7.6 Similarity Transformations

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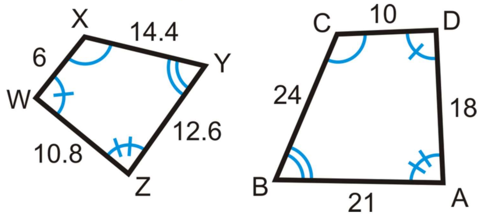
Learning Objectives

- Draw a dilation of a given figure.
- Plot an image when given the center of dilation and scale factor.
- Determine if one figure is the dilation of another.

Review Queue

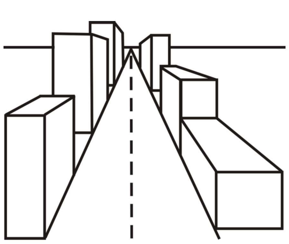
Are the two quadrilaterals similar? How do you know?



[Figure 1]

1. What is the scale factor from $XYZW$ to $CDAB$? *Leave as a fraction.*
2. Quadrilateral $EFGH$ has vertices $E(-4,-2), F(2,8), G(6,2)$ and $H(0,-4)$. Quadrilateral $LMNO$ has vertices $L(-2,-1), M(1,4), N(3,1),$ and $O(0,-2)$. Determine if the two quadrilaterals are similar. Explain your reasoning.

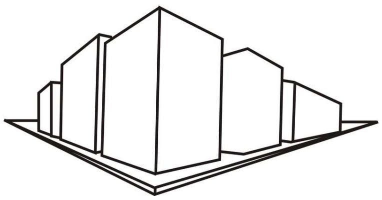
Know What? One practical application of dilations is perspective drawings. These drawings use a *vanishing point* (the point where the road meets the horizon) to trick the eye into thinking the picture is three-dimensional. The picture to the right is a one-point perspective and is typically used to draw streets, train tracks, rivers or anything else that is linear.



[Figure 2]

There are also two-point perspective drawings, which are very often used to draw a street corner or a scale drawing of a building.

Both of these drawing are simple representations of one and two perspective drawings. Your task for this **Know What?** is to draw your own perspective drawing with either one or two vanishing points and at least 5 objects. Each object should have detail (windows, doors, sign, stairs, etc.)



[Figure 3]

Dilations

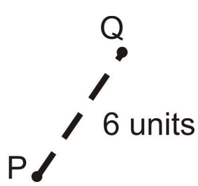
A dilation makes a figure larger or smaller, but has the same shape as the original. In other words, the dilation is similar to the original.

Transformation: An operation that moves, flips, or changes a figure to create a new figure. Transformations that preserve size are ***rigid*** and ones that do not are ***non-rigid.***

Dilation: A non-rigid transformation that preserves shape but not size.

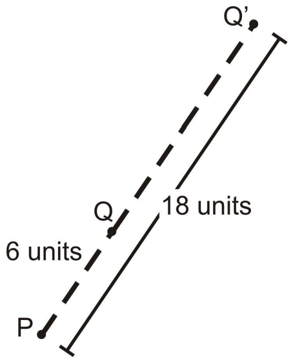
All dilations have a **center** and a **scale factor.** The center is the point of reference for the dilation (like the vanishing point in a perspective drawing) and scale factor tells us how much the figure stretches or shrinks. A scale factor is typically labeled k and is always greater than zero. Also, if the original figure is labeled $\triangle ABC$, for example, the dilation would be $\triangle A'B'C'$. The $'$ indicates that it is a copy. This tic mark is said "prime," so A' is read "A prime." A second dilation would be A'' , read "A double-prime."

Example 1: The center of dilation is P and the scale factor is 3. Find Q' .



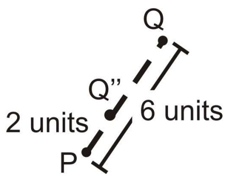
[Figure 4]

Solution: If the scale factor is 3 and Q is 6 units away from P , then Q' is going to be $6 \times 3 = 18$ units away from P . Because we are only dilating a point, the dilation will be collinear with the original and center.



[Figure 5]

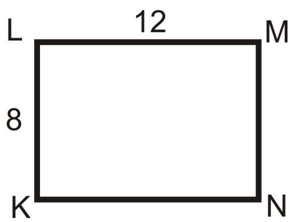
Example 2: Using the picture above, change the scale factor to $\frac{1}{3}$. Find Q''.



[Figure 6]

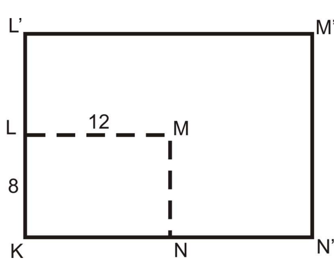
Solution: Now the scale factor is $\frac{1}{3}$, so Q'' is going to be $\frac{1}{3}$ the distance away from P as Q is. In other words, Q'' is going to be $6 \times \frac{1}{3} = 2$ units away from P. Q'' will also be collinear with Q and center.

Example 3: $KLMN$ is a rectangle with length 12 and width 8. If the center of dilation is K with a scale factor of 2, draw $K'L'M'N'$.



[Figure 7]

Solution: If K is the center of dilation, then K and K' will be the same point. From there, L' will be 8 units above L and N' will be 12 units to the right of N.

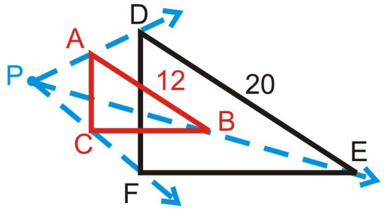


[Figure 8]

Example 4: Find the perimeters of $KLMN$ and $K'L'M'N'$. Compare this to the scale factor.

Solution: The perimeter of $KLMN = 12+8+12+8 = 40$. The perimeter of $K'L'M'N' = 24+16+24+16 = 80$. The ratio of the perimeters is 80:40 or 2:1, which is the same as the scale factor.

Example 5: $\triangle ABC$ is a dilation of $\triangle DEF$. If P is the center of dilation, what is the scale factor?



[Figure 9]

Solution: Because $\triangle ABC$ is a dilation of $\triangle DEF$, we know that the triangles are similar. Therefore the scale factor is the ratio of the sides. Since $\triangle ABC$ is smaller than the original, $\triangle DEF$, the scale factor is going to be a fraction less than one, $\frac{12}{20} = \frac{3}{5}$.

If $\triangle DEF$ was the dilated image, the scale factor would have been $\frac{5}{3}$.

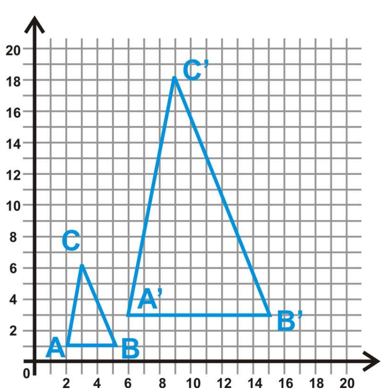
If the dilated image is smaller than the original, then the scale factor is $0 < k < 1$***.***

If the dilated image is larger than the original, then the scale factor is $k > 1$***.***

Dilations in the Coordinate Plane

In this text, the center of dilation will always be the origin, unless otherwise stated.

Example 6: Determine the coordinates of $\triangle ABC$ and $\triangle A'B'C'$ and find the scale factor.



[Figure 10]

Solution: The coordinates of $\triangle ABC$ are $A(2,1), B(5,1)$ and $C(3,6)$. The coordinates of $\triangle A'B'C'$ are $A'(6,3), B'(15,3)$ and $C'(9,18)$. By looking at the corresponding coordinates, each is three times the original. That means $k = 3$.

Again, the center, original point, and dilated point are collinear. Therefore, you can draw a ray from the origin to $C', B',$ and A' such that the rays pass through $C, B,$ and A, respectively.

Let's show that dilations are a similarity transformation (preserves shape). Using the distance formula, we will find the lengths of the sides of both triangles in Example 6 to demonstrate this.

$\underline{\triangle ABC}$

$$AB = \sqrt{(2-5)^2 + (1-1)^2} = \sqrt{9} = 3$$
$$AC = \sqrt{(2-3)^2 + (1-6)^2} = \sqrt{26}$$
$$CB = \sqrt{(3-5)^2 + (6-1)^2} = \sqrt{29}$$

$\underline{\triangle A'B'C'}$

$$A'B' = \sqrt{(6-15)^2 + (3-3)^2} = \sqrt{81} =$$
$$A'C' = \sqrt{(6-9)^2 + (3-18)^2} = \sqrt{234} =$$
$$C'B' = \sqrt{(9-15)^2 + (18-3)^2} = \sqrt{261}$$

From this, we also see that all the sides of $\triangle A'B'C'$ are three times larger than $\triangle ABC$. Therefore, ***a dilation will always produce a similar shape to the original***.

In the coordinate plane, we say that A' is a "mapping" of A. So, if the scale factor is 3, then $A(2,1)$ is mapped to (usually drawn with an arrow) $A'(6,3)$. The entire mapping of $\triangle ABC$ can be written $(x,y) \rightarrow (3x,3y)$ because $k = 3$. ***For any dilation the mapping will be*** $(x,y) \rightarrow (kx,ky)$.

Know What? Revisited Answers to this project will vary depending on what you decide to draw. Make sure that you have at least five objects with some sort of detail. If you are having trouble getting started, go to the website: http://www.drawing-and-painting-techniques.com/drawing-perspective.html

Review Questions

Given A and the scale factor, determine the coordinates of the dilated point, A'. You may assume the center of dilation is the origin.

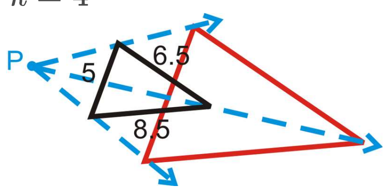
1. $A(3,9), k = \frac{2}{3}$
2. $A(-4,6), k = 2$
3. $A(9,-13), k = \frac{1}{2}$

Given A and A', find the scale factor. You may assume the center of dilation is the origin.

4. $A(8,2), A'(12,3)$
5. $A(-5,-9), A'(-45,-81)$
6. $A(22,-7), A(11,-3.5)$

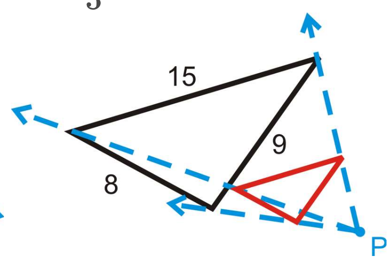
In the two questions below, you are told the scale factor. Determine the dimensions of the dilation. In each diagram, the **black** figure is the original and P is the center of dilation.

$k = 4$



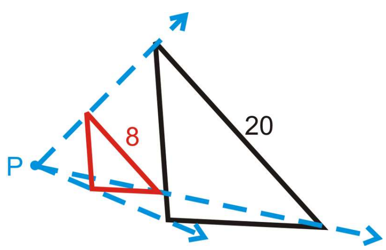
[Figure 11]

$k = \frac{1}{3}$

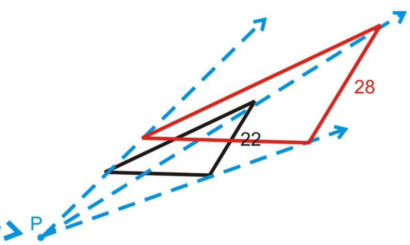


[Figure 12]

In the two questions below, find the scale factor, given the corresponding sides. In each diagram, the **black** figure is the original and P is the center of dilation.



[Figure 13]



[Figure 14]

9. Find the perimeter of both triangles in #7. What is the ratio of the perimeters?
10. ***Writing*** What happens if $k = 1$?

The origin is the center of dilation. Find the coordinates of the dilation of each figure, given the scale factor.

13. $A(2,4), B(-3,7), C(-1,-2); k = 3$
14. $A(12,8), B(-4,-16), C(0,10); k = \frac{3}{4}$

Multi-Step Problem Questions 15-21 build upon each other.

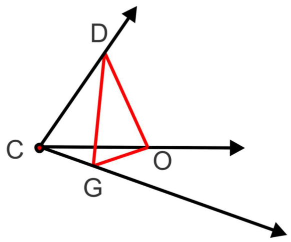
15. Plot $A(1,2), B(12,4), C(10,10)$. Connect to form a triangle.
16. Make the origin the center of dilation. Draw 4 rays from the origin to each point from #15. Then, plot $A'(2,4), B'(24,8), C'(20,20)$. What is the scale factor?

17. Use $k = 4$, to find $A''B''C''$. Plot these points.
18. What is the scale factor from $A'B'C'$ to $A''B''C''$?
19. Find (O is the origin):
    a. OA
    b. AA'
    c. AA''
    d. OA'
    e. OA''
20. Find:
    a. AB
    b. $A'B'$
    c. $A''B''$
21. Compare the ratios:
    a. $OA : OA'$ and $AB : A'B'$
    b. $OA : OA''$ and $AB : A''B''$

Algebra Connection For questions 22-27, use quadrilateral $ABCD$ with $A(1,5), B(2,6), C(3,3)$ and $D(1,3)$ and its transformation $A'B'C'D'$ with $A'(-3,1), B'(0,4), C'(3,-5)$ and $D'(-3,-5)$.

22. Plot the two quadrilaterals in the coordinate plane.
23. Find the equation of $\overleftrightarrow{CC'}$.
24. Find the equation of $\overleftrightarrow{DD'}$.
25. Find the intersection of these two lines algebraically or graphically.
26. What is the significance of this point?
27. What is the scale factor of the dilation?

Construction We can use a compass and straight edge to construct a dilation as well. Copy the diagram below.



[Figure 15]

28. Set your compass to be CG and use this setting to mark off a point 3 times as far from C as G is. Label this point G'. Repeat this process for CO and CD to find O' and D'.
29. Connect G', O' and D' to make $\triangle D'O'G'$. Find the ratios, $\frac{D'O'}{DO}, \frac{O'G'}{OG}$ and $\frac{G'D'}{GD}$.
30. What is the scale factor of this dilation?
31. Describe how you would dilate the figure by a scale factor of 4.
32. Describe how you would dilate the figure by a scale factor of $\frac{1}{2}$.

Review Queue Answers

1. Yes, all the angles are congruent and the corresponding sides are in the same ratio.
2. $\frac{5}{3}$
3. Yes, $LMNO \sim EFGH$ because $LMNO$ is exactly half of $EFGH$.