# 9.4 Inscribed Angles

FlexBooks® 2.0 > American HS Geometry > Inscribed Angles

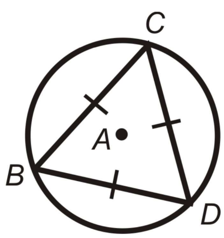
Last Modified: Dec 25, 2014

## Learning Objectives

- Find the measure of inscribed angles and the arcs they intercept.

## Review Queue

We are going to use #14 from the homework in the previous section.

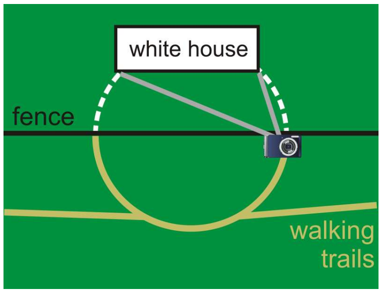



[Figure 1]

1. What is the measure of each angle in the triangle? How do you know?
2. What do you know about the three arcs?
3. What is the measure of each arc?
4. What is the relationship between the angles in the triangles and the measure of each arc?

**Know What?** Your family went to Washington DC over the summer and saw the White House. The closest you can get to the White House are the walking trails on the far right. You got as close as you could (on the trail) to the fence to take a picture (you were not allowed to walk on the grass). Where else could you have taken your picture from to get the same frame of the White House? Where do you think the best place to stand would be? *Your line of sight in the camera is marked in the picture as the grey lines. The white dotted arcs do not actually exist, but were added to help with this problem.*

[Figure 2]




[Figure 3]

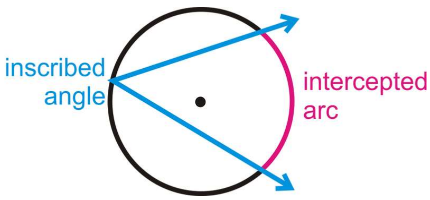
## Inscribed Angles

We have discussed central angles so far in this chapter. We will now introduce another type of angle, the inscribed angle.

**Inscribed Angle:** An angle with its vertex is the circle and its sides contain chords.

**Intercepted Arc:** The arc that is on the interior of the inscribed angle and whose endpoints are on the angle.

The vertex of an inscribed angle can be anywhere on the circle as long as its sides intersect the circle to form an intercepted arc.



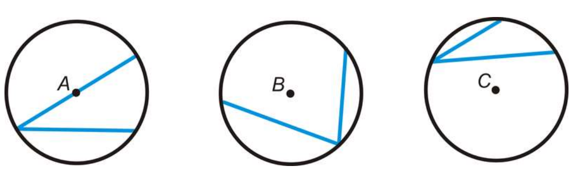

[Figure 4]

Now, we will investigation the relationship between the inscribed angle, the central angle and the arc they intercept.

**Investigation 9-4: Measuring an Inscribed Angle**

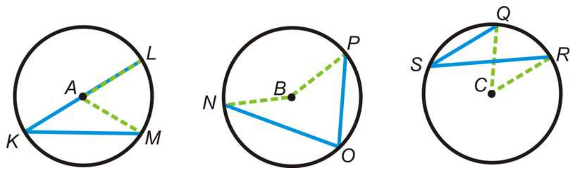
Tools Needed: pencil, paper, compass, ruler, protractor

1. Draw three circles with three different inscribed angles. For $\odot A$, make one side of the inscribed angle a diameter, for $\odot B$, make $B$ inside the angle and for $\odot C$ make $C$ outside the angle. Try to make all the angles different sizes.




[Figure 5]

2. Using your ruler, draw in the corresponding central angle for each angle and label each set of endpoints.




[Figure 6]

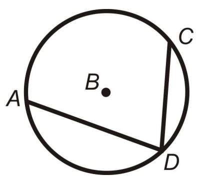
3. Using your protractor measure the six angles and determine if there is a relationship between the central angle, the inscribed angle, and the intercepted arc.

$m\angle LAM =$ _____ $m\angle NBP =$ _____ $m\angle QCR =$ _____

$mLM =$ _____ $mNP =$ _____ $mQR =$ _____

$m\angle LKM =$ _____ $m\angle NOP =$ _____ $m\angle QSR =$ _____

**Inscribed Angle Theorem:** The measure of an inscribed angle is half the measure of its intercepted arc.

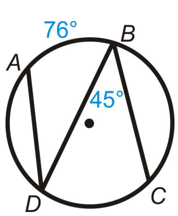



[Figure 7]

In the picture, $m\angle ADC = \frac{1}{2} mAC$ . If we had drawn in the central angle $\angle ABC$ , we could also say that $m\angle ADC = \frac{1}{2} m\angle ABC$ because the measure of the central angle is equal to the measure of the intercepted arc.

To prove the Inscribed Angle Theorem, you would need to split it up into three cases, like the three different angles drawn from Investigation 9-4. We will touch on the algebraic proofs in the review exercises.

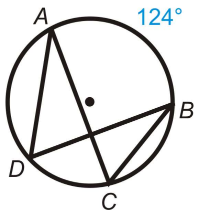
**Example 1:** Find $mDC$ and $m\angle ADB$ .




[Figure 8]

**Solution:** From the Inscribed Angle Theorem, $mDC = 2 \cdot 45^\circ = 90^\circ$ . $m\angle ADB = \frac{1}{2} \cdot 76^\circ = 38^\circ$ .

**Example 2:** Find $m\angle ADB$ and $m\angle ACB$ .




[Figure 9]

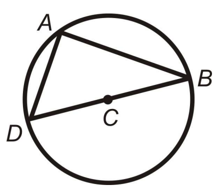
**Solution:** The intercepted arc for both angles is $AB$ . Therefore, $m\angle ADB = m\angle ACB = \frac{1}{2} \cdot 124^\circ = 62^\circ$

This example leads us to our next theorem.

**Theorem 9-8:** Inscribed angles that intercept the same arc are congruent.

To prove Theorem 9-8, you would use the similar triangles that are formed by the chords.

**Example 3:** Find $m\angle DAB$ in $\odot C$ .




[Figure 10]

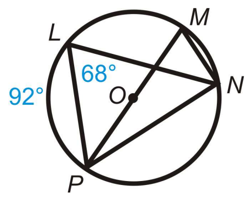
**Solution:** Because $C$ is the center, $DB$ is a diameter. Therefore, $\angle DAB$ inscribes semicircle, or $180^\circ$ . $m\angle DAB = \frac{1}{2} \cdot 180^\circ = 90^\circ$ .

**Theorem 9-9:** An angle that intercepts a semicircle is a right angle.

In Theorem 9-9 we could also say that the angle is inscribed in a semicircle. Anytime a right angle is inscribed in a circle, the endpoints of the angle are the endpoints of a diameter. Therefore, the converse of Theorem 9-9 is also true.

When the three vertices of a triangle are on the circle, like in Example 3, we say that the triangle is *inscribed* in the circle. We can also say that the circle is *circumscribed* around (or about) the triangle. Any polygon can be inscribed in a circle.

**Example 4:** Find $m\angle PMN, mPN, m\angle MNP, m\angle LNP$ , and $mLN$ .




[Figure 11]

**Solution:**

$m\angle PMN = m\angle PLN = 68^\circ$ by Theorem 9-8.

$mPN = 2 \cdot 68^\circ = 136^\circ$ from the Inscribed Angle Theorem.

$m\angle MNP = 90^\circ$ by Theorem 9-9.

$m\angle LNP = \frac{1}{2} \cdot 92^\circ = 46^\circ$ from the Inscribed Angle Theorem.

To find $mLN$, we need to find $m\angle LPN$. $\angle LPN$ is the third angle in $\triangle LPN$, so $68^\circ + 46^\circ + m\angle LPN = 180^\circ$. $m\angle LPN = 66^\circ$, which means that $mLN = 2 \cdot 66^\circ = 132^\circ$.

## Inscribed Quadrilaterals

The last theorem for this section involves inscribing a quadrilateral in a circle.

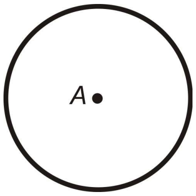
**Inscribed Polygon:** A polygon where every vertex is on a circle.

Note, that not every quadrilateral or polygon can be inscribed in a circle. Inscribed quadrilaterals are also called *cyclic quadrilaterals.* For these types of quadrilaterals, they must have one special property. We will investigate it here.

**Investigation 9-5: Inscribing Quadrilaterals**

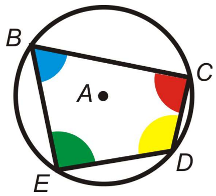
Tools Needed: pencil, paper, compass, ruler, colored pencils, scissors

Draw a circle. Mark the center point $A$.



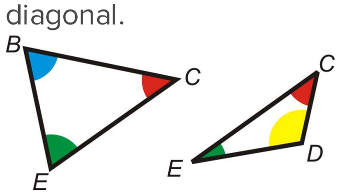

[Figure 12]

Place four points on the circle. Connect them to form a quadrilateral. Color the 4 angles of the quadrilateral 4 different colors.



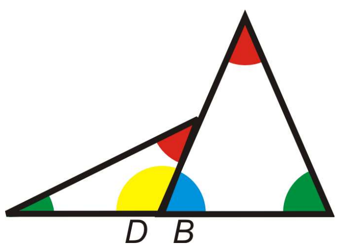

[Figure 13]

Cut out the quadrilateral. Then cut the quadrilateral into two triangles, by cutting on a diagonal.




[Figure 14]

Line up $\angle B$ and $\angle D$ so that they are adjacent angles. What do you notice? What does this show?



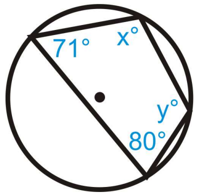

[Figure 15]

This investigation shows that the opposite angles in an inscribed quadrilateral are supplementary. By cutting the quadrilateral in half, through the diagonal, we were able to show that the other two angles (that we did not cut through) formed a linear pair when matched up.

**Theorem 9-10:** A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary.

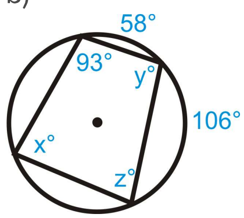
**Example 5:** Find the value of the missing variables.

a)




[Figure 16]

b)




[Figure 17]

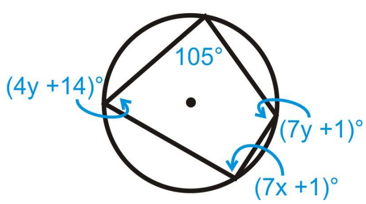
**Solution:**

a) $x+80^\circ = 180^\circ$ by Theorem 9-10. $x = 100^\circ$.

$y+71^\circ = 180^\circ$ by Theorem 9-10. $y = 109^\circ$.

b) It is easiest to figure out $z$ first. It is supplementary with $93^\circ$, so $z = 87^\circ$. Second, we can find $x$. $x$ is an inscribed angle that intercepts the arc $58^\circ + 106^\circ = 164^\circ$. Therefore, by the Inscribed Angle Theorem, $x = 82^\circ$. $y$ is supplementary with $x$, so $y = 98^\circ$.

**Example 6:** ***Algebra Connection*** Find $x$ and $y$ in the picture below.

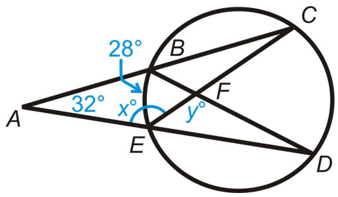



[Figure 18]

**Solution:** The opposite angles are supplementary. Set up an equation for $x$ and $y$.

$$\begin{aligned} (7x+1)^\circ + 105^\circ &= 180^\circ \\ 7x + 106^\circ &= 180^\circ \\ 7x &= 84^\circ \\ x &= 12^\circ \end{aligned} \qquad \begin{aligned} (4y+14)^\circ + (7y+1)^\circ &= 180^\circ \\ 11y + 15^\circ &= 180^\circ \\ 11y &= 165^\circ \\ y &= 15^\circ \end{aligned}$$

**Example 7:** Find $x$ and $y$ in the picture below.




[Figure 19]

**Solution:** To find $x$, use $\triangle ACE$. $m\angle ACE = 14^\circ$ because it is half of $mBE$ by the Inscribed Angle Theorem.

$$\begin{aligned} 32^\circ + 14^\circ + x^\circ &= 180^\circ \\ x &= 134^\circ \end{aligned}$$

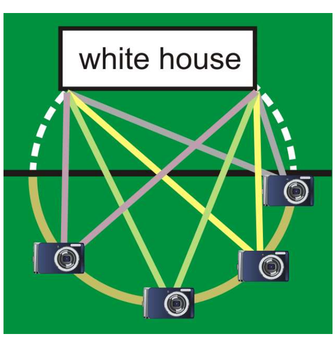
To find $y$, we will use $\triangle EFD$.

$$\begin{aligned} m\angle FED &= 180^\circ - x^\circ \\ m\angle FED &= 180^\circ - 134^\circ = 46^\circ \end{aligned}$$

$m\angle BDE = m\angle ACE = 14^\circ$ because they intercept the same arc, Theorem 9-8. Let's solve for $y$ in $\triangle EFD$, using the Triangle Sum Theorem.

$$\begin{aligned} 46^\circ + 14^\circ + y^\circ &= 180^\circ \\ y &= 120^\circ \end{aligned}$$

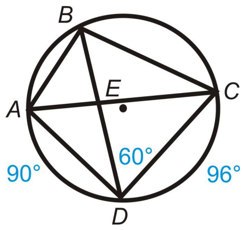
**Know What? Revisited** You can take the picture from anywhere on the semicircular walking path. The best place to take the picture is subjective, but most would think the pale green frame, straight-on, would be the best view.




[Figure 20]

## Review Questions

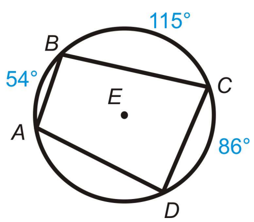
Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:




[Figure 21]

1. $m\angle DBC$
2. $mBC$
3. $mAB$
4. $m\angle ACD$
5. $m\angle ADC$
6. $m\angle ACB$

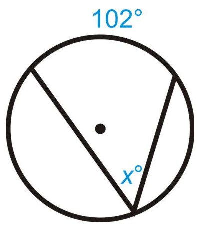
Quadrilateral $ABCD$ is inscribed in $\odot E$ . Find:



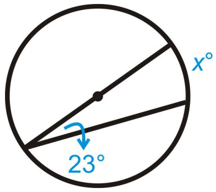

[Figure 22]

7. $m\angle A$
8. $m\angle B$
9. $m\angle C$
10. $m\angle D$

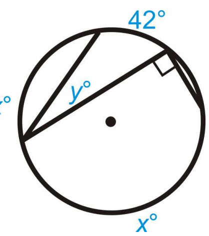
Find the value of $x$ and/or $y$ in $\odot A$ .



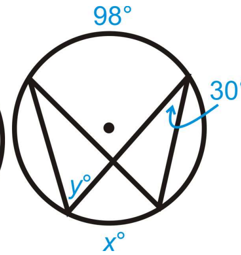
[Figure 23]



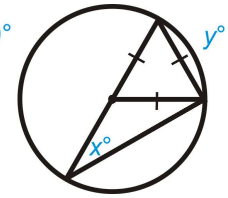
[Figure 24]



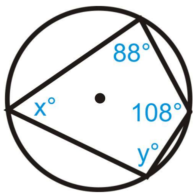
[Figure 25]



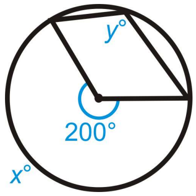
[Figure 26]



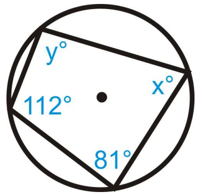
[Figure 27]



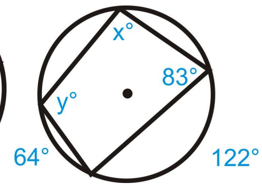
[Figure 28]



[Figure 29]

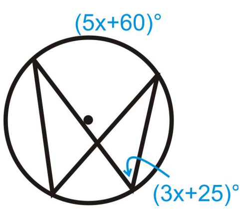


[Figure 30]

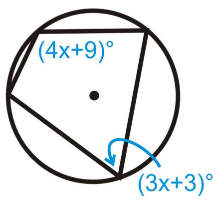


[Figure 31]

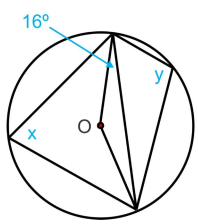
***Algebra Connection*** Solve for the variables.



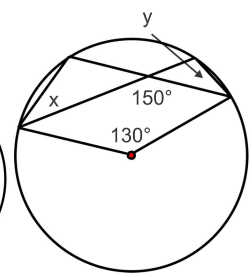
[Figure 32]



[Figure 33]

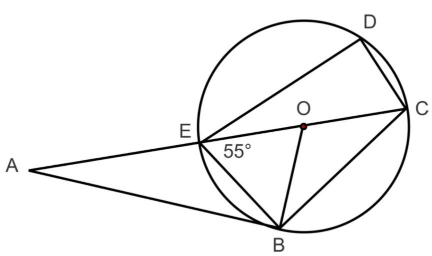


[Figure 34]



[Figure 35]

Use the diagram below to find the measures of the indicated angles and arcs in problems 24-29.



[Figure 36]

24. $m\angle EBO$

25. $m\angle EOB$

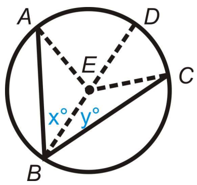
26. $mBC$

27. $m\angle ABO$

28. $m\angle A$

29. $m\angle EDC$

Fill in the blanks of one proof of the Inscribed Angle Theorem.

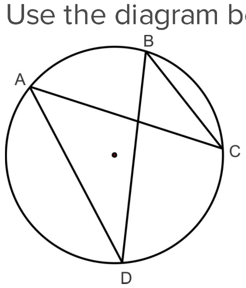



[Figure 37]

Given: Inscribed $\angle ABC$ and diameter $BD$ Prove: $m\angle ABC = \frac{1}{2}mAC$

| *Statement* | *Reason* |
|---|---|
| 1. Inscribed $\angle ABC$ and diameter $BD$ $m\angle ABE = x^\circ$ and $m\angle CBE = y^\circ$ | |
| 2. $x^\circ + y^\circ = m\angle ABC$ | |
| 3. | All radii are congruent |
| 4. | Definition of an isosceles triangle |
| 5. $m\angle EAB = x^\circ$ and $m\angle ECB = y^\circ$ | |
| 6. $m\angle AED = 2x^\circ$ and $m\angle CED = 2y^\circ$ | |
| 7. $mAD = 2x^\circ$ and $mDC = 2y^\circ$ | |
| 8. | Arc Addition Postulate |
| 9. $mAC = 2x^\circ + 2y^\circ$ | |
| 10. | Distributive PoE |
| 11. $mAC = 2m\angle ABC$ | |
| 12. $m\angle ABC = \frac{1}{2}mAC$ | |

Use the diagram below to write a proof of Theorem 9-8.




[Figure 38]

31. Suppose that $AB$ is a diameter of a circle centered at $O$, and $C$ is any other point on the circle. Draw the line through $O$ that is parallel to $AC$, and let $D$ be the point

where it meets $BC$ . Prove that $D$ is the midpoint of $BC$ .

## Review Queue Answers

1. $60^\circ$ , it is an equilateral triangle.
2. They are congruent because the chords are congruent.
3. $\frac{360^\circ}{3} = 120^\circ$
4. The arcs are double each angle.