9.2 Properties of Arcs

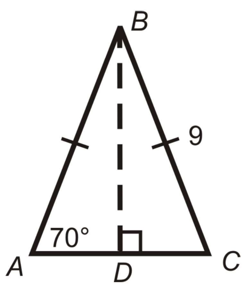
FlexBooks® 2.0 > American HS Geometry > Properties of Arcs

Last Modified: Dec 25, 2014

Learning Objectives

- Define and measure central angles in circles.
- Define minor arcs and major arcs.

Review Queue



[Figure 1]

1. What kind of triangle is $\triangle ABC$?

2. How does $\overline{BD}$ relate to $\triangle ABC$?

3. Find $m\angle ABC$ and $m\angle ABD$.

Round to the nearest tenth.

4. Find AD .

5. Find AC .

Know What? The Ferris wheel to the right has equally spaced seats, such that the central angle is 20° . How many seats are there? Why do you think it is important to have equally spaced seats on a Ferris wheel?

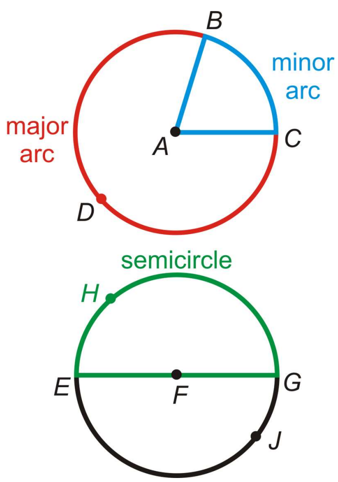
[Figure 2]

If the radius of this Ferris wheel is 25 ft., how far apart are two adjacent seats? Round your answer to the nearest tenth. *The shortest distance between two points is a straight line.*

Central Angles & Arcs

Central Angle: The angle formed by two radii of the circle with its vertex at the center of the circle.

In the picture to the right, the central angle would be $\angle BAC$. Every central angle divides a circle into two ***arcs.*** In this case the arcs are BC and BDC . Notice the $\odot$ above the letters. To label an arc, always use this curve above the letters. Do not confuse BC and BC .



[Figure 3]

Arc: A section of the circle.

If D was not on the circle, we would not be able to tell the difference between BC and BDC . There are 360° in a circle, where a semicircle is half of a circle, or 180° . $m\angle EFG = 180^\circ$, because it is a straight angle, so $mEHG = 180^\circ$ and $mEJG = 180^\circ$.

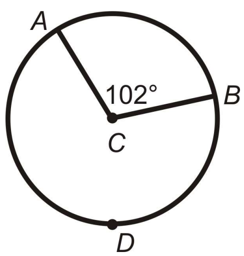
Semicircle: An arc that measures 180° .

Minor Arc: An arc that is less than 180° .

Major Arc: An arc that is greater than 180° . ***Always*** use 3 letters to label a major arc.

An arc can be measured in degrees or in a linear measure (cm, ft, etc.). In this chapter we will use degree measure. ***The measure of the minor arc is the same as the measure of the central angle*** that corresponds to it. The measure of the major arc equals to 360° minus the measure of the minor arc. In order to prevent confusion, major arcs are always named with three letters; the letters that denote the endpoints of the arc and any other point on the major arc. When referring to the measure of an arc, always place an "m" in from of the label.

Example 1: Find mAB and $mADB$ in $\odot C$.

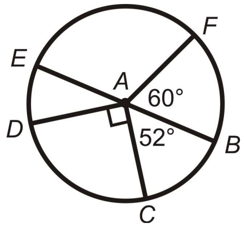


[Figure 4]

Solution: mAB is the same as $m\angle ACB$. So, $mAB = 102^\circ$. The measure of $mADB$, which is the major arc, is equal to 360° minus the minor arc.

$mADB = 360^\circ - mAB = 360^\circ - 102^\circ = 258^\circ$

Example 2: Find the measures of the arcs in $\odot A$. EB is a diameter.



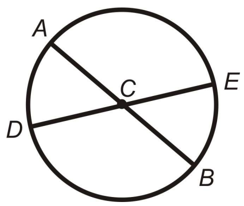
[Figure 5]

Solution: Because EB is a diameter, $m\angle EAB = 180^\circ$. Each arc is the same as its corresponding central angle.

$$
\begin{aligned}
mBF &= m\angle FAB = 60^\circ \\
mEF &= m\angle EAF = 120^\circ && \rightarrow m\angle EAB - m\angle FAB \\
mED &= m\angle EAD = 38^\circ && \rightarrow m\angle EAB - m\angle BAC - m\angle CAD \\
mDC &= m\angle DAC = 90^\circ \\
mBC &= m\angle BAC = 52^\circ
\end{aligned}
$$

Congruent Arcs: Two arcs are congruent if their central angles are congruent.

Example 3: List all the congruent arcs in $\odot C$ below. AB and DE are diameters.

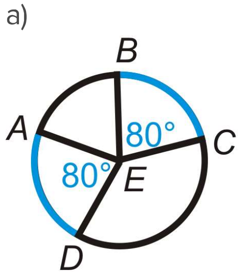


[Figure 6]

Solution: From the picture, we see that $\angle ACD$ and $\angle ECB$ are vertical angles. $\angle DCB$ and $\angle ACE$ are also vertical angles. Because all vertical angles are equal and these four angles are all central angles, we know that $AD \cong EB$ and $AE \cong DB$.

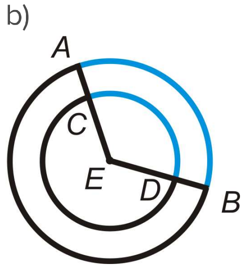
Example 4: Are the blue arcs congruent? Explain why or why not.

a)



[Figure 7]

b)



[Figure 8]

Solution: In part a, $AD \cong BC$ because they have the same central angle measure. In part b, the two arcs do have the same measure, but are not congruent because the circles are not congruent.

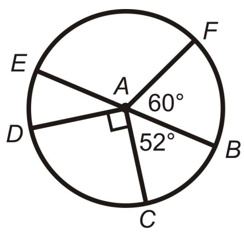
Arc Addition Postulate

Just like the Angle Addition Postulate and the Segment Addition Postulate, there is an Arc Addition Postulate. It is very similar.

Arc Addition Postulate: The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs.

Using the picture from Example 3, we would say $mAE + mEB = mAEB$.

Example 5: Reusing the figure from Example 2, find the measure of the following arcs in $\odot A$. EB is a diameter.



[Figure 9]

a) $mFED$

b) $mCDF$

c) mBD

d) $mDFC$

Solution: Use the Arc Addition Postulate.

a) $mFED = mFE + mED = 120^\circ + 38^\circ = 158^\circ$

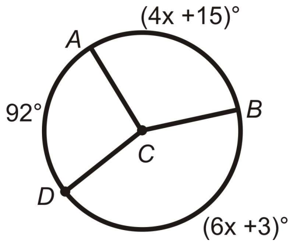
We could have labeled FED as FD because it is less than 180° .

b) $mCDF = mCD + mDE + mEF = 90^\circ + 38^\circ + 120^\circ = 248^\circ$

c) $mBD = mBC + mCD = 52^\circ + 90^\circ = 142^\circ$

d) $mDFC = 38^\circ + 120^\circ + 60^\circ + 52^\circ = 270^\circ$ or $mDFC = 360^\circ - mCD = 360^\circ - 90^\circ = 270^\circ$

Example 6: ***Algebra Connection*** Find the value of x for $\odot C$ below.

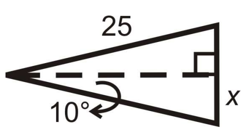


[Figure 10]

Solution: There are 360° in a circle. Let's set up an equation.

$$\begin{aligned} mAB + mAD + mDB &= 360^\circ \\ (4x+15)^\circ + 92^\circ + (6x+3)^\circ &= 360^\circ \\ 10x + 110^\circ &= 360^\circ \\ 10x &= 250^\circ \\ x &= 25^\circ \end{aligned}$$

Know What? Revisited Because the seats are 20° apart, there will be $\frac{360^\circ}{20^\circ} = 18$ seats. It is important to have the seats evenly spaced for balance. To determine how far apart the adjacent seats are, use the triangle to the right. We will need to use sine to find x and then multiply it by 2.



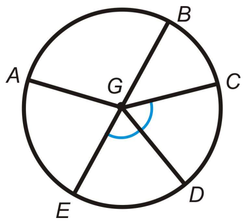
[Figure 11]

$$\begin{aligned} \sin 10^\circ &= \frac{x}{25} \\ x = 25 \sin 10^\circ &= 4.3\ ft. \end{aligned}$$

The total distance apart is 8.6 feet.

Review Questions

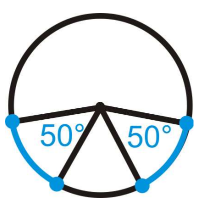
Determine if the arcs below are a minor arc, major arc, or semicircle of $\odot G$. EB is a diameter.



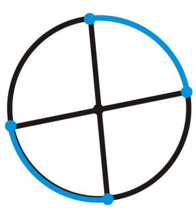
[Figure 12]

1. AB
2. ABD
3. BCE
4. CAE
5. ABC
6. EAB
7. Are there any congruent arcs? If so, list them.
8. If $mBC = 48^\circ$, find mCD .
9. Using #8, find $mCAE$.

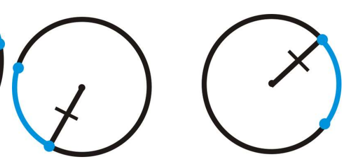
Determine if the blue arcs are congruent. If so, state why.



[Figure 13]

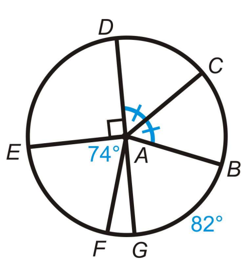


[Figure 14]



[Figure 15]

Find the measure of the indicated arcs or central angles in $\odot A$. DG is a diameter.

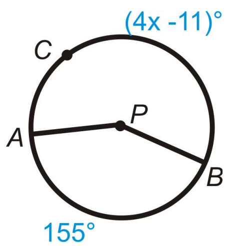


[Figure 16]

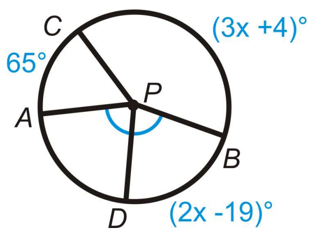
13. DE
14. DC
15. $\angle GAB$

16. FG
17. EDB
18. $\angle EAB$
19. DCF
20. DBE

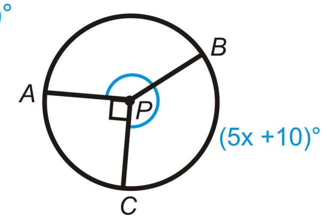
Algebra Connection Find the measure of x in $\odot P$.



[Figure 17]

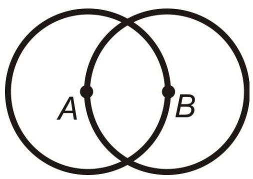


[Figure 18]



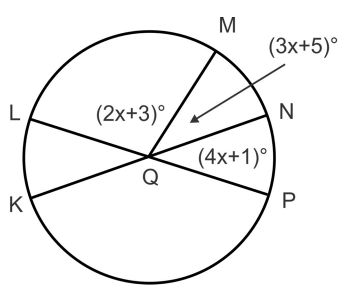
[Figure 19]

What can you conclude about $\odot A$ and $\odot B$?



[Figure 20]

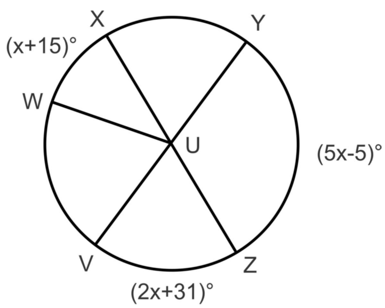
Use the diagram below to find the measures of the indicated arcs in problems 25-30.



[Figure 21]

25. mMN
26. mLK
27. mMP
28. mMK
29. $mNPL$
30. $mLKM$

Use the diagram below to find the measures indicated in problems 31-36.



[Figure 22]

31. $m\angle VUZ$
32. $m\angle YUZ$
33. $m\angle WUV$
34. $m\angle XUV$
35. $mYWZ$
36. $mWYZ$

Review Queue Answers

1. isosceles
2. BD is the angle bisector of $\angle ABC$ and the perpendicular bisector of AC .
3. $m\angle ABC = 40^\circ, m\angle ABD = 25^\circ$
4. $\cos 70^\circ = \frac{AD}{9} \rightarrow AD = 9 \cdot \cos 70^\circ = 3.1$
5. $AC = 2 \cdot AD = 2 \cdot 3.1 = 6.2$

9.3 Properties of Chords

FlexBooks® 2.0 > American HS Geometry > Properties of Chords

Last Modified: Dec 25, 2014

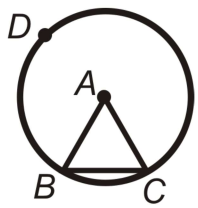
Learning Objectives

- Find the lengths of chords in a circle.
- Discover properties of chords and arcs.

Review Queue

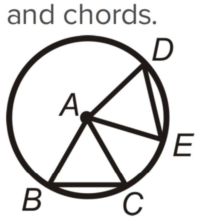
1. Draw a chord in a circle.
2. Draw a diameter in the circle from #1. Is a diameter a chord?

$\triangle ABC$ is an equilateral triangle in $\odot A$. Find $m\widehat{BC}$ and $m\widehat{BDC}$.



[Figure 1]

$\triangle ABC$ and $\triangle ADE$ are equilateral triangles in $\odot A$. List a pair of congruent arcs and chords.



[Figure 2]

Know What? To the right is the Gran Teatro Falla, in Cadiz, Andalucía, Spain. This theater was built in 1905 and hosts several plays and concerts. It is an excellent example of circles in architecture. Notice the five windows, $A-E$. $\odot A \cong \odot E$ and $\odot B \cong \odot C \cong \odot D$. Each window is topped with a 240° arc. The gold chord in each circle connects the rectangular portion of the window to the circle. Which chords are congruent? How do you know?

[Figure 3]

Recall from the first section, that a chord is a line segment whose endpoints are on a circle. A diameter is the longest chord in a circle. There are several theorems that explore the properties of chords.

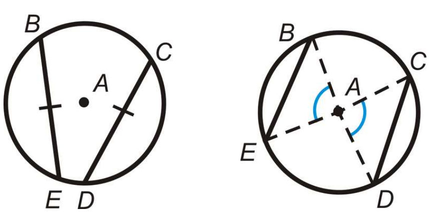
Congruent Chords & Congruent Arcs

From #4 in the Review Queue above, we noticed that $BC \cong DE$ and $\widehat{BC} \cong \widehat{DE}$. This leads to our first theorem.

Theorem 10-3: In the same circle or congruent circles, minor arcs are congruent if and only if their corresponding chords are congruent.

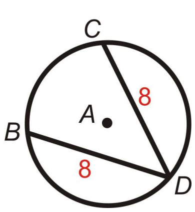
Notice the "if and only if" in the middle of the theorem. This means that Theorem 10-3 is a biconditional statement. Taking this theorem one step further, any time two central angles are congruent, the chords and arcs from the endpoints of the sides of the central angles are also congruent.

In both of these pictures, $BE \cong CD$ and $\widehat{BE} \cong \widehat{CD}$. In the second picture, we have $\triangle BAE \cong \triangle CAD$ because the central angles are congruent and $BA \cong AC \cong AD \cong AE$ because they are all radii (SAS). By CPCTC, $BE \cong CD$.



[Figure 4]

Example 1: Use $\odot A$ to answer the following.



[Figure 5]

a) If $m\widehat{BD} = 125^\circ$, find $m\widehat{CD}$.

b) If $m\widehat{BC} = 80^\circ$, find $m\widehat{CD}$.

Solution:

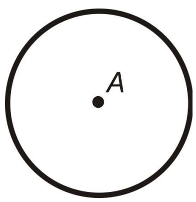
a) From the picture, we know $BD = CD$. Because the chords are equal, the arcs are too. $m\widehat{CD} = 125^\circ$.

b) To find $m\widehat{CD}$, subtract 80° from 360° and divide by 2.
$$m\widehat{CD} = \frac{360^\circ - 80^\circ}{2} = \frac{280^\circ}{2} = 140^\circ$$

Investigation 9-2: Perpendicular Bisector of a Chord

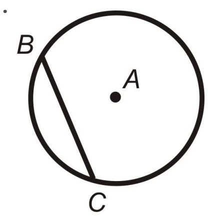
Tools Needed: paper, pencil, compass, ruler

Draw a circle. Label the center A.



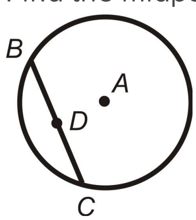
[Figure 6]

Draw a chord in $\odot A$. Label it BC.



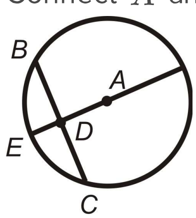
[Figure 7]

Find the midpoint of BC by using a ruler. Label it D.



[Figure 8]

Connect A and D to form a diameter. How does AD relate to the chord, BC ?



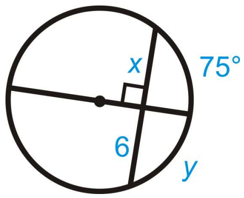
[Figure 9]

Theorem 10-4: The perpendicular bisector of a chord is also a diameter.

In the picture to the left, $AD \perp BC$ and $BD \cong DC$. From this theorem, we also notice that AD also bisects the corresponding arc at E , so $\widehat{BE} \cong \widehat{EC}$.

Theorem 10-5: If a diameter is perpendicular to a chord, then the diameter bisects the chord and its corresponding arc.

Example 2: Find the value of x and y.

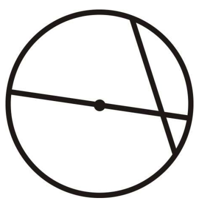


[Figure 10]

Solution: The diameter here is also perpendicular to the chord. From Theorem 10-5, $x = 6$ and $y = 75^\circ$.

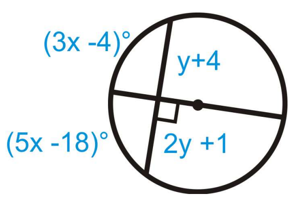
Example 3: Is the converse of Theorem 10-4 true?

Solution: The converse of Theorem 10-4 would be: A diameter is also the perpendicular bisector of a chord. This is not a true statement, see the counterexample to the right.



[Figure 11]

Example 4: ***Algebra Connection*** Find the value of x and y.



[Figure 12]

Solution: Because the diameter is perpendicular to the chord, it also bisects the chord and the arc. Set up an equation for x and y.

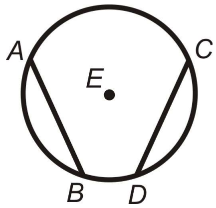
$$\begin{aligned} (3x-4)^\circ &= (5x-18)^\circ \\ 14^\circ &= 2x \\ 7^\circ &= x \end{aligned} \qquad \begin{aligned} y+4 &= 2y+1 \\ 3 &= y \end{aligned}$$

Equidistant Congruent Chords

Investigation 9-3: Properties of Congruent Chords

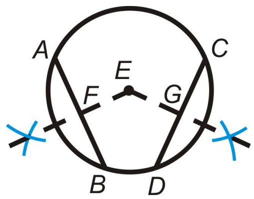
Tools Needed: pencil, paper, compass, ruler

Draw a circle with a radius of 2 inches and two chords that are both 3 inches. Label as in the picture to the right. *This diagram is drawn to scale.*



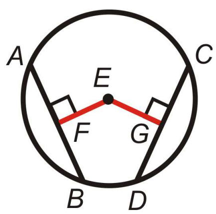
[Figure 13]

From the center, draw the perpendicular segment to AB and CD. You can either use your ruler, a protractor or Investigation 3-2 (Constructing a Perpendicular Line through a Point not on the line. We will show arc marks for Investigation 3-2.



[Figure 14]

Erase the arc marks and lines beyond the points of intersection, leaving FE and EG. Find the measure of these segments. What do you notice?

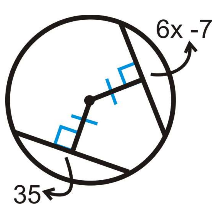


[Figure 15]

Theorem 10-6: In the same circle or congruent circles, two chords are congruent if and only if they are equidistant from the center.

Recall that two lines are equidistant from the same point if and only if the shortest distance from the point to the line is congruent. The shortest distance from any point to a line is the perpendicular line between them. In this theorem, the fact that $FE = EG$ means that AB and CD are equidistant to the center and $AB \cong CD$.

Example 5: *Algebra Connection* Find the value of x.

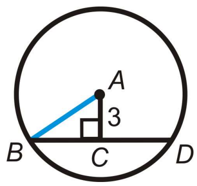


[Figure 16]

Solution: Because the distance from the center to the chords is congruent and perpendicular to the chords, then the chords are equal.

$$\begin{aligned} 6x-7 &= 35 \\ 6x &= 42 \\ x &= 7 \end{aligned}$$

Example 6: $BD = 12$ and $AC = 3$ in $\odot A$. Find the radius and $m\widehat{BD}$.



[Figure 17]

Solution: First find the radius. In the picture, AB is a radius, so we can use the right triangle $\triangle ABC$, such that AB is the hypotenuse. From 10-5, $BC = 6$.

$$\begin{aligned} 3^2+6^2 &= AB^2 \\ 9+36 &= AB^2 \\ AB &= \sqrt{45} = 3\sqrt{5} \end{aligned}$$

In order to find $m\widehat{BD}$, we need the corresponding central angle, $\angle BAD$. We can find half of $\angle BAD$ because it is an acute angle in $\triangle ABC$. Then, multiply the measure by 2 for $m\widehat{BD}$.

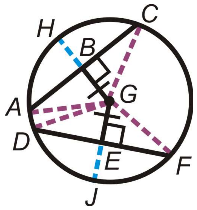
$$\begin{aligned} \tan^{-1}\left(\frac{6}{3}\right) &= m\angle BAC \\ m\angle BAC &\approx 63.43^\circ \end{aligned}$$

This means that $m\angle BAD \approx 126.9^\circ$ and $m\widehat{BD} \approx 126.9^\circ$ as well.

Know What? Revisited In the picture, the chords from $\odot A$ and $\odot E$ are congruent and the chords from $\odot B, \odot C$, and $\odot D$ are also congruent. We know this from Theorem 10-3. All five chords are not congruent because all five circles are not congruent, even though the central angle for the circles is the same.

Review Questions

Two chords in a circle are perpendicular and congruent. Does one of them have to be a diameter? Why or why not? Fill in the blanks.

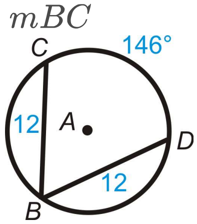


[Figure 18]

1. ____ $\cong DF$
2. $\widehat{AC} \cong$ ____
3. $\widehat{DJ} \cong$ ____
4. ____ $\cong EJ$
5. $\angle AGH \cong$ ____
6. $\angle DGF \cong$ ____
7. List all the congruent radii in $\odot G$.

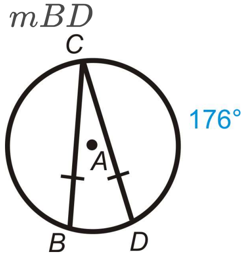
Find the value of the indicated arc in $\odot A$.

$m\widehat{BC}$



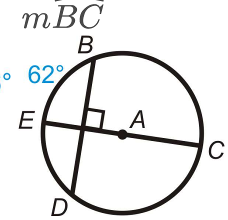
[Figure 19]

$m\widehat{BD}$



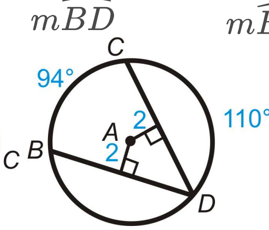
[Figure 20]

$m\widehat{BC}$



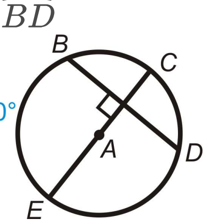
[Figure 21]

$m\widehat{BD}$



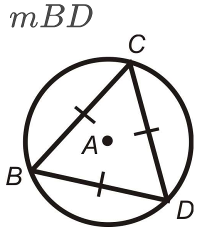
[Figure 22]

$m\widehat{BD}$



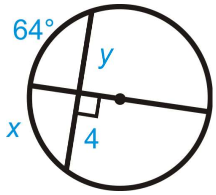
[Figure 23]

$m\widehat{BD}$

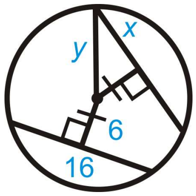


[Figure 24]

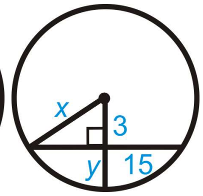
Algebra Connection Find the value of x and/or y.



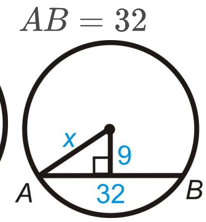
[Figure 25]



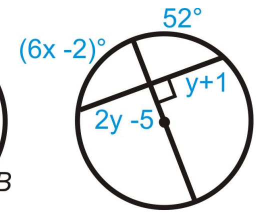
[Figure 26]



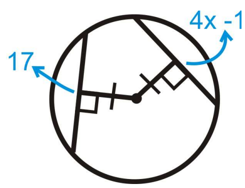
[Figure 27]



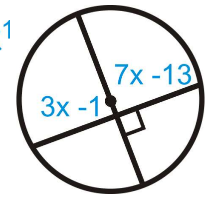
[Figure 28]



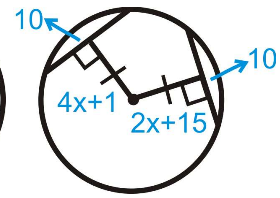
[Figure 29]



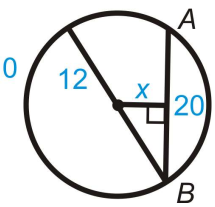
[Figure 30]



[Figure 31]



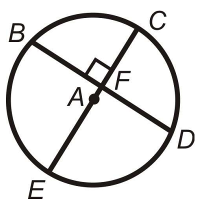
[Figure 32]



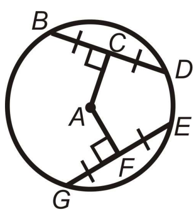
[Figure 33]

15. Find $m\widehat{AB}$ in Question 18. Round your answer to the nearest tenth of a degree.
16. Find $m\widehat{AB}$ in Question 23. Round your answer to the nearest tenth of a degree.

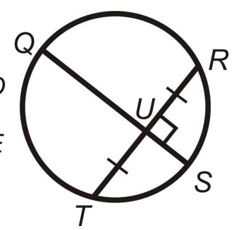
In problems 26-28, what can you conclude about the picture? State a theorem that justifies your answer. You may assume that A is the center of the circle.



[Figure 34]

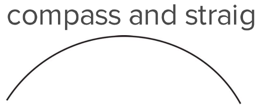


[Figure 35]



[Figure 36]

Trace the arc below onto your paper then follow the steps to locate the center using a compass and straightedge.



[Figure 37]

a. Use your straightedge to make a chord in the arc.
b. Use your compass and straightedge to construct the perpendicular bisector of this chord.
c. Repeat steps a and b so that you have two chords and their perpendicular bisectors.
d. What is the significance of the point where the perpendicular bisectors intersect?
e. Verify your answer to part d by using the point and your compass to draw the rest of the circle.

26. ***Algebra Connection*** Let's repeat what we did in problem 29 using coordinate geometry skills. Given the points $A(-3,5), B(5,5)$ and $C(4,-2)$ on the circle (an arc could be

drawn through these points from A to C). The following steps will walk you through the process to find the equation of the perpendicular bisector of a chord, and use two of these perpendicular bisectors to locate the center of the circle. Let's first find the perpendicular bisector of chord AB .

a. Since the perpendicular bisector passes through the midpoint of a segment we must first find the midpoint between A and B .

b. Now the perpendicular line must have a slope that is the opposite reciprocal of the slope of $\overleftrightarrow{AB}$. Find the slope of $\overleftrightarrow{AB}$ and then its opposite reciprocal.

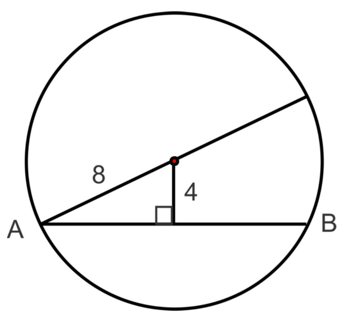
c. Finally, you can write the equation of the perpendicular bisector of AB using the point you found in part a and the slope you found in part b.

d. Repeat steps a-c for chord BC .

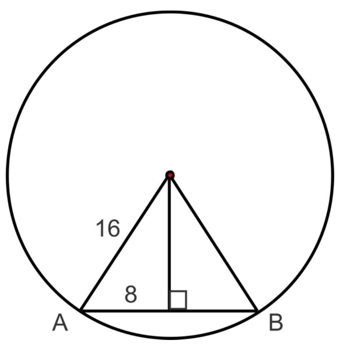
e. Now that we have the two perpendicular bisectors of the chord we can use algebra to find their intersection. Solve the system of linear equations to find the center of the circle.

f. Find the radius of the circle by finding the distance from the center (point found in part e) to any of the three given points on the circle.

27. Find the measure of $\widehat{AB}$ in each diagram below.



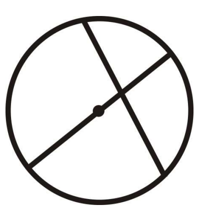
[Figure 38]



[Figure 39]

Review Queue Answers

1 & 2. Answers will vary



[Figure 40]

3. $m\widehat{BC} = 60^\circ, mBDC = 300^\circ$

4. $BC \cong DE$ and $\widehat{BC} \cong \widehat{DE}$