4.5 Isosceles and Equilateral Triangles

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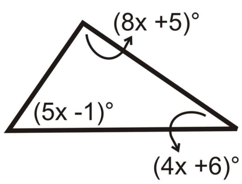
Learning Objectives

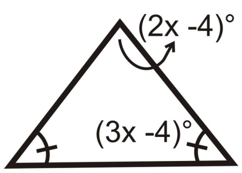
- Understand the properties of isosceles and equilateral triangles.
- Use the Base Angles Theorem and its converse.
- Prove an equilateral triangle is also equiangular.

Review Queue

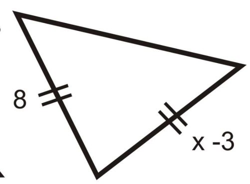
Find the value of x .



[Figure 1]



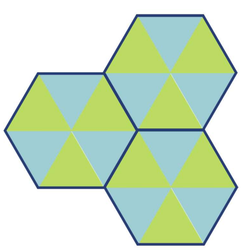
[Figure 2]



[Figure 3]

1. If a triangle is equiangular, what is the measure of each angle?

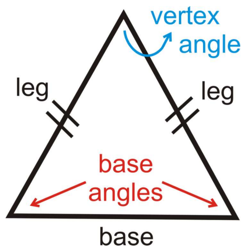
Know What? Your parents now want to redo the bathroom. To the right is the tile they would like to place in the shower. The blue and green triangles are all equilateral. What type of polygon is dark blue outlined figure? Can you determine how many degrees are in each of these figures? Can you determine how many degrees are around a point? HINT: For a "point" you can use a point where the six triangles meet.



[Figure 4]

Isosceles Triangle Properties

An isosceles triangle is a triangle that has ***at least*** two congruent sides. The congruent sides of the isosceles triangle are called the ***legs***. The other side is called the **base** and the angles between the base and the congruent sides are called **base angles**. The angle made by the two legs of the isosceles triangle is called the **vertex angle.**

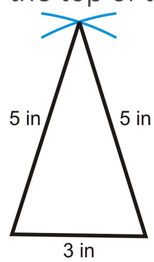


[Figure 5]

Investigation 4-5: Isosceles Triangle Construction

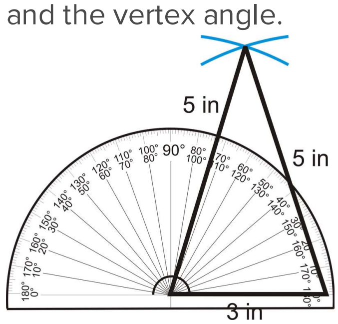
Tools Needed: pencil, paper, compass, ruler, protractor

Refer back to Investigation 4-2. Using your compass and ruler, draw an isosceles triangle with sides of 3 in, 5 in and 5 in. Draw the 3 in side (the base) horizontally 6 inches from the top of the page.



[Figure 6]

Now that you have an isosceles triangle, use your protractor to measure the base angles and the vertex angle.



[Figure 7]

The base angles should each be 72.5° *and the vertex angle should be* 35° .

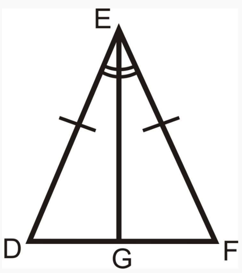
We can generalize this investigation into the Base Angles Theorem.

Base Angles Theorem: The base angles of an isosceles triangle are congruent.

To prove the Base Angles Theorem, we will construct the angle bisector (Investigation 1-5) through the vertex angle of an isosceles triangle.

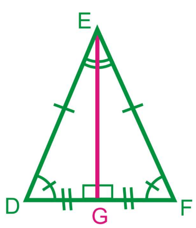
Given: Isosceles triangle $\triangle DEF$ with $DE \cong EF$

Prove: $\angle D \cong \angle F$

Statement	**Reason**
1. Isosceles triangle $\triangle DEF$ with $DE \cong EF$	Given
2. Construct angle bisector EG for $\angle E$  [Figure 8]	Every angle has one angle bisector
3. $\angle DEG \cong \angle FEG$	Definition of an angle bisector
4. $EG \cong EG$	Reflexive PoC
5. $\triangle DEG \cong \triangle FEG$	SAS
6. $\angle D \cong \angle F$	CPCTC

By constructing the angle bisector, EG, we designed two congruent triangles and then used CPCTC to show that the base angles are congruent. Now that we have proven the Base Angles Theorem, you do not have to construct the angle bisector every time. It can now be assumed that base angles of any isosceles triangle are always equal.

Let's further analyze the picture from step 2 of our proof.



[Figure 9]

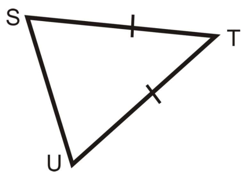
Because $\triangle DEG \cong \triangle FEG$, we know that $\angle EGD \cong \angle EGF$ by CPCTC. Thes two angles are also a linear pair, so they are congruent supplements, or 90° each. Therefore, $EG \perp DF$.

Additionally, $DG \cong GF$ by CPCTC, so G is the midpoint of DF. This means that EG is the **perpendicular bisector** of DF, in addition to being the angle bisector of $\angle DEF$.

Isosceles Triangle Theorem: The angle bisector of the vertex angle in an isosceles triangle is also the perpendicular bisector to the base.

This is ONLY true for the vertex angle. We will prove this theorem in the review questions for this section.

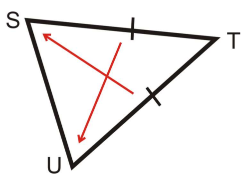
Example 1: Which two angles are congruent?



[Figure 10]

Solution: This is an isosceles triangle. The congruent angles, are opposite the congruent sides.

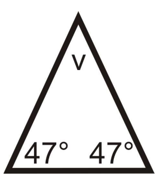
From the arrows we see that $\angle S \cong \angle U$.



[Figure 11]

Example 2: If an isosceles triangle has base angles with measures of 47° , what is the measure of the vertex angle?

Solution: Draw a picture and set up an equation to solve for the vertex angle, v .

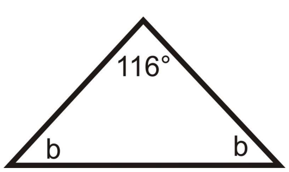


[Figure 12]

$$\begin{aligned} 47^\circ + 47^\circ + v &= 180^\circ \\ v &= 180^\circ - 47^\circ - 47^\circ \\ v &= 86^\circ \end{aligned}$$

Example 3: If an isosceles triangle has a vertex angle with a measure of 116° , what is the measure of each base angle?

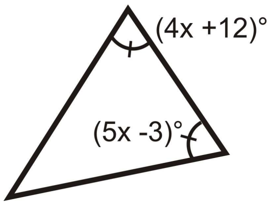
Solution: Draw a picture and set up and equation to solve for the base angles, b . Recall that the base angles are equal.



[Figure 13]

$$\begin{aligned} 116^\circ + b + b &= 180^\circ \\ 2b &= 64^\circ \\ b &= 32^\circ \end{aligned}$$

Example 4: ***Algebra Connection*** Find the value of x and the measure of each angle.



[Figure 14]

Solution: Set the angles equal to each other and solve for x.

$$\begin{aligned} (4x+12)^\circ &= (5x-3)^\circ \\ 15^\circ &= x \end{aligned}$$

If $x = 15^\circ$, then the base angles are $4(15^\circ) + 12^\circ$, or 72°. The vertex angle is $180^\circ - 72^\circ - 72^\circ = 36^\circ$.

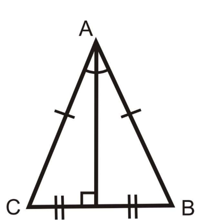
The converses of the Base Angles Theorem and the Isosceles Triangle Theorem are both true.

Base Angles Theorem Converse: If two angles in a triangle are congruent, then the opposite sides are also congruent.

So, for a triangle $\triangle ABC$, if $\angle A \cong \angle B$, then $CB \cong CA$. $\angle C$ would be the vertex angle.

Isosceles Triangle Theorem Converse: The perpendicular bisector of the base of an isosceles triangle is also the angle bisector of the vertex angle.

In other words, if $\triangle ABC$ is isosceles, $AD \perp CB$ and $CD \cong DB$, then $\angle CAD \cong \angle BAD$.



[Figure 15]

Equilateral Triangles

By definition, all sides in an equilateral triangle have exactly the same length. Therefore, ***every equilateral triangle is also an isosceles triangle.***

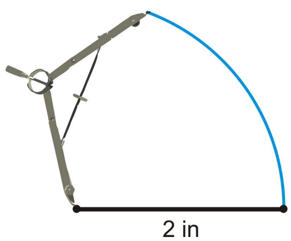
Investigation 4-6: Constructing an Equilateral Triangle

Tools Needed: pencil, paper, compass, ruler, protractor

1. Because all the sides of an equilateral triangle are equal, pick a length to be all the sides of the triangle. Measure this length and draw it horizontally on your paper.

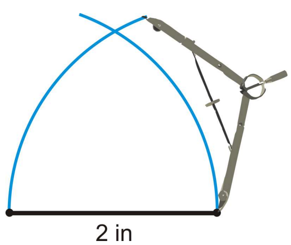
[Figure 16]

2. Put the pointer of your compass on the left endpoint of the line you drew in Step 1. Open the compass to be the same width as this line. Make an arc above the line.



[Figure 17]

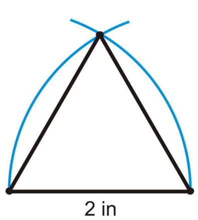
3. Repeat Step 2 on the right endpoint.



[Figure 18]

4. Connect each endpoint with the arc intersections to make the equilateral triangle.

Use the protractor to measure each angle of your constructed equilateral triangle. What do you notice?

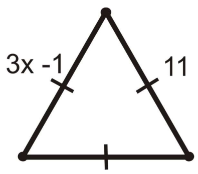


[Figure 19]

From the Base Angles Theorem, the angles opposite congruent sides in an isosceles triangle are congruent. So, if all three sides of the triangle are congruent, then all of the angles are congruent or 60° each.

Equilateral Triangles Theorem: All equilateral triangles are also equiangular. Also, all equiangular triangles are also equilateral.

Example 5: ***Algebra Connection*** Find the value of x .

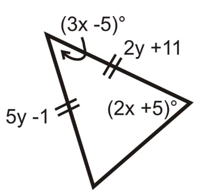


[Figure 20]

Solution: Because this is an equilateral triangle $3x-1=11$. Now, we have an equation, solve for x .

$$\begin{aligned}3x-1&=11\\3x&=12\\x&=4\end{aligned}$$

Example 6: ***Algebra Connection*** Find the values of x and y .



[Figure 21]

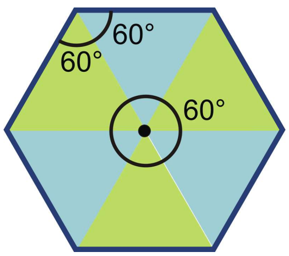
Solution: Let's start with y . Both sides are equal, so set the two expressions equal to each other and solve for y .

$$\begin{aligned}5y-1&=2y+11\\3y&=12\\y&=4\end{aligned}$$

For x , we need to use two $(2x+5)^\circ$ expressions because this is an isosceles triangle and that is the base angle measurement. Set all the angles equal to 180° and solve.

$$\begin{aligned}(2x+5)^\circ+(2x+5)^\circ+(3x-5)^\circ &= 180^\circ\\(7x+5)^\circ &= 180^\circ\\7x &= 175^\circ\\x &= 25^\circ\end{aligned}$$

Know What? Revisited Let's focus on one tile. First, these triangles are all equilateral, so this is an equilateral hexagon (6 sided polygon). Second, we now know that every equilateral triangle is also equiangular, so every triangle within this tile has 360° angles. This makes our equilateral hexagon also equiangular, with each angle measuring 120° . Because there are 6 angles, the sum of the angles in a hexagon are 6.120° or 720° . Finally, the point in the center of this tile, has 660° angles around it. That means there are 360° around a point.



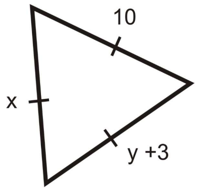
[Figure 22]

Review Questions

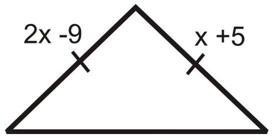
Constructions For questions 1-5, use your compass and ruler to:

1. Draw an isosceles triangle with sides 3.5 in, 3.5 in, and 6 in.
2. Draw an isosceles triangle that has a vertex angle of 100° and legs with length of 4 cm. (you will also need your protractor for this one)
3. Draw an equilateral triangle with sides of length 7 cm.
4. Using what you know about constructing an equilateral triangle, construct (without your protractor) a 60° angle.
5. Draw an isosceles right triangle. What is the measure of the base angles?

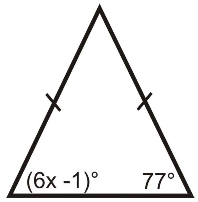
For questions 6-17, find the measure of x and/or y .



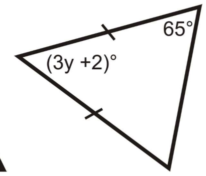
[Figure 23]



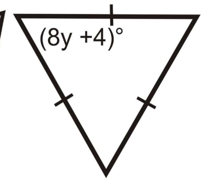
[Figure 24]



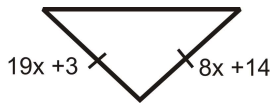
[Figure 25]



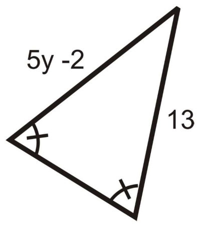
[Figure 26]



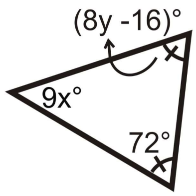
[Figure 27]



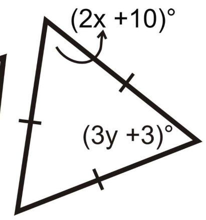
[Figure 28]



[Figure 29]

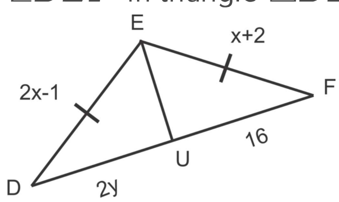


[Figure 30]



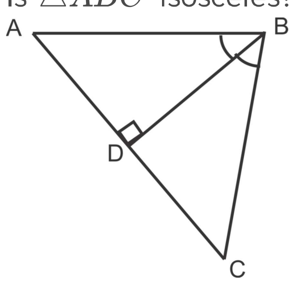
[Figure 31]

$\angle DEF$ in triangle $\triangle DEF$ is bisected by EU. Find x and y.



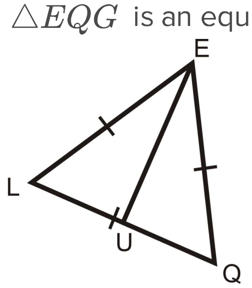
[Figure 32]

Is $\triangle ABC$ isosceles? Explain your reasoning.



[Figure 33]

$\triangle EQG$ is an equilateral triangle. If EU bisects $\angle LEQ$, find:



[Figure 34]

a. $m\angle EUL$

b. $m\angle UEL$

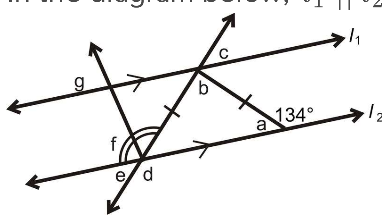
c. $m\angle ELQ$

d. If $EQ = 4$, find LU.

Determine if the following statements are ALWAYS, SOMETIMES, or NEVER true. Explain your reasoning.

18. Base angles of an isosceles triangle are congruent.
19. Base angles of an isosceles triangle are complementary.
20. Base angles of an isosceles triangle can be equal to the vertex angle.
21. Base angles of an isosceles triangle can be right angles.
22. Base angles of an isosceles triangle are acute.

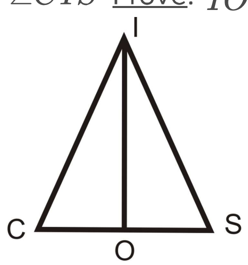
In the diagram below, $l_1 \,||\, l_2$. Find all of the lettered angles.



[Figure 35]

Fill in the blanks in the proofs below.

Given: Isosceles $\triangle CIS$, with base angles $\angle C$ and $\angle S$ IO is the angle bisector of $\angle CIS$ Prove: IO is the perpendicular bisector of CS

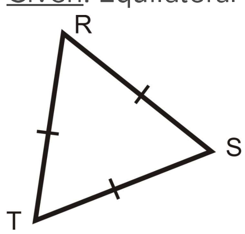


[Figure 36]

Statement	***Reason***
1.	Given
2.	Base Angles Theorem
3. $\angle CIO \cong \angle SIO$	
4.	Reflexive PoC
5. $\triangle CIO \cong \triangle SIO$	
6. $CO \cong OS$	
7.	CPCTC
8. $\angle IOC$ and $\angle IOS$ are supplementary	
9.	Congruent Supplements Theorem
10. IO is the perpendicular bisector of CS	

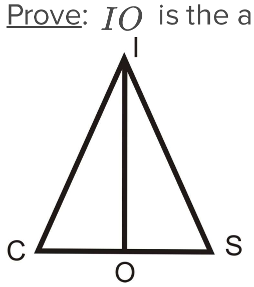
Write a 2-column proof.

Given: Equilateral $\triangle RST$ with $RT \cong ST \cong RS$ Prove: $\triangle RST$ is equiangular



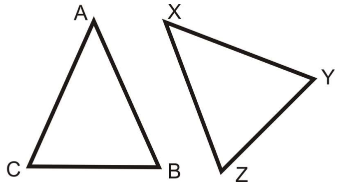
[Figure 37]

Given: Isosceles $\triangle ICS$ with $\angle C$ and $\angle S$ IO is the perpendicular bisector of CS
Prove: IO is the angle bisector of $\angle CIS$



[Figure 38]

Given: Isosceles $\triangle ABC$ with base angles $\angle B$ and $\angle C$ Isosceles $\triangle XYZ$ with base angles $\angle Y$ and $\angle Z$ $\angle C \cong \angle Z, BC \cong YZ$ Prove: $\triangle ABC \cong \triangle XYZ$



[Figure 39]

Constructions

28. Using the construction of an equilateral triangle (investigation 4-6), construct a 30° angle. *Hint: recall how to bisect an angle from investigation 1-4.*
29. Use the construction of a 60° angle to construct a 120° angle.
30. Is there another way to construction a 120° angle? Describe the method.
31. Describe how you could construct a 45° angle (there is more than one possible way).

Review Queue Answers

1. $$\begin{aligned}(5x-1)^\circ+(8x+5)^\circ+(4x+6)^\circ &= 180^\circ\\ 17x+10 &= 180^\circ\\ 17x &= 170^\circ\\ x &= 10^\circ\end{aligned}$$
2. $$\begin{aligned}(2x-4)^\circ+(3x-4)^\circ+(3x-4)^\circ &= 180^\circ\\ 8x-12 &= 180^\circ\\ 8x &= 192^\circ\\ x &= 24^\circ\end{aligned}$$
3. $$\begin{aligned}x-3 &= 8\\ x &= 11\end{aligned}$$
4. Each angle is $\frac{180^\circ}{3}$, or 60°