10.5 Areas of Circles and Sectors

FlexBooks® 2.0 > American HS Geometry > Areas of Circles and Sectors

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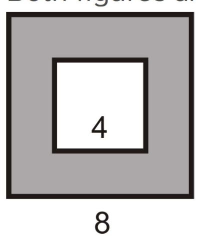
Learning Objectives

- Find the area of circles, sectors, and segments.

Review Queue

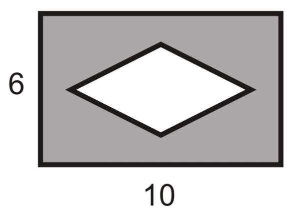
Find the area of the shaded region in the following figures.

Both figures are squares.



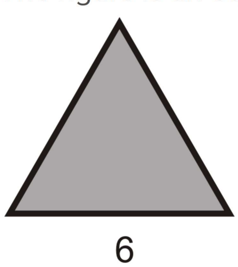
[Figure 1]

Each vertex of the rhombus is 1.5 in from midpoints of the sides of the rectangle.



[Figure 2]

The figure is an equilateral triangle. (find the altitude)



[Figure 3]

1. Find the area of an equilateral triangle with side s .

Know What? Back to the pizza. In the previous section, we found the length of the crust for a 14 in pizza. However, crust typically takes up some area on a pizza. Leave your answers in terms of π and reduced improper fractions.

[Figure 4]

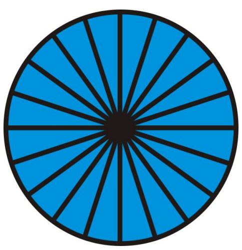
a) Find the area of the crust of a deep-dish 16 in pizza. A typical deep-dish pizza has 1 in of crust around the toppings.

b) A thin crust pizza has $\frac{1}{2}$ - in of crust around the edge of the pizza. Find the area of a thin crust 16 in pizza.

c) Which piece of pizza has more crust? A twelfth of the deep dish pizza or a fourth of the thin crust pizza?

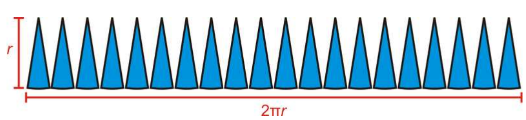
Area of a Circle

Recall in the previous section we derived π as the ratio between the circumference of a circle and its diameter. We are going to use the formula for circumference to derive the formula for area.



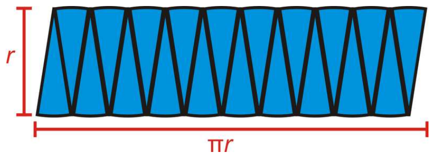
[Figure 5]

First, take a circle and divide it up into several wedges, or sectors. Then, unfold the wedges so they are all on one line, with the points at the top.



[Figure 6]

Notice that the height of the wedges is r , the radius, and the length is the circumference of the circle. Now, we need to take half of these wedges and flip them upside-down and place them in the other half so they all fit together.



[Figure 7]

Now our circle looks like a parallelogram. The area of this parallelogram is $A = bh = \pi r \cdot r = \pi r^2$.

To see an animation of this derivation, see http://www.rkm.com.au/ANIMATIONS/animation-Circle-Area-Derivation.html, by Russell Knightley.

Area of a Circle: If r is the radius of a circle, then $A = \pi r^2$.

Example 1: Find the area of a circle with a diameter of 12 cm.

Solution: If the diameter is 12 cm, then the radius is 6 cm. The area is $A = \pi(6^2) = 36\pi\ cm^2$.

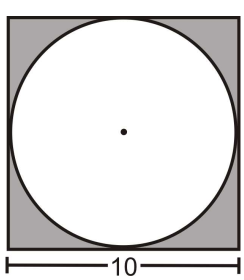
Example 2: If the area of a circle is 20π , what is the radius?

Solution: Work backwards on this problem. Plug in the area and solve for the radius.

$$\begin{aligned} 20\pi &= \pi r^2 \\ 20 &= r^2 \\ r &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

Just like the circumference, we will leave our answers in terms of π , unless otherwise specified. In Example 2, the radius could be $\pm 2\sqrt{5}$, however the radius is always positive, so we do not need the negative answer.

Example 3: A circle is inscribed in a square. Each side of the square is 10 cm long. What is the area of the circle?



[Figure 8]

Solution: The diameter of the circle is the same as the length of a side of the square. Therefore, the radius is half the length of the side, or 5 cm.

$$A = \pi 5^2 = 25\pi \, cm$$

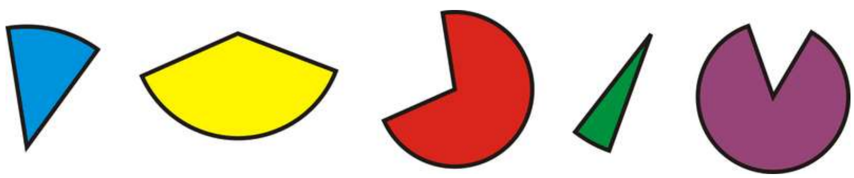
Example 4: Find the area of the shaded region.

Solution: The area of the shaded region would be the area of the square minus the area of the circle.

$$A = 10^2 - 25\pi = 100 - 25\pi \approx 21.46 \, cm^2$$

Area of a Sector

Sector of a Circle: The area bounded by two radii and the arc between the endpoints of the radii.

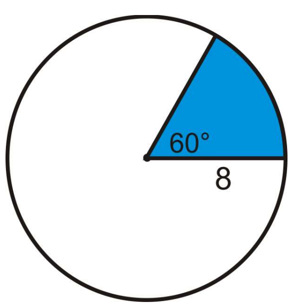


[Figure 9]

The area of a sector is a fractional part of the area of the circle, just like arc length is a fractional portion of the circumference.

Area of a Sector: If r is the radius and $\widehat{AB}$ is the arc bounding a sector, then $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$.

Example 5: Find the area of the blue sector. Leave your answer in terms of π .



[Figure 10]

Solution: In the picture, the central angle that corresponds with the sector is 60° . 60° would be $\frac{1}{6}$ of 360° , so this sector is $\frac{1}{6}$ of the total area.

$$area\ of\ blue\ sector = \frac{1}{6} \cdot \pi 8^2 = \frac{32}{3}\pi$$

Another way to write the sector formula is $A = \frac{central\ angle}{360^\circ} \cdot \pi r^2$.

Example 6: The area of a sector is 8π and the radius of the circle is 12. What is the central angle?

Solution: Plug in what you know to the sector area formula and then solve for the central angle, we will call it x .

$$\begin{aligned} 8\pi &= \frac{x}{360^\circ} \cdot \pi 12^2 \\ 8\pi &= \frac{x}{360^\circ} \cdot 144\pi \\ 8 &= \frac{2x}{5^\circ} \\ x &= 8 \cdot \frac{5^\circ}{2} = 20^\circ \end{aligned}$$

Example 7: The area of a sector of circle is 50π and its arc length is 5π . Find the radius of the circle.

Solution: First plug in what you know to both the sector formula and the arc length formula. In both equations we will call the central angle, "CA ."

$$\begin{aligned} 50\pi &= \frac{CA}{360}\pi r^2 & 5\pi &= \frac{CA}{360} 2\pi r \\ 50 \cdot 360 &= CA \cdot r^2 & 5 \cdot 180 &= CA \cdot r \\ 18000 &= CA \cdot r^2 & 900 &= CA \cdot r \end{aligned}$$

Now, we can use substitution to solve for either the central angle or the radius. Because the problem is asking for the radius we should solve the second equation for the central angle and substitute that into the first equation for the central angle. Then, we can solve for the radius. Solving the second equation for CA , we have: $CA = \frac{900}{r}$. Plug this into the first equation.

$$
\begin{aligned}
18000 &= \frac{900}{r} \cdot r^2 \\
18000 &= 900r \\
r &= 20
\end{aligned}
$$

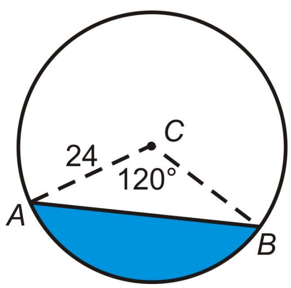
We could have also solved for the central angle in Example 7 once r was found. The central angle is $\frac{900}{20} = 45^\circ$.

Segments of a Circle

The last part of a circle that we can find the area of is called a segment, not to be confused with a line segment.

Segment of a Circle: The area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

Example 8: Find the area of the blue segment below.



[Figure 11]

Solution: As you can see from the picture, the area of the segment is the area of the sector minus the area of the isosceles triangle made by the radii. If we split the isosceles triangle in half, we see that each half is a 30-60-90 triangle, where the radius is the hypotenuse. Therefore, the height of $\triangle ABC$ is 12 and the base would be $2\left(12\sqrt{3}\right) = 24\sqrt{3}$.

$$
\begin{aligned}
A_{sector} &= \frac{120}{360}\pi \cdot 24^2 & A_{\triangle} &= \frac{1}{2}\left(24\sqrt{3}\right)(12) \\
&= 192\pi & &= 144\sqrt{3}
\end{aligned}
$$

The area of the segment is $A = 192\pi - 144\sqrt{3} \approx 353.8$.

In the review questions, make sure you know how the answer is wanted. If the directions say "leave in terms of π and simplest radical form," your answer would be the first one above. If it says "give an approximation," your answer would be the second. It is helpful to leave your answer in simplest radical form and in terms of π because that is the most accurate answer. However, it is also nice to see what the approximation of the answer is, to see how many square units something is.

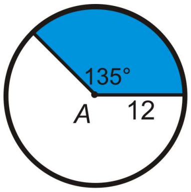
Know What? Revisited The area of the crust for a deep-dish pizza is $8^2\pi - 7^2\pi = 15\pi$. The area of the crust of the thin crust pizza is $8^2\pi - 7.5^2\pi = \frac{31}{4}\pi$. One-twelfth of the deep dish pizza has $\frac{15}{12}\pi$ or $\frac{5}{4}\pi\, in^2$ of crust. One-fourth of the thin crust pizza has $\frac{31}{16}\pi\, in^2$. To compare the two measurements, it might be easier to put them both into decimals. $\frac{5}{4}\pi \approx 3.93\, in^2$ and $\frac{31}{16}\pi \approx 6.09\, in^2$. From this, we see that one-fourth of the thin-crust pizza has more crust than one-twelfth of the deep dish pizza.

Review Questions

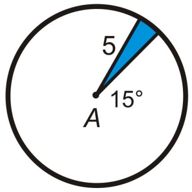
Fill in the following table. Leave all answers in terms of π .

	radius	*Area*	*circumference*
1.	2		
2.		16π	
3.			10π
4.			24π
5.	9		
6.		90π	
7.			35π
8.	$\frac{7}{\pi}$		
9.			60
10.		36	

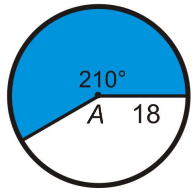
Find the area of the blue sector or segment in $\odot A$. Leave your answers in terms of π . You may use decimals or fractions in your answers, but do not round.



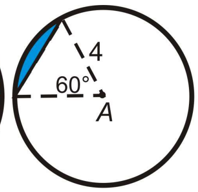
[Figure 12]



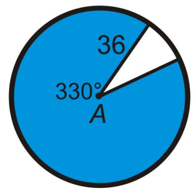
[Figure 13]



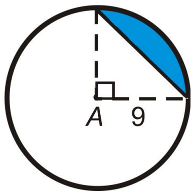
[Figure 14]



[Figure 15]

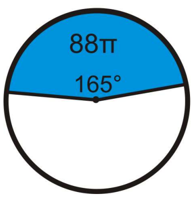


[Figure 16]

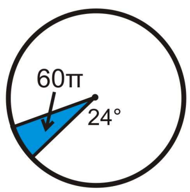


[Figure 17]

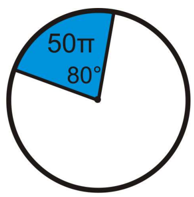
Find the radius of the circle. Leave your answer in simplest radical form.



[Figure 18]

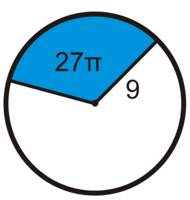


[Figure 19]

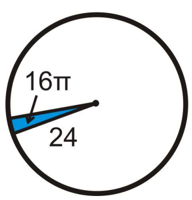


[Figure 20]

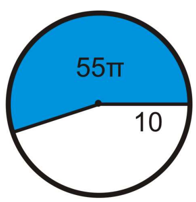
Find the central angle of each blue sector. Round any decimal answers to the nearest tenth.



[Figure 21]



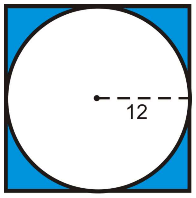
[Figure 22]



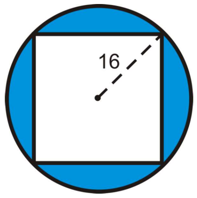
[Figure 23]

Find the area of the shaded region. Round your answer to the nearest hundredth.

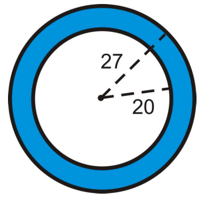
The quadrilateral is a square.



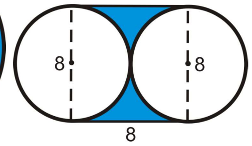
[Figure 24]



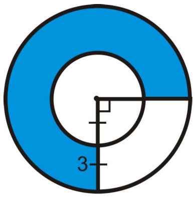
[Figure 25]



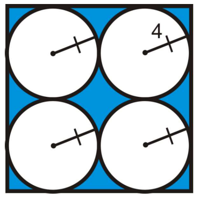
[Figure 26]



[Figure 27]



[Figure 28]



[Figure 29]

23. Carlos has 400 ft of fencing to completely enclose an area on his farm for an animal pen. He could make the area a square or a circle. If he uses the entire 400 ft of fencing, how much area is contained in the square and the circle? Which shape will yield the greatest area?
24. The area of a sector of a circle is 54π and its arc length is 6π. Find the radius of the circle.
25. The area of a sector of a circle is 2304π and its arc length is 32π. Find the central angle of the sector.

Review Queue Answers

1. $8^2-4^2=64-16=48$
2. $6(10)-\frac{1}{2}(7)(3)=60-10.5=49.5$
3. $\frac{1}{2}(6)\left(3\sqrt{3}\right)=9\sqrt{3}$
4. $\frac{1}{2}(s)\left(\frac{1}{2}s\sqrt{3}\right)=\frac{1}{4}s^2\sqrt{3}$