5.3 Angle Bisectors in Triangles

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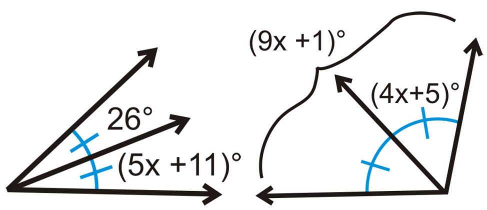
Learning Objectives

- Apply the Angle Bisector Theorem and its converse.
- Understand concurrency for angle bisectors.

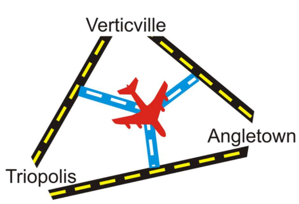
Review Queue

1. Construct the angle bisector of an 80° angle (Investigation 1-4).
2. Draw the following: M is on the interior of $\angle LNO$. O is on the interior of $\angle MNP$. If $\overleftrightarrow{NM}$ and $\overleftrightarrow{NO}$ are the angle bisectors of $\angle LNO$ and $\angle MNP$ respectively, write all the congruent angles.
3. Find the value of x .



[Figure 1] [Figure 2]

Know What? The cities of Verticville, Triopolis, and Angletown are joining their city budgets together to build a centrally located airport. There are freeways between the three cities and they want to have the freeway on the interior of these freeways. Where is the best location to put the airport so that they have to build the least amount of road?

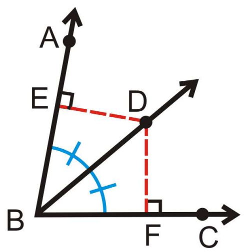


[Figure 3]

In the picture to the right, the blue roads are proposed.

Angle Bisectors

In Chapter 1, you learned that an angle bisector cuts an angle exactly in half. In #1 in the Review Queue above, you constructed an angle bisector of an 80° angle. Let's analyze this figure.



[Figure 4]

$\overrightarrow{BD}$ is the angle bisector of $\angle ABC$. Looking at point D, if we were to draw ED and DF, we would find that they are equal. Recall from Chapter 3 that the shortest distance from a point to a line is the perpendicular length between them. ED and DF are the shortest lengths between D, *which is on the angle bisector*, and each side of the angle.

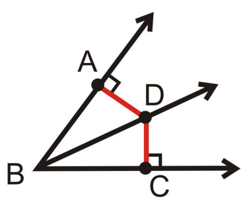
Angle Bisector Theorem: If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

In other words, if $\overleftrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BE} \perp ED$, and $\overrightarrow{BF} \perp DF$, then $ED = DF$.

Proof of the Angle Bisector Theorem

Given: $\overrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BA} \perp AD$, and $\overrightarrow{BC} \perp DC$

Prove: $AD \cong DC$



[Figure 5]

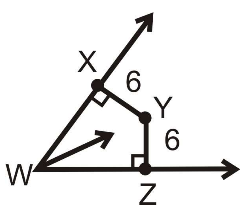
Statement	*Reason*
1. $\overrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BA} \perp AD, \overrightarrow{BC} \perp DC$	Given
2. $\angle ABD \cong \angle DBC$	Definition of an angle bisector
3. $\angle DAB$ and $\angle DCB$ are right angles	Definition of perpendicular lines
4. $\angle DAB \cong \angle DCB$	All right angles are congruent
5. $BD \cong BD$	Reflexive PoC
6. $\triangle ABD \cong \triangle CBD$	AAS
7. $AD \cong DC$	CPCTC

The converse of this theorem is also true. The proof is in the review questions.

Angle Bisector Theorem Converse: If a point is in the interior of an angle and equidistant from the sides, then it lies on the bisector of the angle.

Because the Angle Bisector Theorem and its converse are both true we have a biconditional statement. We can put the two conditional statements together using if and only if. *A point is on the angle bisector of an angle if and only if it is equidistant from the sides of the triangle.*

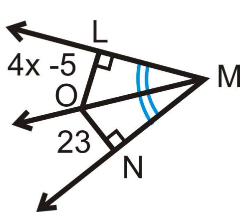
Example 1: Is Y on the angle bisector of $\angle XWZ$?



[Figure 6]

Solution: In order for Y to be on the angle bisector XY needs to be equal to YZ and they both need to be perpendicular to the sides of the angle. From the markings we know $XY \perp \overrightarrow{WX}$ and $ZY \perp \overrightarrow{WZ}$. Second, $XY = YZ = 6$. From this we can conclude that Y is on the angle bisector.

Example 2: $\overrightarrow{MO}$ is the angle bisector of $\angle LMN$. Find the measure of x.



[Figure 7]

Solution: $LO = ON$ by the Angle Bisector Theorem Converse.

$$\begin{aligned} 4x - 5 &= 23 \\ 4x &= 28 \\ x &= 7 \end{aligned}$$

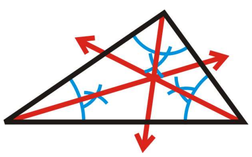
Angle Bisectors in a Triangle

Like perpendicular bisectors, the point of concurrency for angle bisectors has interesting properties.

Investigation 5-2: Constructing Angle Bisectors in Triangles

Tools Needed: compass, ruler, pencil, paper

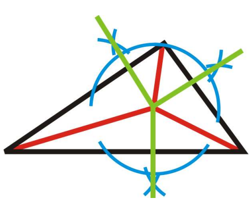
1. Draw a scalene triangle. Construct the angle bisector of each angle. Use Investigation 1-4 and #1 from the Review Queue to help you.



[Figure 8]

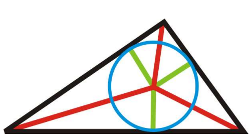
Incenter: The point of concurrency for the angle bisectors of a triangle.

2. Erase the arc marks and the angle bisectors after the incenter. Draw or construct the perpendicular lines to each side, through the incenter.



[Figure 9]

3. Erase the arc marks from #2 and the perpendicular lines beyond the sides of the triangle. Place the pointer of the compass on the incenter. Open the compass to intersect one of the three perpendicular lines drawn in #2. Draw a circle.

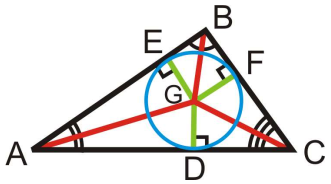


[Figure 10]

Notice that the circle touches all three sides of the triangle. We say that this circle is ***inscribed*** in the triangle because it touches all three sides. The incenter is on all three angle bisectors, so ***the incenter is equidistant from all three sides of the triangle.***

Concurrency of Angle Bisectors Theorem: The angle bisectors of a triangle intersect in a point that is equidistant from the three sides of the triangle.

If AG, BG , and GC are the angle bisectors of the angles in the triangle, then $EG = GF = GD$.

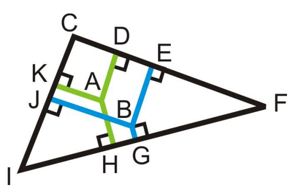


[Figure 11]

In other words, EG, FG , and DG are the radii of the inscribed circle.

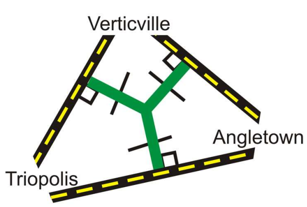
Example 3: If J, E , and G are midpoints and $KA = AD = AH$ what are points A and B called?

Solution: A is the incenter because $KA = AD = AH$, which means that it is equidistant to the sides. B is the circumcenter because JB, BE , and BG are the perpendicular bisectors to the sides.



[Figure 12]

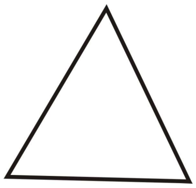
Know What? Revisited The airport needs to be equidistant to the three highways between the three cities. Therefore, the roads are all perpendicular to each side and congruent. The airport should be located at the incenter of the triangle.

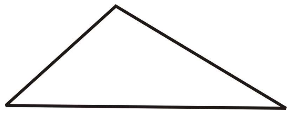


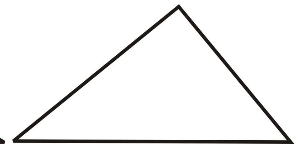
[Figure 13]

Review Questions

Construction Construct the incenter for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-2. Draw the inscribed circle too.

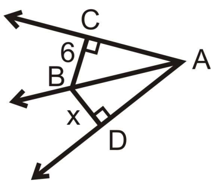

[Figure 14]


[Figure 15]

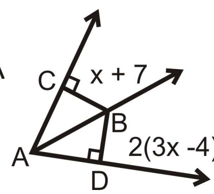

[Figure 16]

1. Is the incenter always going to be inside of the triangle? Why?
2. For an equilateral triangle, what can you conclude about the circumcenter and the incenter?

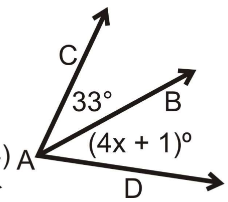
For questions 6-11, $\overrightarrow{AB}$ is the angle bisector of $\angle CAD$. Solve for the missing variable.



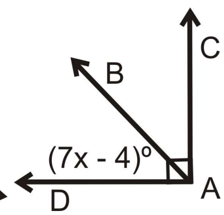
[Figure 17]



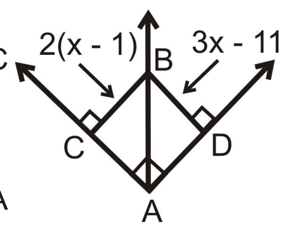
[Figure 18]



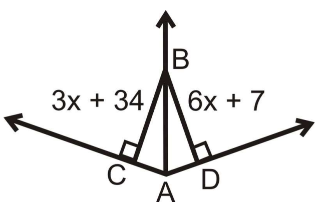
[Figure 19]



[Figure 20]

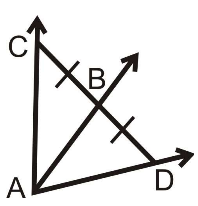


[Figure 21]

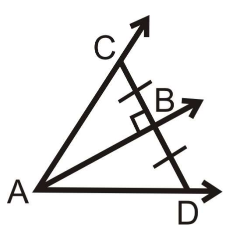


[Figure 22]

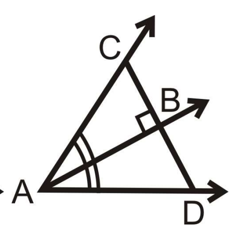
Is there enough information to determine if $\overrightarrow{AB}$ is the angle bisector of $\angle CAD$? Why or why not?



[Figure 23]



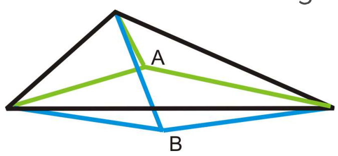
[Figure 24]



[Figure 25]

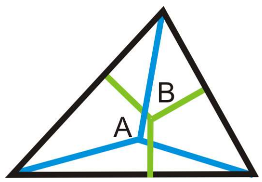
What are points A and B? How do you know?

The blue lines are congruent The green lines are angle bisectors



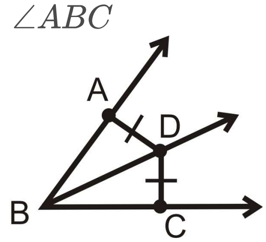
[Figure 26]

Both sets of lines are congruent The green lines are perpendicular to the sides



[Figure 27]

Fill in the blanks in the Angle Bisector Theorem Converse. Given: $AD \cong DC$, such that AD and DC are the shortest distances to $\overrightarrow{BA}$ and $\overrightarrow{BC}$ Prove: $\overrightarrow{BD}$ bisects $\angle ABC$



[Figure 28]

Statement	***Reason***
1.	
2.	The shortest distance from a point to a line is perpendicular.
3. $\angle DAB$ and $\angle DCB$ are right angles	
4. $\angle DAB \cong \angle DCB$	
5. $BD \cong BD$	
6. $\triangle ABD \cong \triangle CBD$	
7.	CPCTC
8. $\overrightarrow{BD}$ bisects $\angle ABC$	

Determine if the following descriptions refer to the incenter or circumcenter of the triangle.

18. A lighthouse on a triangular island is equidistant to the three coastlines.
19. A hospital is equidistant to three cities.
20. A circular walking path passes through three historical landmarks.
21. A circular walking path connects three other straight paths.

Constructions

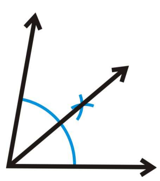
22. Construct an equilateral triangle.
23. Construct the angle bisectors of two of the angles to locate the incenter.
24. Construct the perpendicular bisectors of two sides to locate the circumcenter.
25. What do you notice? Use these points to construct an inscribed circle inside the triangle and a circumscribed circle about the triangle.

Multi- Step Problem

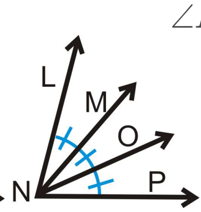
26. Draw $\angle ABC$ through $A(1,3), B(3,-1)$ and $C(7,1)$.
27. Use slopes to show that $\angle ABC$ is a right angle.
28. Use the distance formula to find AB and BC.
29. Construct a line perpendicular to AB through A.
30. Construct a line perpendicular to BC through C.
31. These lines intersect in the interior of $\angle ABC$. Label this point D and draw $\overrightarrow{BD}$.
32. Is $\overrightarrow{BD}$ the angle bisector of $\angle ABC$? Justify your answer.

Review Queue Answers

$$\angle LNM \cong \angle MNO \cong \angle ONP$$
$$\angle LNO \cong \angle MNP$$



[Figure 29]



[Figure 30]

1. Answers:

a. $$\begin{aligned} 5x + 11 &= 26 \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

b. $$\begin{aligned} 9x - 1 &= 2(4x+5) \\ 9x - 1 &= 8x + 10 \\ x &= 11^\circ \end{aligned}$$