. Each number, or point, has a weight, and the sum of the weights is 1.

Media ponderada Una media ponderada es un promedio que favorece a los números, o puntos, con mayor peso. Cada número, o punto, tiene un peso cuya suma es 1.

Example The weighted average of 3 with a weight of $\frac{1}{3}$ and 10 with a weight of $\frac{2}{3}$ is:

$$\frac{1}{3}(3) + \frac{2}{3}(10) = 1 + \frac{20}{3}$$
$$= \frac{23}{3}$$
$$= 7\frac{2}{3}$$

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Acknowledgments

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