7.5 Proportionality Relationships

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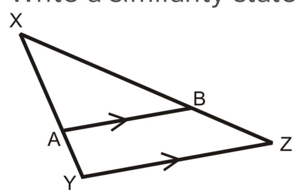
Learning Objectives

- Identify proportional segments when two sides of a triangle are cut by a segment parallel to the third side.
- Extend triangle proportionality to parallel lines.

Review Queue

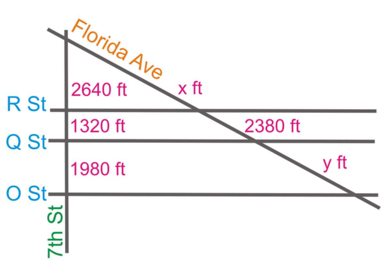
Write a similarity statement for the two triangles in the diagram. Why are they similar?



[Figure 1]

1. If $XA = 16, XY = 18, XB = 32,$ find XZ.
2. If $YZ = 27$, find AB.
3. Find AY and BZ.
4. Is $\frac{AY}{AX} = \frac{BZ}{BX}$?

Know What? To the right is a street map of part of Washington DC. R Street, Q Street, and O Street are parallel and 7^{th} Street is perpendicular to all three. R and Q are one "city block" (usually $\frac{1}{4}$ mile or 1320 feet) apart. The other given measurements are on the map. What are x and y?



[Figure 2]

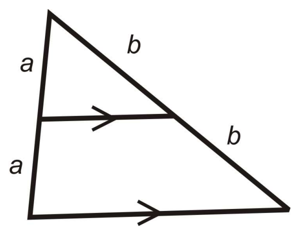
What is the distance from:

- R and 7^{th} to R and Florida?
- Q and 7^{th} to Q and Florida?
- O and 7^{th} to O and Florida?

Triangle Proportionality

Think about a midsegment of a triangle. A midsegment is parallel to one side of a triangle and divides the other two sides into congruent halves. The midsegment divides those two sides ***proportionally.***

Example 1: A triangle with its midsegment is drawn below. What is the ratio that the midsegment divides the sides into?



[Figure 3]

Solution: The midsegment's endpoints are the midpoints of the two sides it connects. The midpoints split the sides evenly. Therefore, the ratio would be $a : a$ or $b : b$. Both of these reduce to 1:1.

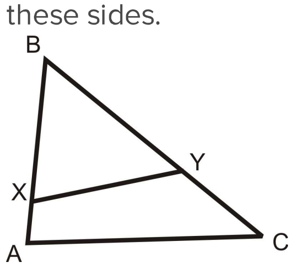
The midsegment divides the two sides of the triangle proportionally, but what about other segments?

Investigation 7-4: Triangle Proportionality

Tools Needed: pencil, paper, ruler

1. Draw $\triangle ABC$. Label the vertices.

   Draw XY so that X is on AB and Y is on BC. X and Y can be ***anywhere*** on these sides.

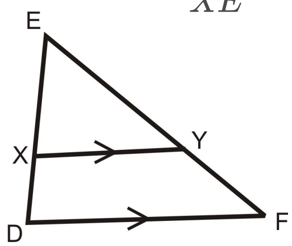


[Figure 4]

2. Is $\triangle XBY \sim \triangle ABC$? Why or why not? Measure $AX, XB, BY,$ and YC. Then set up the ratios $\frac{AX}{XB}$ and $\frac{YC}{YB}$. Are they equal?

3. Draw a second triangle, $\triangle DEF$. Label the vertices.
4. Draw XY so that X is on DE and Y is on EF AND $XY \parallel DF$.

   Is $\triangle XEY \sim \triangle DEF$? Why or why not? Measure $DX, XE, EY,$ and YF . Then set up the ratios $\frac{DX}{XE}$ and $\frac{FY}{YE}$. Are they equal?



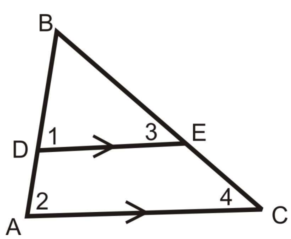
[Figure 5]

From this investigation, it is clear that if the line segments are parallel, then XY divides the sides proportionally.

Triangle Proportionality Theorem: If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally.

Triangle Proportionality Theorem Converse: If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Proof of the Triangle Proportionality Theorem



[Figure 6]

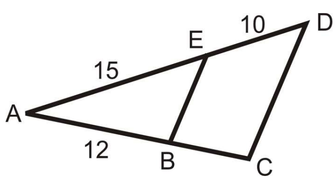
Given: $\triangle ABC$ with $DE \parallel AC$

Prove: $\frac{AD}{DB} = \frac{CE}{EB}$

Statement	***Reason***
1. $DE \mid\mid AC$	Given
2. $\angle 1 \cong \angle 2, \angle 3 \cong \angle 4$	Corresponding Angles Postulate
3. $\triangle ABC \sim \triangle DBE$	AA Similarity Postulate
4. $AD + DB = AB$ $EC + EB = BC$	Segment Addition Postulate
5. $\frac{AB}{BD} = \frac{BC}{BE}$	Corresponding sides in similar triangles are proportional
6. $\frac{AD+DB}{BD} = \frac{EC+EB}{BE}$	Substitution PoE
7. $\frac{AD}{BD} + \frac{DB}{DB} = \frac{EC}{BE} + \frac{BE}{BE}$	Separate the fractions
8. $\frac{AD}{BD} + 1 = \frac{EC}{BE} + 1$	Substitution PoE (something over itself always equals 1)
9. $\frac{AD}{BD} = \frac{EC}{BE}$	Subtraction PoE

We will not prove the converse, it is essentially this proof but in the reverse order. Using the corollaries from earlier in this chapter, $\frac{BD}{DA} = \frac{BE}{EC}$ is also a true proportion.

Example 2: In the diagram below, $EB \mid\mid BD$. Find BC.

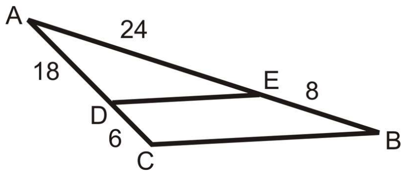


[Figure 7]

Solution: Use the Triangle Proportionality Theorem.

$$\frac{10}{15} = \frac{BC}{12} \longrightarrow 15(BC) = 120$$
$$BC = 8$$

Example 3: Is $DE \mid\mid CB$?



[Figure 8]

Solution: Use the Triangle Proportionality Converse. If the ratios are equal, then the lines are parallel.

$\frac{6}{18} = \frac{1}{3}$ and $\frac{8}{24} = \frac{1}{3}$

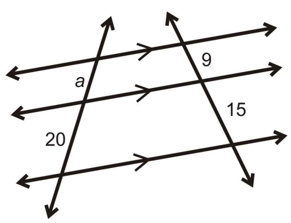
Because the ratios are equal, $DE \, || \, CB$.

Parallel Lines and Transversals

We can extend the Triangle Proportionality Theorem to multiple parallel lines.

Theorem 7-7: If three parallel lines are cut by two transversals, then they divide the transversals proportionally.

Example 4: Find a.



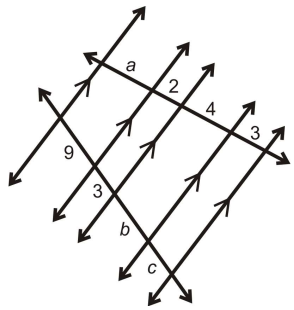
[Figure 9]

Solution: The three lines are marked parallel, so you can set up a proportion.

$$\begin{aligned} \frac{a}{20} &= \frac{9}{15} \\ 180 &= 15a \\ a &= 12 \end{aligned}$$

Theorem 7-7 can be expanded to ***any*** number of parallel lines with ***any*** number of transversals. When this happens all corresponding segments of the transversals are proportional.

Example 5: Find $a, b,$ and c.



[Figure 10]

Solution: Look at the corresponding segments. Only the segment marked "2" is opposite a number, all the other segments are opposite variables. That means we will be using this ratio, 2:3 in all of our proportions.

$$\begin{array}{ccc} \frac{a}{2}=\frac{9}{3} & \frac{2}{4}=\frac{3}{b} & \frac{2}{3}=\frac{3}{c} \\ 3a=18 & 2b=12 & 2c=9 \\ a=6 & b=6 & c=4.5 \end{array}$$

There are several ratios you can use to solve this example. To solve for b , you could have used the proportion $\frac{6}{4}=\frac{9}{b}$, which will still give you the same answer.

Proportions with Angle Bisectors

[Figure 11]

The last proportional relationship we will explore is how an angle bisector intersects the opposite side of a triangle. By definition, $\overrightarrow{AC}$ divides $\angle BAD$ equally, so $\angle BAC \cong \angle CAD$. The proportional relationship is $\frac{BC}{CD}=\frac{AB}{AD}$. The proof is in the review exercises.

Theorem 7-8: If a ray bisects an angle of a triangle, then it divides the opposite side into segments that are proportional to the lengths of the other two sides.

Example 6: Find x .

[Figure 12]

Solution: Because the ray is the angle bisector it splits the opposite side in the same ratio as the sides. So, the proportion is:

$$\frac{9}{x} = \frac{21}{14}$$
$$21x = 126$$
$$x = 6$$

Example 7: *Algebra Connection* Determine the value of x that would make the proportion true.

[Figure 13]

Solution: You can set up this proportion just like the previous example.

$$\frac{5}{3} = \frac{4x+1}{15}$$
$$75 = 3(4x+1)$$
$$75 = 12x+3$$
$$72 = 12x$$
$$6 = x$$

Know What? Revisited To find x and y, you need to set up a proportion using parallel the parallel lines.

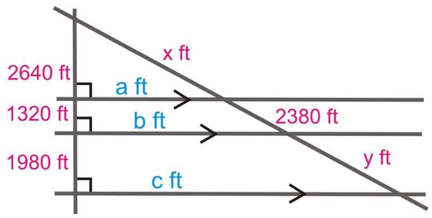
$$\frac{2640}{x} = \frac{1320}{2380} = \frac{1980}{y}$$

From this, $x = 4760\ ft$ and $y = 3570\ ft$.

To find $a, b,$ and c, use the Pythagorean Theorem.

$$2640^2 + a^2 = 4760^2$$
$$3960^2 + b^2 = 7140^2$$
$$5940^2 + c^2 = 10710^2$$

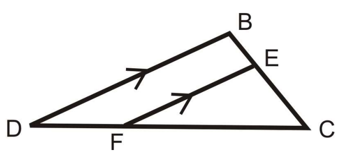
$$a = 3960.81,\ b = 5941.21,\ c = 8911.82$$



[Figure 14]

Review Questions

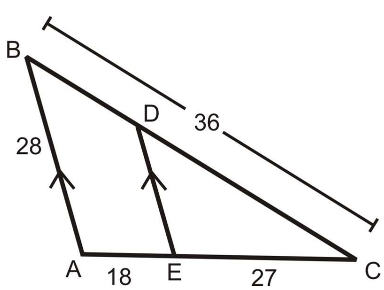
Use the diagram to answers questions 1-5. $DB \parallel FE$.



[Figure 15]

1. Name the similar triangles. Write the similarity statement.
2. $\frac{BE}{EC} = \frac{?}{FC}$
3. $\frac{EC}{CB} = \frac{CF}{?}$
4. $\frac{DB}{?} = \frac{BC}{EC}$
5. $\frac{FC+?}{FC} = \frac{?}{FE}$

Use the diagram to answer questions 6-10. $AB \parallel DE$.

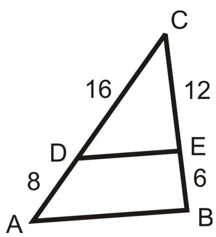


[Figure 16]

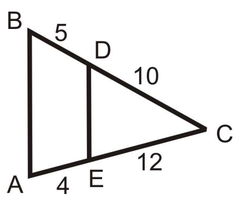
6. Find BD.
7. Find DC.
8. Find DE.
9. Find AC.

10. We know that $\frac{BD}{DC} = \frac{AE}{EC}$ and $\frac{BA}{DE} = \frac{BC}{DC}$. Why is $\frac{BA}{DE} \neq \frac{BD}{DC}$?

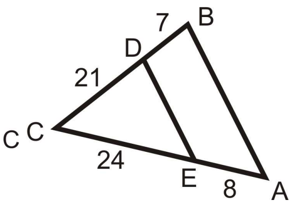
Use the given lengths to determine if $AB \parallel DE$.



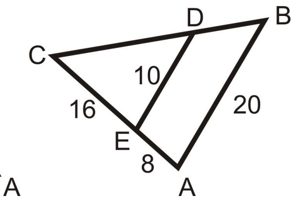
[Figure 17]



[Figure 18]

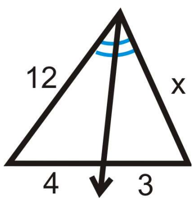


[Figure 19]

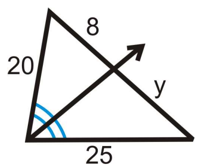


[Figure 20]

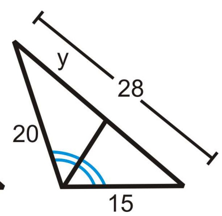
Algebra Connection Find the value of the missing variable(s).



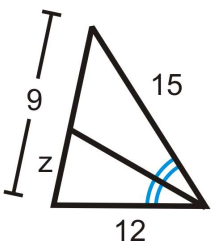
[Figure 21]



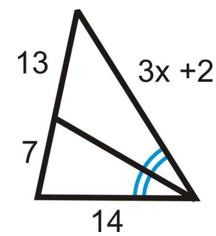
[Figure 22]



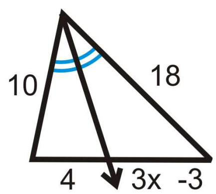
[Figure 23]



[Figure 24]

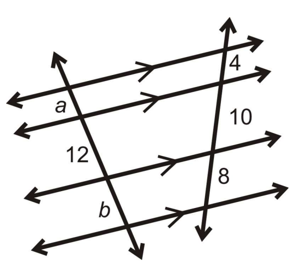


[Figure 25]

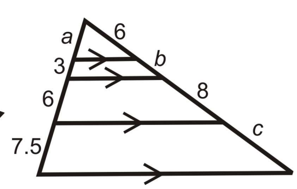


[Figure 26]

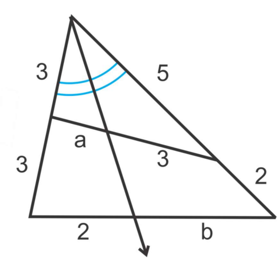
Find the value of each variable in the pictures below.



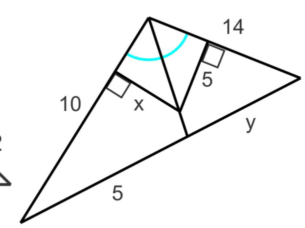
[Figure 27]



[Figure 28]

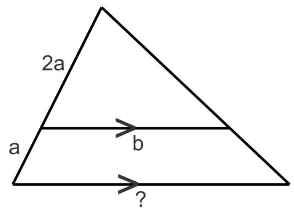


[Figure 29]

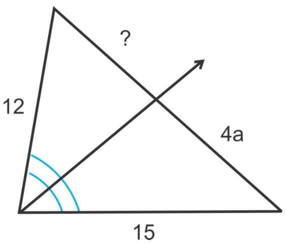


[Figure 30]

Find the unknown lengths.

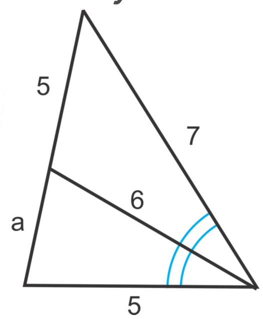


[Figure 31]



[Figure 32]

Error Analysis



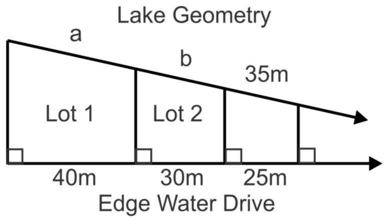
[Figure 33]

Casey attempts to solve for a in the diagram using the proportion

$$\frac{5}{a} = \frac{6}{5}$$

What did Casey do wrong? Write the correct proportion and solve for a.

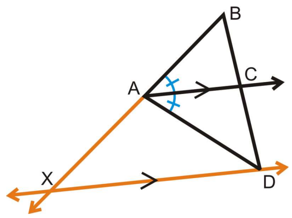
25. Michael has a triangular shaped garden with sides of length 3, 5 and 6 meters. He wishes to make a path along the perpendicular bisector of the angle between the sides of length 3 m and 5 m. Where will the path intersect the third side?



[Figure 34]

This is a map of lake front properties. Find a and b, the length of the edge of Lot 1 and Lot 2 that is adjacent to the lake.

Fill in the blanks of the proof of Theorem 7-8.



[Figure 35]

Given: $\triangle BAD$ with $\overrightarrow{AC}$ is the angle bisector of $\angle BAD$ Auxiliary lines $\overrightarrow{AX}$ and $\overleftrightarrow{XD}$, such that X, A, B are collinear and $\overrightarrow{AC} \parallel \overleftrightarrow{XD}$. Prove: $\frac{BC}{CD} = \frac{BA}{AD}$

Statement	***Reason***
1. $\overrightarrow{AC}$ is the angle bisector of $\angle BAD$ X, A, B are collinear and $\overrightarrow{AC} \parallel \overleftrightarrow{XD}$	
2. $\angle BAC \cong \angle CAD$	
3.	Corresponding Angles Postulate
4. $\angle CAD \cong \angle ADX$	
5. $\angle X \cong \angle ADX$	
6. $\triangle XAD$ is isosceles	
7.	Definition of an Isosceles Triangle
8.	Congruent segments are also equal
9.	Theorem 7-7
10.	

Review Queue Answers

1. $\triangle AXB \sim \triangle YXZ$ by AA Similarity Postulate
2. $\frac{16}{18} = \frac{32}{XZ}, XZ = 36$
3. $\frac{16}{18} = \frac{AB}{27}, AB = 24$
4. $AY = 18 - 16 = 2, BZ = 36 - 32 = 4$
5. $\frac{2}{16} = \frac{4}{32}$. Yes, this is a true proportion.