5.2 Perpendicular Bisectors in Triangles

FlexBooks® 2.0 > American HS Geometry > Perpendicular Bisectors in Triangles

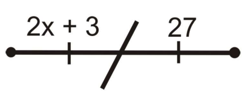
Last Modified: Dec 25, 2014

Learning Objectives

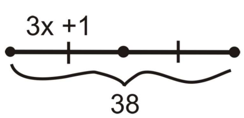
- Understand points of concurrency.
- Apply the Perpendicular Bisector Theorem and its converse to triangles.
- Understand concurrency for perpendicular bisectors.

Review Queue

1. Construct the perpendicular bisector of a 3 inch line. Use Investigation 1-3 from Chapter 1 to help you.
2. Find the value of x.

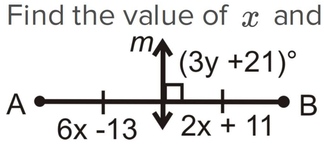


[Figure 1]



[Figure 2]

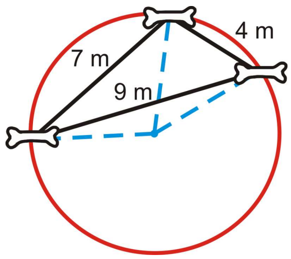
Find the value of x and y. Is m the perpendicular bisector of AB? How do you know?



[Figure 3]

Know What? An archeologist has found three bones in Cairo, Egypt. The bones are 4 meters apart, 7 meters apart and 9 meters apart (to form a triangle). The likelihood that more bones are in this area is very high. The archeologist wants to dig in an appropriate circle around these bones. If these bones are on the edge of the digging circle, where is the center of the circle?

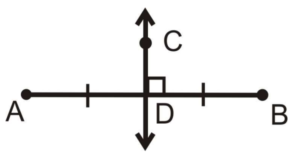
Can you determine how far apart each bone is from the center of the circle? What is this length?



[Figure 4]

Perpendicular Bisectors

In Chapter 1, you learned that a perpendicular bisector intersects a line segment at its midpoint and is perpendicular. In #1 in the Review Queue above, you constructed a perpendicular bisector of a 3 inch segment. Let's analyze this figure.



[Figure 5]

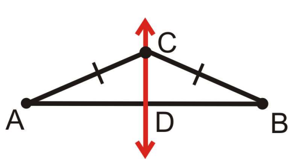
$\overleftrightarrow{CD}$ is the perpendicular bisector of AB. If we were to draw in AC and CB, we would find that they are equal. Therefore, any point on the perpendicular bisector of a segment is the same distance from each endpoint.

Perpendicular Bisector Theorem: If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

The proof of the Perpendicular Bisector Theorem is in the exercises for this section. In addition to the Perpendicular Bisector Theorem, we also know that its converse is true.

Perpendicular Bisector Theorem Converse: If a point is equidistant from the endpoints of a segment, then the point is on the perpendicular bisector of the segment.

Proof of the Perpendicular Bisector Theorem Converse



[Figure 6]

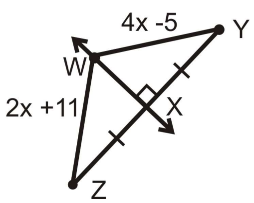
Given: $AC \cong CB$

Prove: $\overleftrightarrow{CD}$ is the perpendicular bisector of AB

Statement	Reason
1. $\overline{AC} \cong \overline{CB}$	Given
2. $\triangle ACB$ is an isosceles triangle	Definition of an isosceles triangle
3. $\angle A \cong \angle B$	Isosceles Triangle Theorem
4. Draw point D, such that D is the midpoint of $\overline{AB}$.	Every line segment has exactly one midpoint
5. $\overline{AD} \cong \overline{DB}$	Definition of a midpoint
6. $\triangle ACD \cong \triangle BCD$	SAS
7. $\angle CDA \cong \angle CDB$	CPCTC
8. $m\angle CDA = m\angle CDB = 90^\circ$	Congruent Supplements Theorem
9. $\overleftrightarrow{CD} \perp \overline{AB}$	Definition of perpendicular lines
10. $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$	Definition of perpendicular bisector

Let's use the Perpendicular Bisector Theorem and its converse in a few examples.

Example 1: ***Algebra Connection*** Find x and the length of each segment.



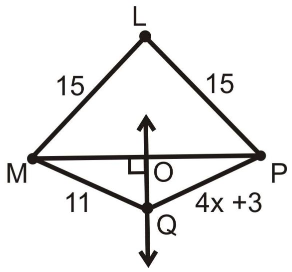
[Figure 7]

Solution: From the markings, we know that $\overleftrightarrow{WX}$ is the perpendicular bisector of XY. Therefore, we can use the Perpendicular Bisector Theorem to conclude that $WZ = WY$. Write an equation.

$$\begin{aligned} 2x + 11 &= 4x - 5 \\ 16 &= 2x \\ 8 &= x \end{aligned}$$

To find the length of WZ and WY, substitute 8 into either expression, $2(8) + 11 = 16 + 11 = 27$.

Example 2: $\overleftrightarrow{OQ}$ is the perpendicular bisector of MP.



[Figure 8]

a) Which segments are equal?

b) Find x .

c) Is L on $\overleftrightarrow{OQ}$? How do you know?

Solution:

a) $ML = LP$ because they are both 15.

$MO = OP$ because O is the midpoint of MP

$MQ = QP$ because Q is on the perpendicular bisector of MP .

b) $$\begin{aligned} 4x + 3 &= 11 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

c) Yes, L is on $\overleftrightarrow{OQ}$ because $ML = LP$ (Perpendicular Bisector Theorem Converse).

Perpendicular Bisectors and Triangles

Two lines intersect at a point. If more than two lines intersect at the same point, it is called a point of concurrency.

Point of Concurrency: When three or more lines intersect at the same point.

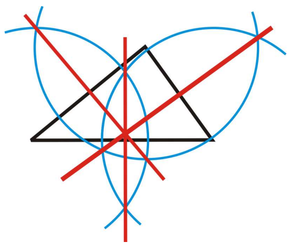
Investigation 5-1: Constructing the Perpendicular Bisectors of the Sides of a Triangle

Tools Needed: paper, pencil, compass, ruler

1. Draw a scalene triangle.

2. Construct the perpendicular bisector (Investigation 1-3) for all three sides.

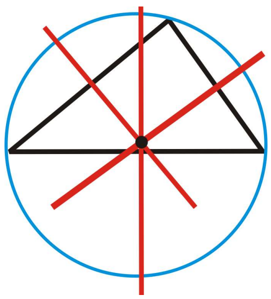
The three perpendicular bisectors all intersect at the same point, called the circumcenter.



[Figure 9]

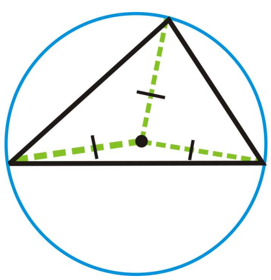
Circumcenter: The point of concurrency for the perpendicular bisectors of the sides of a triangle.

3. Erase the arc marks to leave only the perpendicular bisectors. Put the pointer of your compass on the circumcenter. Open the compass so that the pencil is on one of the vertices. Draw a circle. What happens?



[Figure 10]

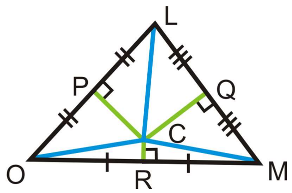
The circumcenter is the center of a circle that passes through all the vertices of the triangle. We say that this circle ***circumscribes*** the triangle. This means that ***the circumcenter is equidistant to the vertices.***



[Figure 11]

Concurrency of Perpendicular Bisectors Theorem: The perpendicular bisectors of the sides of a triangle intersect in a point that is equidistant from the vertices.

If PC, QC , and RC are perpendicular bisectors, then $LC = MC = OC$.

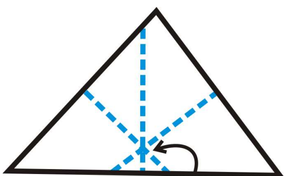


[Figure 12]

Example 3: For further exploration, try the following:

1. Cut out an acute triangle from a sheet of paper.
2. Fold the triangle over one side so that the side is folded in half. Crease.
3. Repeat for the other two sides. What do you notice?

Solution: The folds (blue dashed lines)are the perpendicular bisectors and cross at the circumcenter.

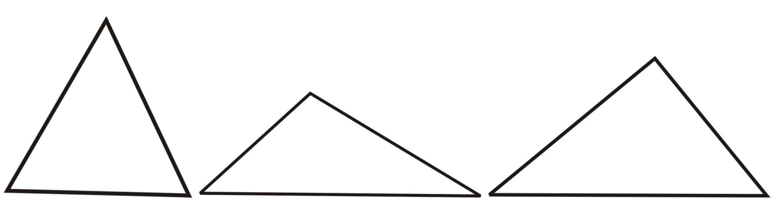


[Figure 13]

Know What? Revisited The center of the circle will be the circumcenter of the triangle formed by the three bones. Construct the perpendicular bisector of at least two sides to find the circumcenter. After locating the circumcenter, the archeologist can measure the distance from each bone to it, which would be the radius of the circle. This length is approximately 4.7 meters.

Review Questions

Construction Construct the circumcenter for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-1.

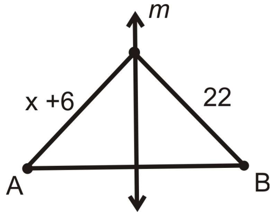


[Figure 14] [Figure 15] [Figure 16]

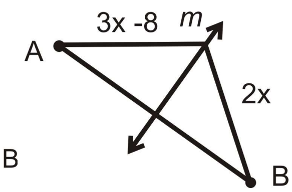
1. Can you use the method in Example 3 to locate the circumcenter for these three triangles?
2. Based on your constructions in 1-3, state a conjecture about the relationship between a triangle and the location of its circumcenter.

3. Construct equilateral triangle $\triangle ABC$ (Investigation 4-6). Construct the perpendicular bisectors of the sides of the triangle and label the circumcenter X. Connect the circumcenter to each vertex. Your original triangle is now divided into six triangles. What can you conclude about the six triangles? Why?

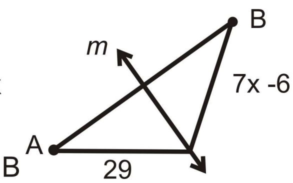
Algebra Connection For questions 7-12, find the value of x. m is the perpendicular bisector of AB.



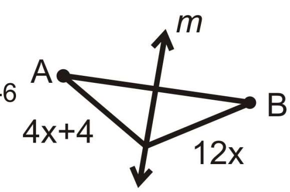
[Figure 17]



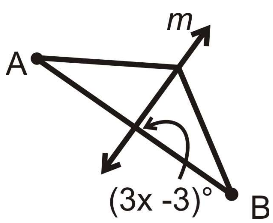
[Figure 18]



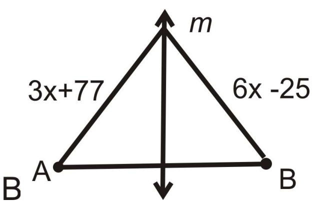
[Figure 19]



[Figure 20]



[Figure 21]



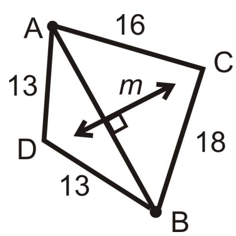
[Figure 22]

7. m is the perpendicular bisector of AB.

a. List all the congruent segments.

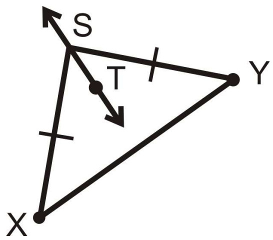
b. Is C on AB ? Why or why not?

c. Is D on AB ? Why or why not?

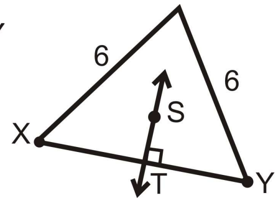


[Figure 23]

For Questions 14 and 15, determine if $\overleftrightarrow{ST}$ is the perpendicular bisector of XY. Explain why or why not.



[Figure 24]



[Figure 25]

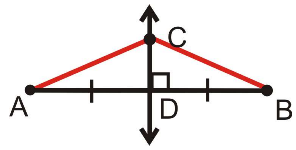
For Questions 16-20, consider line segment AB with endpoints $A(2,1)$ and $B(6,3)$.

16. Find the slope of AB.
17. Find the midpoint of AB.
18. Find the equation of the perpendicular bisector of AB.
19. Find AB. Simplify the radical, if needed.
20. Plot A, B, and the perpendicular bisector. Label it m. How could you find a point C on m, such that C would be the third vertex of equilateral triangle $\triangle ABC$? *You do not have to find the coordinates, just describe how you would do it.*

For Questions 21-25, consider $\triangle ABC$ with vertices $A(7,6), B(7,-2)$ and $C(0,5)$. Plot this triangle on graph paper.

21. Find the midpoint and slope of AB and use them to draw the perpendicular bisector of AB. You do not need to write the equation.
22. Find the midpoint and slope of BC and use them to draw the perpendicular bisector of BC. You do not need to write the equation.
23. Find the midpoint and slope of AC and use them to draw the perpendicular bisector of AC. You do not need to write the equation.
24. Are the three lines concurrent? What are the coordinates of their point of intersection (what is the circumcenter of the triangle)?
25. Use your compass to draw the circumscribed circle about the triangle with your point found in question 24 as the center of your circle.
26. Repeat questions 21-25 with $\triangle LMO$ where $L(2,9), M(3,0)$ and $O(-7,0)$.
27. Repeat questions 21-25 with $\triangle REX$ where $R(4,2), E(6,0)$ and $X(0,0)$.
28. Can you explain why the perpendicular bisectors of the sides of a triangle would all pass through the center of the circle containing the vertices of the triangle? Think about the definition of a circle: The set of all point equidistant from a given center.
29. Fill in the blanks: There is exactly __________ circle which contains any ___________ points.

Fill in the blanks of the proof of the Perpendicular Bisector Theorem.

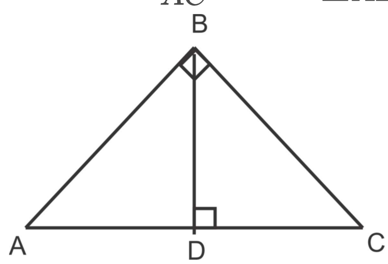


[Figure 26]

Given: $\overleftrightarrow{CD}$ is the perpendicular bisector of AB Prove: $AC \cong CB$

Statement	*Reason*
1.	
2. D is the midpoint of AB	
3.	Definition of a midpoint
4. $\angle CDA$ and $\angle CDB$ are right angles	
5. $\angle CDA \cong \angle CDB$	
6.	Reflexive PoC
7. $\triangle CDA \cong \triangle CDB$	
8. $AC \cong CB$	

Write a two column proof. Given: $\triangle ABC$ is a right isosceles triangle and BD is the $\perp$ bisector of AC Prove: $\triangle ABD$ and $\triangle CBD$ are congruent.



[Figure 27]

31. Write a paragraph explaining why the two smaller triangles in question 31 are also isosceles right triangles.

Review Queue Answers

1. Reference Investigation 1-3.
2. Answers:

a.
$$\begin{aligned} 2x+3 &= 27 \\ 2x &= 24 \\ x &= 12 \end{aligned}$$

b. $$\begin{aligned} 3x+1&=19 \\ 3x&=18 \\ x&=6 \end{aligned}$$

3. $$\begin{aligned} 6x-13&=2x+11 \\ 4x&=24 \\ x&=6 \end{aligned} \qquad \begin{aligned} 3y+21&=90^\circ \\ 3y&=69^\circ \\ y&=23^\circ \end{aligned}$$

Yes, m is the perpendicular bisector of AB because it is perpendicular to AB and passes through the midpoint.

5.3 Angle Bisectors in Triangles

FlexBooks® 2.0 > American HS Geometry > Angle Bisectors in Triangles

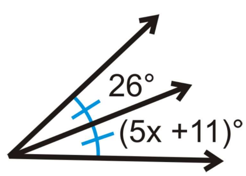
Last Modified: Dec 25, 2014

Learning Objectives

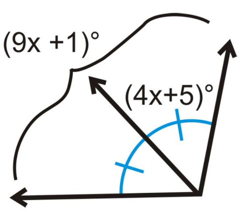
- Apply the Angle Bisector Theorem and its converse.
- Understand concurrency for angle bisectors.

Review Queue

1. Construct the angle bisector of an 80° angle (Investigation 1-4).
2. Draw the following: M is on the interior of $\angle LNO$. O is on the interior of $\angle MNP$. If $\overleftrightarrow{NM}$ and $\overleftrightarrow{NO}$ are the angle bisectors of $\angle LNO$ and $\angle MNP$ respectively, write all the congruent angles.
3. Find the value of x .

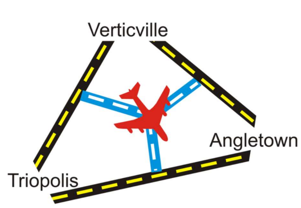


[Figure 1]



[Figure 2]

Know What? The cities of Verticville, Triopolis, and Angletown are joining their city budgets together to build a centrally located airport. There are freeways between the three cities and they want to have the freeway on the interior of these freeways. Where is the best location to put the airport so that they have to build the least amount of road?

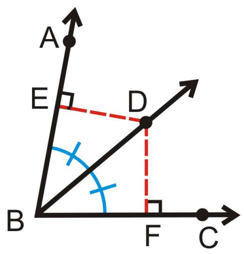


[Figure 3]

In the picture to the right, the blue roads are proposed.

Angle Bisectors

In Chapter 1, you learned that an angle bisector cuts an angle exactly in half. In #1 in the Review Queue above, you constructed an angle bisector of an 80° angle. Let's analyze this figure.



[Figure 4]

$\overrightarrow{BD}$ is the angle bisector of $\angle ABC$. Looking at point D , if we were to draw ED and DF , we would find that they are equal. Recall from Chapter 3 that the shortest distance from a point to a line is the perpendicular length between them. ED and DF are the shortest lengths between D , *which is on the angle bisector*, and each side of the angle.

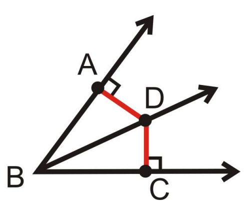
Angle Bisector Theorem: If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

In other words, if $\overleftrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BE} \perp ED$, and $\overrightarrow{BF} \perp DF$, then $ED = DF$.

Proof of the Angle Bisector Theorem

Given: $\overrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BA} \perp AD$, and $\overrightarrow{BC} \perp DC$

Prove: $AD \cong DC$



[Figure 5]

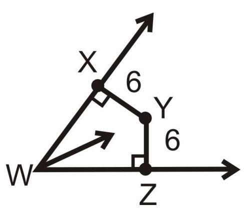
Statement	Reason
1. $\overrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BA} \perp AD, \overrightarrow{BC} \perp DC$	Given
2. $\angle ABD \cong \angle DBC$	Definition of an angle bisector
3. $\angle DAB$ and $\angle DCB$ are right angles	Definition of perpendicular lines
4. $\angle DAB \cong \angle DCB$	All right angles are congruent
5. $BD \cong BD$	Reflexive PoC
6. $\triangle ABD \cong \triangle CBD$	AAS
7. $AD \cong DC$	CPCTC

The converse of this theorem is also true. The proof is in the review questions.

Angle Bisector Theorem Converse: If a point is in the interior of an angle and equidistant from the sides, then it lies on the bisector of the angle.

Because the Angle Bisector Theorem and its converse are both true we have a biconditional statement. We can put the two conditional statements together using if and only if. *A point is on the angle bisector of an angle if and only if it is equidistant from the sides of the triangle.*

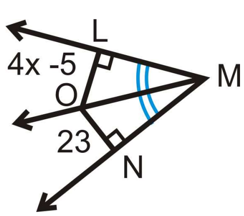
Example 1: Is Y on the angle bisector of $\angle XWZ$?



[Figure 6]

Solution: In order for Y to be on the angle bisector XY needs to be equal to YZ and they both need to be perpendicular to the sides of the angle. From the markings we know $XY \perp \overrightarrow{WX}$ and $ZY \perp \overrightarrow{WZ}$. Second, $XY = YZ = 6$. From this we can conclude that Y is on the angle bisector.

Example 2: $\overrightarrow{MO}$ is the angle bisector of $\angle LMN$. Find the measure of x .



[Figure 7]

Solution: $LO = ON$ by the Angle Bisector Theorem Converse.

$$\begin{aligned}4x - 5 &= 23\\ 4x &= 28\\ x &= 7\end{aligned}$$

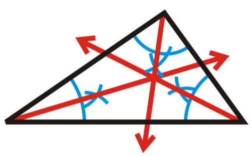
Angle Bisectors in a Triangle

Like perpendicular bisectors, the point of concurrency for angle bisectors has interesting properties.

Investigation 5-2: Constructing Angle Bisectors in Triangles

Tools Needed: compass, ruler, pencil, paper

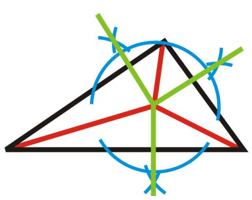
1. Draw a scalene triangle. Construct the angle bisector of each angle. Use Investigation 1-4 and #1 from the Review Queue to help you.



[Figure 8]

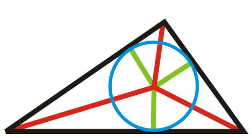
Incenter: The point of concurrency for the angle bisectors of a triangle.

2. Erase the arc marks and the angle bisectors after the incenter. Draw or construct the perpendicular lines to each side, through the incenter.



[Figure 9]

3. Erase the arc marks from #2 and the perpendicular lines beyond the sides of the triangle. Place the pointer of the compass on the incenter. Open the compass to intersect one of the three perpendicular lines drawn in #2. Draw a circle.

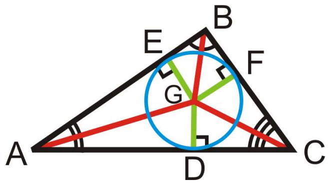


[Figure 10]

Notice that the circle touches all three sides of the triangle. We say that this circle is ***inscribed*** in the triangle because it touches all three sides. The incenter is on all three angle bisectors, so ***the incenter is equidistant from all three sides of the triangle.***

Concurrency of Angle Bisectors Theorem: The angle bisectors of a triangle intersect in a point that is equidistant from the three sides of the triangle.

If AG, BG , and GC are the angle bisectors of the angles in the triangle, then $EG = GF = GD$.

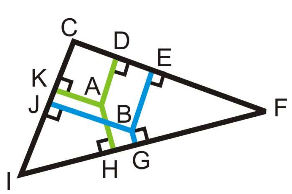


[Figure 11]

In other words, EG, FG , and DG are the radii of the inscribed circle.

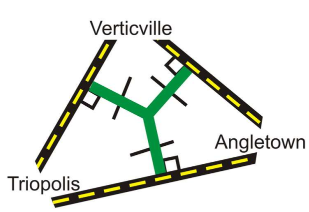
Example 3: If J, E , and G are midpoints and $KA = AD = AH$ what are points A and B called?

Solution: A is the incenter because $KA = AD = AH$, which means that it is equidistant to the sides. B is the circumcenter because JB, BE , and BG are the perpendicular bisectors to the sides.



[Figure 12]

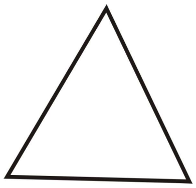
Know What? Revisited The airport needs to be equidistant to the three highways between the three cities. Therefore, the roads are all perpendicular to each side and congruent. The airport should be located at the incenter of the triangle.

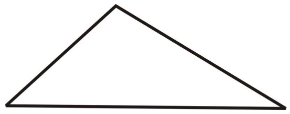


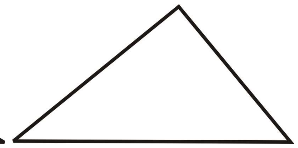
[Figure 13]

Review Questions

Construction Construct the incenter for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-2. Draw the inscribed circle too.

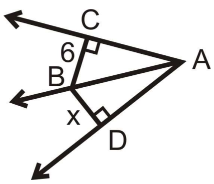

[Figure 14]


[Figure 15]

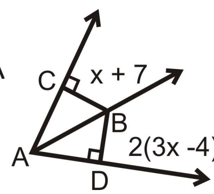

[Figure 16]

1. Is the incenter always going to be inside of the triangle? Why?
2. For an equilateral triangle, what can you conclude about the circumcenter and the incenter?

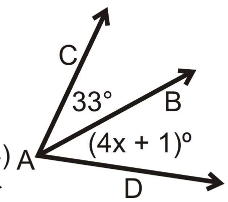
For questions 6-11, $\overrightarrow{AB}$ is the angle bisector of $\angle CAD$. Solve for the missing variable.



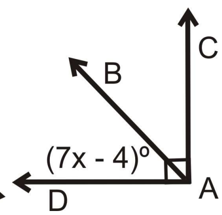
[Figure 17]



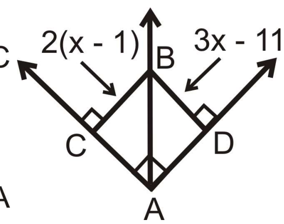
[Figure 18]



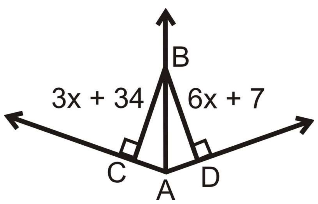
[Figure 19]



[Figure 20]

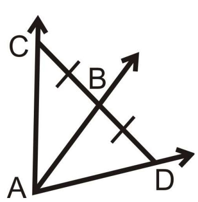


[Figure 21]

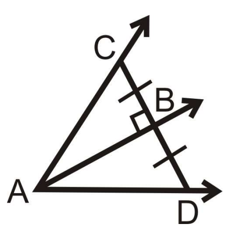


[Figure 22]

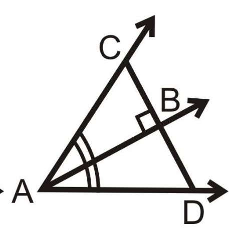
Is there enough information to determine if $\overrightarrow{AB}$ is the angle bisector of $\angle CAD$? Why or why not?



[Figure 23]



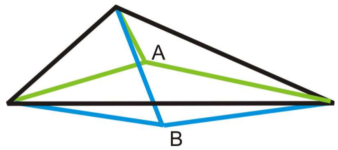
[Figure 24]



[Figure 25]

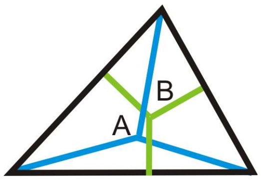
What are points A and B? How do you know?

The blue lines are congruent The green lines are angle bisectors



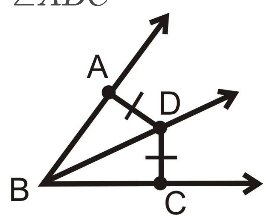
[Figure 26]

Both sets of lines are congruent The green lines are perpendicular to the sides



[Figure 27]

Fill in the blanks in the Angle Bisector Theorem Converse. Given: $AD \cong DC$, such that AD and DC are the shortest distances to $\overrightarrow{BA}$ and $\overrightarrow{BC}$ Prove: $\overrightarrow{BD}$ bisects $\angle ABC$



[Figure 28]

Statement	*Reason*
1.	
2.	The shortest distance from a point to a line is perpendicular.
3. $\angle DAB$ and $\angle DCB$ are right angles	
4. $\angle DAB \cong \angle DCB$	
5. $BD \cong BD$	
6. $\triangle ABD \cong \triangle CBD$	
7.	CPCTC
8. $\overrightarrow{BD}$ bisects $\angle ABC$	

Determine if the following descriptions refer to the incenter or circumcenter of the triangle.

18. A lighthouse on a triangular island is equidistant to the three coastlines.
19. A hospital is equidistant to three cities.
20. A circular walking path passes through three historical landmarks.
21. A circular walking path connects three other straight paths.

Constructions

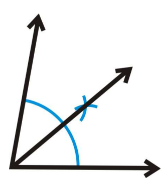
22. Construct an equilateral triangle.
23. Construct the angle bisectors of two of the angles to locate the incenter.
24. Construct the perpendicular bisectors of two sides to locate the circumcenter.
25. What do you notice? Use these points to construct an inscribed circle inside the triangle and a circumscribed circle about the triangle.

Multi- Step Problem

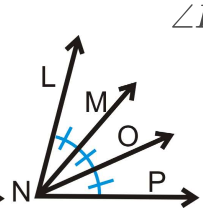
26. Draw $\angle ABC$ through $A(1,3), B(3,-1)$ and $C(7,1)$.
27. Use slopes to show that $\angle ABC$ is a right angle.
28. Use the distance formula to find AB and BC.
29. Construct a line perpendicular to AB through A.
30. Construct a line perpendicular to BC through C.
31. These lines intersect in the interior of $\angle ABC$. Label this point D and draw $\overrightarrow{BD}$.
32. Is $\overrightarrow{BD}$ the angle bisector of $\angle ABC$? Justify your answer.

Review Queue Answers

$$\angle LNM \cong \angle MNO \cong \angle ONP$$
$$\angle LNO \cong \angle MNP$$



[Figure 29]



[Figure 30]

1. Answers:

a. $$\begin{aligned} 5x+11 &= 26 \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

b. $$\begin{aligned} 9x-1 &= 2(4x+5) \\ 9x-1 &= 8x+10 \\ x &= 11^\circ \end{aligned}$$