2.1 Inductive Reasoning

Difficulty Level: **Basic** | Created by: CK-12
Last Modified: Dec 25, 2014

Learning Objectives

- Recognize visual and number patterns.
- Extend and generalize patterns.
- Write a counterexample.

Review Queue

1. Look at the patterns of numbers below. Determine the next three numbers in the list. Describe the pattern.
   a. 1, 2, 3, 4, 5, 6, ______, ______, ______
   b. 3, 6, 9, 12, 15, ______, ______, ______
   c. 1, 4, 9, 16, 25, ______, ______, ______
2. Are the statements below true or false? If they are false, state why.
   a. Perpendicular lines form four right angles.
   b. Angles that are congruent are also equal.
   c. Linear pairs are always congruent.
3. For the line, $y = 3x + 1$, make an $x - y$ table for $x = 1, 2, 3, 4$, and 5. What do you notice? How does it relate to $1b$?

Know What? This is the “famous” locker problem:

A new high school has just been completed. There are 1000 lockers in the school and they have been numbered from 1 through 1000. During recess, the students decide to try an experiment. When recess is over each student walks into the school one at a time. The first student will open all of the locker doors. The second student will close all of the locker doors with even numbers. The third student will change all of the locker doors that are multiples of 3 (*change means closing lockers that are open, and opening lockers that are closed*). The fourth student will change the position of all locker doors numbered with multiples of four and so on.

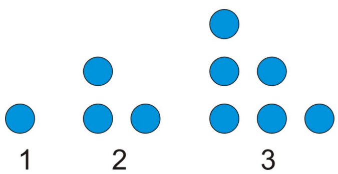
Imagine that this continues until the 1000 students have followed the pattern with the 1000 lockers. At the end, which lockers will be open and which will be closed? Which lockers were touched the most often? Which lockers were touched exactly 5 times?

Visual Patterns

Inductive Reasoning: Making conclusions based upon observations and patterns.

Let's look at some visual patterns to get a feel for what inductive reasoning is.

Example 1: A dot pattern is shown below. How many dots would there be in the bottom row of the 4^{th} figure? What would the *total number* of dots be in the 6^{th} figure?

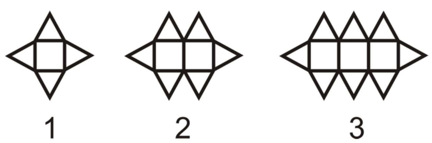


[Figure 1]

Solution: There will be 4 dots in the bottom row of the 4^{th} figure. There is one more dot in the bottom row of each figure than in the previous figure.

There would be a total of 21 dots in the 6^{th} figure, $6+5+4+3+2+1$.

Example 2: How many *triangles* would be in the 10^{th} figure?



[Figure 2]

Solution: There are 10 squares, with a triangle above and below each square. There is also a triangle on each end of the figure. That makes $10+10+2=22$ triangles in all.

Example 2b: If one of these figures contains 34 triangles, how many *squares* would be in that figure?

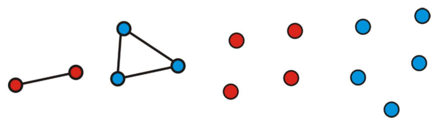
Solution: First, the pattern has a triangle on each end. Subtracting 2, we have 32 triangles. Now, divide 32 by 2 because there is a row of triangles above and below each square. $32 \div 2 = 16$ squares.

Example 2c: How can we find the number of triangles if we know the figure number?

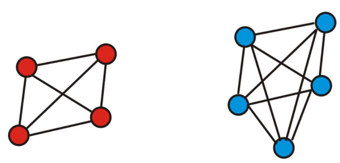
Solution: Let n be the figure number. This is also the number of squares. $2n$ is the number of triangles above and below the squares. Add 2 for the triangles on the ends.

If the figure number is n, then there are $2n+2$ triangles in all.

Example 3: For two points, there is one line segment between them. For three non-collinear points, there are three line segments with those points as endpoints. For four points, no three points being collinear, how many line segments are between them? If you add a fifth point, how many line segments are between the five points?


[Figure 3]

Solution: Draw a picture of each and count the segments.


[Figure 4]

For 4 points there are 6 line segments and for 5 points there are 10 line segments.

Number Patterns

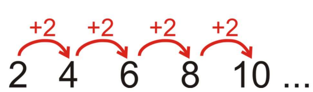
Let's look at a few examples.

Example 4: Look at the pattern 2, 4, 6, 8, 10,...

a) What is the 19^{th} term in the pattern?

b) Describe the pattern and try and find an equation that works for every term in the pattern.

Solution: For part a, each term is 2 more than the previous term.



[Figure 5]

You could count out the pattern until the 19^{th} term, but that could take a while. The easier way is to recognize the pattern. Notice that the 1^{st} term is $2 \cdot 1$, the 2^{nd} term is $2 \cdot 2$, the 3^{rd} term is $2 \cdot 3$, and so on. So, the 19^{th} term would be $2 \cdot 19$ or 38.

For part b, we can use this pattern to generate a formula. Typically with number patterns we use n to represent the term number. So, this pattern is 2 times the term number, or $2n$.

Example 5: Look at the pattern 1, 3, 5, 7, 9, 11,...

a) What is the 34^{th} term in the pattern?

b) What is the n^{th} term?

Solution: The pattern increases by 2 and is odd. From the previous example, we know that if a pattern increases by 2, you would multiply n by 2. However, this pattern is odd, so we need to add or subtract a number. Let's put what we know into a table:

n	$2n$	-1	Pattern
1	2	-1	1
2	4	-1	3
3	6	-1	5
4	8	-1	7
5	10	-1	9
6	12	-1	11

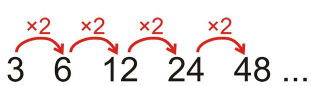
From this we can reason that the 34^{th} term would be $34 \cdot 2$ minus 1, which is 67. Therefore, the n^{th} term would be $2n - 1$.

Example 6: Look at the pattern: 3, 6, 12, 24, 48,...

a) What is the next term in the pattern? The 10^{th} term?

b) Make a rule for the n^{th} term.

Solution: This pattern is different than the previous two examples. Here, each term is multiplied by 2 to get the next term.



[Figure 6]

Therefore, the next term will be $48 \cdot 2$ or 96. To find the 10^{th} term, we need to work on the pattern, let's break apart each term into the factors to see if we can find the rule.

n	Pattern	Factors	Simplify
1	3	3	$3 \cdot 2^0$
2	6	$3 \cdot 2$	$3 \cdot 2^1$
3	12	$3 \cdot 2 \cdot 2$	$3 \cdot 2^2$
4	24	$3 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2^3$
5	48	$3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2^4$

Using this equation, the 10^{th} term will be $3 \cdot 2^9$, or 1536. Notice that the exponent is one less than the term number. So, for the n^{th} term, the equation would be $3 \cdot 2^{n-1}$.

Example 7: Find the 8^{th} term in the list of numbers as well as the rule.

$$2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25} \dots$$

Solution: First, change 2 into a fraction, or $\frac{2}{1}$. So, the pattern is now $\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25} \dots$ Separate the top and the bottom of the fractions into two different patterns. The top is 2, 3, 4, 5, 6. It increases by 1 each time, so the 8^{th} term's numerator is 9. The denominators are the square numbers, so the 8^{th} term's denominator is 10^2 or 100. Therefore, the 8^{th} term is $\frac{9}{100}$. The rule for this pattern is $\frac{n+1}{n^2}$.

To summarize:

- If the same number is **added** from one term to the next, then you multiply n by it.
- If the same number is **multiplied** from one term to the next, then you would multiply the first term by increasing powers of this number. n or $n-1$ is in the exponent of the rule.
- If the pattern has **fractions**, separate the numerator and denominator into two different patterns. Find the rule for each separately.

Conjectures and Counterexamples

Conjecture: An "educated guess" that is based on examples in a pattern.

Numerous examples may make you believe a conjecture. However, no number of examples can actually *prove* a conjecture. It is always possible that the next example would show that the conjecture is false.

Example 8: Here's an algebraic equation and a table of values for n and with the result for t
.

$$t = (n-1)(n-2)(n-3)$$

n	$(n-1)(n-2)(n-3)$	t
1	$(0)(-1)(-2)$	0
2	$(1)(0)(-1)$	0
3	$(2)(1)(0)$	0

After looking at the table, Pablo makes this conjecture:

The value of $(n-1)(n-2)(n-3)$ is 0 for any whole number value of n.

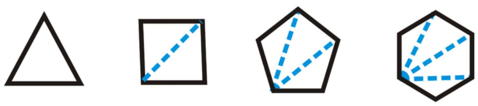
Is this a valid, or true, conjecture?

Solution: No, this is not a valid conjecture. If Pablo were to continue the table to $n = 4$, he would have seen that $(n-1)(n-2)(n-3) = (4-1)(4-2)(4-3) = (3)(2)(1) = 6$.

In this example $n = 4$ is called a counterexample.

Counterexample: An example that disproves a conjecture.

Example 9: Arthur is making figures for a graphic art project. He drew polygons and some of their diagonals.



[Figure 7]

Based on these examples, Arthur made this conjecture:

If a convex polygon has n sides, then there are $n-3$ triangles drawn from any given vertex of the polygon.

Is Arthur's conjecture correct? Can you find a counterexample to the conjecture?

Solution: The conjecture appears to be correct. If Arthur draws other polygons, in every case he will be able to draw $n-3$ triangles if the polygon has n sides.

Notice that we have ***not proved*** Arthur's conjecture, but only found several examples that hold true. This type of conjecture would need to be proven by induction.

Know What? Revisited Start by looking at the pattern. Red numbers are OPEN lockers.

Student 1 changes every locker:

$$1, 2, 3, 4, 5, 6, 7, 8, \ldots 1000$$

Student 2 changes every 2^{nd} locker:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots 1000$$

Student 3 changes every 3^{rd} locker:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots 1000$$

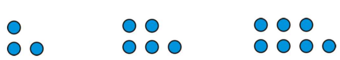
Student 4 changes every 4^{th} locker:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots 1000$$

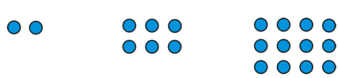
If you continue on in this way, the only lockers that will be left open are the numbers with an odd number of factors, or the square numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, and 961. The lockers that were touched the most are the numbers with the most factors. The one locker that was touched the most was 840, which has 32 factors and thus, touched 32 times. There are three lockers that were touched exactly five times: 16, 81, and 625.

Review Questions

For questions 1 and 2, determine how many dots there would be in the 4^{th} and the 10^{th} pattern of each figure below.



[Figure 8]



[Figure 9]

Use the pattern below to answer the questions.

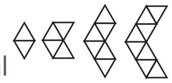
[Figure 10]

a. Draw the next figure in the pattern.

b. How does the number of points in each star relate to the figure number?

c. Use part b to determine a formula for the n^{th} figure.

Use the pattern below to answer the questions. All the triangles are equilateral triangles.



[Figure 11]

a. Draw the next figure in the pattern. How many triangles does it have?

b. Determine how many triangles are in the 24^{th} figure.

c. How many triangles are in the n^{th} figure?

For questions 5-16, determine: 1) the next two terms in the pattern, 2) the 35^{th} figure and 3) the formula for the n^{th} figure.

5. 5, 8, 11, 14, 17,...
6. 6, 1, -4, -9, -14,...
7. 2, 4, 8, 16, 32,...
8. 67, 56, 45, 34, 23,...
9. 9, -4, 6, -8, 3,...
10. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$
11. $\frac{2}{3}, \frac{4}{7}, \frac{6}{11}, \frac{8}{15}, \frac{10}{19}, \ldots$
12. 3, -5, 7, -9, 11,...
13. -1, 5, -9, 13, -17,...
14. $\frac{-1}{2}, \frac{1}{4}, \frac{-1}{6}, \frac{1}{8}, \frac{-1}{10}, \ldots$
15. 5, 12, 7, 10, 9,...
16. 1, 4, 9, 16, 25,...

For questions 17-20, determine the next two terms and describe the pattern.

17. 3, 6, 11, 18, 27,...
18. 3, 8, 15, 24, 35,...

19. 1, 8, 27, 64, 125,...
20. 1, 1, 2, 3, 5,...

We all use inductive reasoning in our daily lives. The process consists of making observations, recognizing a pattern and making a generalization or conjecture. Read the following examples of reasoning in the real world and determine if they are examples of Inductive reasoning. Do you think the conjectures are true or can you give a counterexample?

21. For the last three days Tommy has gone for a walk in the woods near his house at the same time of day. Each time he has seen at least one deer. Tommy reasons that if he goes for a walk tomorrow at the same time, he will see deer again.
22. Maddie likes to bake. She especially likes to take recipes and make substitutions to try to make them healthier. She might substitute applesauce for sugar or oat flour for white flour. She has noticed that she needs to add more baking powder or baking soda than the recipe indicates in these situations in order for the baked goods to rise appropriately.
23. One evening Juan saw a chipmunk in his backyard. He decided to leave a slice of bread with peanut butter on it for the creature to eat. The next morning the bread was gone. Juan concluded that chipmunks like to eat bread with peanut butter.
24. Describe an instance in your life when either you or someone you know used inductive reasoning to correctly make a conclusion.
25. Describe an instance when you observed someone using invalid reasoning skills.

Challenge For the following patterns find a) the next two terms, b) the 40^{th} term and c) the n^{th} term rule. You will need to think about each of these in a different way. *Hint: Double all the values and look for a pattern in their factors. Once you come up with the rule remember to divide it by two to undo the doubling.*

26. 2, 5, 9, 14,...
27. 3, 6, 10, 15,...
28. 3, 12, 30, 60,...

Connections to Algebra

29. Plot the values of the terms in the sequence 3, 8, 13,... against the term numbers in the coordinate plane. In other words, plot the points (1, 3), (2, 8), and (3, 13). What do you notice? Could you use algebra to figure out the “rule” or equation which maps each term number (x) to the correct term value (y)? Try it.

30. Which sequences in problems 5-16 follow a similar pattern to the one you discovered in #29? Can you use inductive reasoning to make a conclusion about which sequences follow the same type of rule?

Review Queue Answers

1. Answers:

   a. 7, 8, 9

   b. 18, 21, 24

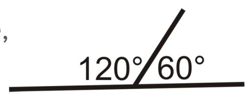
   c. 36, 49, 64

2. Answers:

   a. true

   b. true

   false,



[Figure 12]