7.1 Ratios and Proportions

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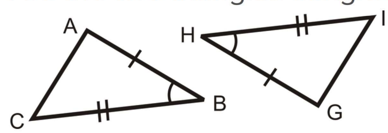
Learning Objectives

- Write, simplify, and solve ratios and proportions.
- Use ratios and proportions in problem solving.

Review Queue

Are the two triangles congruent? If so, how do you know?



[Figure 1]

1. If $AC = 5$, what is GI? What is the reason?
2. How many inches are in a foot? In a yard? In 3 yards?
3. How many cups are in a pint? In a quart? In a gallon? In 7 quarts?

Know What? You want to make a scale drawing of your room and furniture for a little redecorating. Your room measures 12 feet by 12 feet. Also in your room is a twin bed (36 in by 75 in), a desk (4 feet by 2 feet), and a chair (3 feet by 3 feet). You decide to scale down your room to 8 in by 8 in, so the drawing fits on a piece of paper. What size should the bed, desk and chair be? Draw an appropriate layout for the furniture within the room. *Do not round your answers.*

Using Ratios

Ratio: A way to compare two numbers. Ratios can be written: $\frac{a}{b}, a : b,$ and a to b.

Example 1: The total bagel sales at a bagel shop for Monday is in the table below. What is the ratio of cinnamon raisin bagels to plain bagels?

Type of Bagel	*Number Sold*
Plain	80
Cinnamon Raisin	30
Sesame	25
Jalapeno Cheddar	20
Everything	45
Honey Wheat	50

Solution: The ratio is $\frac{30}{80}$, 30:80, or 30 to 80. Depending on the problem, ratios are usually written in simplest form, which means to reduce the ratio. The answer is then $\frac{3}{8}$, 3:8, or 3 to 8.

Example 2: What is the ratio, in simplest form, of Honey Wheat bagels to total bagels sold?

Solution: Remember that order matters. Because the Honey Wheat is listed first, that is the number that comes first in the ratio (on in the numerator of the fraction). Find the total number of bagels sold.

$$80+30+25+20+45+50=250$$

The ratio is then $\frac{50}{250}=\frac{1}{5}$, 1:5, or 1 to 5.

We call the ratio 50:250 and 1:5 ***equivalent*** because one reduces to the other.

In some problems you may need to write a ratio of more than two numbers. For example, the ratio of the number of cinnamon raisin bagels to sesame bagels to jalapeno cheddar bagels is 30:25:20 or 6:5:4.

Measurements are used a lot with ratios and proportions. For example, how many feet are in 2 miles? How many inches are in 4 feet? You will need to know these basic measurements.

Example 3: Simplify the following ratios.

a) $\frac{7\,ft}{14\,in}$

b) $9m : 900cm$

c) $\frac{4\,gal}{16\,gal}$

Solution: Change these so that they are in the same units.

a) $\frac{7\ \cancel{ft}}{14\ \cancel{in}} \cdot \frac{12\ \cancel{in}}{1\ \cancel{ft}} = \frac{84}{14} = \frac{6}{1}$

Notice that the inches cancel each other out. ***All ratios should not have units once simplified.***

b) It is easier to simplify ratios when they are written as a fraction.
$\frac{9\ m}{900\ cm} \cdot \frac{100\ cm}{1\ m} = \frac{900}{900} = \frac{1}{1}$

c) $\frac{4\ gal}{16\ gal} = \frac{1}{4}$

Example 4: A talent show features dancers and singers. The ratio of dancers to singers is 3:2. There are 30 performers total, how many singers are there?

Solution: 3:2 is a reduced ratio, so there is a whole number, n, that we can multiply both by to find the total number in each group.

$$\begin{aligned} \text{dancers} = 3n, \text{singers} = 2n \ \longrightarrow \ 3n + 2n &= 30 \\ 5n &= 30 \\ n &= 6 \end{aligned}$$

Therefore, there are $3 \cdot 6 = 18$ dancers and $2 \cdot 6 = 12$ singers. To double-check, $18 + 12 = 30$ total performers.

Proportions

Proportion: When two ratios are set equal to each other.

Example 4: Solve the proportions.

a) $\frac{4}{5} = \frac{x}{30}$

b) $\frac{y+1}{8} = \frac{5}{20}$

c) $\frac{6}{5} = \frac{2x+5}{x-2}$

Solution: To solve a proportion, you need to ***cross-multiply***.

a)

$$\frac{4}{5} = \frac{x}{30}$$
$$4 \cdot 30 = 5 \cdot x$$
$$120 = 5x$$
$$24 = x$$

[Figure 2]

b)

$$\frac{y+1}{8} = \frac{5}{20}$$
$$(y+1) \cdot 20 = 5 \cdot 8$$
$$20y + 20 = 40$$
$$20y = 20$$
$$y = 1$$

[Figure 3]

c)

$$\frac{6}{5} = \frac{2x+4}{x-2}$$
$$6 \cdot (x-2) = 5 \cdot (2x+4)$$
$$6x-12 = 10x+20$$
$$-32 = 4x$$
$$-8 = x$$

[Figure 4]

In proportions, the blue numbers are called the ***means*** and the orange numbers are called the ***extremes***. For the proportion to be true, the product of the means must equal the product of the extremes. This can be generalized in the Cross-Multiplication Theorem.

Cross-Multiplication Theorem: Let $a, b, c,$ and d be real numbers, with $b \neq 0$ and $d \neq 0$. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

The proof of the Cross-Multiplication Theorem is an algebraic proof. Recall that multiplying by $\frac{2}{2}, \frac{b}{b},$ or $\frac{d}{d} = 1$ because it is the same number divided by itself $(b \div b = 1)$.

Proof of the Cross-Multiplication Theorem

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} && \text{Multiply the left side by } \frac{d}{d} \text{ and the right side by } \frac{b}{b}. \\ \frac{a}{b} \cdot \frac{d}{d} &= \frac{c}{d} \cdot \frac{b}{b} \\ \frac{ad}{bd} &= \frac{bc}{bd} && \text{The denominators are the same, so the numerators are equal.} \\ ad &= bc \end{aligned}$$

Think of the Cross-Multiplication Theorem as a shortcut. Without this theorem, you would have to go through all of these steps every time to solve a proportion.

Example 5: Your parents have an architect's drawing of their home. On the paper, the house's dimensions are 36 in by 30 in. If the shorter length of your parents' house is actually 50 feet, what is the longer length?

Solution: Set up a proportion. If the shorter length is 50 feet, then it will line up with 30 in. It does not matter which numbers you put in the numerators of the fractions, as long as they line up correctly.

$$\frac{30}{36} = \frac{50}{x} \longrightarrow 1800 = 30x$$
$$60 = x$$

So, the dimension of your parents' house is 50 ft by 60 ft.

Properties of Proportions

The Cross-Multiplication Theorem has several sub-theorems that follow from its proof. The formal term is ***corollary.***

Corollary: A theorem that follows quickly, easily, and directly from another theorem.

Below are three corollaries that are immediate results of the Cross Multiplication Theorem and the fundamental laws of algebra.

Corollary 7-1: If $a, b, c,$ and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Corollary 7-2: If $a, b, c,$ and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{d}{b} = \frac{c}{a}$.

Corollary 7-3: If $a, b, c,$ and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

In other words, a true proportion is also true if you switch the means, switch the extremes, or flip it upside down. Notice that you will still end up with $ad = bc$ after cross-multiplying for all three of these corollaries.

Example 6: Suppose we have the proportion $\frac{2}{5} = \frac{14}{35}$. Write down the other three true proportions that follow from this one.

Solution: First of all, we know this is a true proportion because you would multiply $\frac{2}{5}$ by $\frac{7}{7}$ to get $\frac{14}{35}$. Using the three corollaries, we would get:

1. $\frac{2}{14} = \frac{5}{35}$
2. $\frac{35}{5} = \frac{14}{2}$

3. $\frac{5}{2} = \frac{35}{14}$

If you cross-multiply all four of these proportions, you would get $70 = 70$ for each one.

Corollary 7-4: If $a, b, c,$ and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Corollary 7-5: If $a, b, c,$ and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Example 7: In the picture, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$.

Find the measures of AC and XY.

[Figure 5]

Solution: This is an example of an *extended* proportion. Substituting in the numbers for the sides we know, we have $\frac{4}{XY} = \frac{3}{9} = \frac{AC}{15}$. Separate this into two different proportions and solve for XY and AC.

$$\begin{aligned} \frac{4}{XY} &= \frac{3}{9} & \frac{3}{9} &= \frac{AC}{15} \\ 36 &= 3(XY) & 9(AC) &= 45 \\ XY &= 12 & AC &= 5 \end{aligned}$$

Example 8: In the picture, $\frac{ED}{AD} = \frac{BC}{AC}$. Find y.

[Figure 6]

Solution: Substituting in the numbers for the sides we know, we have

$$\frac{6}{y} = \frac{8}{12+8} . \longrightarrow 8y = 6(20)$$
$$y = 15$$

Example 9: If $\frac{AB}{BE} = \frac{AC}{CD}$ in the picture above, find BE.

Solution:

$$\frac{12}{BE} = \frac{20}{25} \longrightarrow 20(BE) = 12(25)$$
$$BE = 15$$

Know What? Revisited Everything needs to be scaled down by a factor of $\frac{1}{18}\,(144\,in \div 8\,in)$. Change everything into inches and then multiply by the scale factor.

Bed: 36 in by 75 in $\longrightarrow$ 2 in by 4.167 in

Desk: 48 in by 24 in $\longrightarrow$ 2.67 in by 1.33 in

Chair: 36 in by 36 in $\longrightarrow$ 2 in by 2 in

There are several layout options for these three pieces of furniture. Draw an 8 in by 8 in square and then the appropriate rectangles for the furniture. Then, cut out the rectangles and place inside the square.

Review Questions

1. The votes for president in a club election were: $\text{Smith}: 24 \qquad \text{Munoz}: 32 \qquad \text{Park}: 20$ Find the following ratios and write in simplest form.

   a. Votes for Munoz to Smith

   b. Votes for Park to Munoz

   c. Votes for Smith to total votes

   d. Votes for Smith to Munoz to Park

Use the picture to write the following ratios for questions 2-6.

[Figure 7]

$AEFD$ is a square

$ABCD$ is a rectangle

2. $AE : EF$
3. $EB : AB$
4. $DF : FC$
5. $EF : BC$
6. Perimeter $ABCD$: Perimeter $AEFD$: Perimeter $EBCF$

7. The measures of the angles of a triangle are have the ratio 3:3:4. What are the measures of the angles?
8. The lengths of the sides in a triangle are in a 3:4:5 ratio. The perimeter of the triangle is 36. What are the lengths of the sides?
9. The length and width of a rectangle are in a 3:5 ratio. The perimeter of the rectangle is 64. What are the length and width?
10. The length and width of a rectangle are in a 4:7 ratio. The perimeter of the rectangle is 352. What are the length and width?
11. The ratio of the short side to the long side in a parallelogram is 5:9. The perimeter of the parallelogram is 112. What are the lengths of the sides?
12. The length and width of a rectangle are in a 3:11 ratio. The area of the rectangle is 528. What are the length and width of the rectangle?
13. ***Writing*** Explain why $\frac{a+b}{b}=\frac{c+d}{d}$ is a valid proportion. HINT: Cross-multiply and see if it equals $ad=bc$.
14. ***Writing*** Explain why $\frac{a-b}{b}=\frac{c-d}{d}$ is a valid proportion. HINT: Cross-multiply and see if it equals $ad=bc$.

Solve each proportion.

15. $\frac{x}{10}=\frac{42}{35}$
16. $\frac{x}{x-2}=\frac{5}{7}$
17. $\frac{6}{9}=\frac{y}{24}$
18. $\frac{x}{9}=\frac{16}{x}$
19. $\frac{y-3}{8}=\frac{y+6}{5}$
20. $\frac{20}{z+5}=\frac{16}{7}$
21. Shawna drove 245 miles and used 8.2 gallons of gas. At the same rate, if she drove 416 miles, how many gallons of gas will she need? Round to the nearest tenth.
22. The president, vice-president, and financial officer of a company divide the profits is a 4:3:2 ratio. If the company made $1,800,000 last year, how much did each person receive?
23. Many recipes describe ratios between ingredients. For example, one recipe for paper mache paste suggests 3 parts flour to 5 parts water. If we have one cup of flour, how

much water should we add to make the paste?

24. A recipe for krispy rice treats calls for 6 cups of rice cereal and 40 large marshmallows. You want to make a larger batch of goodies and have 9 cups of rice cereal. How many large marshmallows do you need? However, you only have the miniature marshmallows at your house. You find a list of substitution quantities on the internet that suggests 10 large marshmallows are equivalent to 1 cup miniatures. How many cups of miniatures do you need?

Given the true proportion, $\frac{10}{6}=\frac{15}{d}=\frac{x}{y}$ and $d, x,$ and y are nonzero, determine if the following proportions are also true.

25. $\frac{10}{y}=\frac{x}{6}$
26. $\frac{15}{10}=\frac{d}{6}$
27. $\frac{6+10}{10}=\frac{y+x}{x}$
28. $\frac{15}{x}=\frac{y}{d}$

For questions 29-32, $\frac{AE}{ED}=\frac{BC}{CD}$ and $\frac{ED}{AD}=\frac{CD}{DB}=\frac{EC}{AB}$.

[Figure 8]

29. Find DB.
30. Find EC.
31. Find CB.
32. Find AD.

Review Queue Answers

1. Yes, they are congruent by SAS.
2. $GI=5$ by CPCTC
3. 12 in = 1 ft, 36 in = 3 ft, 108 in = 3 yrds
4. 2c = 1 pt, 4c = 1 qt, 16 c = 4 qt = 1 gal, 28c = 7 qt