# 5.5 Inequalities in Triangles

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## Learning Objectives

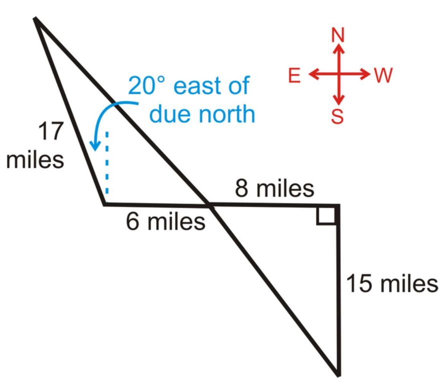
- Determine relationships among the angles and sides of a triangle.
- Understand the Triangle Inequality Theorem.
- Understand the Hinge Theorem and its converse.

## Review Queue

Solve the following inequalities.

1. $4x-9 \leq 19$
2. $-5 > -2x+13$
3. $\frac{2}{3}x+1 \geq 13$
4. $-7 < 3x-1 < 14$

**Know What?** Two mountain bike riders leave from the same parking lot and headin opposite directions, on two different trails. The first ridergoes 8 miles due west, then rides due south for 15 miles. The second rider goes 6 miles due east, then changes direction and rides $20^\circ$ east of due north for 17 miles. Both riders have been travelling for 23 miles, but which one is further from the parking lot?

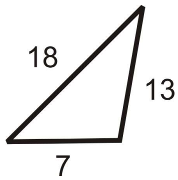



[Figure 1]

## Comparing Angles and Sides

Look at the triangle to the right. The sides of the triangle are given. Can you determine which angle is the largest?

As you might guess, the largest angle will be opposite 18 because it is the longest side. Similarly, the smallest angle will be opposite the shortest side, 7. Therefore, the angle measure in the middle will be opposite 13.

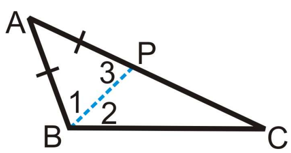



[Figure 2]

**Theorem 5-9:** If one side of a triangle is longer than another side, then the angle opposite the longer side will be larger than the angle opposite the shorter side.

**Converse of Theorem 5-9:** If one angle in a triangle is larger than another angle in a triangle, then the side opposite the larger angle will be longer than the side opposite the smaller angle.

***Proof of Theorem 5-9***




[Figure 3]

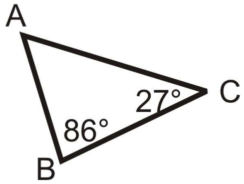
Given: $AC > AB$

Prove: $m\angle ABC > m\angle C$

| ***Statement*** | ***Reason*** |
|---|---|
| 1. $AC > AB$ | Given |
| 2. Locate point $P$ such that $AB = AP$ | Ruler Postulate |
| 3. $\triangle ABP$ is an isosceles triangle | Definition of an isosceles triangle |
| 4. $m\angle 1 = m\angle 3$ | Base Angles Theorem |
| 5. $m\angle 3 = m\angle 2 + m\angle C$ | Exterior Angle Theorem |
| 6. $m\angle 1 = m\angle 2 + m\angle C$ | Substitution PoE |
| 7. $m\angle ABC = m\angle 1 + m\angle 2$ | Angle Addition Postulate |
| 8. $m\angle ABC = m\angle 2 + m\angle 2 + m\angle C$ | Substitution PoE |
| 9. $m\angle ABC > m\angle C$ | Definition of "greater than" (from step 8) |

To prove the converse, we will need to do so indirectly. This will be done in the extension at the end of this chapter.

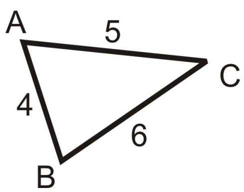
**Example 1:** List the sides in order, from shortest to longest.




[Figure 4]

**Solution:** First, we need to find $m\angle A$ . From the Triangle Sum Theorem, $m\angle A+86^\circ+27^\circ=180^\circ$ . So, $m\angle A=67^\circ$ . From Theorem 5-9, we can conclude that the longest side is opposite the largest angle. $86^\circ$ is the largest angle, so $AC$ is the longest side. The next largest angle is $67^\circ$ , so $BC$ would be the next longest side. $27^\circ$ is the smallest angle, so $AB$ is the shortest side. In order from shortest to longest, the answer is: $AB, BC, AC$ .

**Example 2:** List the angles in order, from largest to smallest.

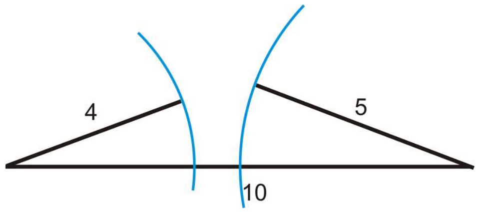



[Figure 5]

**Solution:** Just like with the sides, the largest angle is opposite the longest side. The longest side is $BC$ , so the largest angle is $\angle A$ . Next would be $\angle B$ and finally $\angle A$ is the smallest angle.

## Triangle Inequality Theorem

Can any three lengths make a triangle? The answer is no. There are limits on what the lengths can be. For example, the lengths 1, 2, 3 cannot make a triangle because $1+2=3$ , so they would all lie on the same line. The lengths 4, 5, 10 also cannot make a triangle because $4+5=9$ .




[Figure 6]

The arc marks show that the two sides would never meet to form a triangle.

**Triangle Inequality Theorem:** The sum of the lengths of any two sides of a triangle must be greater than the length of the third.

**Example 3:** Do the lengths below make a triangle?

a) 4.1, 3.5, 7.5

b) 4, 4, 8

c) 6, 7, 8

**Solution:** Even though the Triangle Inequality Theorem says "the sum of the length of any two sides," really, it is referring to the sum of the lengths of the two shorter sides must be longer than the third.

a) $4.1+3.5>7.5$ Yes, these lengths could make a triangle.

b) $4+4=8$ No, not a triangle. Two lengths cannot equal the third.

c) $6+7>8$ Yes, these lengths could make a triangle.

**Example 4:** Find the possible lengths of the third side of a triangle if the other two sides are 10 and 6.

**Solution:** The Triangle Inequality Theorem can also help you determine the possible range of the third side of a triangle. The two given sides are 6 and 10, so the third side, $s$, can either be the shortest side or the longest side. For example $s$ could be 5 because $6+5>10$. It could also be 15 because $6+10>15$. Therefore, we write the possible values of $s$ as a range, $4<s<16$.

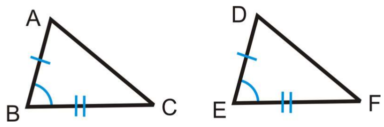

[Figure 7]

Notice the range is no less than 4, and not equal to 4. The third side could be 4.1 because $4.1+6$ would be greater than the third side, 10. For the same reason, $s$ cannot be greater than 16, but it could 15.9. In this case, $s$ would be the longest side and $10+6$ must be greater than $s$ to form a triangle.

***If two sides are lengths $a$ and $b$, then the third side, $s$, has the range $a-b<s<a+b$.***

**The SAS Inequality Theorem** (also called the Hinge Theorem)

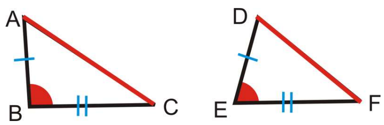
The Hinge Theorem is an extension of the Triangle Inequality Theorem using two triangles. If we have two congruent triangles $\triangle ABC$ and $\triangle DEF$, marked below:




[Figure 8]

Therefore, if $AB = DE$ and $BC = EF$ and $m\angle B > m\angle E$ , then $AC > DF$ .

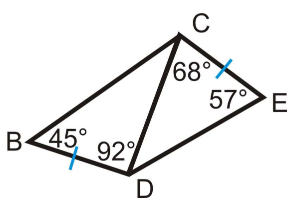
Now, let's adjust $m\angle B > m\angle E$ . Would that make $AC > DF$ ? Yes. See the picture below.




[Figure 9]

**The SAS Inequality Theorem (Hinge Theorem):** If two sides of a triangle are congruent to two sides of another triangle, but the included angle of one triangle has greater measure than the included angle of the other triangle, then the third side of the first triangle is longer than the third side of the second triangle.

**Example 5:** List the sides in order, from least to greatest.




[Figure 10]

**Solution:** Let's start with $\triangle DCE$ . The missing angle is $55^\circ$ . By Theorem 5-9, the sides, in order are $CE, CD$ , and $DE$ .

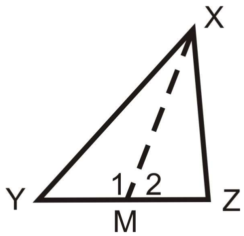
For $\triangle BCD$ , the missing angle is $43^\circ$ . Again, by Theorem 5-9, the order of the sides is $BD, CD$ , and $BC$ .

By the SAS Inequality Theorem, we know that $BC > DE$ , so the order of all the sides would be: $BD = CE, CD, DE, BC$ .

**SSS Inequality Theorem** (also called the Converse of the Hinge Theorem)

**SSS Inequality Theorem:** If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle.

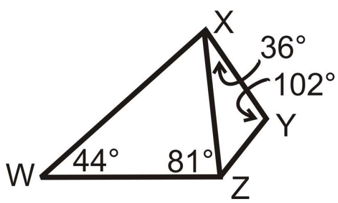
**Example 6:** If $XM$ is a median of $\triangle XYZ$ and $XY > XZ$ , what can we say about $m\angle 1$ and $m\angle 2$ ? What we can deduce from the following diagrams.




[Figure 11]

**Solution:** By the definition of a median, $M$ is the midpoint of $YZ$ . This means that $YM = MZ$ . $MX = MX$ by the Reflexive Property and we know that $XY > XZ$ . Therefore, we can use the SSS Inequality Theorem to conclude that $m\angle 1 > m\angle 2$ .

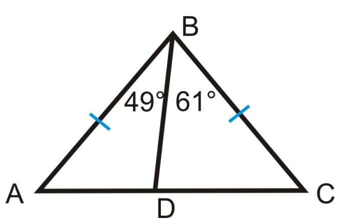
**Example 7:** List the sides of the two triangles in order, from least to greatest.




[Figure 12]

**Solution:** Here we have no congruent sides or angles. So, let's look at each triangle separately. Start with $\triangle XYZ$ . First the missing angle is $42^\circ$ . By Theorem 5-9, the order of the sides is $YZ, XY$ , and $XZ$ . For $\triangle WXZ$ , the missing angle is $55^\circ$ . The order of these sides is $XZ, WZ$ , and $WX$ . Because the longest side in $\triangle XYZ$ is the shortest side in $\triangle WXZ$ , we can put all the sides together in one list: $YZ, XY, XZ, WZ, WX$ .

**Example 8:** Below is isosceles triangle $\triangle ABC$ . List everything you can about the triangle and why.




[Figure 13]

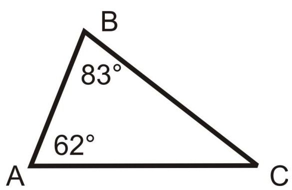
**Solution:**

- $AB = BC$ because it is given.
- $m\angle A = m\angle C$ by the Base Angle Theorem.
- $AD < DC$ because $m\angle ABD < m\angle CBD$ and the SAS Triangle Inequality Theorem.

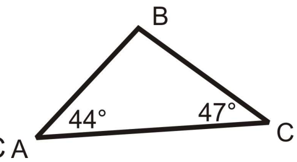
**Know What? Revisited** Even though the two sets of lengths are not equal, they both add up to 23. Therefore, the second rider is further away from the parking lot because $110^\circ > 90^\circ$ .

## Review Questions

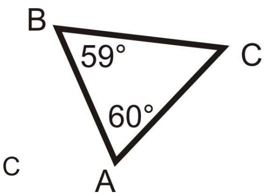
For questions 1-3, list the sides in order from shortest to longest.




[Figure 14]

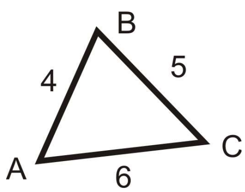



[Figure 15]

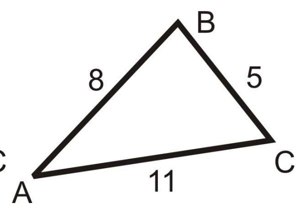



[Figure 16]

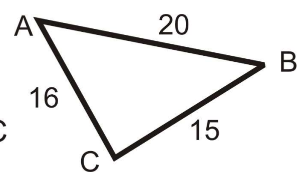
For questions 4-6, list the angles from largest to smallest.




[Figure 17]




[Figure 18]




[Figure 19]

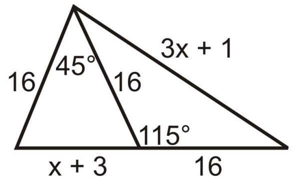
Determine if the sets of lengths below can make a triangle. If not, state why.

7. 6, 6, 13
8. 1, 2, 3
9. 7, 8, 10
10. 5, 4, 3
11. 23, 56, 85
12. 30, 40, 50

If two lengths of the sides of a triangle are given, determine the range of the length of the third side.

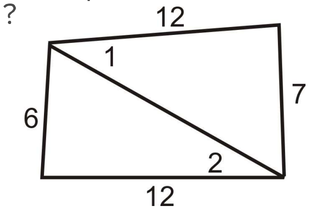
13. 8 and 9
14. 4 and 15
15. 20 and 32
16. The base of an isosceles triangle has length 24. What can you say about the length of each leg?

What conclusions can you draw about $x$?



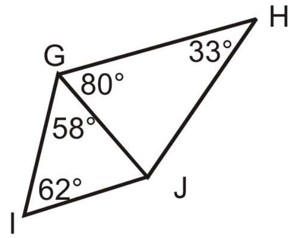

[Figure 20]

Compare $m\angle 1$ and $m\angle 2$.



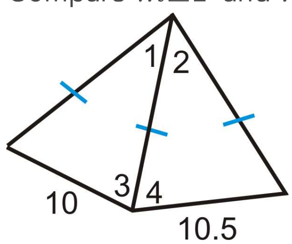

[Figure 21]

List the sides from shortest to longest.



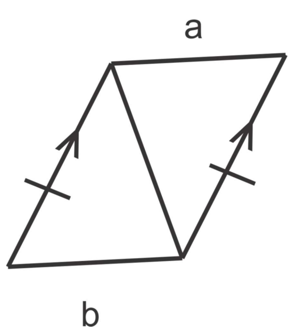

[Figure 22]

Compare $m\angle 1$ and $m\angle 2$. What can you say about $m\angle 3$ and $m\angle 4$?

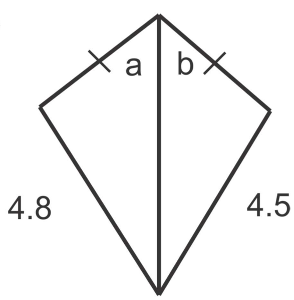



[Figure 23]

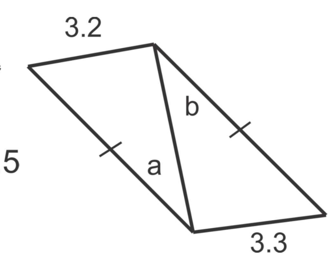
In questions 21-23, compare the measures of $a$ and $b$.




[Figure 24]

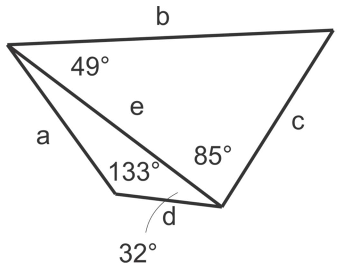



[Figure 25]

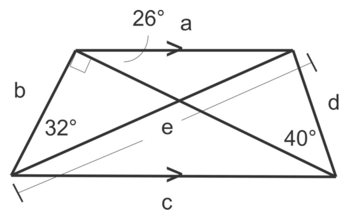



[Figure 26]

In questions 24 and 25, list the measures of the sides in order from least to greatest

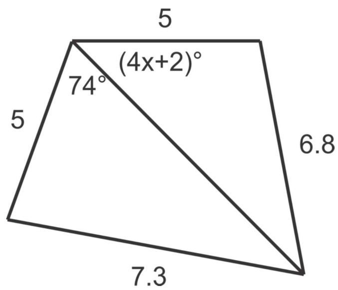



[Figure 27]

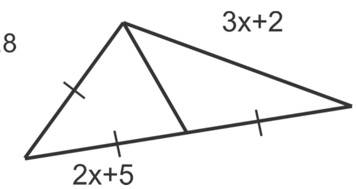



[Figure 28]

In questions 26 and 27 determine the range of possible values for $x$.



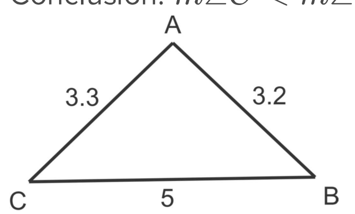

[Figure 29]




[Figure 30]

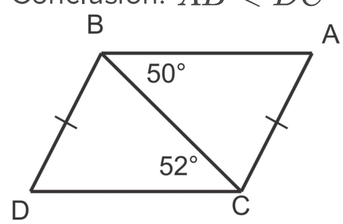
In questions 28 and 29 explain why the conclusion is false.

Conclusion: $m\angle C < m\angle B < m\angle A$




[Figure 31]

Conclusion: $AB < DC$




[Figure 32]

28. If $AB$ is a median of $\triangle CAT$ and $CA > AT$, explain why $\angle ABT$ is acute. You may wish to draw a diagram.

## Review Queue Answers

1. $$\begin{aligned} 4x-9 &\le 19 \\ 4x &\le 28 \\ x &\le 7 \end{aligned}$$

2. $$\begin{aligned} -5 &> -2x+13 \\ -18 &> -2x \\ 9 &< x \end{aligned}$$

3. $$\begin{aligned}\frac{2}{3}x+1 &\geq 13\\ \frac{2}{3}x &\geq 12\\ x &\geq 18\end{aligned}$$

4. $$\begin{aligned}-7 < 3x-1 &< 14\\ -6 < 3x &< 15\\ -2 < x &< 5\end{aligned}$$