9.5 Angles of Chords, Secants, and Tangents

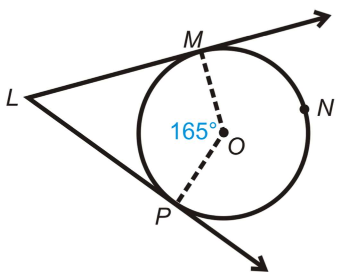
FlexBooks® 2.0 > American HS Geometry > Angles of Chords, Secants, and Tangents

Last Modified: Dec 25, 2014

Learning Objectives

- Find the measures of angles formed by chords, secants, and tangents.

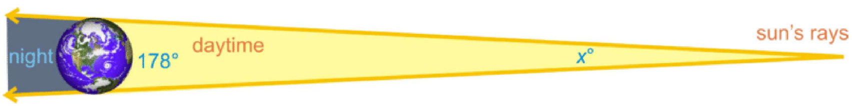
Review Queue



[Figure 1]

1. What is $m\angle OML$ and $m\angle OPL$? How do you know?
2. Find $m\angle MLP$.
3. Find $m\widehat{MNP}$.
4. Find $\frac{m\widehat{MNP}-m\widehat{MP}}{2}$. What is it the same as?

Know What? The sun's rays hit the Earth such that the tangent rays determine when daytime and night time are. The time and Earth's rotation determine when certain locations have sun. If the arc that is exposed to sunlight is 178°, what is the angle at which the sun's rays hit the earth (x°)?



[Figure 2]

Angle *on* a Circle

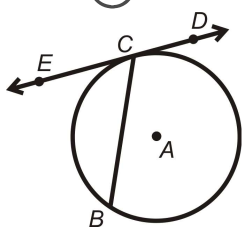
When an angle is on a circle, the vertex is on the circumference of the circle. One type of angle *on* a circle is the inscribed angle, from the previous section. Recall that ***an inscribed***

angle is formed by two chords and is half the measure of the intercepted arc. Another type of angle *on* a circle is one formed by a tangent and a chord.

Investigation 9-6: The Measure of an Angle formed by a Tangent and a Chord

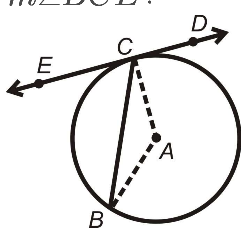
Tools Needed: pencil, paper, ruler, compass, protractor

Draw $\odot A$ with chord BC and tangent line $\overleftrightarrow{ED}$ with point of tangency C.



[Figure 3]

Draw in central angle $\angle CAB$. Then, using your protractor, find $m\angle CAB$ and $m\angle BCE$.



[Figure 4]

1. Find mBC (the minor arc). How does the measure of this arc relate to $m\angle BCE$?

What other angle that you have learned about is this type of angle similar to?

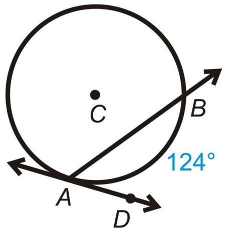
This investigation proves Theorem 9-11.

Theorem 9-11: The measure of an angle formed by a chord and a tangent that intersect on the circle is half the measure of the intercepted arc.

From Theorem 9-11, we now know that there are two types of angles that are half the measure of the intercepted arc; an inscribed angle and an angle formed by a chord and a tangent. Therefore, ***any angle with its vertex on a circle will be half the measure of the intercepted arc***.

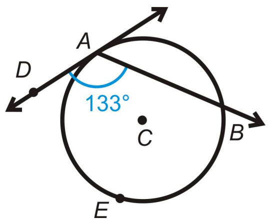
Example 1: Find:

a) $m\angle BAD$



[Figure 5]

b) $mAEB$



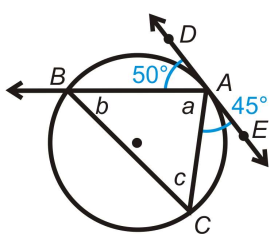
[Figure 6]

Solution: Use Theorem 9-11.

a) $m\angle BAD = \frac{1}{2}mAB = \frac{1}{2}\cdot 124^\circ = 62^\circ$

b) $mAEB = 2\cdot m\angle DAB = 2\cdot 133^\circ = 266^\circ$

Example 2: Find a, b , and c .



[Figure 7]

Solution: To find a , it is in line with 50° and 45° . The three angles add up to 180° . $50^\circ + 45^\circ + m\angle a = 180^\circ, m\angle a = 85^\circ$.

b is an inscribed angle, so its measure is half of mAC . From Theorem 9-11, $mAC = 2\cdot m\angle EAC = 2\cdot 45^\circ = 90^\circ$.

$m\angle b = \frac{1}{2}\cdot mAC = \frac{1}{2}\cdot 90^\circ = 45^\circ$.

To find c , you can either use the Triangle Sum Theorem or Theorem 9-11. We will use the Triangle Sum Theorem. $85^\circ + 45^\circ + m\angle c = 180^\circ, m\angle c = 50^\circ$.

From this example, we see that Theorem 9-8, from the previous section, is also true for angles formed by a tangent and chord with the vertex on the circle. If two angles, with their vertices *on* the circle, intercept the same arc then the angles are congruent.

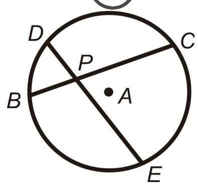
Angles *inside* a Circle

An angle is considered *inside* a circle when the vertex is somewhere inside the circle, but not on the center. All angles inside a circle are formed by two intersecting chords.

Investigation 9-7: Find the Measure of an Angle *inside* a Circle

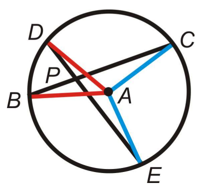
Tools Needed: pencil, paper, compass, ruler, protractor, colored pencils (optional)

Draw $\odot A$ with chord BC and DE. Label the point of intersection P.



[Figure 8]

Draw central angles $\angle DAB$ and $\angle CAE$. Use colored pencils, if desired.

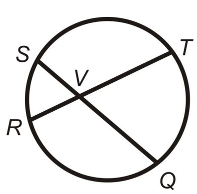


[Figure 9]

1. Using your protractor, find $m\angle DPB, m\angle DAB$, and $m\angle CAE$. What is mDB and mCE?
2. Find $\frac{mDB + mCE}{2}$.
3. What do you notice?

Theorem 9-12: The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs.

In the picture to the left:



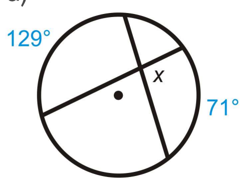
[Figure 10]

$$m\angle SVR = \frac{1}{2}\left(mSR + mTQ\right) = \frac{mSR + mTQ}{2} = m\angle TVQ$$
$$m\angle SVT = \frac{1}{2}\left(mST + mRQ\right) = \frac{mST + mRQ}{2} = m\angle RVQ$$

The proof of this theorem is in the review exercises.

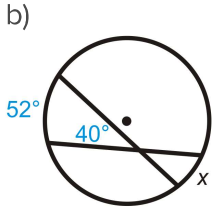
Example 3: Find x.

a)



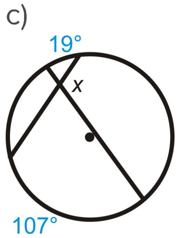
[Figure 11]

b)



[Figure 12]

c)



[Figure 13]

Solution: Use Theorem 9-12 and write an equation.

a) The intercepted arcs for x are 129° and 71°.

$$x = \frac{129^\circ + 71^\circ}{2} = \frac{200^\circ}{2} = 100^\circ$$

b) Here, x is one of the intercepted arcs for 40°.

$$
\begin{aligned}
40^\circ &= \frac{52^\circ + x}{2} \\
80^\circ &= 52^\circ + x \\
38^\circ &= x
\end{aligned}
$$

c) x is supplementary to the angle that the average of the given intercepted arcs. We will call this supplementary angle y.

$y = \frac{19^\circ + 107^\circ}{2} = \frac{126^\circ}{2} = 63^\circ$ This means that $x = 117^\circ; 180^\circ - 63^\circ$

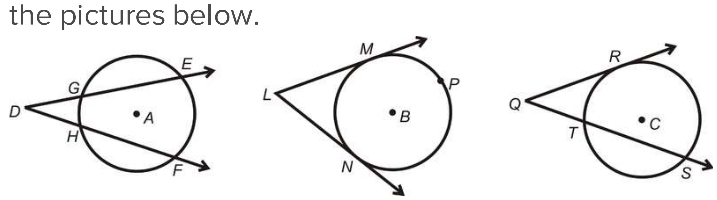
Angles *outside* a Circle

An angle is considered to be outside a circle if the vertex of the angle is outside the circle and the sides are tangents or secants. There are three types of angles that are outside a circle: an angle formed by two tangents, an angle formed by a tangent and a secant, and an angle formed by two secants. Just like an angle inside or on a circle, an angle outside a circle has a specific formula, involving the intercepted arcs.

Investigation 9-8: Find the Measure of an Angle outside a Circle

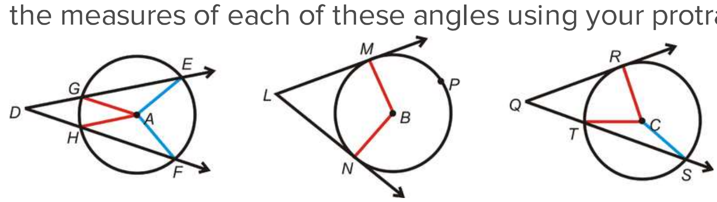
Tools Needed: pencil, paper, ruler, compass, protractor, colored pencils (optional)

Draw three circles and label the centers A, B, and C. In $\odot A$ draw two secant rays with the same endpoint, $\overrightarrow{DE}$ and $\overrightarrow{DF}$. In $\odot B$, draw two tangent rays with the same endpoint, $\overrightarrow{LM}$ and $\overrightarrow{LN}$. In $\odot C$, draw a tangent ray and a secant ray with the same endpoint, $\overrightarrow{QR}$ and $\overrightarrow{QS}$. Label the points of intersection with the circles like they are in the pictures below.



[Figure 14]

Draw in all the central angles: $\angle GAH, \angle EAF, \angle MBN, \angle RCT, \angle RCS$. Then, find the measures of each of these angles using your protractor. Use color to differentiate.



[Figure 15]

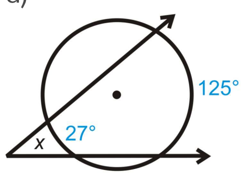
1. Find $m\angle EDF, m\angle MLN$, and $m\angle RQS$.

2. Find $\frac{mEF - mGH}{2}$, $\frac{mMPN - mMN}{2}$, and $\frac{mRS - mRT}{2}$. What do you notice?

Theorem 9-13: The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point outside the circle is equal to half the difference of the measures of the intercepted arcs.

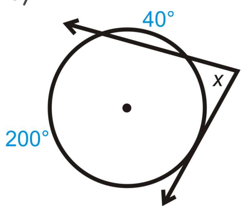
Example 4: Find the measure of x.

a)



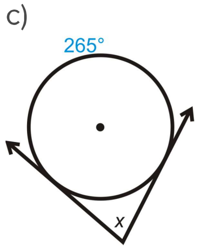
[Figure 16]

b)



[Figure 17]

c)



[Figure 18]

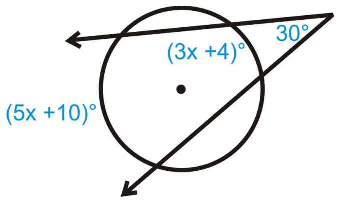
Solution: For all of the above problems we can use Theorem 9-13.

a) $x = \frac{125^\circ - 27^\circ}{2} = \frac{98^\circ}{2} = 49^\circ$

b) 40° is not the intercepted arc. Be careful! The intercepted arc is 120°, $(360^\circ - 200^\circ - 40^\circ)$. Therefore, $x = \frac{200^\circ - 120^\circ}{2} = \frac{80^\circ}{2} = 40^\circ$.

c) First, we need to find the other intercepted arc, $360^\circ - 265^\circ = 95^\circ$.
$x = \frac{265^\circ - 95^\circ}{2} = \frac{170^\circ}{2} = 85^\circ$

Example 5: ***Algebra Connection*** Find the value of x. You may assume lines that look tangent, are.



[Figure 19]

Solution: Set up an equation using Theorem 9-13.

$$\begin{aligned}
\frac{(5x+10)^\circ-(3x+4)^\circ}{2} &= 30^\circ\\
(5x+10)^\circ-(3x+4)^\circ &= 60^\circ\\
5x+10^\circ-3x-4^\circ &= 60^\circ\\
2x+6^\circ &= 60^\circ\\
2x &= 54^\circ\\
x &= 27^\circ
\end{aligned}$$

Know What? Revisited If 178° of the Earth is exposed to the sun, then the angle at which the sun's rays hit the Earth is 2°. From Theorem 9-13, these two angles are supplementary. From this, we also know that the other 182° of the Earth is not exposed to sunlight and it is probably night time.

Review Questions

1. Draw two secants that intersect:

   a. inside a circle.

   b. on a circle.

   c. outside a circle.

2. Can two tangent lines intersect inside a circle? Why or why not?
3. Draw a tangent and a secant that intersect:

   a. on a circle.

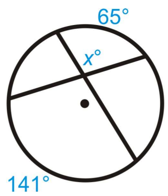
   b. outside a circle.

Fill in the blanks.

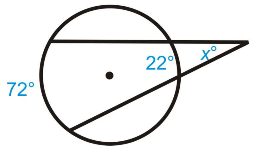
4. If the vertex of an angle is on the ________________ of a circle, then its measure is ________________ to the intercepted arc.
5. If the vertex of an angle is ________________ a circle, then its measure is the average of the ____________________ arcs.

6. If the vertex of an angle is __________ a circle, then its measure is _________________ the intercepted arc.
7. If the vertex of an angle is _______________ a circle, then its measure is ______________ the difference of the intercepted arcs.

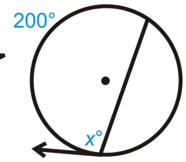
For questions 8-19, find the value of the missing variable(s).



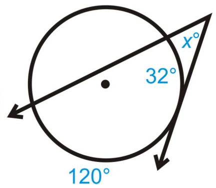
[Figure 20]



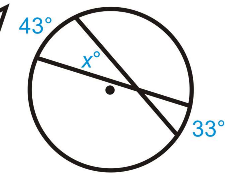
[Figure 21]



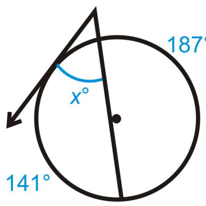
[Figure 22]



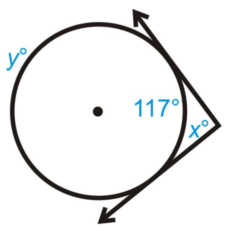
[Figure 23]



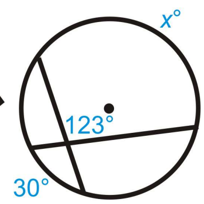
[Figure 24]



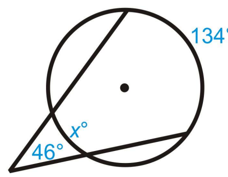
[Figure 25]



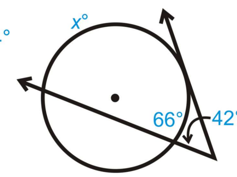
[Figure 26]



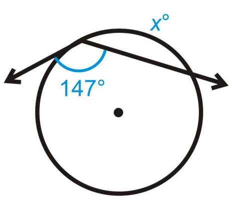
[Figure 27]



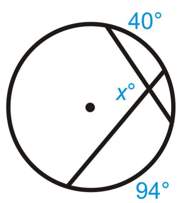
[Figure 28]



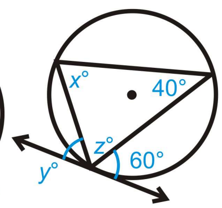
[Figure 29]



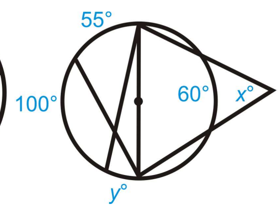
[Figure 30]



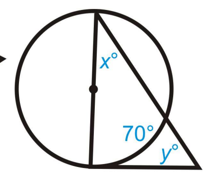
[Figure 31]



[Figure 32]

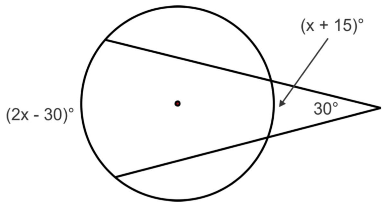


[Figure 33]

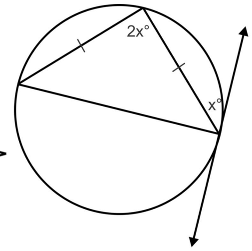


[Figure 34]

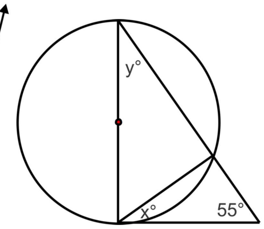
Algebra Connection Solve for the variable(s).



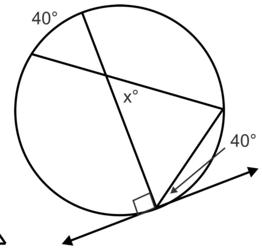
[Figure 35]



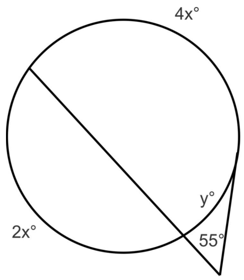
[Figure 36]



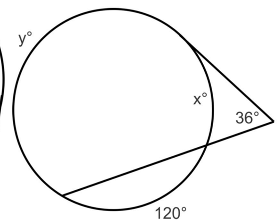
[Figure 37]



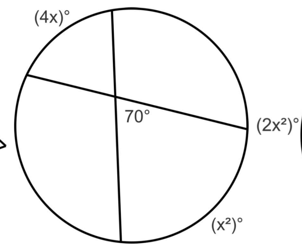
[Figure 38]



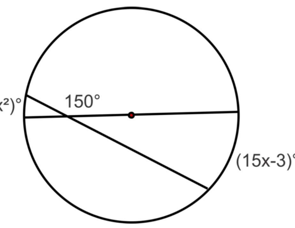
[Figure 39]



[Figure 40]

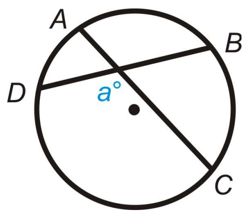


[Figure 41]



[Figure 42]

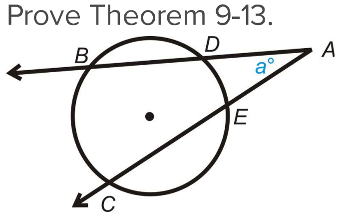
Prove Theorem 9-12.



[Figure 43]

Given: Intersecting chords AC and BD. Prove: $m\angle a = \frac{1}{2}\left(mDC + mAB\right)$ *HINT:* Draw BC and use inscribed angles.

Prove Theorem 9-13.



[Figure 44]

Given: Secant rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ Prove: $m\angle a = \frac{1}{2}\left(mBC - mDE\right)$ *HINT:* Draw BE and use inscribed angles.

Review Queue Answers

1. $m\angle OML = m\angle OPL = 90^\circ$ because a tangent line and a radius drawn to the point of tangency are perpendicular.
2. $$\begin{aligned} 165^\circ + m\angle OML + m\angle OPL + m\angle MLP &= 360^\circ \\ 165^\circ + 90^\circ + 90^\circ + m\angle MLP &= 360^\circ \\ m\angle MLP &= 15^\circ \end{aligned}$$

3. $mMNP = 360^\circ - 165^\circ = 195^\circ$
4. $\frac{195^\circ - 165^\circ}{2} = \frac{30^\circ}{2} = 15^\circ$, this is the same as $m\angle MLP$.

9.6 Segments of Chords, Secants, and Tangents

FlexBooks® 2.0 > American HS Geometry > Segments of Chords, Secants, and Tangents

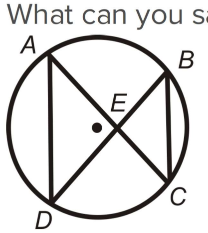
Last Modified: Dec 25, 2014

Learning Objectives

- Find the lengths of segments associated with circles.

Review Queue

What can you say about $m\angle DAC$ and $m\angle DBC$? What theorem do you use?



[Figure 1]

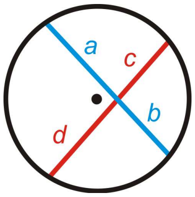
1. What do you know about $m\angle AED$ and $m\angle BEC$? Why?
2. Is $\triangle AED \sim \triangle BEC$? How do you know?
3. If $AE = 8, ED = 7$, and $BE = 6$, find EC .
4. If $\overline{AD}$ and $\overline{BC}$ are not in the circle, would the ratios from #4 still be valid?

Know What? As you know, the moon orbits the earth. At a particular time, the moon is 238,857 miles from Beijing, China. On the same line, Yukon is 12,451 miles from Beijing. Drawing another line from the moon to Cape Horn (the southernmost point of South America), we see that Jakarta, Indonesia is collinear. If the distance from Cape Horn to Jakarta is 9849 miles, what is the distance from the moon to Jakarta?

[Figure 2]

Segments from Chords

In the Review Queue above, we have two chords that intersect inside a circle. The two triangles are similar, making the sides of each triangle in proportion with each other. If we remove AD and BC the ratios between AE, EC, DE , and EB will still be the same. This leads us to our first theorem.



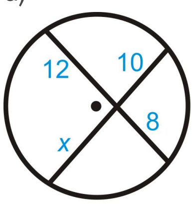
[Figure 3]

Theorem 9-14: If two chords intersect inside a circle so that one is divided into segments of length a and b and the other into segments of length c and d then $ab = cd$.

The product of the segments of one chord is equal to the product of segments of the second chord.

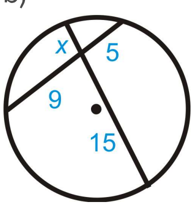
Example 1: Find x in each diagram below.

a)



[Figure 4]

b)



[Figure 5]

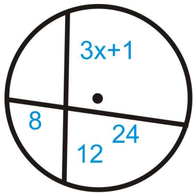
Solution: Use the ratio from Theorem 9-13. The product of the segments of one chord is equal to the product of the segments of the other.

a)
$$\begin{aligned} 12 \cdot 8 &= 10 \cdot x \\ 96 &= 10x \\ 9.6 &= x \end{aligned}$$

b)
$$\begin{aligned} x \cdot 15 &= 5 \cdot 9 \\ 15x &= 45 \\ x &= 3 \end{aligned}$$

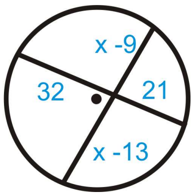
Example 2: ***Algebra Connection*** Solve for x .

a)



[Figure 6]

b)



[Figure 7]

Solution: Again, we can use Theorem 9-13. Set up an equation and solve for x.

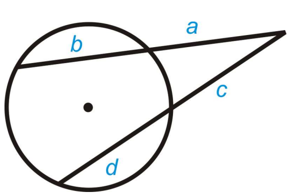
a)
$$\begin{aligned} 8 \cdot 24 &= (3x+1) \cdot 12 \\ 192 &= 36x + 12 \\ 180 &= 36x \\ 5 &= x \end{aligned}$$

b)
$$\begin{aligned} 32 \cdot 21 &= (x-9)(x-13) \\ 672 &= x^2 - 22x + 117 \\ 0 &= x^2 - 22x - 555 \\ 0 &= (x-37)(x+15) \\ x &= 37, -15 \end{aligned}$$

However, $x \neq -15$ because length cannot be negative, so $x = 37$.

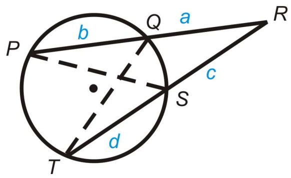
Segments from Secants

In addition to forming an angle outside of a circle, the circle can divide the secants into segments that are proportional with each other.



[Figure 8]

If we draw in the intersecting chords, we will have two similar triangles.



[Figure 9]

From the inscribed angles and the Reflexive Property $(\angle R \cong \angle R), \triangle PRS \sim \triangle TRQ$.

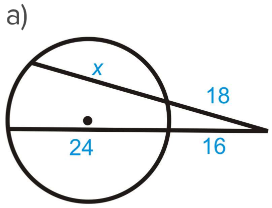
Because the two triangles are similar, we can set up a proportion between the corresponding sides. Then, cross-multiply. $\frac{a}{c+d} = \frac{c}{a+b} \Rightarrow a(a+b) = c(c+d)$

Theorem 9-15: If two secants are drawn from a common point outside a circle and the segments are labeled as above, then $a(a+b) = c(c+d)$.

In other words, the product of the outer segment and the whole of one secant is equal to the product of the outer segment and the whole of the other secant.

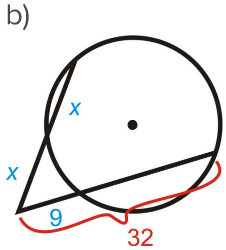
Example 3: Find the value of the missing variable.

a)



[Figure 10]

b)



[Figure 11]

Solution: Use Theorem 9-15 to set up an equation. For both secants, you multiply the outer portion of the secant by the whole.

a)
$$\begin{aligned} 18 \cdot (18+x) &= 16 \cdot (16+24) \\ 324 + 18x &= 256 + 384 \\ 18x &= 316 \\ x &= 17\frac{5}{9} \end{aligned}$$

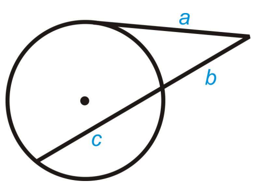
b)

$$\begin{aligned} x \cdot (x+x) &= 9 \cdot 32 \\ 2x^2 &= 288 \\ x^2 &= 144 \\ x &= 12 \end{aligned}$$

$x \neq -12$ because length cannot be negative.

Segments from Secants and Tangents

If a tangent and secant meet at a common point outside a circle, the segments created have a similar relationship to that of two secant rays in Example 3. Recall that the product of the outer portion of a secant and the whole is equal to the same of the other secant. If one of these segments is a tangent, it will still be the product of the outer portion and the whole. However, for a tangent line, the outer portion and the whole are equal.



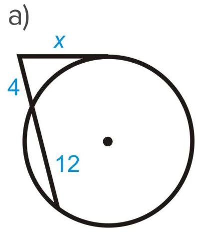
[Figure 12]

Theorem 9-16: If a tangent and a secant are drawn from a common point outside the circle (and the segments are labeled like the picture to the left), then $a^2 = b(b+c)$.

This means that the product of the outside segment of the secant and the whole is equal to the square of the tangent segment.

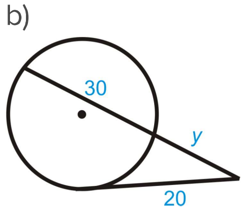
Example 4: Find the value of the missing segment.

a)



[Figure 13]

b)



[Figure 14]

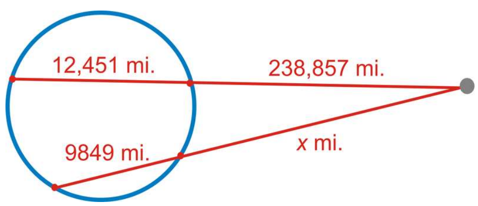
Solution: Use Theorem 9-16. Square the tangent and set it equal to the outer part times the whole secant.

a)
$$x^2 = 4(4+12)$$
$$x^2 = 4 \cdot 16 = 64$$
$$x = 8$$

b)
$$20^2 = y(y+30)$$
$$400 = y^2 + 30y$$
$$0 = y^2 + 30y - 400$$
$$0 = (y+40)(y-10)$$
$$y = \cancel{-40}, 10$$

When you have to factor a quadratic equation to find an answer, always eliminate the negative answer(s). Length is never negative.

Know What? Revisited The given information is to the left. Let's set up an equation using Theorem 9-15.

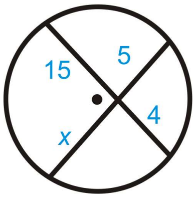


[Figure 15]

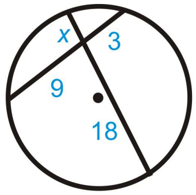
$$238857 \cdot 251308 = x \cdot (x + 9849)$$
$$60026674956 = x^2 + 9849x$$
$$0 = x^2 + 9849x - 60026674956$$
$$Use\ the\ Quadratic\ Formula\ x \approx \frac{-9849 \pm \sqrt{9849^2 - 4(-60026674956)}}{2}$$
$$x \approx 240128.4\ miles$$

Review Questions

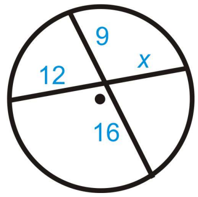
Find x in each diagram below. Simplify any radicals.



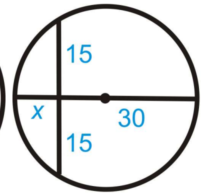
[Figure 16]



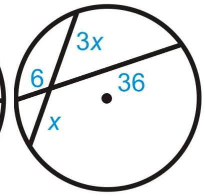
[Figure 17]



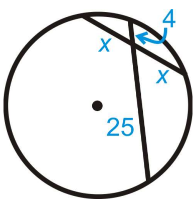
[Figure 18]



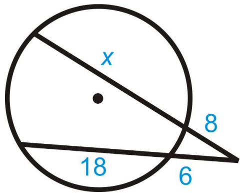
[Figure 19]



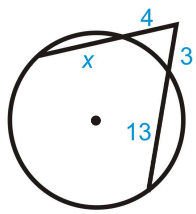
[Figure 20]



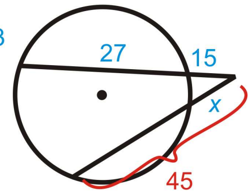
[Figure 21]



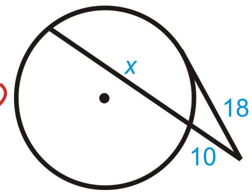
[Figure 22]



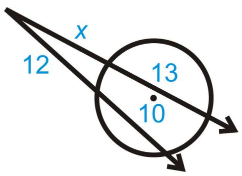
[Figure 23]



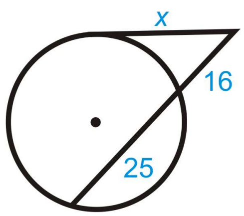
[Figure 24]



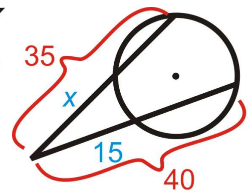
[Figure 25]



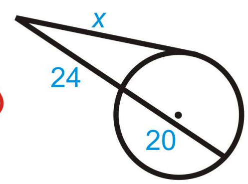
[Figure 26]



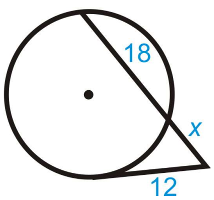
[Figure 27]



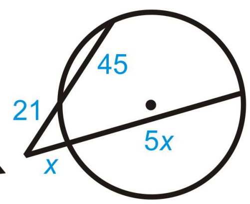
[Figure 28]



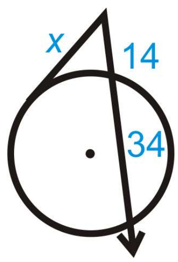
[Figure 29]



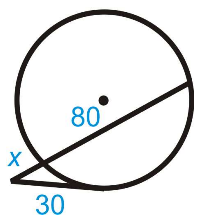
[Figure 30]



[Figure 31]

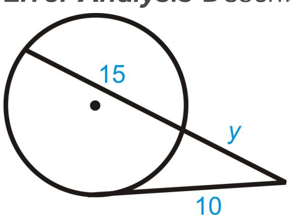


[Figure 32]



[Figure 33]

Error Analysis Describe and correct the error in finding y.

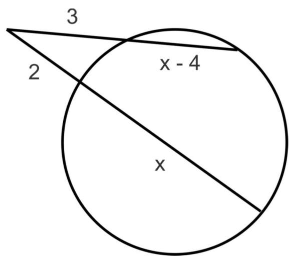


[Figure 34]

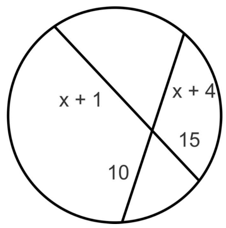
$$\begin{aligned} 10 \cdot 10 &= y \cdot 15y \\ 100 &= 15y^2 \\ \frac{20}{3} &= y^2 \\ \frac{2\sqrt{15}}{3} &= y \quad \leftarrow y \text{ is \textbackslash underline\{not\} correct} \end{aligned}$$

1. Suzie found a piece of a broken plate. She places a ruler across two points on the rim, and the length of the chord is found to be 6 inches. The distance from the midpoint of this chord to the nearest point on the rim is found to be 1 inch. Find the diameter of the plate.

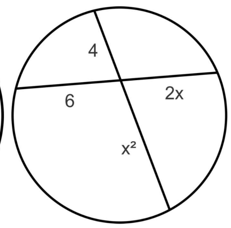
Algebra Connection For problems 21-30, solve for x.



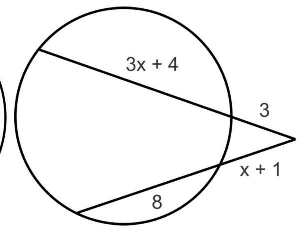
[Figure 35]



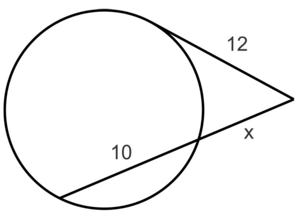
[Figure 36]



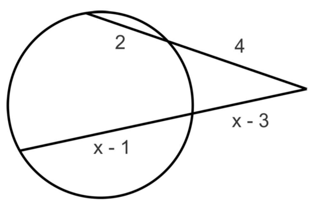
[Figure 37]



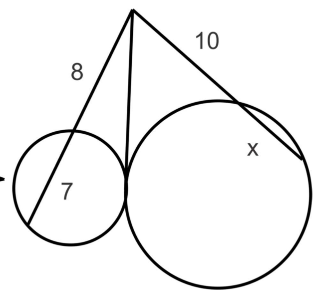
[Figure 38]



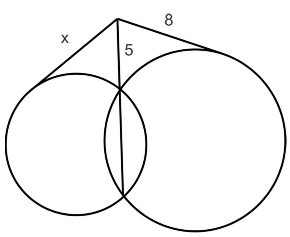
[Figure 39]



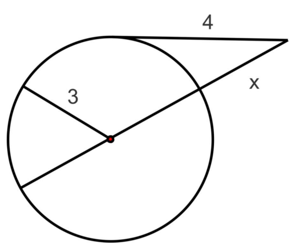
[Figure 40]



[Figure 41]

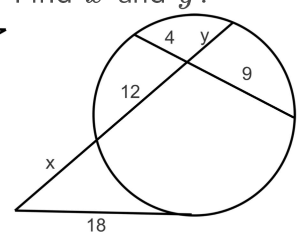


[Figure 42]



[Figure 43]

Find x and y.



[Figure 44]

Review Queue Answers

1. $m\angle DAC = m\angle DBC$ by Theorem 9-8, they are inscribed angles and intercept the same arc.
2. $m\angle AED = m\angle BEC$ by the Vertical Angles Theorem.
3. Yes, by AA Similarity Postulate.
4. $$\frac{8}{6} = \frac{7}{EC}$$ $$8 \cdot EC = 42$$ $$EC = \frac{21}{4} = 5.25$$

5. Yes, the EC would be the same and the ratio would still be valid.