8.1 The Pythagorean Theorem

Last Modified: Dec 25, 2014

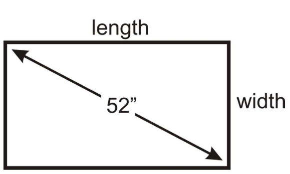
Learning Objectives

- Prove and use the Pythagorean Theorem.
- Identify common Pythagorean triples.
- Use the Pythagorean Theorem to find the area of isosceles triangles.
- Use the Pythagorean Theorem to derive the distance formula on a coordinate grid.

Review Queue

1. Draw a right scalene triangle.
2. Draw an isosceles right triangle.
3. Simplify the radical.
   a. $\sqrt{50}$
   b. $\sqrt{27}$
   c. $\sqrt{272}$
4. Perform the indicated operations on the following numbers. Simplify all radicals.
   a. $2\sqrt{10}+\sqrt{160}$
   b. $5\sqrt{6}\cdot 4\sqrt{18}$
   c. $\sqrt{8}\cdot 12\sqrt{2}$

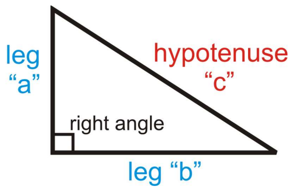
Know What? All televisions dimensions refer to the diagonal of the rectangular viewing area. Therefore, for a 52" TV, 52" is the length of the diagonal. High Definition Televisions (HDTVs) have sides in the ratio of 16:9. What is the length and width of a 52" HDTV? What is the length and width of an HDTV with a y'' long diagonal?



[Figure 1]

The Pythagorean Theorem

We have used the Pythagorean Theorem already in this text, but we have never proved it. Recall that the sides of a right triangle are called legs (the sides of the right angle) and the side opposite the right angle is the hypotenuse. For the Pythagorean Theorem, the legs are " a " and " b " and the hypotenuse is " c ".



[Figure 2]

Pythagorean Theorem: Given a right triangle with legs of lengths a and b and a hypotenuse of length c, then $a^2+b^2=c^2$.

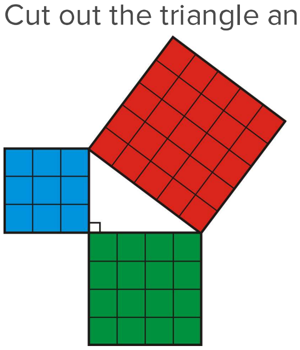
There are several proofs of the Pythagorean Theorem. We will provide one proof within the text and two others in the review exercises.

Investigation 8-1: Proof of the Pythagorean Theorem

Tools Needed: pencil, 2 pieces of graph paper, ruler, scissors, colored pencils (optional)

1. On the graph paper, draw a 3 in. square, a 4 in. square, a 5 in square and a right triangle with legs of 3 and 4 inches.

   Cut out the triangle and square and arrange them like the picture on the right.

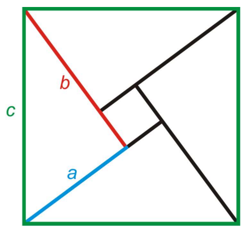


[Figure 3]

2. This theorem relies on area. Recall from a previous math class, that the area of a square is length times width. But, because the sides are the same you can rewrite this formula as $A_{square}=length\times width=side\times side=side^2$. So, the Pythagorean Theorem can be interpreted as $(square\ with\ side\ a)^2+(square\ with\ side\ b)^2=(square\ with\ side\ c)^2$. In this Investigation, the sides are 3, 4 and 5 inches. What is the area of each square?
3. Now, we know that $9+16=25$, or $3^2+4^2=5^2$. Cut the smaller squares to fit into the larger square, thus proving the areas are equal.

Another Proof of the Pythagorean Theorem

This proof is "more formal," meaning that we will use letters, $a, b,$ and c to represent the sides of the right triangle. In this particular proof, we will take four right triangles, with legs a and b and hypotenuse c and make the areas equal.



[Figure 4]

$$A_{green\ square} = c^2$$
$$A_{green\ square} = 4\left(\frac{1}{2}ab\right) + (b-a)^2$$
$$= 2ab + b^2 - 2ab + a^2$$
$$= b^2 + a^2$$

area of the four triangles

area of small, central square

FOIL $(b-a)^2$

[Figure 5]

For two animated proofs, go to http://www.mathsisfun.com/pythagoras.html and scroll down to "And You Can Prove the Theorem Yourself."

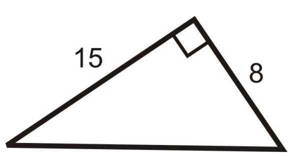
Using the Pythagorean Theorem

The Pythagorean Theorem can be used to find a missing side of any right triangle, to prove that three given lengths can form a right triangle, to find Pythagorean Triples, to derive the Distance Formula, and to find the area of an isosceles triangle. Here are several examples. ***Simplify all radicals***.

Example 1: Do 6, 7, and 8 make the sides of a right triangle?

Solution: Plug in the three numbers into the Pythagorean Theorem. ***The largest length will always be the hypotenuse***. $6^2 + 7^2 = 36 + 49 = 85 \neq 8^2$. Therefore, these lengths do not make up the sides of a right triangle.

Example 2: Find the length of the hypotenuse of the triangle below.



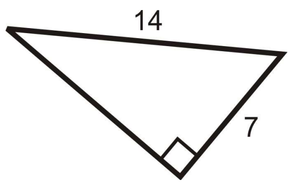
[Figure 6]

Solution: Let's use the Pythagorean Theorem. Set a and b equal to 8 and 15 and solve for c , the hypotenuse.

$$\begin{aligned} 8^2 + 15^2 &= c^2 \\ 64 + 225 &= c^2 \\ 289 &= c^2 \qquad \textit{Take the square root of both sides.} \\ 17 &= c \end{aligned}$$

When you take the square root of an equation, usually the answer is +17 or -17. Because we are looking for length, we only use the positive answer. ***Length is never negative***.

Example 3: Find the missing side of the right triangle below.

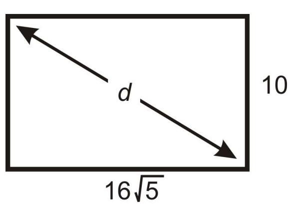


[Figure 7]

Solution: Here, we are given the hypotenuse and a leg. Let's solve for b.

$$\begin{aligned} 7^2 + b^2 &= 14^2 \\ 49 + b^2 &= 196 \\ b^2 &= 147 \\ b &= \sqrt{147} = \sqrt{7 \cdot 7 \cdot 3} = 7\sqrt{3} \end{aligned}$$

Example 4: What is the diagonal of a rectangle with sides 10 and $16\sqrt{5}$?



[Figure 8]

Solution: For any square and rectangle, you can use the Pythagorean Theorem to find the length of a diagonal. Plug in the sides to find d.

$$\begin{aligned} 10^2 + (16\sqrt{5})^2 &= d^2 \\ 100 + 1280 &= d^2 \\ 1380 &= d^2 \\ d &= \sqrt{1380} = 2\sqrt{345} \end{aligned}$$

Pythagorean Triples

In Example 2, the sides of the triangle were 8, 15, and 17. This combination of numbers is referred to as a ***Pythagorean triple***.

Pythagorean Triple: A set of three whole numbers that makes the Pythagorean Theorem true.

The most frequently used Pythagorean triple is 3, 4, 5, as in Investigation 8-1. Any multiple of a Pythagorean triple is also considered a triple because it would still be three whole numbers. Therefore, 6, 8, 10 and 9, 12, 15 are also sides of a right triangle. Other Pythagorean triples are:

$3, 4, 5 \qquad 5, 12, 13 \qquad 7, 24, 25 \qquad 8, 15, 17$

There are infinitely many Pythagorean triples. To see if a set of numbers makes a triple, plug them into the Pythagorean Theorem.

Example 5: Is 20, 21, 29 a Pythagorean triple?

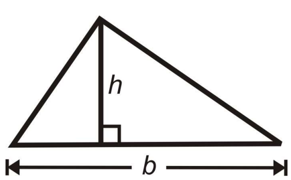
Solution: If $20^2 + 21^2$ is equal to 29^2, then the set is a triple.

$$\begin{aligned} 20^2 + 21^2 &= 400 + 441 = 841 \\ 29^2 &= 841 \end{aligned}$$

Therefore, 20, 21, and 29 is a Pythagorean triple.

Area of an Isosceles Triangle

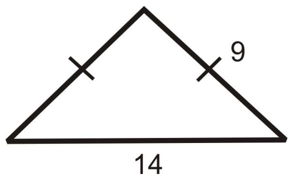
There are many different applications of the Pythagorean Theorem. One way to use The Pythagorean Theorem is to identify the heights in isosceles triangles so you can calculate the area. The area of a triangle is $\frac{1}{2}bh$, where b is the base and h is the height (or altitude).



[Figure 9]

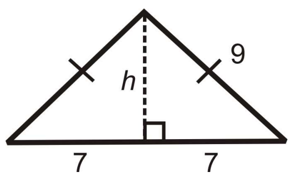
If you are given the base and the sides of an isosceles triangle, you can use the Pythagorean Theorem to calculate the height.

Example 6: What is the area of the isosceles triangle?



[Figure 10]

Solution: First, draw the altitude from the vertex between the congruent sides, which will bisect the base (Isosceles Triangle Theorem). Then, find the length of the altitude using the Pythagorean Theorem.



[Figure 11]

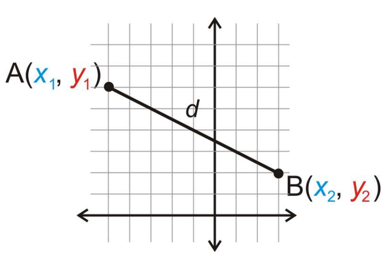
$$\begin{aligned} 7^2 + h^2 &= 9^2 \\ 49 + h^2 &= 81 \\ h^2 &= 32 \\ h &= \sqrt{32} = 4\sqrt{2} \end{aligned}$$

Now, use h and b in the formula for the area of a triangle.

$$A = \frac{1}{2}bh = \frac{1}{2}(14)\left(4\sqrt{2}\right) = 28\sqrt{2}\,units^2$$

The Distance Formula

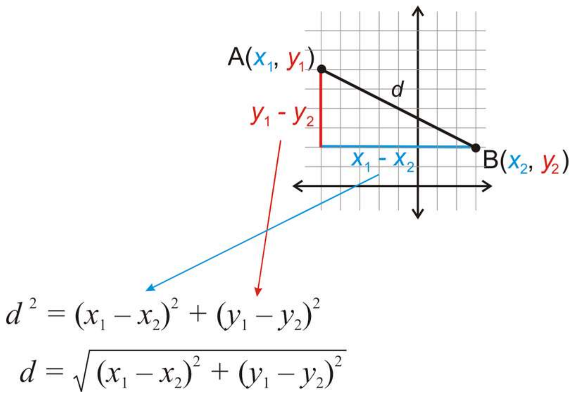
Another application of the Pythagorean Theorem is the Distance Formula. We have already been using the Distance Formula in this text, but we can prove it here.



[Figure 12]

First, draw the vertical and horizontal lengths to make a right triangle. Then, use the differences to find these distances.

Now that we have a right triangle, we can use the Pythagorean Theorem to find d.



[Figure 13]

Distance Formula: The distance $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 7: Find the distance between (1, 5) and (5, 2).

Solution: Make $A(1, 5)$ and $B(5, 2)$. Plug into the distance formula.

$$\begin{aligned} d &= \sqrt{(1-5)^2 + (5-2)^2} \\ &= \sqrt{(-4)^2 + (3)^2} \\ &= \sqrt{16+9} = \sqrt{25} = 5 \end{aligned}$$

You might recall that the distance formula was presented as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, with the first and second points switched. It does not matter which point is first as long as x and y are both first in each parenthesis. In Example 7, we could have switched A and B and would still get the same answer.

$$\begin{aligned} d &= \sqrt{(5-1)^2 + (2-5)^2} \\ &= \sqrt{(4)^2 + (-3)^2} \\ &= \sqrt{16+9} = \sqrt{25} = 5 \end{aligned}$$

Also, just like the lengths of the sides of a triangle, distances are always positive.

Know What? Revisited To find the length and width of a 52" HDTV, plug in the ratios and 52 into the Pythagorean Theorem. We know that the sides are going to be a multiple of 16 and 9, which we will call n.

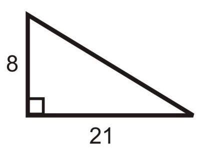
$$\begin{aligned}(16n)^2+(9n)^2 &= 52^2\\ 256n^2+81n^2 &= 2704\\ 337n^2 &= 2704\\ n^2 &= 8.024\\ n &= 2.83\end{aligned}$$

[Figure 14]

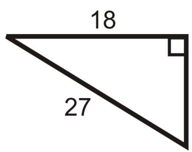
Therefore, the dimensions of the TV are $16(2.83'')$ by $9(2.833'')$, or $45.3''$ by $25.5''$. If the diagonal is y'' long, it would be $n\sqrt{337}''$ long. The extended ratio is $9:16:\sqrt{337}$.

Review Questions

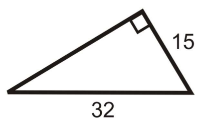
Find the length of the missing side. Simplify all radicals.



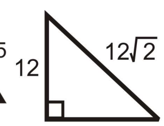
[Figure 15]



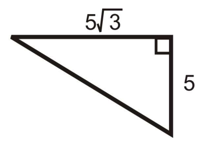
[Figure 16]



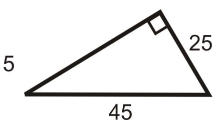
[Figure 17]



[Figure 18]



[Figure 19]



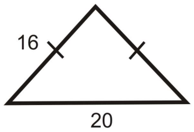
[Figure 20]

1. If the legs of a right triangle are 10 and 24, then the hypotenuse is _______________.
2. If the sides of a rectangle are 12 and 15, then the diagonal is _______________.
3. If the legs of a right triangle are x and y, then the hypotenuse is _______________.
4. If the sides of a square are 9, then the diagonal is _______________.

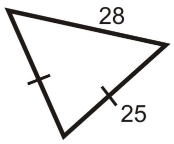
Determine if the following sets of numbers are Pythagorean Triples.

11. 12, 35, 37
12. 9, 17, 18
13. 10, 15, 21
14. 11, 60, 61
15. 15, 20, 25
16. 18, 73, 75

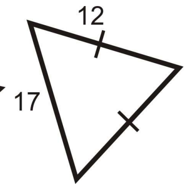
Find the area of each triangle below. Simplify all radicals.



[Figure 21]



[Figure 22]

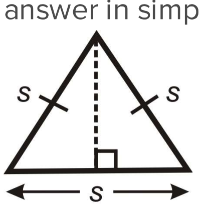


[Figure 23]

Find the length between each pair of points.

20. (-1, 6) and (7, 2)
21. (10, -3) and (-12, -6)
22. (1, 3) and (-8, 16)
23. What are the length and width of a $42''$ HDTV? Round your answer to the nearest tenth.
24. Standard definition TVs have a length and width ratio of 4:3. What are the length and width of a $42''$ Standard definition TV? Round your answer to the nearest tenth.

Challenge An equilateral triangle is an isosceles triangle. If all the sides of an equilateral triangle are s, find the area, using the technique learned in this section. Leave your answer in simplest radical form.

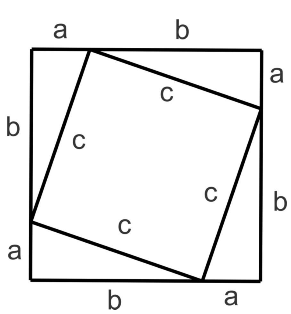


[Figure 24]

25. Find the area of an equilateral triangle with sides of length 8.

Pythagorean Theorem Proofs

The first proof below is similar to the one done earlier in this lesson. Use the picture below to answer the following questions.

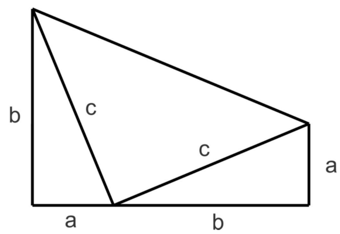


[Figure 25]

27. Find the area of the square with sides $(a+b)$.

28. Find the sum of the areas of the square with sides c and the right triangles with legs a and b.
29. The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

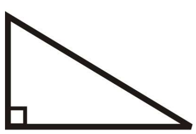
Major General James A. Garfield (and former President of the U.S) is credited with deriving this next proof of the Pythagorean Theorem using a trapezoid.



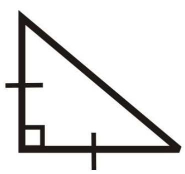
[Figure 26]

30. Find the area of the trapezoid using the trapezoid area formula: $A=\frac{1}{2}(b_1+b_2)h$
31. Find the sum of the areas of the three right triangles in the diagram.
32. The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

Review Queue Answers



[Figure 27]



[Figure 28]

1. Answers:
   a. $\sqrt{50}=\sqrt{25\cdot 2}=5\sqrt{2}$
   b. $\sqrt{27}=\sqrt{9\cdot 3}=3\sqrt{3}$
   c. $\sqrt{272}=\sqrt{16\cdot 17}=4\sqrt{17}$
2. Answers:
   a. $2\sqrt{10}+\sqrt{160}=2\sqrt{10}+4\sqrt{10}=4\sqrt{10}$
   b. $5\sqrt{6}\cdot 4\sqrt{18}=5\sqrt{6}\cdot 12\sqrt{2}=60\sqrt{12}=120\sqrt{3}$
   c. $\sqrt{8}\cdot 12\sqrt{2}=12\sqrt{16}=12\cdot 4=48$

8.2 Converse of the Pythagorean Theorem

FlexBooks® 2.0 > American HS Geometry > Converse of the Pythagorean Theorem

Last Modified: Dec 25, 2014

Learning Objectives

- Understand the converse of the Pythagorean Theorem.
- Identify acute and obtuse triangles from side measures.

Review Queue

1. Determine if the following sets of numbers are Pythagorean triples.

   a. 14, 48, 50

   b. 9, 40, 41

   c. 12, 43, 44

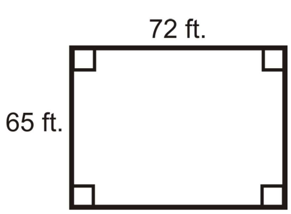
2. Do the following lengths make a right triangle? How do you know?

   a. $\sqrt{5}, 3, \sqrt{14}$

   b. $6, 2\sqrt{3}, 8$

   c. $3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

Know What? A friend of yours is designing a building and wants it to be rectangular. One wall 65 ft. long and the other is 72 ft. long. How can he ensure the walls are going to be perpendicular?



[Figure 1]

Converse of the Pythagorean Theorem

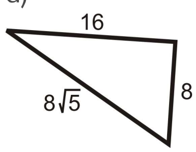
In the last lesson, you learned about the Pythagorean Theorem and how it can be used. The converse of the Pythagorean Theorem is also true. We touched on this in the last section with Example 1.

Pythagorean Theorem Converse: If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any of the triangle's angle measurements.

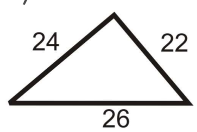
Example 1: Determine if the triangles below are right triangles.

a)



[Figure 2]

b)



[Figure 3]

Solution: Check to see if the three lengths satisfy the Pythagorean Theorem. Let the longest sides represent c, in the equation.

a)
$$\begin{aligned} a^2+b^2 &= c^2 \\ 8^2+16^2 &\stackrel{?}{=} (8\sqrt{5})^2 \\ 64+256 &\stackrel{?}{=} 64\cdot 5 \\ 320 &= 320 \end{aligned}$$

The triangle is a right triangle.

b)
$$\begin{aligned} a^2+b^2 &= c^2 \\ 22^2+24^2 &\stackrel{?}{=} 26^2 \\ 484+576 &= 676 \\ 1060 &\neq 676 \end{aligned}$$

The triangle is not a right triangle.

Identifying Acute and Obtuse Triangles

We can extend the converse of the Pythagorean Theorem to determine if a triangle has an obtuse angle or is acute. We know that if the sum of the squares of the two smaller sides equals the square of the larger side, then the triangle is right. We can also interpret the outcome if the sum of the squares of the smaller sides does not equal the square of the third.

Theorem 8-3: If the sum of the squares of the two shorter sides in a right triangle is ***greater*** than the square of the longest side, then the triangle is ***acute***.

Theorem 8-4: If the sum of the squares of the two shorter sides in a right triangle is ***less*** than the square of the longest side, then the triangle is ***obtuse***.

In other words: The sides of a triangle are a, b , and c and $c > b$ and $c > a$.

If $a^2 + b^2 > c^2$, then the triangle is acute.

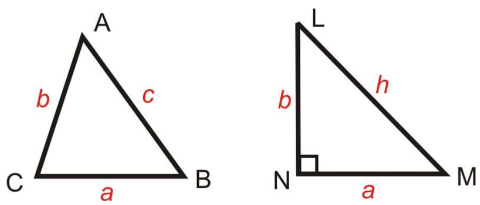
If $a^2 + b^2 = c^2$, then the triangle is right.

If $a^2 + b^2 < c^2$, then the triangle is obtuse.

Proof of Theorem 8-3

Given: In $\triangle ABC, a^2 + b^2 > c^2$, where c is the longest side.

In $\triangle LMN, \angle N$ is a right angle.



[Figure 4]

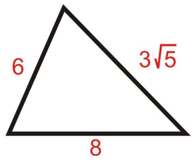
Prove: $\triangle ABC$ is an acute triangle. (all angles are less than 90°)

Statement	***Reason***
1. In $\triangle ABC, a^2 + b^2 > c^2$, and c is the longest side. In $\triangle LMN, \angle N$ is a right angle.	Given
2. $a^2 + b^2 = h^2$	Pythagorean Theorem
3. $c^2 < h^2$	Transitive PoE
4. $c < h$	Take the square root of both sides
5. $\angle C$ is the largest angle in $\triangle ABC$.	The largest angle is opposite the longest side.
6. $m\angle N = 90^\circ$	Definition of a right angle
7. $m\angle C < m\angle N$	SSS Inequality Theorem
8. $m\angle C < 90^\circ$	Transitive PoE
9. $\angle C$ is an acute angle.	Definition of an acute angle
10. $\triangle ABC$ is an acute triangle.	If the largest angle is less than 90° , then all the angles are less than 90° .

The proof of Theorem 8-4 is very similar and is in the review questions.

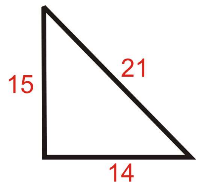
Example 2: Determine if the following triangles are acute, right or obtuse.

a)



[Figure 5]

b)



[Figure 6]

Solution: Set the shorter sides in each triangle equal to a and b and the longest side equal to c.

a)
$$\begin{aligned} 6^2 + (3\sqrt{5})^2 &\ ?\ 8^2 \\ 36 + 45 &\ ?\ 64 \\ 81 &> 64 \end{aligned}$$

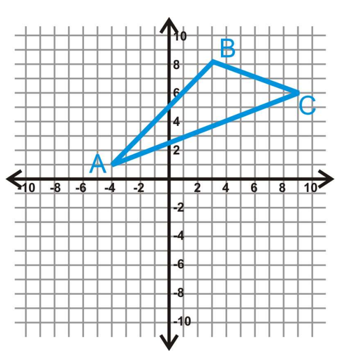
The triangle is acute.

b)
$$\begin{aligned} 15^2 + 14^2 &\ ?\ 21^2 \\ 225 + 196 &\ ?\ 441 \\ 421 &< 441 \end{aligned}$$

The triangle is obtuse.

Example 3: Graph $A(-4,1), B(3,8)$, and $C(9,6)$. Determine if $\triangle ABC$ is acute, obtuse, or right.

Solution: This looks like an obtuse triangle, but we need proof to draw the correct conclusion. Use the distance formula to find the length of each side.



[Figure 7]

$$
\begin{aligned}
AB &= \sqrt{(-4-3)^2+(1-8)^2} = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2} \\
BC &= \sqrt{(3-9)^2+(8-6)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10} \\
AC &= \sqrt{(-4-9)^2+(1-6)^2} = \sqrt{169+25} = \sqrt{194}
\end{aligned}
$$

Now, let's plug these lengths into the Pythagorean Theorem.

$$
\begin{aligned}
(\sqrt{98})^2 + \left(\sqrt{40}\right)^2 \ ?\ \left(\sqrt{194}\right)^2 \\
98+40 \ ?\ 194 \\
138 < 194
\end{aligned}
$$

$\triangle ABC$ is an obtuse triangle.

Know What? Revisited To make the walls perpendicular, find the length of the diagonal.

$$
\begin{aligned}
65^2+72^2 &= c^2 \\
4225+5184 &= c^2 \\
9409 &= c^2 \\
97 &= c
\end{aligned}
$$

In order to make the building rectangular, both diagonals must be 97 feet.

Review Questions

1. The two *shorter* sides of a triangle are 9 and 12.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?
2. The two *longer* sides of a triangle are 24 and 25.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?
3. The lengths of the sides of a triangle are $8x, 15x,$ and $17x$. Determine if the triangle is acute, right, or obtuse.

Determine if the following lengths make a right triangle.

4. 15, 20, 25
5. 20, 25, 30
6. $8\sqrt{3}, 6, 2\sqrt{39}$

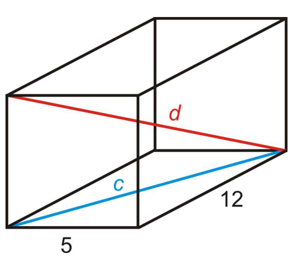
Determine if the following triangles are acute, right or obtuse.

7. 7, 8, 9
8. 14, 48, 50
9. 5, 12, 15
10. 13, 84, 85
11. 20, 20, 24
12. 35, 40, 51
13. 39, 80, 89
14. 20, 21, 38
15. 48, 55, 76

Graph each set of points and determine if $\triangle ABC$ is acute, right, or obtuse.

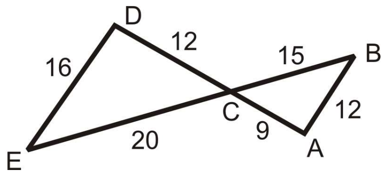
16. $A(3,-5), B(-5,-8), C(-2,7)$
17. $A(5,3), B(2,-7), C(-1,5)$
18. ***Writing*** Explain the two different ways you can show that a triangle in the coordinate plane is a right triangle.

The figure to the right is a rectangular prism. All sides (or faces) are either squares (the front and back) or rectangles (the four around the middle). All sides are perpendicular.



[Figure 8]

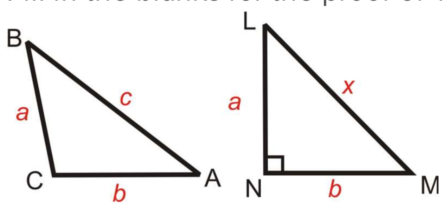
19. Find c.
20. Find d.



[Figure 9]

21. ***Writing*** Explain why $m\angle A = 90^\circ$.

Fill in the blanks for the proof of Theorem 8-4.



[Figure 10]

Given: In $\triangle ABC, a^2 + b^2 < c^2$, where c is the longest side. In $\triangle LMN, \angle N$ is a right angle. Prove: $\triangle ABC$ is an obtuse triangle. (one angle is greater than 90°)

Statement	***Reason***
1. In $\triangle ABC, a^2 + b^2 < c^2$, and c is the longest side. In $\triangle LMN, \angle N$ is a right angle.	
2. $a^2 + b^2 = h^2$	
3. $c^2 > h^2$	
4.	
5. $\angle C$ is the largest angle in $\triangle ABC$.	
6. $m\angle N = 90^\circ$	
7. $m\angle C > m\angle N$	
8.	Transitive PoE
9. $\angle C$ is an obtuse angle.	
10. $\triangle ABC$ is an obtuse triangle.	

Given AB , with $A(3,3)$ and $B(2,-3)$ determine whether the given point C in problems 23-25 makes an acute, right or obtuse triangle.

23. $C(3,-3)$
24. $C(4,-1)$
25. $C(5,-2)$

Given AB , with $A(-2,5)$ and $B(1,-3)$ find at least two possible points, C , such that $\triangle ABC$ is

26. right, with right $\angle C$.
27. acute, with acute $\angle C$.

28. obtuse, with obtuse $\angle C$.
29. ***Construction***
    a. Draw AB , such that $AB = 3\,in$.
    b. Draw $\overrightarrow{AD}$ such that $\angle BAD < 90^\circ$.
    c. Construct a line through B which is perpendicular to $\overrightarrow{AD}$, label the intersection C .
    d. $\triangle ABC$ is a right triangle with right $\angle C$.
30. Is the triangle you made unique? In other words, could you have multiple different outcomes with the same AB ? Why or why not? You may wish to experiment to find out.
31. Why do the instructions specifically require that $\angle BAD < 90^\circ$?
32. Describe how this construction could be changed so that $\angle B$ is the right angle in the triangle.

Review Queue Answers

1. Answers:
    a. Yes
    b. Yes
    c. No
2. Answers:
    a. Yes
    b. No
    c. Yes