11.3 Surface Area of Pyramids and Cones

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Learning Objectives

- Find the surface area of a pyramid.
- Find the surface area of a cone.

Review Queue

1. A rectangular prism has sides of 5 cm, 6 cm, and 7 cm. What is the surface area?
2. Triple the dimensions of the rectangular prism from #1. What is its surface area?
3. A cylinder has a diameter of 10 in and a height of 25 in. What is the surface area?
4. A cylinder has a circumference of $72\pi\ ft$. and a height of 24 ft. What is the surface area?
5. Draw the net of a square pyramid.

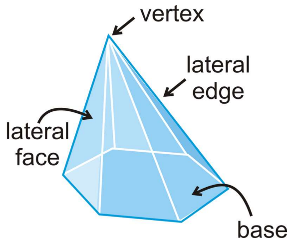
Know What? A typical waffle cone is 6 inches tall and has a diameter of 2 inches. This happens to be your friend Jeff's favorite part of his ice cream dessert. You decide to use your mathematical prowess to figure out exactly how much waffle cone Jeff is eating. What is the surface area of the waffle cone? (You may assume that the cone is straight across at the top)

Jeff decides he wants a "king size" cone, which is 8 inches tall and has a diameter of 4 inches. What is the surface area of this cone?

[Figure 1]

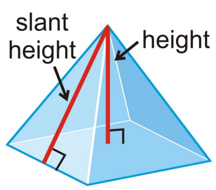
Parts of a Pyramid

A pyramid has one ***base*** and all the ***lateral faces*** meet at a common ***vertex***. The edges between the lateral faces are ***lateral edges***. The edges between the base and the lateral faces are called ***base edges***. If we were to draw the height of the pyramid to the right, it would be off to the left side.



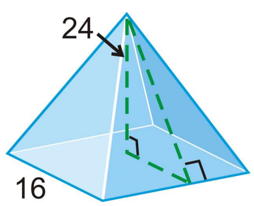
[Figure 2]

When a pyramid has a height that is directly in the center of the base, the pyramid is said to be regular. These pyramids have a regular polygon as the base. All regular pyramids also have a ***slant height*** that is the height of a lateral face. Because of the nature of regular pyramids, ***all slant heights are congruent***. A non-regular pyramid does not have a slant height.



[Figure 3]

Example 1: Find the slant height of the square pyramid.



[Figure 4]

Solution: Notice that the slant height is the hypotenuse of a right triangle formed by the height and half the base length. Use the Pythagorean Theorem.

$$\begin{aligned}8^2+24^2 &= l^2\\64+576 &= l^2\\640 &= l^2\\l &= \sqrt{640}=8\sqrt{10}\end{aligned}$$

Surface Area of a Regular Pyramid

Using the slant height, which is usually labeled l, the area of each triangular face is $A = \frac{1}{2}bl$.

Example 2: Find the surface area of the pyramid from Example 1.

Solution: The surface area of the four triangular faces are $4\left(\frac{1}{2}bl\right) = 2(16)\left(8\sqrt{10}\right) = 256\sqrt{10}$. To find the total surface area, we also need the area of the base, which is $16^2 = 256$. The total surface area is $256\sqrt{10} + 256 \approx 1065.54$.

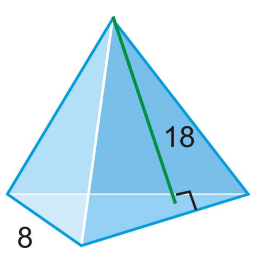
From this example, we see that the formula for a square pyramid is:

$$
\begin{aligned}
SA &= (area\ of\ the\ base) + 4(area\ of\ triangular\ faces) \\
SA &= B + n\left(\frac{1}{2}bl\right) && B \text{ is the area of the base and } n \text{ is the number of triangle} \\
SA &= B + \frac{1}{2}l(nb) && \text{Rearranging the variables, } nb = P, \text{ the perimeter of the} \\
SA &= B + \frac{1}{2}Pl
\end{aligned}
$$

Surface Area of a Regular Pyramid: If B is the area of the base and P is the perimeter of the base and l is the slant height, then $SA = B + \frac{1}{2}Pl$.

If you ever forget this formula, use the net. Each triangular face is congruent, plus the area of the base. This way, you do not have to remember a formula, just a process, which is the same as finding the area of a prism.

Example 3: Find the area of the regular triangular pyramid.



[Figure 5]

Solution: The area of the base is $A = \frac{1}{4}s^2\sqrt{3}$ because it is an equilateral triangle.

$$B = \frac{1}{4}8^2\sqrt{3} = 16\sqrt{3}$$
$$SA = 16\sqrt{3} + \frac{1}{2}(24)(18) = 16\sqrt{3} + 216 \approx 243.71$$

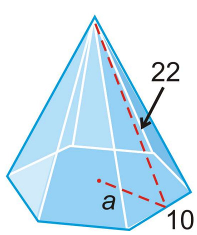
Example 4: If the lateral surface area of a square pyramid is $72\ ft^2$ and the base edge is equal to the slant height, what is the length of the base edge?

Solution: In the formula for surface area, the lateral surface area is $\frac{1}{2}Pl$ or $\frac{1}{2}nbl$. We know that $n = 4$ and $b = l$. Let's solve for b.

$$\frac{1}{2}nbl = 72\ ft^2$$
$$\frac{1}{2}(4)b^2 = 72$$
$$2b^2 = 72$$
$$b^2 = 36$$
$$b = 6$$

Therefore, the base edges are all 6 units and the slant height is also 6 units.

Example 4: Find the area of the regular hexagonal pyramid below.



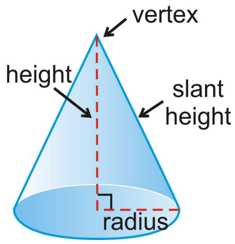
[Figure 6]

Solution: To find the area of the base, we need to find the apothem. If the base edges are 10 units, then the apothem is $5\sqrt{3}$ for a regular hexagon. The area of the base is $\frac{1}{2}asn = \frac{1}{2}(5\sqrt{3})(10)(6) = 150\sqrt{3}$. The total surface area is:

$$SA = 150\sqrt{3} + \frac{1}{2}(6)(10)(22)$$
$$= 150\sqrt{3} + 660 \approx 919.81\ units^2$$

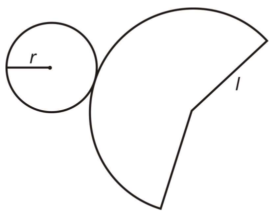
Surface Area of a Cone

Cone: A solid with a circular base and sides taper up towards a common vertex.



[Figure 7]

It is said that a cone is generated from rotating a right triangle around one leg in a circle. Notice that a cone has a slant height, just like a pyramid. The surface area of a cone is a little trickier, however. We know that the base is a circle, but we need to find the formula for the curved side that tapers up from the base. Unfolding a cone, we have the net:



[Figure 8]

From this, we can see that the lateral face's edge is $2\pi r$ and the sector of a circle with radius l. We can find the area of the sector by setting up a proportion.

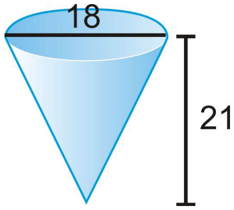
$$\frac{Area\ of\ circle}{Area\ of\ sector} = \frac{Circumference}{Arc\ length}$$
$$\frac{\pi l^2}{Area\ of\ sector} = \frac{2\pi l}{2\pi r} = \frac{l}{r}$$

Cross multiply:

$$l(Area\ of\ sector) = \pi r l^2$$
$$Area\ of\ sector = \pi r l$$

Surface Area of a Right Cone: The surface area of a right cone with slant height l and base radius r is $SA = \pi r^2 + \pi r l$.

Example 5: What is the surface area of the cone?



[Figure 9]

Solution: In order to find the surface area, we need to find the slant height. Recall from a pyramid, that the slant height forms a right triangle with the height and the radius. Use the Pythagorean Theorem.

$$\begin{aligned} l^2 &= 9^2 + 21^2 \\ &= 81 + 441 \\ l &= \sqrt{522} \approx 22.85 \end{aligned}$$

The surface area would be $SA = \pi 9^2 + \pi(9)(22.85) \approx 900.54\ units^2$.

Example 6: The surface area of a cone is 36π and the slant height is 5 units. What is the radius?

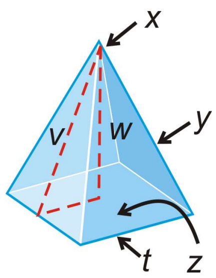
Solution: Plug in what you know into the formula for the surface area of a cone and solve for r .

$$\begin{aligned} 36\pi &= \pi r^2 + \pi r(5) && \text{Because every term has } \pi, \text{ we can cancel it out.} \\ 36 &= r^2 + 5r && \text{Set one side equal to zero, and this becomes a facto} \\ r^2 + 5r - 36 &= 0 \\ (r-4)(r+9) &= 0 && \text{The possible answers for } r \text{ are 4 and } -9. \text{ The radiu} \\ & && \text{so our answer is 4.} \end{aligned}$$

Know What? Revisited The standard cone has a surface area of $\pi + 6\pi = 7\pi \approx 21.99\ in^2$. The "king size" cone has a surface area of $4\pi + 16\pi = 20\pi \approx 62.83$, almost three times as large as the standard cone.

Review Questions

Fill in the blanks about the diagram to the left.



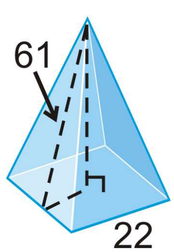
[Figure 10]

1. x is the ______________.
2. The slant height is __________.
3. y is the ______________.
4. The height is __________.
5. The base is _________.
6. The base edge is __________.
7. Sketch a right cone. Label the *height, slant height*, and *radius*.

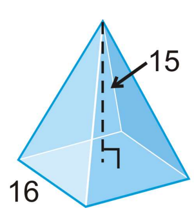
For questions 8-10, sketch each of the following solids and answer the question. Your drawings should be to scale, but not one-to-one. Leave your answer in simplest radical form.

8. Draw a right cone with a radius of 5 cm and a height of 15 cm. What is the slant height?
9. Draw a square pyramid with an edge length of 9 in and a 12 in height. Find the slant height.
10. Draw an equilateral triangle pyramid with an edge length of 6 cm and a height of 6 cm. ***Describe*** how you would find the slant height and then find it.

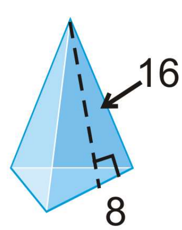
Find the area of a lateral face of the regular pyramid. Leave your answer in simplest radical form.



[Figure 11]

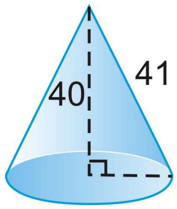


[Figure 12]

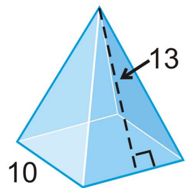


[Figure 13]

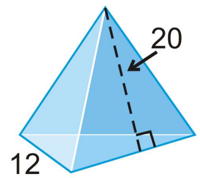
Find the surface area of the regular pyramids and right cones. Round your answers to 2 decimal places.



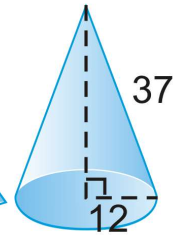
[Figure 14]



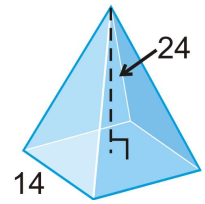
[Figure 15]



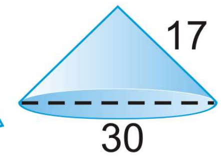
[Figure 16]



[Figure 17]



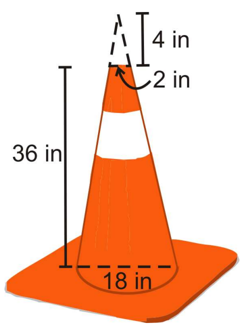
[Figure 18]



[Figure 19]

14. From these pictures, we see that a regular triangle pyramid does not have to have four congruent faces. How many faces must be congruent?
15. A ***regular tetrahedron*** has four equilateral triangles as its faces. Find the surface area of a regular tetrahedron with edge length of 6 units.
16. Using the formula for the area of an equilateral triangle, what is the surface area of a regular tetrahedron, with edge length s?

Challenge Find the surface area of the traffic cone with the given information. The gone is cut off at the top (4 inch cone) and the base is a square with sides of length 24 inches. Round answers to the nearest hundredth.



[Figure 20]

23. Find the area of the entire square. Then, subtract the area of the base of the cone.
24. Find the lateral area of the cone portion (include the 4 inch cut off top of the cone).
25. Now, subtract the cut-off top of the cone, to only have the lateral area of the cone portion of the traffic cone.
26. Combine your answers from #23 and #25 to find the entire surface area of the traffic cone.

For questions 27-30, consider the sector of a circle with radius 25 cm and arc length 14π.

27. What is the central angle of this sector?
28. If this sector is rolled into a cone, what are the radius and area of the base of the cone?
29. What is the height of this cone?

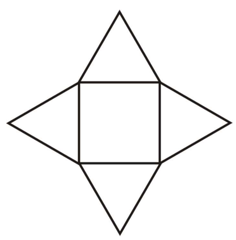
30. What is the total surface are of the cone?

For questions 31-33, consider a square with diagonal length $10\sqrt{2}\,in$.

31. What is the length of a side of the square?
32. If this square is the base of a right pyramid with height 12, what is the slant height of the pyramid?
33. What is the surface area of the pyramid?

Review Queue Answers

1. $2(5\cdot 6)+2(5\cdot 7)+2(6\cdot 7)=214\,cm^2$
2. $2(15\cdot 18)+2(15\cdot 21)+2(18\cdot 21)=1926\,cm^2$
3. $2\cdot 25\pi+250\pi=300\pi\,in^2$
4. $36^2(2\pi)+72\pi(24)=4320\pi\,ft^2$



[Figure 21]