6.5 Trapezoids and Kites

FlexBooks® 2.0 > American HS Geometry > Trapezoids and Kites

Last Modified: Dec 25, 2014

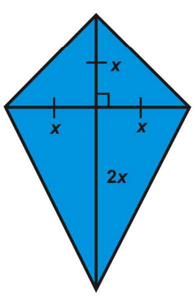
Learning Objectives

- Define and find the properties of trapezoids, isosceles trapezoids, and kites.
- Discover the properties of midsegments of trapezoids.
- Plot trapezoids, isosceles trapezoids, and kites in the coordinate plane.

Review Queue

1. Draw a quadrilateral with one set of parallel lines.
2. Draw a quadrilateral with one set of parallel lines and two right angles.
3. Draw a quadrilateral with one set of parallel lines and two congruent sides.
4. Draw a quadrilateral with one set of parallel lines and three congruent sides.
5. Draw a quadrilateral with two sets of congruent sides and no parallel sides.

Know What? A traditional kite, seen at the right, is made by placing two pieces of wood perpendicular to each other and one piece of wood is bisected by the other. The typical dimensions are included in the picture. If you have two pieces of wood, 36 inches and 54 inches, determine the values of x and $2x$. Then, determine how large a piece of canvas you would need to make the kite (find the perimeter of the kite).



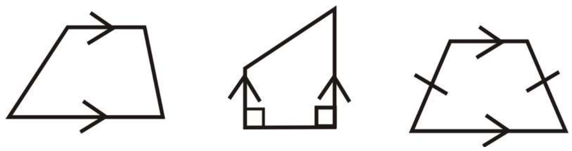
[Figure 1]

Trapezoids

Unlike parallelograms, trapezoids have only one set of parallel lines. The other two sides have no restrictions.

Trapezoid: A quadrilateral with exactly one pair of parallel sides.

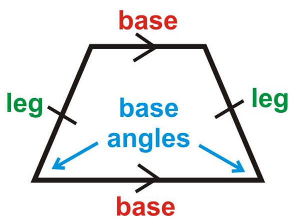
Examples look like:



[Figure 2]

Isosceles Trapezoid: A trapezoid where the non-parallel sides are congruent.

The third trapezoid above is an example of an isosceles trapezoid. Think of it as an isosceles triangle with the top cut off. Isosceles trapezoids also have parts that are labeled much like an isosceles triangle. Both parallel sides are called bases.



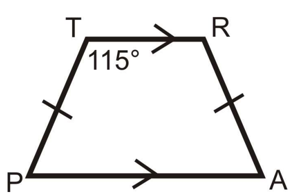
[Figure 3]

Isosceles Trapezoids

Previously, we introduced the Base Angles Theorem with isosceles triangles. The theorem states that in an isosceles triangle, the two base angles are congruent. This property holds true for isosceles trapezoids. ***The two angles along the same base in an isosceles trapezoid will also be congruent.*** This creates two pairs of congruent angles.

Theorem 6-17: The base angles of an isosceles trapezoid are congruent.

Example 1: Look at trapezoid $TRAP$ below. What is $m\angle A$?



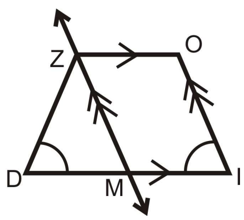
[Figure 4]

Solution: $TRAP$ is an isosceles trapezoid. So, $m\angle R = 115^\circ$, by Theorem 6-17. To find $m\angle A$, set up an equation.

$$\begin{aligned} 115^\circ + 115^\circ + m\angle A + m\angle P &= 360^\circ \\ 230^\circ + 2m\angle A &= 360^\circ \rightarrow m\angle A = m\angle P \\ 2m\angle A &= 130^\circ \\ m\angle A &= 65^\circ \end{aligned}$$

Notice that $m\angle R + m\angle A = 115^\circ + 65^\circ = 180^\circ$. These angles will always be supplementary because of the Consecutive Interior Angles Theorem from Chapter 3. Therefore, the two angles along the same leg (or non-parallel side) are always going to be supplementary. Only in isosceles trapezoids will opposite angles also be supplementary.

Example 2: Write a two-column proof.



[Figure 5]

Given: Trapezoid $ZOID$ and parallelogram $ZOIM$

$\angle D \cong \angle I$

Prove: $ZD \cong OI$

Solution:

Statement	***Reason***
1. Trapezoid $ZOID$ and parallelogram $ZOIM$, $\angle D \cong \angle I$	Given
2. $ZM \cong OI$	Opposite Sides Theorem
3. $\angle I \cong \angle ZMD$	Corresponding Angles Postulate
4. $\angle D \cong \angle ZMD$	Transitive PoC
5. $ZM \cong ZD$	Base Angles Converse
6. $ZD \cong OI$	Transitive PoC

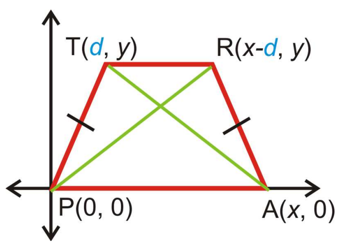
In this example we proved the converse of Theorem 6-17.

Theorem 6-17 Converse: If a trapezoid has congruent base angles, then it is an isosceles trapezoid.

Next, we will investigate the diagonals of an isosceles triangle. Recall, that the diagonals of a rectangle are congruent AND they bisect each other. The diagonals of an isosceles trapezoid are also congruent, but they do NOT bisect each other.

Isosceles Trapezoid Diagonals Theorem: The diagonals of an isosceles trapezoid are congruent.

Example 3: Show $TA = RP$.



[Figure 6]

Solution: This is an example of a coordinate proof. Here, we will use the distance formula to show that $TA = RP$, but with letters instead of numbers for the coordinates.

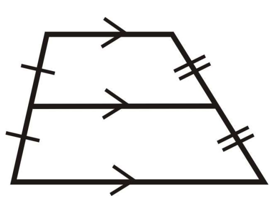
$$\begin{aligned} TA &= \sqrt{(x-d)^2+(0-y)^2} \\ &= \sqrt{(x-d)^2+(-y)^2} \\ &= \sqrt{(x-d)^2+y^2} \end{aligned} \qquad \begin{aligned} RP &= \sqrt{(x-d-0)^2+(y-0)^2} \\ &= \sqrt{(x-d)^2+y^2} \end{aligned}$$

Notice that we end up with the same thing for both diagonals. This means that the diagonals are equal and we have proved the theorem.

Midsegment of a Trapezoid

Midsegment (of a trapezoid): A line segment that connects the midpoints of the non-parallel sides.

There is only one midsegment in a trapezoid. It will be parallel to the bases because it is located halfway between them. Similar to the midsegment in a triangle, where it is half the length of the side it is parallel to, the midsegment of a trapezoid also has a link to the bases.

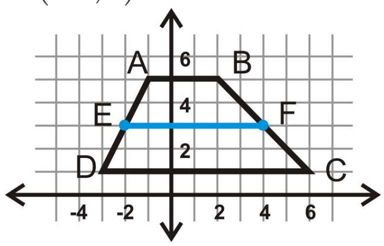


[Figure 7]

Investigation 6-5: Midsegment Property

Tools Needed: graph paper, pencil, ruler

Draw a trapezoid on your graph paper with vertices $A(-1,5), B(2,5), C(6,1)$ and $D(-3,1)$. Notice this is NOT an isosceles trapezoid.

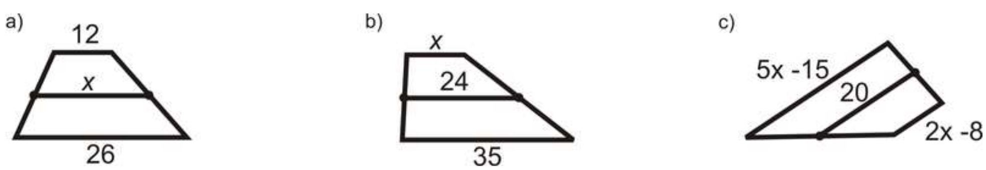


[Figure 8]

1. Find the midpoint of the non-parallel sides either by using slopes or the midpoint formula. Label them E and F. Connect the midpoints to create the midsegment.
2. Find the lengths of AB, EF, and CD. Can you write a formula to find the midsegment?

Midsegment Theorem: The length of the midsegment of a trapezoid is the average of the lengths of the bases, or $EF = \frac{AB+CD}{2}$.

Example 4: ***Algebra Connection*** Find x. All figures are trapezoids with the midsegment.



[Figure 9]

Solution:

a) x is the average of 12 and 26. $\frac{12+26}{2} = \frac{38}{2} = 19$

b) 24 is the average of x and 35.

$$\begin{aligned} \frac{x+35}{2} &= 24 \\ x+35 &= 48 \\ x &= 13 \end{aligned}$$

c) 20 is the average of $5x-15$ and $2x-8$.

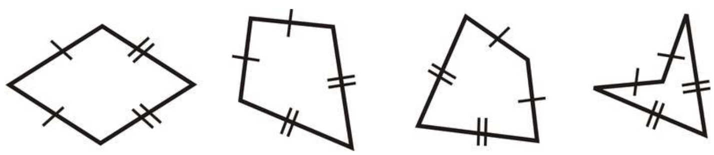
$$\frac{5x-15+2x-8}{2} = 20$$
$$7x-23 = 40$$
$$7x = 63$$
$$x = 9$$

Kites

The last quadrilateral we will study is a kite. Like you might think, it looks like a traditional kite that is flown in the air.

Kite: A quadrilateral with two sets of adjacent congruent sides.

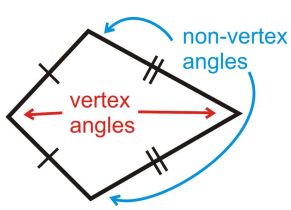
A few examples:



[Figure 10]

From the definition, a kite is the only quadrilateral that we have discussed that could be concave, as with the case of the last kite. If a kite is concave, it is called a ***dart***.

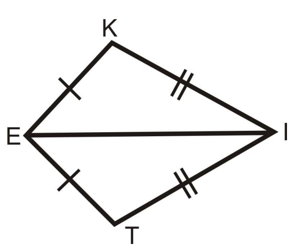
The angles between the congruent sides are called ***vertex angles***. The other angles are called ***non-vertex angles***. If we draw the diagonal through the vertex angles, we would have two congruent triangles.



[Figure 11]

Given: $KITE$ with $KE \cong TE$ and $KI \cong TI$

Prove: $\angle K \cong \angle T$



[Figure 12]

Statement	***Reason***
1. $KE \cong TE$ and $KI \cong TI$	Given
2. $EI \cong EI$	Reflexive PoC
3. $\triangle EKI \cong \triangle ETI$	SSS
4. $\angle K \cong \angle T$	CPCTC

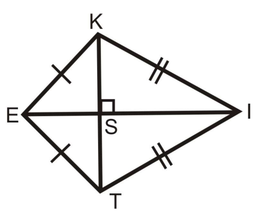
Theorem 6-21: The non-vertex angles of a kite are congruent.

Theorem 6-22: The diagonal through the vertex angles is the angle bisector for both angles.

The proof of Theorem 6-22 is very similar to the proof above for Theorem 6-21. If we draw in the other diagonal in $KITE$ we find that the two diagonals are perpendicular.

Kite Diagonals Theorem: The diagonals of a kite are perpendicular.

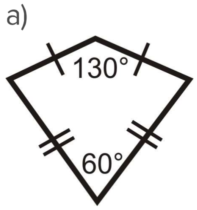
To prove that the diagonals are perpendicular, look at $\triangle KET$ and $\triangle KIT$. Both of these triangles are isosceles triangles, which means EI is the perpendicular bisector of KT (the Isosceles Triangle Theorem, Chapter 4). Use this information to help you prove the diagonals are perpendicular in the review questions.



[Figure 13]

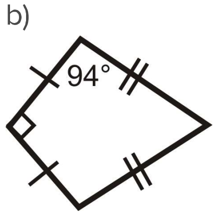
Example 5: Find the other two angle measures in the kites below.

a)



[Figure 14]

b)



[Figure 15]

Solution:

a) The two angles left are the non-vertex angles, which are congruent.

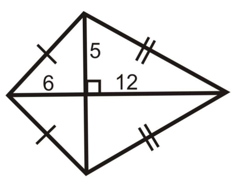
$$\begin{aligned} 130^\circ + 60^\circ + x + x &= 360^\circ \\ 2x &= 170^\circ \\ x &= 85^\circ \qquad \text{Both angles are } 85^\circ. \end{aligned}$$

b) The other non-vertex angle is also 94° . To find the fourth angle, subtract the other three angles from 360° .

$$\begin{aligned} 90^\circ + 94^\circ + 94^\circ + x &= 360^\circ \\ x &= 82^\circ \end{aligned}$$

Be careful with the definition of a kite. The congruent pairs are distinct. This means that ***a rhombus and square cannot be a kite***.

Example 6: Use the Pythagorean Theorem to find the length of the sides of the kite.



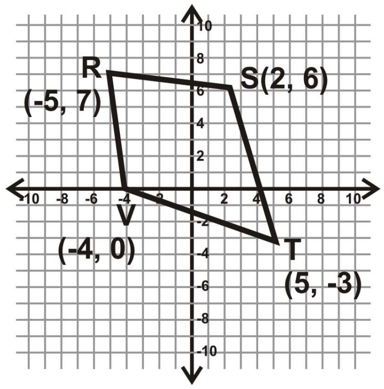
[Figure 16]

Solution: Recall that the Pythagorean Theorem is $a^2 + b^2 = c^2$, where c is the hypotenuse. In this kite, the sides are all hypotenuses.

$$\begin{aligned} 6^2 + 5^2 &= h^2 & 12^2 + 5^2 &= j^2 \\ 36 + 25 &= h^2 & 144 + 25 &= j^2 \\ 61 &= h^2 & 169 &= j^2 \\ \sqrt{61} &= h & 13 &= j \end{aligned}$$

Kites and Trapezoids in the Coordinate Plane

Example 7: Determine what type of quadrilateral $RSTV$ is. Simplify all radicals.



[Figure 17]

Solution: There are two directions you could take here. First, you could determine if the diagonals bisect each other. If they do, then it is a parallelogram and you could proceed like the previous section. Or, you could find the lengths of all the sides. Let's do this option.

$$\begin{aligned} RS &= \sqrt{(-5-2)^2+(7-6)^2} \\ &= \sqrt{(-7)^2+1^2} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(2-5)^2+(6-(-3))^2} \\ &= \sqrt{(-3)^2+9^2} \\ &= \sqrt{90} = 3\sqrt{10} \end{aligned}$$

$$\begin{aligned} RV &= \sqrt{(-5-(-4))^2+(7-0)^2} \\ &= \sqrt{(-1)^2+7^2} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned} \qquad \begin{aligned} VT &= \sqrt{(-4-5)^2+(0-(-3))^2} \\ &= \sqrt{(-9)^2+3^2} \\ &= \sqrt{90} = 3\sqrt{10} \end{aligned}$$

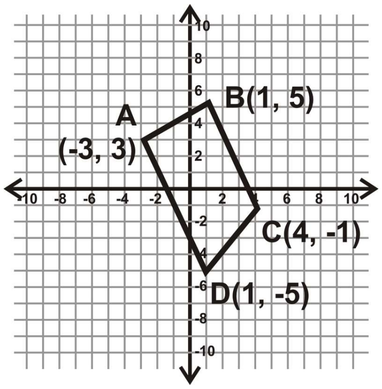
From this we see that the adjacent sides are congruent. Therefore, $RSTV$ is a kite.

Algebra Review: From now on, this text will ask you to "simplify the radical." From Algebra, this means that you pull all square numbers (1, 4, 9, 16, 25, ...) out of the radical. Above $\sqrt{50} = \sqrt{25 \cdot 2}$. We know $\sqrt{25} = 5$, so $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.

Hint: If you are only given a set of points when determining what type of quadrilateral a figure is, always plot the points and graph. The visual will help you decide which direction to go.

Example 8: Determine what type of quadrilateral $ABCD$ is. $A(-3,3),\ B(1,5),\ C(4,-1),\ D(1,-5)$. Simplify all radicals.

Solution: First, graph $ABCD$. This will make it easier to figure out what type of quadrilateral it is. From the graph, we can tell this is not a parallelogram. Find the slopes of BC and AD to see if they are parallel.



[Figure 18]

Slope of $BC = \frac{5-(-1)}{1-4} = \frac{6}{-3} = -2$

Slope of $AD = \frac{3-(-5)}{-3-1} = \frac{8}{-4} = -2$

We now know $BC \,||\, AD$. This is a trapezoid. To determine if it is an isosceles trapezoid, find AB and CD.

$$\begin{aligned} AB &= \sqrt{(-3-1)^2+(3-5)^2} \\ &= \sqrt{(-4)^2+(-2)^2} \\ &= \sqrt{20} = 2\sqrt{5} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(4-1)^2+(-1-(-5))^2} \\ &= \sqrt{3^2+4^2} \\ &= \sqrt{25} = 5 \end{aligned}$$

$AB \neq CD$, therefore this is only a trapezoid.

Example 9: Determine what type of quadrilateral $EFGH$ is.

$$E(5,-1),\ F(11,-3),\ G(5,-5),\ H(-1,-3)$$

Solution: To contrast with Example 8, we will not graph this example. Let's find the length of all four sides.

$$\begin{aligned} EF &= \sqrt{(5-11)^2+(-1-(-3))^2} \\ &= \sqrt{(-6)^2+2^2} \\ &= \sqrt{40} = 2\sqrt{10} \end{aligned} \qquad \begin{aligned} FG &= \sqrt{(11-5)^2+(-3-(-5))^2} \\ &= \sqrt{6^2+2^2} \\ &= \sqrt{40} = 2\sqrt{10} \end{aligned}$$

$$GH = \sqrt{(5-(-1))^2+(-5-(-3))^2} \qquad HE = \sqrt{(-1-5)^2+(-3-(-1))^2}$$
$$= \sqrt{6^2+(-2)^2} \qquad = \sqrt{(-6)^2+(-2)^2}$$
$$= \sqrt{40} = 2\sqrt{10} \qquad = \sqrt{40} = 2\sqrt{10}$$

All four sides are equal. That means, this quadrilateral is either a rhombus or a square. The difference between the two is that a square has four 90° angles and congruent diagonals. Let's find the length of the diagonals.

$$EG = \sqrt{(5-5)^2+(-1-(-5))^2} \qquad FH = \sqrt{(11-(-1))^2+(-3-(-3))^2}$$
$$= \sqrt{0^2+4^2} \qquad = \sqrt{12^2+0^2}$$
$$= \sqrt{16} = 4 \qquad = \sqrt{144} = 12$$

The diagonals are not congruent, so $EFGH$ is a rhombus.

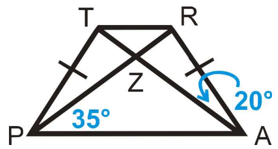
Know What? Revisited If the diagonals (pieces of wood) are 36 inches and 54 inches, x is half of 36, or 18 inches. Then, $2x$ is 36. To determine how large a piece of canvas to get, find the length of each side of the kite using the Pythagorean Theorem.

$$18^2+18^2 = s^2 \qquad 18^2+36^2 = t^2$$
$$324 = s^2 \qquad 1620 = t^2$$
$$18\sqrt{2} \approx 25.5 \approx s \qquad 18\sqrt{5} \approx 40.25 \approx t$$

The perimeter of the kite would be $25.5+25.5+40.25+40.25 = 131.5$ inches or 11 ft, 10.5 in.

Review Questions

$TRAP$ an isosceles trapezoid. Find:



[Figure 19]

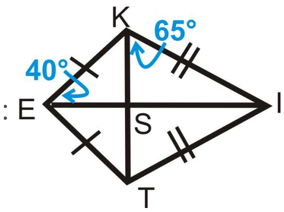
a. $m\angle TPA$

b. $m\angle PTR$

c. $m\angle ZRA$

d. $m\angle PZA$

$KITE$ is a kite. Find:



[Figure 20]

a. $m\angle ETS$

b. $m\angle KIT$

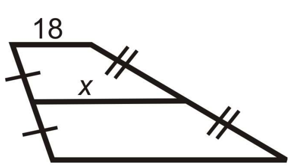
c. $m\angle IST$

d. $m\angle SIT$

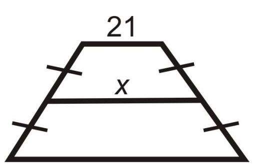
e. $m\angle ETI$

1. ***Writing*** Can the parallel sides of a trapezoid be congruent? Why or why not?
2. ***Writing*** Besides a kite and a rhombus, can you find another quadrilateral with perpendicular diagonals? Explain and draw a picture.
3. ***Writing*** Describe how you would draw or construct a kite.

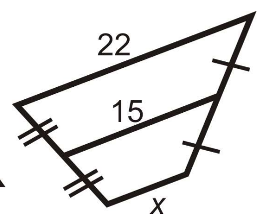
For questions 6-11, find the length of the midsegment or missing side.



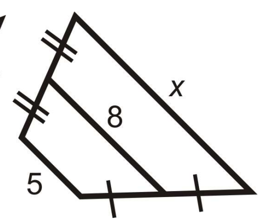
[Figure 21]



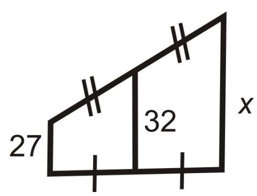
[Figure 22]



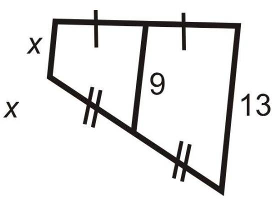
[Figure 23]



[Figure 24]

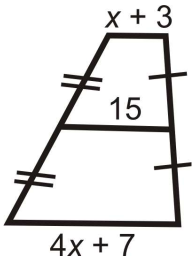


[Figure 25]

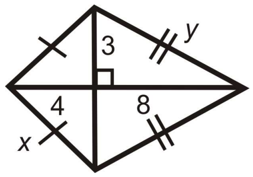


[Figure 26]

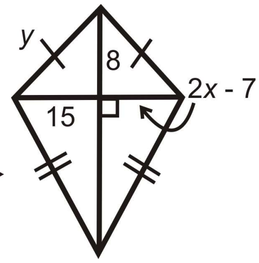
Algebra Connection For questions 12-17, find the value of the missing variable(s). Simplify all radicals.



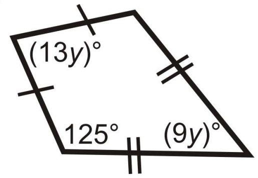
[Figure 27]



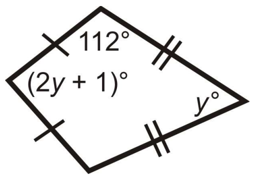
[Figure 28]



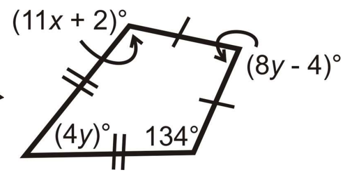
[Figure 29]



[Figure 30]



[Figure 31]

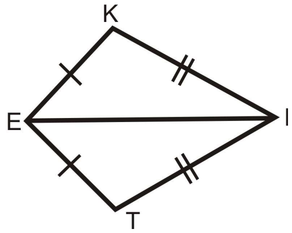


[Figure 32]

For questions 18-25, determine what type of quadrilateral $ABCD$ is. $ABCD$ could be any quadrilateral that we have learned in this chapter. If it is none of these, write *none*.

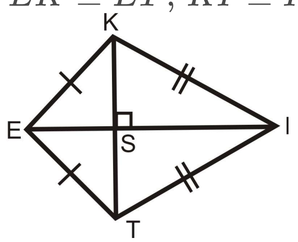
18. $A(1,-2),\ B(7,-5),\ C(4,-8),\ D(-2,-5)$
19. $A(6,6),\ B(10,8),\ C(12,4),\ D(8,2)$
20. $A(-1,8),\ B(1,4),\ C(-5,-4),\ D(-5,6)$
21. $A(5,-1),\ B(9,-4),\ C(6,-10),\ D(3,-5)$
22. $A(-2,2),\ B(0,1),\ C(2,2),\ D(1,5)$
23. $A(-7,4),\ B(-4,4),\ C(0,0),\ D(0,-3)$
24. $A(3,3),\ B(5,-1),\ C(7,0),\ D(5,4)$
25. $A(-4,4),\ B(-1,2),\ C(2,4),\ D(-1,6)$

Write a two-column proof of Theorem 6-22. Given: $KE \cong TE$ and $KI \cong TI$ Prove: EI is the angle bisector of $\angle KET$ and $\angle KIT$



[Figure 33]

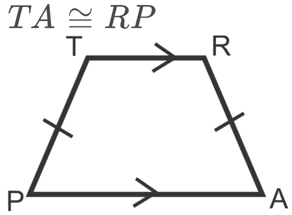
Write a two-column proof of the Kite Diagonal Theorem. Given: $EK \cong ET, KI \cong IT$ Prove: $KT \perp EI$



[Figure 34]

* Use the hint given earlier in this section.

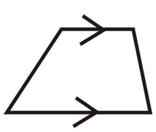
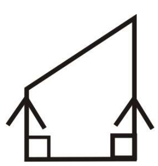
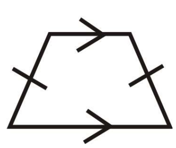
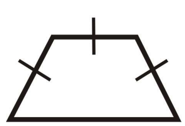
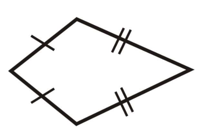
Write a two-column proof of the Isosceles Trapezoid Diagonals Theorem using congruent triangles. Given: $TRAP$ is an isosceles trapezoid with $TR \parallel AP$. Prove: $TA \cong RP$



[Figure 35]

26. Explain why the segments connecting the midpoints of the consecutive sides in a kite will always form a rectangle.
27. Explain why the segments connecting the midpoints of the consecutive sides in an isosceles trapezoid will always form a rhombus.

Review Queue Answers

[Figure 36] [Figure 37] [Figure 38] [Figure 39] [Figure 40]